Calc. Var. 18, 77–94 (2003)

Calculus of Variations

DOI: 10.1007/s00526-002-0181-x

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C^∞ regularity of the free boundary for a two–dimensional optimal compliance problem

Received: 5 March 2002 / Accepted: 3 September 2002 /
Published online: 17 December 2002 – © Springer-Verlag 2002

Abstract. We study the regularizing effect of perimeter penalties for a problem of optimal compliance in two dimensions. In particular, we consider minimizers of

$$\mathcal{E}(\Omega) = J(\Omega) + \lambda|\Omega| + \mu\mathcal{H}^1(\partial\Omega)$$

where

$$J(\Omega) = -2\inf\left\{\frac{1}{2}\int_\Omega \mathbf{A}e(u) : e(u) - \int_\Gamma f\cdot u \,:\, u \in LD(\Omega),\ u \equiv 0 \text{ on } D\right\}.$$

The sets $D \subset \Omega$, $\Gamma \subset \overline{\Omega}$, and the force f are given. We show that if we consider only scalar valued u and constant $\mathbf{A}$, or if we consider the elastic energy $|\nabla u|^2$, then $\partial\Omega$ is C^∞ away from where Ω is pinned. In the scalar case, we also show that, for any $\mathbf{A}$ of class $C^{k,\theta}$, $\partial\Omega$ is $C^{k+2,\theta}$. The proofs rely on a notion of weak outward curvature of $\partial\Omega$, which we can bound without considering properties of the minimizing fields, together with a bootstrap argument.

1 Introduction

A standard problem in optimal design is the so–called optimal compliance problem. The situation is usually the following: an object is fixed on part of its boundary and a force is exerted on another part of the boundary. One wants to design the object that best resists this force. One way to measure this stiffness is to compute the total work of the force at equilibrium, and try to optimize the shape of the object in order to minimize this work. This criterion is called the compliance. Of course, some other constraint has to be added, since usually the larger the object, the lower the compliance. One usually limits the quantity of material available, the idea being to build the best shape for a given weight. We refer to [3] for a monograph on optimal design.

In general, the optimal compliance problem has no solution. As was observed in [13], minimizing sequences of sets tend to *homogenize* and form complex mixtures of void and solid. (A numerical method taking into account this homogenization phenomenon is proposed in [1].) One way around this problem, studied

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for instance in [2, 12], consists in penalizing the perimeter of the unknown set. In this paper we study such an approach, in dimension two, and show the existence of smooth solutions. Let us mention the fact that the regularity (also in dimension two) for the problem introduced in [2] has already been established by the second author in [14].

Note that from an applications point of view, there is no loss of generality in analyzing the problem with a perimeter penalty, since in practice designs will have finite perimeter. The problem is then to find the optimal design subject to a constraint on perimeter, which is roughly equivalent to including a perimeter penalty. A numerical method based on this idea is detailed in [4], and gives satisfactory results.

For technical reasons we show smoothness only for the elastic energy density $|\nabla u|^2$, although we believe that the situation should not be very different for a class of linear elastic energy densities $\mathbf{A}e(u) : e(u)$, which is the subject of future study.

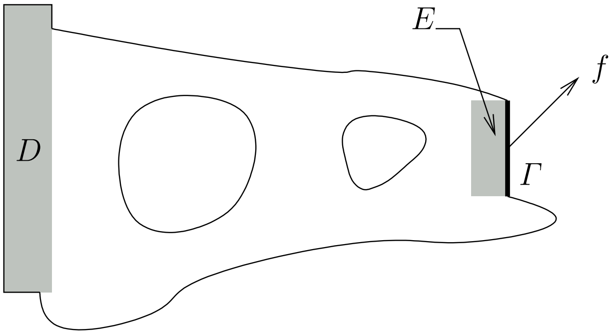

Fig. 1. The unknown domain Ω contains D and E

The setting is similar to the situation described in [8, Sect. 4.1], see Fig. 1. We fix two open subsets D and E of $\mathbb{R}^2$, bounded with Lipschitz-boundary and a finite number of components (throughout the whole paper the word “component” designates a “connected component”). D is the set where we impose a Dirichlet boundary condition on our displacement, whereas the boundary of E contains an arc (or a finite number of arcs) Γ on which the external force will be exerted. An admissible shape will be an open set $\Omega \supset D \cup E$, and an admissible displacement of the shape Ω is a $u \in L^2_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$ such that $e(u) := (\nabla u + \nabla u^T)/2 \in L^2(\Omega; \mathcal{S}^{2\times 2})$ and $u \equiv 0$ on D. Here $\mathcal{S}^{2\times 2}$ denotes the 3–dimensional vector space of symmetric matrices. Given such an Ω, the elastic equilibrium $\overline{u}$ of the solid is the minimizer of the energy

$$\frac{1}{2}\int_\Omega \mathbf{A}e(u) : e(u) \; - \int_\Gamma f \cdot u$$

where $\mathbf{A}$ is the *Hooke's law* of the linear elastic material and $f \in H^{-1/2}(\Gamma; \mathbb{R}^2)$ is the external force. If $\overline{u}$ exists, the compliance $J(\Omega)$ is the work of the force f under the displacement $\overline{u}$, that is, $J(\Omega) = \int_\Gamma f \cdot \overline{u}$. One checks easily that in this case, $J(\Omega) = -2\left(\frac{1}{2}\int_\Omega \mathbf{A}e(\overline{u}) : e(\overline{u}) - \int_\Gamma f\cdot\overline{u}\right)$, so that in general we define

$$J(\Omega) \,=\, -2\inf\left\{\frac{1}{2}\int_\Omega \mathbf{A}e(u) : e(u) \,-\, \int_\Gamma f\cdot u \,:\, u \in LD(\Omega),\ u \equiv 0 \text{ on } D\right\}. \tag{1}$$

Here we have introduced the space

$$LD(\Omega) \,=\, \left\{u \in L^2_{\text{loc}}(\Omega;\mathbb{R}^2) \,:\, e(u) \in L^2(\Omega;\mathcal{S}^{2\times 2})\right\};$$

we also point out that in the integrals over Γ, u has to be understood as the trace of $u_{|E}$ on Γ (this precision is meaningful when some part of Γ lies in the interior of the set $\overline{\Omega}$).

The global criterion we want to minimize is the sum of J and of two terms, one penalizing the total quantity of material used, which is proportional to $|\Omega|$ (the Lebesgue measure of the set Ω), and the other penalizing the perimeter of the design, proportional to $\mathcal{H}^1(\partial\Omega)$ (the Hausdorff one–dimensional measure, or length, of $\partial\Omega$). We introduce thus two positive parameters λ, μ and define

$$\mathcal{E}(\Omega) \,=\, J(\Omega) \,+\, \lambda|\Omega| \,+\, \mu\mathcal{H}^1(\partial\Omega).$$

For technical reasons, only the results in Sects. 2 and 3 will be valid for the compliance J as it has been just defined. The general result will hold only in the *scalar case*, *i.e.*, when J is defined by

$$J(\Omega) = -2\inf\left\{\frac{1}{2}\int_\Omega |\nabla u(x)|^2\,dx \,-\, \int_\Gamma f(x)u(x)\,d\mathcal{H}^1(x) \,:\, u \in L^{1,2}(\Omega),\right.$$
$$\left. u \equiv 0 \text{ on } D\right\}, \tag{2}$$

where $f \in H^{-1/2}(\Gamma)$, and the space $L^{1,2}(\Omega)$ is defined by

$$L^{1,2}(\Omega) \,=\, \left\{u \in L^2_{\text{loc}}(\Omega) \,:\, \nabla u \in L^2(\Omega;\mathbb{R}^2)\right\}.$$

We consider scalar–valued u and f, although all our results hold if these are vector–valued functions, as long as the internal energy is of the form $\int_\Omega |\nabla u|^2$.

In the next section we list some elementary properties of sets Ω with finite energy, and of the corresponding function u. In particular we show that in general the displacement field u exists, as well as a minimizer Ω for $\mathcal{E}$. We then state our main result, concerning the regularity of a minimizer Ω.

2 Preliminary remarks and main result

We begin by enumerating the following list of remarks:

1. If $\Omega \subset \Omega'$, $J(\Omega) \geq J(\Omega')$.
2. If $\mathcal{E}(\Omega) < +\infty$ and $\Omega \setminus \overline{D}$ has some components that do not intersect E, then if Ω' is the union of D and of the components intersecting E, $J(\Omega') = J(\Omega)$ and hence $\mathcal{E}(\Omega') < \mathcal{E}(\Omega)$. The set Ω' has at most K components, where K is the total number of components of $D \cup E$.

