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Partial symmetry and asymptotic behavior for some elliptic variational problems

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Abstract. A short elementary proof based on polarizations yields a useful (new) rearrangement inequality for symmetrically weighted Dirichlet type functionals. It is then used to answer some symmetry related open questions in the literature. The non symmetry of the Hénon equation ground states (previously proved in [19]) as well as their asymptotic behavior are analyzed more in depth. A special attention is also paid to the minimizers of the Caffarelli-Kohn-Nirenberg [8] inequalities.

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1. Introduction

The symmetry properties of positive solutions of symmetric PDE's is a wide and crucial question. Symmetrization as well as moving plane methods are among the most useful tools in this direction. When they do not apply, symmetry breaking can sometimes be proved, but weaker symmetries than the ones present in the problem may still remain and be useful even in analyzing the symmetry breaking.

Numerous model problems in the literature involve radially symmetrically weighted L^p or Dirichlet type norms. In the next section, using a fairly simple approach based on polarizations, we prove a rearrangement inequality for such kind of functionals. The main important consequence of this inequality is that ground states of these problems, when they are not radial, keep a strong degree of symmetry : they essentially depend on only two variables, the radial one and one of the angular ones. The simplicity and usefulness of polarizations for rearrangement inequalities was discovered by Ahlfors [1] for functions on $\mathbb{C}$ and Baernstein [3,4] (see also [6] for a detailed study of polarizations in many different settings.)

In Sect. 3, we use this partial symmetry result to analyze more in depth the asymptotic behavior, as $\alpha \to +\infty$, of the ground states of the Hénon [14] equation,

$$\begin{cases} -\Delta u = |x|^\alpha u^{p-1} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

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where $N \geq 3$, $2 < p < 2^*$ and $\alpha > 0$. The symmetry breaking for these ground states was previously studied by the authors in [19].

In Sect. 4, we analyze the symmetry properties of the minimizers for the Caffarelli-Kohn-Nirenberg inequality in the case where the weights are increasing in $|x|$. These minimizers satisfy, up to some Lagrange multiplier, the equation

$$-\text{div}(|x|^{-qa}|\nabla u|^{q-2}\nabla u) = |x|^{-bp}u^{p-1} \text{ in } \mathbb{R}^N,$$

where $0 < \frac{1}{q} - \frac{1}{p} = \frac{1+a-b}{N}$, $q > 1$, $p > 1$, and $a \leq b < \frac{N}{q}$, and hence fall in the setting of Sect. 2. Symmetry breaking for the CKN ground states was studied by Catrina and Wang in [10]. We extend it somewhat here and answer some questions left open in [10], mainly the existence of a region where full radial symmetry remains even though moving planes or Schwarz symmetrization techniques do not apply.

We have considered only basic model problems in order to stress the simplicity of the approach, but it makes no doubt that the partial symmetry which follows from the rearrangement inequality of Sect. 2 can be proved in many different nearby settings, like invariance under subgroups of $SO(N)$ as cylindrical symmetry in [2]. This particular example will be treated elsewhere [18]. We also would like to point out an interesting recent work by Pacella [16] where the maximum principle is used to prove partial symmetry results for index 1 solutions of problems where moving plane fails.

2. Polarization and foliated Schwarz symmetrization

Let us consider first the case of $\mathbb{R}^N$ equipped with its Lebesgue measure. We denote by $\mathcal{H}$ the set of all half spaces in $\mathbb{R}^N$ (or equivalently the set of all $(N-1)$-dimensional affine hyperplanes with orientation), and by $\mathcal{H}_0$ the subset of $\mathcal{H}$ corresponding to $N-1$-dimensional Euclidean hyperplanes.

Let $H \in \mathcal{H}$ be a half space, we denote by σ_H the reflexion with respect to ∂H.

Definition 2.1. *The polarization (with respect to H) of a measurable positive function u is defined by*

$$u_H(x) := \begin{cases} \max(u(x), u(\sigma_H(x))) \text{ if } x \in H, \\ \min(u(x), u(\sigma_H(x))) \text{ if } x \in \mathbb{R}^N \setminus H. \end{cases}$$

Similarly, the polarization (with respect to H) of a measurable set $A \subseteq \mathbb{R}^N$ is defined by $\chi_{A_H} = (\chi_A)_H$.