3. If $\mathcal{E}(\Omega) < +\infty$ and all the components of Ω touch $D \cup E$, then Ω lies inside the bounded set $\{\mathrm{dist}\,(x, D \cup E) \le \mathcal{E}(\Omega)/\mu\}$.
4. If $\mathcal{E}(\Omega) < +\infty$, then $\mathcal{E}(\overset{\circ}{\overline{\Omega}}) \le \mathcal{E}(\Omega)$.
5. Let $\mathcal{E}(\Omega) < +\infty$ and assume Ω has at most K components. Then Ω is bounded. We will show that if Ω has holes whose area is below a certain threshold, then filling in these holes reduces the energy. Let $(C_i)_{i=0}^N$ ($N \in \mathbb{N}$ or $N = +\infty$) be the components of $\overline{\Omega}^c = \mathbb{R}^2 \setminus \overline{\Omega}$. We assume $|C_i| \ge |C_{i+1}|$ for every $i < N$. Let $i_0 \ge 1$ be such that $i \ge i_0$ iff $|C_i| < 4\pi(\mu/\lambda)^2$. We let

$$\Omega' = \overset{\circ}{\overline{\Omega \cup \bigcup_{i=i_0}^N C_i}}.$$

Notice first that $\partial C_i \cap \partial C_j$ has at most $K-1$ points for every $i \ne j$. We have that $\overset{\circ}{\overline{\Omega'}} = \Omega'$, $\overline{\Omega'}^c = \bigcup_{i<i_0} C_i$, $\partial\Omega' = \bigcup_{i<i_0} \partial C_i$, whereas $\partial\Omega \supseteq \bigcup_{i=0}^N \partial C_i$. Hence

$$\mathcal{H}^1(\partial\Omega) \ge \mathcal{H}^1(\partial\Omega') + \sum_{i=i_0}^N \mathcal{H}^1(\partial C_i).$$

Also, $\Omega' \subset \Omega \cup \bigcup_{i=i_0}^N C_i \cup \partial\Omega$ and $|\partial\Omega| = 0$, hence $|\Omega'| = |\Omega| + \sum_{i=i_0}^N |C_i|$. Since $J(\Omega') \le J(\Omega)$ by remark 1 above, we get

$$\begin{aligned}\mathcal{E}(\Omega') &\le \lambda\left(|\Omega| + \textstyle\sum_{i=i_0}^N |C_i|\right) + \mu\left(\mathcal{H}(\partial\Omega) - \textstyle\sum_{i=i_0}^N \mathcal{H}^1(\partial C_i)\right) + J(\Omega)\\ &= \mathcal{E}(\Omega) + \textstyle\sum_{i=i_0}^N (\lambda|C_i| - \mu\mathcal{H}^1(\partial C_i)).\end{aligned}$$

By the isoperimetric inequality, $|C_i| \le \mathcal{H}^1(\partial C_i)^2/4\pi$ for every i, hence

$$\lambda|C_i| - \mu\mathcal{H}^1(\partial C_i) \le \lambda|C_i| - 2\mu\sqrt{\pi}\sqrt{|C_i|} = \lambda\sqrt{|C_i|}(\sqrt{|C_i|} - 2\sqrt{\pi}\mu/\lambda),$$

which is negative if $i \ge i_0$, so that $\mathcal{E}(\Omega') < \mathcal{E}(\Omega)$ unless $|C_i| \ge 4\pi(\mu/\lambda)^2$ for every i.
6. We claim that $J(\Omega) < +\infty$ if and only if each component A of Ω such that $A \cap E \ne \emptyset$ satisfies either $A \cap D \ne \emptyset$, or $\int_{\Gamma\cap\overline{A}} f = 0$, and it is equivalent to the existence of a function u realizing the inf in (2) (or in (1) in the vectorial case). Indeed, assume $A \cap D = \emptyset$ and $\alpha = \int_{\Gamma\cap\overline{A}} f \ne 0$. Then, choosing in problem (2) the function equal to zero in $\Omega' \setminus A$ and to $t\alpha$, $t > 0$, on A, we find that $J(\Omega') \ge t\alpha^2$. Sending t to infinity we get $J(\Omega') = +\infty$.
Now for each component A with $A \cap E \ne \emptyset$, we can find a path in A connecting all components of D and E intersecting A. There exists a (smooth) neighborhood of this path that is strictly inside A and such that the union G of this neighborhood and of $A \cap E$ has a Lipschitz boundary, that contains $\Gamma \cap \overline{A}$. If $D \cap A \ne \emptyset$ we have a Poincaré inequality on G: $\int_G |v|^2 \le c\int_G |\nabla v|^2$ for every $v \in H^1(G)$ with $v \equiv 0$ on $G \cap D$, so that $\int_{\Gamma\cap\overline{A}} fv \le c\|\nabla v\|_{L^2(G)} \le c\|\nabla v\|_{L^2(A)}$. If $D \cap A = \emptyset$ and $\int_{\Gamma\cap\overline{A}} f = 0$, from the Poincaré inequality $\int_G |v - ⨍_G v|^2 \le c\int_G |\nabla v|^2$ (here $⨍_G v$ is the average of v on G) we also get $\int_{\Gamma\cap\overline{A}} fv \le c\|\nabla v\|_{L^2(G)} \le c\|\nabla v\|_{L^2(A)}$.

Repeating this argument in every component of Ω intersecting E, we find that $\int_\Gamma fv \le c\|\nabla v\|_{L^2(\Omega)}$ for every $v \in L^{1,2}(\Omega)$. This is enough to get the compactness of minimizing sequences for problem (2) and hence the existence of a limit u minimizing the energy. The proof for the vectorial case is not different. We refer to the proof of [8, Lemma 2] for more details.

If we consider now a minimizing sequence $(\Omega_n)_{n\ge1}$ for $\mathcal{E}$, we can assume without loss of generality that the Ω_n are all contained in some ball B_R, and that for all n, Ω_n has at most K components, $\Omega_n = \overset{\circ}{\overline{\Omega_n}}$, and each component of $\overline{\Omega}_n^c$ has measure at least $4\pi(\mu/\lambda)^2$. In particular, for every n, $\overline{\Omega}_n^c$ has at most $|B_R|/(4\pi(\mu/\lambda)^2)$ components, and Ω_n^c has the same or fewer. We deduce that we can reproduce the proof in [8, Thm 2] and show that (up to a subsequence), the complements Ω_n^c converge in the Hausdorff distance to the complement Ω^c of a set Ω that minimizes the energy $\mathcal{E}$. The semicontinuity of the term $\mathcal{H}^1(\partial\Omega_n)$ is a consequence of Gołąb's theorem [11], which holds because the number of components of the boundaries $\partial\Omega_n$ is uniformly bounded. By remark 6 above, we also get the existence of a minimizer u for problem (2) (scalar case) or (1) (vectorial case).

From remark 4, we will assume that $\Omega = \overset{\circ}{\overline{\Omega}}$. Notice that if we introduce as in remark 5 the components $C_1, \dots, C_N$ of Ω^c (that are finite in number), by remarks 2 and 5 we have the properties that $\partial C_i \cap \partial C_j$ has at most $K-1$ points for every i, j and that $\partial\Omega = \bigcup_{i=1}^N \partial C_i$. Therefore

$$\mathcal{H}^1(\partial\Omega) = \sum_{i=1}^N \mathcal{H}^1(\partial C_i) \tag{3}$$

and a similar additivity property holds for $\overset{\circ}{\overline{\Omega \cup C_j}}$, for all j.

The following result holds for the scalar case only, which we consider in the sequel. Notice though that the lemmas in Sect. 3 hold in both the scalar and vectorial cases.

Theorem 1 *$\partial\Omega \setminus \overline{D \cup E}$ is C^∞.*

Remark. In general, $\partial\Omega \setminus \overline{D \cup E}$ is a *finite* number of smooth curves. Notice however that one may imagine a set D or E with a pathological boundary so that this is not true anymore (for instance, if the boundary of D is locally like the graph of an oscillating function such as $t \mapsto \int_0^t \text{sign } \sin(1/s)\, ds$).

In the next section we first show a basic regularity estimate on the boundary of Ω. Up to some technical details, it states that each point of $\partial\Omega$ belongs to the boundary of a disc exterior to Ω. Since the radius of this disc is uniformly estimated (as a function of μ/λ), this *external sphere condition* can be seen as a weak curvature bound on the free boundary. This regularity yields regularity of the field u itself, that is shown in Sect. 4 to be locally Lipschitz up to the boundary. Then, the C^∞–regularity is shown through a bootstrap argument. In the last section we explain how the proofs need to be adapted in order to treat the case of a nonuniform energy (where the principal term $\int_\Omega |\nabla u(x)|^2\, dx$ is replaced with $\int_\Omega (\mathbf{A}(x)\nabla u(x))\cdot\nabla u(x)\, dx$).