Observe that if $v := u \circ \sigma_H$, then

$$u_H(x) := \begin{cases} \frac{1}{2}(u(x) + v(x)) + \frac{1}{2}|u(x) - v(x)| \text{ if } x \in H, \\ \frac{1}{2}(u(x) + v(x)) - \frac{1}{2}|u(x) - v(x)| \text{ if } x \in \mathbb{R}^N \setminus H. \end{cases}$$

If μ is a Radon measure over $\mathbb{R}^N$, and $\mathcal{M}(\mu)$ the measurable sets for μ, then a mapping $* : \mathcal{M}(\mu) \to \mathcal{M}(\mu)$ is called a rearrangement provided it satisfies both

the monotonicity property ($A \subset B \Rightarrow A^* \subset B^*$) and the conservation property ($\mu(A^*) = \mu(A)$).
The following lemma is straightforward.

Lemma 2.2. *Let $f : \mathbb{R}^+ \to R^+$ positive measurable. The mapping $A \mapsto A_H$ is a rearrangement for the Lebesgue measure for any $H \in \mathcal{H}$, and is a rearrangement for the measure $d\mu := f(|x|)dx$ for any $H \in \mathcal{H}_0$. In particular, $\|u_H\|_{L^p(d\mu)} = \|u\|_{L^p(d\mu)}$ for each $H \in \mathcal{H}_0$ and $u \in L^p(d\mu)$.*

We now concentrate essentially on weighted norms, the unweighted case being already well known leading to Schwarz symmetrization. In the sequel, we will assume that f is a positive bounded measurable function on R^+, and write $d\mu := f(|x|)dx$ where dx is the Lebesgue measure. The boundedness condition on f can be easily removed by working in weighted Sobolev spaces, we do not include this here for simplicity. For a functional space E, we will denote by E_+ the positive cone of non negative functions in E.

The following simple lemma is the key to the subsequent rearrangement inequalities.

Proposition 2.3. *Let $1 \leq p < +\infty$, $d\mu := f(|x|)dx$ and $H \in \mathcal{H}_0$. If $u \in W^{1,p}_+(\mathbb{R}^N)$, then $u_H \in W^{1,p}_+(\mathbb{R}^N)$ and $\|\nabla u_H\|_{L^p(d\mu)} = \|\nabla u\|_{L^p(d\mu)}$.*

Let $v := u \circ \sigma_H$, so that clearly $u - v \in W^{1,p}_0(H)$. By a standard result, $|u-v| \in W^{1,p}_0(H)$ and similarly $|u-v| \in W^{1,p}_0(\sigma_H(H))$. Now let $f := \chi_H|u-v|$ and $g := (\chi_H - 1)|u - v|$, then both f and g belong to $W^{1,p}(\mathbb{R}^N)$, and since $u_H = \frac{1}{2}(u + v + f + g)$ we obtain $u_H \in W^{1,p}(\mathbb{R}^N)$.
Almost everywhere u_H satisfies,

$$\nabla u_H(x) = \begin{cases} \nabla u(x) & \text{on } (H \cap \{u \geq v\}) \cup (\sigma_H(H) \cap \{u < v\}) \\ \nabla u(\sigma_H(x)) & \text{on } (H \cap \{u < v\}) \cup (\sigma_H(H) \cap \{u \geq v\}, \end{cases} \tag{1}$$

but notice also that

$$\begin{aligned} &\{x \in H \text{ s.t. } u(x) < v(x)\} = \sigma_H\left(\{x \in \sigma_H(H) \text{ s.t. } u(x) > v(x)\}\right) \\ \text{and} \quad &\{x \in \sigma_H(H) \text{ s.t. } u(x) \geq v(x)\} = \sigma_H\left(\{x \in H \text{ s.t. } u(x) \leq v(x)\}\right). \end{aligned} \tag{2}$$

Taking the modulus of (1) to the power p and integrating over $\mathbb{R}^N$, we get the desired result. Indeed, the part of the integrals on $(H \cap \{u \geq v\}) \cup (\sigma_H(H) \cap \{u < v\})$ are clearly the same,while on the complement one uses the change of variable formula with $y = \sigma_H(x)$, taking into account (2) and the fact that since $H \in \mathcal{H}_0$, the measure $f(|x|)dx$ is transported into itself under σ_H. □