3 An external sphere condition and its consequences

Lemma 1 *There exist two radii r_0 and ρ_0 such that for every $x \in \partial\Omega$ and every ball $B(x,r)$ with $r \le r_0$, if C is the component of $\overline{\Omega} \cap \overline{B}(x,r)$ containing x, then every ball $B(y,\rho) \subset C^c$ with $\rho \le \rho_0$ meets ∂C no more than once.*

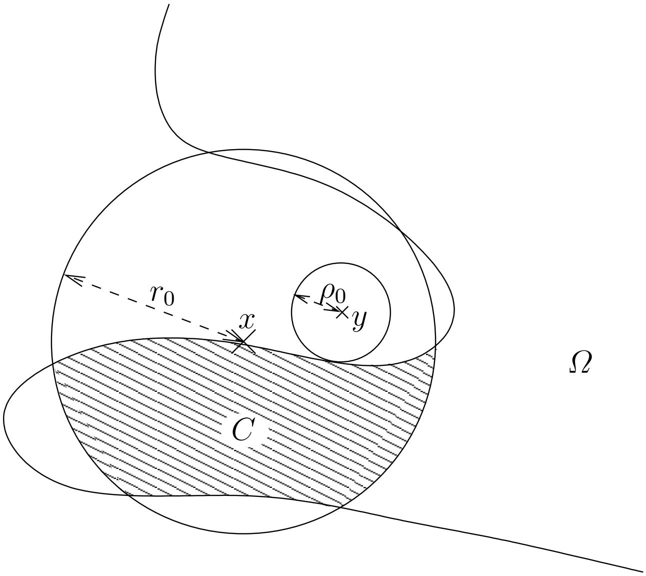

Fig. 2. By Lemma 1, the small ball touches C at most in one point

Proof. Fix $r > 0$ and consider $x \in \partial\Omega$ and the component C of $\overline{\Omega} \cap \overline{B}(x,r)$ containing x. Suppose there exists $B(y,\rho) \subset C^c$ such that $\partial B(y,\rho) \cap \partial C$ has at least two points a and b. We consider the chord $[a,b]$ of length l. We claim that $C^c \setminus [a,b]$ has at least one bounded component that intersects the ball $B(y,\rho)$. Otherwise, choose two points inside $B(y,\rho)$ on either side of the chord, which are both in the unbounded component of $C^c \setminus [a,b]$. These points must be connected by a path in $C^c \setminus [a,b]$. The union of this path with the segment joining them is a Jordan curve in C^c dividing the plane into two components, one containing a and the other b, which is impossible since C is connected.

Let A be a bounded component of $C^c \setminus [a,b]$ that intersects $B(y,\rho)$ (see Fig. 3). Notice that its boundary contains the chord $[a,b]$. Let $L = \mathcal{H}^1(\partial A) - l$. We will consider adding the set A to Ω, forming $\Omega' = \overset{\circ}{\overline{\Omega \cup A}}$, and comparing the energies of Ω and Ω' we will deduce from the minimality of Ω lower bounds for r and ρ.

By equation (3), replacing Ω with Ω' reduces the perimeter by exactly $L - \mathcal{H}^1([a,b] \setminus \overline{\Omega})$, so that the reduction is at least $L - l$, while it increases the area by $|A \setminus \Omega| \le |A|$. It also reduces the compliance due to remark 1. Hence, by

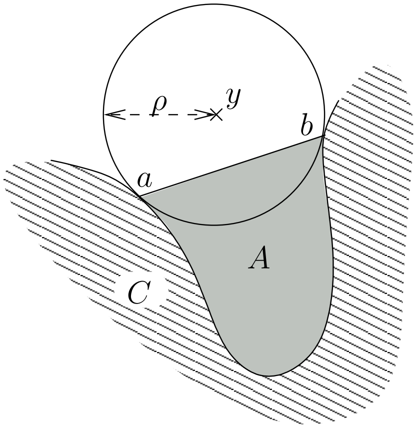

Fig. 3. Merging the area A with C must increase the energy

minimality of Ω it must be that

$$\mu(L - l) \le \lambda |A|. \tag{4}$$

Introducing $\theta = L/l > 1$, we can rewrite (4) as $l \le (\lambda/\mu)|A|/(\theta - 1)$. The isoperimetric inequality yields $|A| \le (L + l)^2/4\pi = (\theta + 1)^2 l^2/4\pi$, hence

$$|A| \le \frac{\lambda^2}{4\pi\mu^2}\left(\frac{\theta+1}{\theta-1}\right)^2 |A|^2.$$

Since $|A| \le \pi r^2$, we find that

$$r \ge \frac{2\mu}{\lambda}\left(\frac{\theta-1}{\theta+1}\right). \tag{5}$$

If r is small enough, we see that θ must be close to 1.

Now, if θ is close to 1, we introduce the arc of circle of length $L = \theta l$ and with extremities a and b, which lies outside of $B(y, \rho)$. The area $\mathcal{A}$ enclosed by this arc of circle and the chord $[a, b]$ is the largest area bounded by the chord and a piece of curve of length L, hence $|A| \le \mathcal{A}$. This arc lies on a circle $\partial B(z, \sigma)$ and defines an angle of measure 2α, i.e., $L = l\theta = 2\sigma\alpha$. If θ is small enough (in fact, $\theta < \pi/2$), then $\sigma < \rho$ since z must lie somewhere between y and the middle of $[a, b]$, and $\alpha < \pi/2$. Since $l = 2\sigma \sin\alpha$, we have $(\sin\alpha)/\alpha = 1/\theta$. On the other hand, $\mathcal{A} = \alpha\sigma^2 - \sigma^2 \cos\alpha \sin\alpha = (\sigma^2/2)(2\alpha - \sin 2\alpha)$, and we deduce from (4)

$$2\mu\sigma(\alpha - \sin\alpha) \le \frac{\lambda\sigma^2}{2}(2\alpha - \sin 2\alpha),$$

hence

$$\rho > \sigma \ge \frac{4\mu}{\lambda}\left(\frac{\alpha - \sin\alpha}{2\alpha - \sin 2\alpha}\right).$$

This quantity approaches its infimum, $\mu/2\lambda$, as $\alpha \to 0$.

Fix $r_0 < (2\mu/\lambda)(\pi - 2)/(\pi + 2)$, so that if $r \leq r_0$, (5) yields that $\theta < \pi/2$. If now we fix $\rho_0 < \mu/2\lambda$, we get that $\rho > \rho_0$. Hence if $\rho \leq \rho_0$, any ball $B(y,\rho) \subset C^c$ can meet ∂C no more than once, and the lemma is proven. $\square$

Lemma 2 *Let $r_0 < 2\mu/\lambda$ and ρ_0 be given by Lemma 1, $x \in \partial\Omega$, $r \leq r_0$ and C the component of $\overline{\Omega} \cap \overline{B}(x,r)$ containing x. Then at each $y \in \partial C$ there exists a ball $B(z,\rho_0) \subset C^c$ such that $\partial B(z,\rho_0) \cap \partial C = \{y\}$.*

Proof. Notice that $r_0 < 2\mu/\lambda$ implies that C is simply connected, otherwise C^c would contain components of $\overline{\Omega}^c$ of measure less than $4\pi(\mu/\lambda)^2$, which is ruled out by remark 5.

We claim that C^c is exactly the union U of all balls with radius ρ_0 that are contained in C^c. Otherwise, let $x \in \partial U \setminus \partial C$. If we choose a sequence $x_n \to x$, $x_n \in U$, then there exists for every n a ball $B(y_n,\rho_0) \subset C^c$ containing x_n. The limit of a subsequence of these balls, in the Hausdorff metric, is a ball $B(y,\rho_0) \subset C^c$ with x at its boundary. $\partial B(y,\rho_0)$ must intersect ∂C exactly once, since if the intersection were empty we could translate the ball slightly towards x while still not intersecting C, which would imply $x \in U$.

Let $\{z\} = \partial B(y,\rho_0) \cap \partial C$. It must be that $(z-y)\cdot(x-y) \geq 0$, otherwise we can translate the ball as above, implying $x \in U$. If we rotate the ball around x, slightly away from z, we get a new ball that does not touch the boundary of C but still has x on its boundary. Translating it towards x as above gives $x \in U$, a contradiction. Hence $\partial U \subseteq \partial C$.

Now if $x \in C^c \setminus U$, there exists a continuous path in C^c (which is connected) connecting x to some point in U, and this path must contain a boundary point in $\partial U \cap C^c$, which is not in the boundary of C. We just showed that this is impossible, so that $U = C^c$ as claimed.

We deduce that $\partial U = \partial C$, and this shows the lemma. $\square$

Lemma 3 *Let $x_0 \in \partial\Omega$. Then there exists an orthonormal basis η,ν of $\mathbb{R}^2$, and a rectangle $Q = \{x_0 + t\eta + s\nu \,:\, -\alpha < t < \alpha, -\beta < s < \beta\}$, $\alpha,\beta > 0$, such that $\Omega \cap Q$ has one of the two following representations:*

- *There exists a Lipschitz function $h : (-\alpha,\alpha) \to (-\beta,\beta)$ such that $\Omega \cap Q = \{x_0 + t\eta + s\nu \,:\, -\alpha < t < \alpha, -\beta < s < h(t)\}$; moreover, we may assume that h has at 0 opposite left and right derivatives, respectively p and $-p$, with $p \geq 0$.*
- *There exist two Lipschitz functions $h_1, h_2 : (-\alpha,\alpha) \to (-\beta,\beta)$, with $h_i(0) = h_i'(0) = 0$, $i = 1,2$, and $h_1(t) < h_2(t)$ if $t \neq 0$, such that $\Omega \cap Q$ is either $\{x_0 + t\eta + s\nu \,:\, -\alpha < t < \alpha, h_1(t) < s < h_2(t)\}$ or $\{x_0 + t\eta + s\nu \,:\, 0 < t < \alpha, h_1(t) < s < h_2(t)\}$.*

The functions h or h_1, h_2 have left and right derivatives at every point, that are respectively left and right continuous.

Proof. Choose $x_0 \in \partial\Omega$, $r \leq r_0$, and let C be the component of $\overline{\Omega} \cap B(x_0,r)$ containing x_0, as in Lemmas 1 and 2. For every $y \in \partial C$ define the set of normal vectors $n(y) = \{\nu \in \mathbb{S}^1 : B(y+\rho_0\nu, \rho_0) \subset C^c\}$, which is not empty by Lemma 2. We claim that either $n(y) = \{-\nu, \nu\}$ for a given $\nu \in \mathbb{S}^1$, or $n(y)$ is connected and has measure less than π. Indeed, if $n(y)$ contains two vectors ν_1 and ν_2 with $\nu_1 + \nu_2 \neq 0$, for any vector ν between ν_1 and ν_2 (i.e., on the shortest path in $\mathbb{S}^1$ between these two points), there exists $\rho > 0$ such that

$$B(y+\rho\nu, \rho) \subset B(y+\rho_0\nu_1, \rho_0) \cup B(y+\rho_0\nu_2, \rho_0) \subset C^c.$$

But then also $B(y+\rho_0\nu, \rho_0)$ (otherwise there would exist a radius $\rho' \in [\rho, \rho_0]$ such that $B(y+\rho'\nu, \rho')$ meets twice ∂C), and $\nu \in n(y)$. If $|n(y)| > \pi$, the same argument shows that $n(y) = \mathbb{S}^1$, which is impossible since it would yield $C = \{y\}$.

Consider now $y \in \partial C$ and a sequence $(y_n)_{n\geq 1} \subset \partial C$, $y_n \to y$. If $\nu_n \in n(y_n)$, we can assume (possibly extracting a subsequence) that ν_n converges to some $\nu \in \mathbb{S}^1$ as $n \to \infty$. The balls $\overline{B}(y_n+\rho_0\nu_n, \rho_0) \subset \overline{C^c}$ converge in the Hausdorff distance to the ball $\overline{B}(y+\rho_0\nu, \rho_0)$, hence this ball is also in $\overline{C^c}$ and $B(y+\rho_0\nu, \rho_0) \subset C^c$. This shows that $\nu \in n(y)$, in particular $n(y)$ is closed.

Additionally, we can show that if $n(y)$ is the arc $[\nu_1, \nu_2]$ on $\mathbb{S}^1$, and $y_n \neq y$ for every n, then the limit ν can only be either ν_1 or ν_2. Indeed, if we let $\rho_n = 2|y - y_n|$ and define in $B(0,1)$ $\Omega_n = \{z \in B(0,1) : y + \rho_n z \in \Omega\}$, $z_n = (y_n - y)/\rho_n$, then up to a subsequence $\overline{\Omega}_n$ converges as $n \to \infty$ to some closed set $A \ni 0$ (in the Hausdorff distance), z_n converges to some $z_0 \in A$ with $|z_0| = 1/2$, and it is easy to check that A must be contained simultaneously in $\{z\cdot\nu_1 \leq 0\}$, $\{z\cdot\nu_2 \leq 0\}$, and $\{(z - z_0)\cdot\nu \leq 0\}$. Hence $z_0\cdot\nu_1 \leq 0$, $z_0\cdot\nu_2 \leq 0$, and $-z_0\cdot\nu \leq 0$, which is impossible unless $\nu \in \{\nu_1, \nu_2\}$ since ν has to be between ν_1 and ν_2 in $\mathbb{S}^1$.

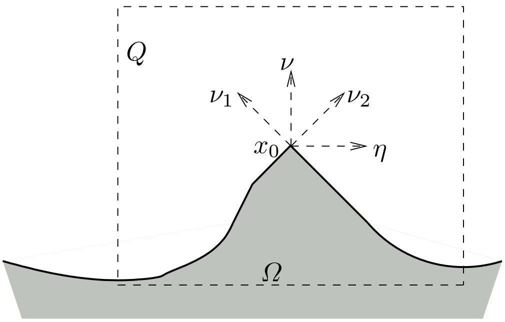

Fig. 4. In the first case, $\Omega \cap Q$ is a Lipschitz subgraph

Assume first we are in the situation where $n(x_0) = [\nu_1, \nu_2]$ (possibly $\nu_1 = \nu_2$) and $|n(x_0)| < \pi$ (see Fig. 4). Let $\nu = (\nu_1 + \nu_2)/\|\nu_1 + \nu_2\|$ and $\delta \in (0, \nu\cdot\nu_1)$. By the continuity property stated above, there exists $r' > 0$, $r' \leq r$, such that if $y \in B(x_0, r')$ and $\xi \in n(y)$, then $\xi\cdot\nu > \delta$. If $y \in \partial C \cap B(x_0, r')$ and $\xi \in n(y)$, then $\xi\cdot\nu > \delta$ and $x_0 \notin \overline{B}(y+\rho_0\xi, \rho_0)$, hence $y \notin \overline{B}(x_0 - \rho_0\xi, \rho_0)$. In particular, we have that

$$\partial C \cap B(x_0, r') \subset \left(\bigcap_{\substack{\xi \in n(y) \\ y \in \partial C \cap B(x_0, r')}} \overline{B}(x_0 - \rho_0 \xi, \rho_0) \right)^c$$

(which excludes at least a small triangle below x_0). On the other hand, $B(x_0 + \rho_0 \nu, \rho_0) \subset C^c$. Hence, choosing $\eta \perp \nu$ (for instance such that $\eta{\cdot}\nu_1 \le 0$), there exist $\alpha, \beta > 0$ such that $Q = \{x_0 + t\eta + s\nu \,:\, -\alpha < t < \alpha, -\beta < s < \beta\} \subset B(x_0, r')$, $\{x_0 + t\eta + \beta\nu \,:\, -\alpha < t < \alpha\} \subset C^c$ and $\{x_0 + t\eta - \beta\nu \,:\, -\alpha < t < \alpha\} \subset \overset{\circ}{C}$.

If now $t \in (-\alpha, \alpha)$, the intersection $\partial C \cap (x_0 + t\eta + (-\beta, \beta)\nu)$, which is not empty, contains exactly one point, otherwise for at least one intersection point y there would exist $\xi \in n(y)$ with $\xi{\cdot}\nu \le 0$. Defining $h(t)$ by $x_0 + t\eta + h(t)\nu \in \partial C$, we see that $\overset{\circ}{C} \cap Q$ is the subgraph of h in Q. We now show that h is Lipschitz.