We arbitrarily choose $(1, 0, \cdots, 0)$ as a fixed direction in $\mathbb{R}^N$ which we will refer to as the north pole direction, and we denote by $\mathcal{H}_1$ the subset of $\mathcal{H}_0$ consisting of half spaces containing the direction of the north pole.
Let $R > 0$ and $d\sigma$ denote the standard measure on $\partial B(R)$. The symmetrization A^* of a set $A \subset \partial B(R)$ is defined as the closed geodesic ball in $\partial B(R)$ centered at the north pole and whose $d\sigma$-measure equals that of A.
If $f \in L^1(\partial B(R))$ is positive, the symmetric decreasing rearrangement f^* of f is defined in such a way that $\{f > t\}^* = \{f^* > t\}$ for all $t > 0$.

Definition 2.4. *Let $1 \le p < +\infty$. The foliated Schwarz symmetrization u^* of u in $W_+^{1,p}(\mathbb{R}^N)$ is defined on any sphere $\partial B(R)$ by the symmetric decreasing rearrangement of the restriction of u to the same sphere.*

The definition clearly makes sense for a broader class of functions, and extends to sets, but we shall not focus on this. It seems that it was first introduced in [17] in dimension 2. The following is also straightforward.

Lemma 2.5. *The foliated Schwarz symmetrization is a rearrangement of $\mathbb{R}^N$ for any measure of the form $d\mu := f(|x|)dx$. Moreover, if $H \in \mathcal{H}_1$ then $(u^*)_H = u^* = (u_H)^*$.*

We denote by $\mathcal{K}(\mathbb{R}^N)$ the set of continuous functions with compact support in $\mathbb{R}^N$. The following two lemmas borrow and simplify some ideas from [6].

Lemma 2.6. *Let $1 \le p < +\infty$ and $u \in \mathcal{K}_+(\mathbb{R}^N)$. If $u \neq u^*$ then there exists $H \in \mathcal{H}_0$ such that*

$$\|u_H - u^*\|_p < \|u - u^*\|_p.$$

If $u \neq u^*$, there exists $R > 0$ and $t > 0$ such that $\{u > t\} \cap \partial B(R) \neq \{u^* > t\} \cap \partial B(R)$. Since $*$ is a rearrangement, $d\sigma(\{u > t\} \cap \partial B(R)) = d\sigma(\{u^* > t\} \cap \partial B(R))$, and there exist $y, z \in \partial B(R)$ satisfying

$$u^*(y) > t \ge u(y) \ \text{ and } \ u(z) > t \ge u^*(z).$$

Let $H \in \mathcal{H}_0$ with $y \in H$ and $z = \sigma_H(y)$. Since $u^*(y) > u^*(z)$, it follows that y is closer to the north pole than z, and hence $H \in \mathcal{H}_1$. For all $x \in H$, using the fact that $(u^*)_H = u^*$ we have :

$$|u_H(x) - u^*(x)|^p + |u_H(\sigma_H(x)) - u^*(\sigma_H(x))|^p$$
$$\le |u(x) - u^*(x)|^p + |u(\sigma_H(x)) - u^*(\sigma_H(x))|^p,$$

and by continuity the inequality is strict in neighbourhood of y. Integrating over H yields the result. □

Clearly for $u \in \mathcal{K}_+(\mathbb{R}^N)$, the mapping $H \mapsto u_H$ is continuous form $\mathcal{H}_1 \sim SO(N)/\mathbb{Z}_2$ to L^p. By compactness, the minimization problem

$$c := \inf_{H \in \mathcal{H}_1} \|u_H - u^*\|p$$

is achieved by some $H := H(u)$.

Lemma 2.7. *Let $1 \le p < +\infty$ and $u \in \mathcal{D}_+(\mathbb{R}^N)$. The sequence (u_n) defined by $u_0 = u$, $u_{n+1} = (u_n)_{H_n}$ and*

$$\|u_{n+1} - u^*\|_p = \min_{H \in \mathcal{H}_1} \|(u_n)_H - u^*\|_p$$

converges to u^* in $L^p(\mathbb{R}^N)$.