Given $y = x_0 + t\eta + h(t)\nu \in \partial C$ and $\xi \in n(y)$, the sphere $\partial B(y + \rho_0 \xi, \rho_0)$ has equation (near y) $x_0 + t'\eta + \psi(t')\nu$ with

$$\psi(t') \;=\; h(t) + \rho_0(\xi{\cdot}\nu) \left(1 - \sqrt{1 + 2\frac{\xi{\cdot}\eta}{\rho_0(\xi{\cdot}\nu)^2}(t' - t) - \frac{(t' - t)^2}{\rho_0^2(\xi{\cdot}\nu)^2}} \right),$$

and we know that near t we must have $h(t') \le \psi(t')$. The function $\psi(t')$ is C^∞ near t and its first and second derivatives at $t' = t$ are

$$\psi'(t) \;=\; -\frac{\xi{\cdot}\eta}{\xi{\cdot}\nu} \;\text{ and }\; \psi''(t) \;=\; \frac{1}{\rho_0\, \xi{\cdot}\nu}\left(1 + \left(\frac{\xi{\cdot}\eta}{\xi{\cdot}\nu}\right)^2\right) \;=\; \frac{1}{\rho_0(\xi{\cdot}\nu)^3} \;\le\; \frac{1}{\rho_0\delta^3}.$$

This shows that for every $\varepsilon > 0$, there exists a neighborhood of t on which the function $\psi(t') - (1+\varepsilon)/(\rho_0\delta^3)\,({t'}^2/2)$ is concave, with

$$\psi(t') - \left(\frac{1+\varepsilon}{\rho_0\delta^3}\right)\frac{{t'}^2}{2} \;\le\; \psi(t) - \left(\frac{1+\varepsilon}{\rho_0\delta^3}\right)\frac{t^2}{2} \;-\; \left(\frac{\xi{\cdot}\eta}{\xi{\cdot}\nu} + \frac{1+\varepsilon}{\rho_0\delta^3}t\right)(t' - t).$$

Since $h(t') \le \psi(t')$ and $h(t) = \psi(t)$, the last inequality also holds with replacing ψ with h. We deduce that the function $h(t) - (1+\varepsilon)/(\rho_0\delta^3)\,(t^2/2)$ is concave on $(-\alpha, \alpha)$ for every $\varepsilon > 0$ (hence also for $\varepsilon = 0$), so that h is (at least locally) Lipschitz, has a left and right derivative at every $t \in (-\alpha, \alpha)$, with continuity (from the left and the right) of this derivative. We recall that the superdifferential $\partial^+ h(t)$ of h at t is the set of $p \in \mathbb{R}$ such that $h(t') - h(t) \le p(t' - t) + o(t' - t)$ as $t' \to t$: the equation above shows that for every $t \in (-\alpha, \alpha)$, $\partial^+ h(t) \subseteq \{-(\xi{\cdot}\eta)/(\xi{\cdot}\nu) \,:\, \xi \in n(x_0 + t\nu + h(t)\eta)\} \ne \emptyset$, and the reverse inclusion is easy to establish (for instance using the left and right continuity of the derivatives, since we also have $\partial^+ h(t) = [h'(t+0), h'(t-0)]$). In particular, $|h'(t)| \le \sqrt{1 - \delta^2}/\delta$ on $(-\alpha, \alpha)$. Setting $p = -(\nu_1{\cdot}\eta)/(\nu_1{\cdot}\nu) \ge 0$, we also have $\partial^+ h(0) = [-p, p]$, and the left and right derivatives of h at 0 must be respectively p and $-p$.

Eventually, notice that if α and β were chosen small enough, $Q \cap \overset{\circ}{C} = Q \cap \Omega$, hence Ω is defined as the subgraph of h near x_0. This shows the first part of Lemma 3.

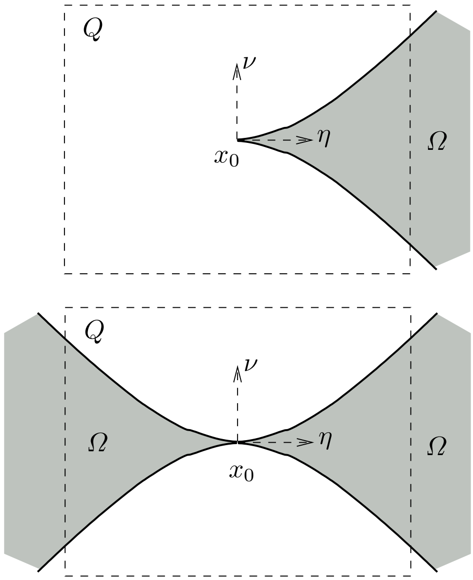

Fig. 5. In the other cases (cusp and flat point), $\Omega \cap Q$ lies between two Lipschitz graphs

The second situation is when there exists $\nu \in n(x_0)$ such that also $-\nu \in n(x_0)$. Hence either $n(x_0) = \{-\nu, \nu\}$, or $n(x_0)$ is a half circle delimited by ν and $-\nu$ (see Fig. 5). The set C is in the complement of $B(x_0 + \rho_0\nu, \rho_0) \cup B(x_0 - \rho_0\nu, \rho_0)$. Consider r' small enough so that for every $y \neq x_0$ in $B(x_0, r') \cap \partial C$ and $\xi \in n(y)$, we have $|\xi \cdot \nu| \geq 1/2$. We assume $r' \leq \rho_0$, so that $B(x_0, r') \setminus (\overline{B}(x_0 + \rho_0\nu, \rho_0) \cup \overline{B}(x_0 - \rho_0\nu, \rho_0))$ has two components. If $n(x_0)$ is a half-circle, then $C \cap B(x_0, r')$ lies in one of these two components. First assume we are in this case. We choose $\eta \perp \nu$ pointing towards C. We assume that C meets the boundary of $B(x_0, r')$, otherwise choose a smaller r'. Let $\alpha > 0$ and $\beta > 0$ be such that $Q = \{x_0 + t\eta + s\nu \,:\, -\alpha < t < \alpha, -\beta < s < \beta\} \subset B(x_0, r')$, $\{x_0 + t\eta + \beta\nu \,:\, -\alpha < t < \alpha\} \subset B(x_0 + \rho_0\nu, \rho_0)$ and $\{x_0 + t\eta - \beta\nu \,:\, -\alpha < t < \alpha\} \subset B(x_0 - \rho_0\nu, \rho_0)$.

If $t \in (0, \alpha)$ and s is such that $y = x_0 + t\eta + s\nu \in \partial C$, then if $\xi \in n(y)$, either $\xi \cdot \nu \geq 1/2$ or $\xi \cdot \nu \leq -1/2$. In the first case, $B(y + \rho_0\xi, \rho_0) \subset C^c$ shows that for every $s' > s$, $x_0 + t\eta + s'\nu \in C^c$ (at least if r' was chosen small enough), whereas in the second case, for every $s' < s$, $x_0 + t\eta + s'\nu \in C^c$. This shows that there exist exactly two values $h_1(t) < h_2(t)$ such that $y_i = x_0 + t\eta + h_i(t)\nu \in \partial C$, $i = 1, 2$. If $\xi \in n(y_1)$, we have $\xi \cdot \nu < -1/2$, while if $\xi \in n(y_2)$, we have $\xi \cdot \nu > 1/2$. We then have $\mathring{C} \cap Q = \{x_0 + t\eta + s\nu \,:\, 0 < t < \alpha, h_1(t) < s < h_2(t)\}$. The proof that h_1, h_2 are Lipschitz and that $h_1'(0) = h_2'(0) = 0$ are similar to the demonstrations in the first situation.

If C lies in both components of $B(x_0, r') \setminus (\overline{B}(x_0 + \rho_0\nu, \rho_0) \cup \overline{B}(x_0 - \rho_0\nu, \rho_0))$ (that is, if $n(x_0) = \{-\nu, \nu\}$), we just reproduce the previous proof in each component and then glue together the two functions h_1 and the two functions h_2 thus obtained on both sides of x_0.

Eventually, observe that if α and β were chosen small enough, we have again that $Q \cap \overset{\circ}{C} = Q \cap \Omega$, so that the description we have found of C near x_0 is in fact a complete description of Ω. Lemma 3 is proven. □

4 An estimate for $|\nabla u|$

Let us first show that we can associate to u a conjugate function v satisfying a Dirichlet boundary condition on $\partial\Omega$.

Lemma 4 *In each situation of Lemma 3, if $Q \cap \overline{D \cup E} = \emptyset$, then the function u has a conjugate v satisfying $\Delta v = 0$, and such that $v \in H^1(\Omega \cap Q) \cap C^0(\overline{\Omega} \cap Q) \cap C^\infty(\Omega \cap \overline{Q})$, $v = 0$ on $\partial\Omega \cap Q$, and $|\nabla u| \equiv |\nabla v|$.*

Proof. We claim that ∇u is the limit in $L^2(\Omega;\mathbb{R}^2)$ of a sequence $(\phi^n)_{n\geq 1}$ of C^∞ fields with compact support in $\Omega \cap \overline{Q}$ and zero divergence. A reason for this is that the orthogonal of these fields in $L^2(\Omega;\mathbb{R}^2)$ is exactly the set $\{\nabla v : v \in H^1(\Omega\cap Q), v = 0 \text{ on } \Omega\cap\partial Q\}$ and ∇u is orthogonal to this set (cf [8, Lemma 1]). Since $\operatorname{div}\phi^n = 0$ and $\partial\Omega \cap Q$ is connected, there exists $v_n \in C_c^\infty(\Omega \cap \overline{Q})$ such that $\nabla v_n = (-\phi_2^n, \phi_1^n)$. Let v be the limit of v_n in $H_0^1(\Omega\cap\overline{Q})$. (This space denotes the closure of $C_c^\infty(\Omega \cap \overline{Q})$ in $H^1(\Omega \cap Q)$.) Clearly $\nabla v = (-\partial_2 u, \partial_1 u)$, $\Delta v = 0$. The conjugate v is the minimizer of $\int_{\Omega\cap Q} |\nabla w|^2$ among all $w \in H_0^1(\Omega \cap \overline{Q})$ satisfying a Dirichlet condition $w(x_0 \pm \alpha\eta + t\nu) = \psi_\pm(t)$ on $\Omega \cap \partial Q$, with $\psi_\pm$ given by $\psi_\pm = 0$ on $\partial\Omega$ and $\psi'_\pm(t) = \nabla u(x_0 \pm \alpha\eta + t\nu)\cdot\eta$ for every $t < h(\pm\alpha)$ or $h_1(\pm\alpha) < t < h_2(\pm\alpha)$ (depending on the situation). The functions ψ_+, ψ_- are continuous and C^∞ away from the boundary. We deduce that v is bounded, $v \in C^\infty(\Omega \cap \overline{Q})$, and by a barrier argument as in [10, Sect. 2.8] (since $\Omega \cap Q$ satisfies an exterior sphere condition) that $v(x)$ goes to zero as x goes to $\partial\Omega \cap Q$. □

Lemma 5 *Let $B(x_0, r)$ be such that $\overline{B}(x_0, r) \cap \overline{D \cup E} = \emptyset$. Then $u \in W^{1,\infty}(\Omega \cap B(x_0, r))$.*

Proof. It is enough to show that $|\nabla u|$ remains bounded near each point of the boundary $\partial\Omega \setminus \overline{D \cup E}$. We consider $x_0 \in \partial\Omega \setminus \overline{D \cup E}$ and Q as in Lemma 3, not intersecting $\overline{D \cup E}$. Let v be the conjugate of u as in Lemma 4. It is the same to show that $|\nabla v|$ is bounded near x_0.

We let $Q' = \{x_0 + t\eta + s\nu : |t| < \alpha/2, |s| < \beta/2\}$ and choose $\rho < \operatorname{dist}(Q', \partial Q)/4$, such that Ω satisfies an exterior sphere condition of radius ρ at each point of $\partial\Omega \cap Q$. Let $M = \max\{|\nabla v(x)| : x \in \overline{Q'}, \operatorname{dist}(x, \Omega^c) \geq \rho\} < +\infty$. Choose $x \in \overline{Q'}$ such that $\delta = \operatorname{dist}(x, \partial\Omega) < \rho$ and $y \in \partial\Omega$ with $|y - x| = \delta$. There exists ξ such that $B(y + \rho\xi, \rho) \subset \Omega^c$. We consider the harmonic barrier $w(z) = k\log(|z - y - \rho\xi|/\rho)$, which is positive if $k > 0$ (except at y where it vanishes) on $\partial\Omega \cap Q$, hence greater than $|v| = 0$, and if k is larger than $(\max_{\Omega\cap\partial Q} |v|)/\log 2 < +\infty$, then w is also greater than $|v|$ on $\Omega \cap \partial Q$ (since by construction $\operatorname{dist}(y + \rho\xi, \partial Q) \geq 2\rho$). Hence, by the maximum principle, $|v| \leq w$ on $\Omega \cap Q$.

Now, we have $\max_{\overline{B}(x,\delta/2)} |\nabla v| \le (c/\delta)\max_{\overline{B}(x,\delta)} |v|$, with a constant c that does not depend on the data. A standard way to check this is to use the fact that $v(z) = \frac{4}{\pi\delta^2}\int_{B(z,\delta/2)} v$ for each $z \in \overline{B}(x,\delta/2)$, hence

$$v(z') - v(z) \le (4/\pi\delta^2)|B(z,\delta/2)\triangle B(z',\delta/2)| \max_{\overline{B}(x,\delta)} |v|$$

for $z, z' \in \overline{B}(x,\delta/2)$. We deduce that $\max_{\overline{B}(x,\delta/2)} |\nabla v| \le (c/\delta)k\log(1+2\delta/\rho) \le 2ck/\rho$, hence $|\nabla v| \le \max\{M, 2ck/\rho\}$ on $\overline{Q'} \cap \overline{\Omega}$ and the lemma is proven. □

5 Variations of the boundary

In order to get local information on the boundary of the minimizing set Ω, we will now consider performing small variations of this set. This is related to the computation of "shape derivatives" introduced in [15, Sect. V] for deriving the Euler–Lagrange equation of some shape optimization problems.

Lemma 6 *Let $B(x_0,r)$ be such that $\overline{B}(x_0,r) \cap \overline{D\cup E} = \emptyset$. Let $\phi \in C_c^\infty(B(x_0,r);\mathbb{R}^2)$ and for $\varepsilon > 0$ small, define $\Omega_\varepsilon = \{x + \varepsilon\phi(x) : x \in \Omega\}$. Then*

$$\begin{aligned}&\mu \limsup_{\varepsilon\to 0} \frac{\mathcal{H}^1(\partial\Omega) - \mathcal{H}^1(\partial\Omega_\varepsilon)}{\varepsilon}\\ &\le \int_{B(x_0,\,r)\,\cap\,\Omega} (\lambda - |\nabla u(x)|^2)\mathrm{div}\,\phi(x) + 2(\nabla\phi(x)\nabla u(x))\cdot\nabla u(x)\,dx. \qquad (6)\end{aligned}$$

Proof. Clearly, we have $\mathcal{E}(\Omega) \le \mathcal{E}(\Omega_\varepsilon)$ so that

$$\mu\frac{\mathcal{H}^1(\partial\Omega) - \mathcal{H}^1(\partial\Omega_\varepsilon)}{\varepsilon} \le \lambda\frac{|\Omega_\varepsilon| - |\Omega|}{\varepsilon} + \frac{J(\Omega_\varepsilon) - J(\Omega)}{\varepsilon}.$$

It is not hard to show that $(|\Omega_\varepsilon| - |\Omega|)/\varepsilon \to \int_\Omega \mathrm{div}\,\phi$. On the other hand, if we introduce the solution u_ε of the minimization problem defining $J(\Omega_\varepsilon)$ and define on Ω the function v_ε by $v_\varepsilon(x) = u_\varepsilon(x + \varepsilon\phi(x))$ for every $x \in \Omega$, we have

$$J(\Omega_\varepsilon) - J(\Omega) \le J(\Omega_\varepsilon) + \int_\Omega |\nabla v_\varepsilon(x)|^2\,dx - 2\int_\Gamma f(x)v_\varepsilon(x)\,d\mathcal{H}^1(x)$$

and since $v_\varepsilon = u_\varepsilon$ on Γ, this reduces to $J(\Omega_\varepsilon) - J(\Omega) \le \int_\Omega |\nabla v_\varepsilon|^2 - \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2$.

We have for every $x \in \Omega$, $\nabla v_\varepsilon(x) = (I + \varepsilon\nabla\phi(x))\nabla u_\varepsilon(x + \varepsilon\phi(x))$, and with a change of variable we get, letting T_ε be the inverse of $x \mapsto x + \varepsilon\phi(x)$ (for ε sufficiently small),

$$\int_\Omega |\nabla v_\varepsilon(x)|^2\,dx = \int_{\Omega_\varepsilon} |(I + \varepsilon\nabla\phi(T_\varepsilon(y)))\nabla u_\varepsilon(y)|^2\,|\det\nabla T_\varepsilon(y)|\,dy.$$

This yields

$$\frac{J(\Omega_\varepsilon)-J(\Omega)}{\varepsilon} \le \int_{\Omega_\varepsilon} |\nabla u_\varepsilon(y)|^2 \left(\frac{|\det \nabla T_\varepsilon(y)|-1}{\varepsilon}\right)$$
$$+\ \big(2(\nabla\phi(T_\varepsilon(y))\nabla u_\varepsilon(y))\cdot\nabla u_\varepsilon(y) + \varepsilon|\nabla\phi(T_\varepsilon(y))\nabla u_\varepsilon(y)|^2\big)\, |\det \nabla T_\varepsilon(y)|\, dy.$$

Assume we know that ∇u_ε (extended with the value zero outside of Ω_ε) converges strongly in $L^2(\mathbb{R}^2)$ to ∇u. Since $\phi \in C_c^\infty(B(x_0,r))$, we have that $(|\det \nabla T_\varepsilon(y)| - 1)/\varepsilon$ goes uniformly to $-\mathrm{div}\,\phi(y)$ as $\varepsilon \to 0$, as well as $\nabla\phi(T_\varepsilon(y))$ to $\nabla\phi(y)$, so that the last integral converges to

$$\int_\Omega -|\nabla u(y)|^2 \mathrm{div}\,\phi(y)\ +\ 2(\nabla\phi(y)\nabla u(y))\cdot\nabla u(y)\, dy$$

as ε goes to zero, and (6) holds.

To show that ∇u_ε converges strongly to ∇u in $L^2(\mathbb{R}^2)$, it is enough to show that it is bounded. Then, the methods in [6–9] will yield the convergence. Since $J(\Omega_\varepsilon) = \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2$, it is the same to show that the $J(\Omega_\varepsilon)$ are uniformly bounded. A way to do so is to reproduce the argument in remark 6, for the construction of the open set G: we thus build a set $\Omega' \subset \Omega$ such that $\Omega' \setminus \overline{D \cup E} \subset\subset \Omega$ and $J(\Omega') < +\infty$. Then, for ε small, we will also have $\Omega_\varepsilon \supset \Omega'$, so that $J(\Omega_\varepsilon) \le J(\Omega')$. $\square$

Remark 1 If around x_0 the boundary is described as in Lemma 3 by $Q \cap \Omega = \{x_0 + t\eta + s\nu \,:\, -\alpha < t < \alpha, h_1(t) < s < h_2(t)\}$, then the previous lemma also holds if the displacement $\varepsilon\phi$ is only applied on one side of x_0. More precisely, we can let $\Omega_\varepsilon = \{x \in \Omega \,:\, (x-x_0)\cdot\eta \le 0\} \cup \{x + \varepsilon\phi(x) \,:\, x \in \Omega, (x-x_0)\cdot\eta > 0\}$ provided $\phi \in C_c^\infty(Q)$ and $\phi(x)\cdot\eta \ge 0$ for every x (so that the right-hand side and the left-hand side of Ω_ε do not overlap). Of course, the integral in (6) must be computed in this case only on $\Omega \cap Q \cap \{(x-x_0)\cdot\eta > 0\}$.

We then show the following lemma.

Lemma 7 *Let $B(x_0,r)$, ϕ and Ω_ε, $\varepsilon > 0$ as in the previous lemma. Let also $k = \sup_{B(x_0,r)\cap\Omega} |\nabla u|$. Then*

$$\mu \limsup_{\varepsilon\to 0} \frac{\mathcal{H}^1(\partial\Omega) - \mathcal{H}^1(\partial\Omega_\varepsilon)}{\varepsilon} \le (\lambda + 3k^2) \int_{B(x_0,r)\cap\partial\Omega} |\phi(x)|\, d\mathcal{H}^1(x). \quad (7)$$

Proof. First notice that $k < +\infty$ by Lemma 5. Consider an increasing sequence Ω_n of open sets with $\Omega_n \subset\subset \Omega$, $\partial\Omega_n$ of class C^∞, and such that $\bigcup_{n\ge1} \Omega_n = \Omega$ and $\mathcal{H}^1(\partial\Omega_n) \to \mathcal{H}^1(\partial\Omega)$ as $n \to \infty$. By (6),

$$\mu \limsup_{\varepsilon\to 0} \frac{\mathcal{H}^1(\partial\Omega) - \mathcal{H}^1(\partial\Omega_\varepsilon)}{\varepsilon} \le \lim_{n\to\infty} \int_{B(x_0,\,r)\,\cap\,\Omega_n} (\lambda - |\nabla u|^2)\mathrm{div}\,\phi + 2(\nabla\phi\nabla u)\cdot\nabla u.$$

Integrating by parts (since $u \in C^\infty(B(x_0,r) \cap \overline{\Omega}_n)$), we see that the integral is equal to

$$\int_{B(x_0,r)\cap\partial\Omega_n} (\lambda - |\nabla u|^2)\phi\cdot\nu\ +\ \int_{B(x_0,r)\cap\Omega_n} (\nabla|\nabla u|^2)\cdot\phi + 2(\nabla\phi\nabla u)\cdot\nabla u$$

$$= \int_{B(x_0,r)\cap\partial\Omega_n} (\lambda - |\nabla u|^2)\phi\cdot\nu \,+\, 2\int_{B(x_0,r)\cap\Omega_n} \nabla(\phi\cdot\nabla u)\cdot\nabla u$$

$$= \int_{B(x_0,r)\cap\partial\Omega_n} (\lambda - |\nabla u|^2)\phi\cdot\nu \,+\, 2\int_{B(x_0,r)\cap\partial\Omega_n} (\phi\cdot\nabla u)(\nabla u\cdot\nu),$$

with ν the outer normal to Ω_n, and using $\Delta u = 0$. Hence

$$\int_{B(x_0,\,r)\,\cap\,\Omega_n} (\lambda - |\nabla u|^2)\mathrm{div}\,\phi \,+\, 2(\nabla\phi\nabla u)\cdot\nabla u \,\le\, (\lambda + 3k^2)\int_{B(x_0,\,r)\,\cap\,\partial\Omega_n} |\phi|,$$

since that last integral converges as $n \to \infty$ to $\int_{B(x_0,r)\cap\partial\Omega} |\phi|$, we get the result. □

6 Regularity of the boundary

Lemma 8 *$\partial\Omega \setminus \overline{D \cup E}$ is $C^{1,1}$.*

Proof. Let $x_0 \in \partial\Omega\setminus\overline{D\cup E}$ and $Q = \{x_0 + t\eta + s\nu \,:\, -\alpha < t < \alpha,\ -\beta < s < \beta\}$, $\alpha, \beta > 0$ given by Lemma 3. Assume first that there exists a Lipschitz function $h : (-\alpha,\alpha) \to (-\beta,\beta)$ such that $\Omega \cap Q = \{x_0 + t\eta + s\nu \,:\, -\alpha < t < \alpha,\ -\beta < s < h(t)\}$. We assume $Q \cap \overline{D\cup E} = \emptyset$ and consider $\psi \in C_c^\infty(-\alpha,\alpha)$. Defining Ω_ε (in Q) as the subgraph of $h - \varepsilon\psi$ we get from Lemma 7 that

$$\int_{-\alpha}^{\alpha} \frac{h'(t)\psi'(t)}{\sqrt{1+h'(t)^2}}\,dt \,\le\, c\int_{-\alpha}^{\alpha} |\psi(t)|\sqrt{1+h'(t)^2}\,dt \,\le\, c\int_{-\alpha}^{\alpha} |\psi(t)|\,dt$$

(with c denoting a generic constant). Since ψ is arbitrary, this shows that the function $h'/\sqrt{1+h'^2}$ is in $W^{1,\infty}(-\alpha,\alpha)$. We easily deduce that h is C^1 and that h' is Lipschitz.

In the second situation, there exist h_1 and h_2 such that $\Omega\cap Q$ is either $\{x_0 + t\eta + s\nu \,:\, -\alpha < t < \alpha,\ h_1(t) < s < h_2(t)\}$, or $\{x_0 + t\eta + s\nu \,:\, 0 < t < \alpha,\ h_1(t) < s < h_2(t)\}$. In both cases, we will consider (following the remark after Lemma 6) variations only of the boundary $\{x_0 + t\eta + h_i\nu \,:\, i \in \{1,2\},\ 0 < t < \alpha\}$, by smooth displacements ϕ with $\phi\cdot\eta \ge 0$.

We choose $\psi \in C_c^\infty(-\alpha,\alpha)$, $\psi \ge 0$, and for $\varepsilon > 0$ small we let $\Omega_\varepsilon \cap Q = \{x_0 + (t + \varepsilon\psi(t))\eta + s\nu \,:\, 0 < t < \alpha,\ h_1(t) < s < h_2(t)\}$. Again defining T_ε to be the inverse of $t \mapsto t + \varepsilon\psi(t)$, the length $\mathcal{H}^1(\partial\Omega_\varepsilon)$ in Q is

$$\sum_{i=1,2}\int_{\varepsilon\psi(0)}^{\alpha} \sqrt{1 + (h_i(T_\varepsilon(s)))'^2}\,ds$$

$$= \sum_{i=1,2}\int_0^{\alpha} \sqrt{1 + \left(\frac{h_i'(t)}{1+\varepsilon\psi'(t)}\right)^2}\,(1 + \varepsilon\psi'(t))\,dt,$$

hence differentiating at $\varepsilon = 0$ together with Lemma 7 yields

$$\sum_{i=1,2}\int_0^{\alpha} \frac{-\psi'}{\sqrt{1+h_i'^2}} \,\le\, c\int_0^{\alpha} |\psi|.$$

This holds for every $\psi \in C_c^\infty(-\alpha,\alpha)$, $\psi \geq 0$, hence also (by approximation) for a non-negative $\psi \in W^{1,\infty}(-\alpha,\alpha)$. Choose $\delta > 0$ small and $\psi(t) = (1-t/\delta)^+$. We get

$$\sum_{i=1,2} \frac{1}{\delta}\int_0^\delta \frac{1}{\sqrt{1+{h_i'}^2}} \leq \frac{c\delta}{2}.$$

Sending $\delta \to 0$ yields $2 \leq 0$, a contradiction. Hence the second situation described in Lemma 3 cannot occur at x_0. This shows the lemma. $\square$

Proof of Theorem 1. Now we prove the main theorem. A consequence of the previous lemma is that $u \in H^2$ up to $\partial\Omega \setminus \overline{D \cup E}$, hence $\phi\cdot\nabla u \in H^1$ for each $\phi \in C_c^\infty(B(x_0,r))$, and equation (6) in Lemma 6 becomes

$$\mu \limsup_{\varepsilon\to 0} \frac{\mathcal{H}^1(\partial\Omega) - \mathcal{H}^1(\partial\Omega_\varepsilon)}{\varepsilon} \leq \int_{B(x_0,r)\cap\partial\Omega} (\lambda - |\nabla u(x)|^2)\phi(x)\cdot n(x)\, d\mathcal{H}^1(x), \tag{8}$$

with $n(x)$ the exterior normal to Ω at x. The previous lemma states that near each point $x_0 \in \partial\Omega \setminus \overline{D\cup E}$, there is a rectangle $Q = \{x_0 + t\eta + s\nu \,:\, -\alpha < t < \alpha,\ -\beta < s < \beta\}$ and a $C^{1,1}$ function h such that $\Omega \cap Q$ coincides with the subgraph $\{s < h(t)\}$. In this setting, (8) is equivalent to

$$-\mu \int_{-\alpha}^{\alpha} \frac{h'(t)\psi'(t)}{\sqrt{1+h'(t)^2}}\, dt \leq \int_{-\alpha}^{\alpha} (\lambda - |\nabla u(x_0 + t\eta + h(t)\nu)|^2)\psi(t)\, dt$$

for every $\psi \in C_c^\infty(-\alpha,\alpha)$. Replacing ψ by $-\psi$ we see that this is in fact an equality. If we consider the conjugate function v of Lemma 4, by [10, Cor. 8.36], we have that (for any $\theta < 1$) $v \in C^{1,\theta}(Q \cap \overline{\Omega})$, hence also u (since $\nabla v = (-\partial_2 u, \partial_1 u)$). We deduce that $h \in C^{2,\theta}(-\alpha,\alpha)$ with

$$\mu\left(\frac{h'(t)}{\sqrt{1+h'(t)^2}}\right)' = \lambda - |\nabla u(x_0 + t\eta + h(t)\nu)|^2. \tag{9}$$

Now we can invoke [10, Thm 9.19]: if $\partial\Omega\setminus\overline{A\cup E}$ is of class $C^{k,\theta}$ with $k \geq 2$ (which we have just checked for $k=2$), then u is $C^{k,\theta}$ up to the boundary $\partial\Omega \setminus \overline{A\cup E}$, hence in (9) we get that h'' is $C^{k-1,\theta}$ which shows that $\partial\Omega\setminus\overline{A\cup E}$ is of class $C^{k+1,\theta}$. Hence $\partial\Omega \setminus \overline{A\cup E}$ is of class C^∞ (and u as well, up to the boundary). This proves Theorem 1. $\square$

An alternate way to write (8) and (9) is

$$\mu\kappa = \lambda - |\nabla u|^2$$

on $\partial\Omega \setminus \overline{A\cup E}$ where κ is the curvature of the boundary of Ω, positive whenever Ω is locally concave. We see that the curvature is bounded from above by λ/μ: this corresponds to the external sphere condition shown in Lemma 2.

7 A more general result

In this section we generalize our study (in the scalar case) to nonuniform internal energies. We assume that given Ω an admissible configuration, the energy is given by

$$E_\Omega(u) \;=\; \frac{1}{2}\int_\Omega (\mathbf{A}(x)\nabla u(x))\cdot\nabla u(x)\,dx \;-\; \int_\Gamma f(x)u(x)\,d\mathcal{H}^1(x) \;:$$

for any $u \in L^{1,2}(\Omega)$ with $u \equiv 0$ on D. The 2×2 symmetric matrix $\mathbf{A}(x)$, defined for every $x \in \mathbb{R}^2$ (of course it does not need to be really defined very far away from D and E) is Lipschitz–regular in x and positive. We consider again the compliance $J(\Omega) = -2\inf_u E_\Omega(u)$, where the inf is among all u admissible, and consider the minimization of $\mathcal{E}$. All remarks 1–6 apply, and again there exists a minimizer Ω with $\overset{\circ}{\overline{\Omega}} = \Omega$. Now, we have

Theorem 2 $\partial\Omega\setminus\overline{D\cup E}$ *is* $C^{2,\theta}$*, for any* $\theta < 1$*. If moreover* $\mathbf{A}$ *is of class* $C^{k,\theta}$*, for* $k \ge 1$ *and* $0<\theta<1$*, then* $\partial\Omega\setminus\overline{D\cup E}$ *is* $C^{k+2,\theta}$.

Proof. We just mention the differences with the proof of Theorem 1. In Lemma 4, the conjugate v, which is such that $\nabla v = (\mathbf{A}\nabla u)^\perp$, satisfies the equation $\operatorname{div}\mathbf{A}'\nabla v = 0$, with $\mathbf{A}' = (1/\det\mathbf{A})\mathbf{A}$. In Lemma 5, we can still (following [10, Sect. 14.1]) define a barrier of the form $w(z) = k\log(1+k'\operatorname{dist}(z,\partial B(y+\rho\xi,\rho)))$, that will be a supersolution (and not an exact solution as before) of the equation, i.e., $\operatorname{div}\mathbf{A}'\nabla w \le 0$. Then, we also have the estimate $\max_{\overline{B(x,\delta/2)}}|\nabla v| \le (c/\delta)\max_{\overline{B(x,\delta)}}|v|$ (*cf.* [5, Thm. 9.1]). Thus Lemma 5 also holds in this case.

Then, in Lemma 6, an additional term appears: instead of (6), we get

$$\begin{aligned}\mu\limsup_{\varepsilon\to 0}\frac{\mathcal{H}^1(\partial\Omega)-\mathcal{H}^1(\partial\Omega_\varepsilon)}{\varepsilon} \le \int_{B(x_0,r)\,\cap\,\Omega} & (\lambda-(\mathbf{A}(x)\nabla u(x))\cdot\nabla u(x))\operatorname{div}\phi(x)\\ &-((\phi(x)\cdot\nabla\mathbf{A}(x))\nabla u(x))\cdot\nabla u(x)\\ &+2(\nabla\phi(x)\nabla u(x))\cdot\nabla u(x)\,dx.\end{aligned}$$

This still yields a bound as in Lemma 7. We deduce once more that $\partial\Omega\setminus\overline{A\cup E}$ is $C^{1,1}$ and that $u\in H^2$ near $\partial\Omega\setminus\overline{A\cup E}$, and after an integration by parts we find the following variant of (8):

$$\begin{aligned}&\mu\limsup_{\varepsilon\to 0}\frac{\mathcal{H}^1(\partial\Omega)-\mathcal{H}^1(\partial\Omega_\varepsilon)}{\varepsilon}\\ &\le \int_{B(x_0,r)\,\cap\,\partial\Omega}(\lambda-(\mathbf{A}(x)\nabla u(x))\cdot\nabla u(x))\phi(x)\cdot n(x)\,d\mathcal{H}^1(x).\end{aligned}$$

We conclude as in the previous section: invoking [10, Cor. 8.36], we get that the conjugate function v of Lemma 4 is $C^{1,\theta}$ (for every $\theta<1$) up to the boundary so that $|\nabla u| = |\mathbf{A}'\nabla v|$ is $C^{0,\theta}$ and the boundary itself is $C^{2,\theta}$.

If moreover $\mathbf{A}\in C^{k,\theta}(\overline{\Omega})$, for $k\ge 1$, we invoke [10, Thm 9.19] and find that if $u\in C^{k,\theta}$ up to the boundary, so that $\partial\Omega\setminus\overline{A\cup E}$ is $C^{k+1,\theta}$ (which is true for $k=1$), then $u\in C^{k+1,\theta}$ up to the boundary hence $\partial\Omega\setminus\overline{A\cup E}$ is $C^{k+2,\theta}$. This achieves the proof of Theorem 2. □

We see that now the Euler equation for the boundary is

$$\mu\kappa = \lambda - (\mathbf{A}\nabla u)\cdot\nabla u.$$

Acknowledgement. This research began while the second author was visiting University of Paris-Nord and was substantially completed while he was visiting University of Paris-Dauphine. The first author is supported by CNRS.

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