Let $R > 0$ such that the support of u is contained in $B(R)$, clearly the same holds for each u_n. Take $q > N$, by Lemma 2.3 the sequence (u_n) is bounded in $W_0^{1,q}(B(R)$ and by the Rellich theorem we can assume that up to a subsequence subsequence $u_n \to v$ uniformly. Since $(u_n)^* = u^*$ it follows that $v^* = u^*$. For each $H \in \mathcal{H}_1$ we have $(u^*)_H = u^*$ and

$$\|u_{n+1} - u^*\|_p \leq \|(u_n)_H - u^*\|_p \leq \|u_n - u^*\|_p.$$

Taking the limit along the subsequence in the preceding inequality we obtain

$$\|v - u^*\|_p \leq \|v_H - u^*\|_p \leq \|v - u^*\|_p.$$

But $v^* = u^*$ and H is arbitrary, the conclusion then follows from Lemma 2.6. □

The following rearrangement inequality is now an easy consequence.

Theorem 2.8. *Let* $1 < p < +\infty$ *and* $d\mu := f(|x|)dx$ *for some positive bounded measurable function* f*. If* $u \in W_+^{1,p}(\mathbb{R}^N)$ *then* $u^* \in W_+^{1,p}(\mathbb{R}^N)$ *and*

$$\int_{\mathbb{R}^N} f(|x|)|\nabla u^*|^p \, dx \leq \int_{\mathbb{R}^N} f(|x|)|\nabla u|^p \, dx.$$

Assume first that $u \in \mathcal{D}_+(\mathbb{R}^N)$. The sequence (u_n) associated to u as in Lemma 2.7 is such that $u_n \to u^*$ in $L^p(\mathbb{R}^N)$ and by Lemma 2.3

$$\|u_n\|_p = \|u\|_p, \;\; \|\nabla u_n\|_{L^p(d\mu)} = \|\nabla u\|_{L^p(d\mu)}.$$

Hence, $u^* \in W_+^{1,p}(\mathbb{R}^N)$ and by weak lower semi-continuity $\|\nabla u^*\|_{L^p(d\mu)} \leq \|\nabla u\|_{L^p(d\mu)}$.

If $u \in W_+^{1,p}(\mathbb{R}^N)$, there exist a sequence $(u_n) \in \mathcal{D}_+(\mathbb{R}^N)$ such that $u_n \to u$ in $W_+^{1,p}(\mathbb{R}^N)$. Since any rearrangement is a contraction in L^p, $u_n^* \to u^*$ in $L^p(\mathbb{R}^N)$ and by what precedes $\|\nabla u_n^*\|_{L^p(d\mu)} \leq \|\nabla u_n\|_{L^p(d\mu)}$, so that

$$\|\nabla u^*\|_{L^p(d\mu)} \leq \liminf_{n\to+\infty} \|\nabla u_n^*\|_{L^p(d\mu)} \leq \liminf_{n\to+\infty} \|\nabla u_n\|_{L^p(d\mu)} = \|\nabla u\|_{L^p(d\mu)}.$$

The proof is complete. □

Remark 2.9. Notice the interesting property of the approximating scheme of the symmetrized u^* in Lemma 2.7 : the Dirichlet norm is constant along the full sequence and finally just decreases in the weak limit. As $n \to +\infty$, the functions u_n thus look like the limit u^* plus some small edged oscillation.

3. Ground states for the Hénon equation

The Hénon equation

$$\begin{cases} -\Delta u = |x|^\alpha u^{p-1} & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

was introduced in [14] in the context of stellar clusters. Here, α is a positive constant, $2 < p < 2^* := 2N/(N-2)$ and Ω stands for the unit ball in $\mathbb{R}^N$.

An interesting feature of Eq. (3) is the presence of the weight $|x|^\alpha$ in the nonlinearity, which is increasing in $|x|$ since $\alpha > 0$. Because of this, the Gidas-Ni-Nirenberg [12] theorem cannot be applied to positive solutions of (3).

In [19], we studied the symmetry of ground state solutions of (3), depending on both the parameters α and p. By a ground state, or least energy solution, we mean any solution which also minimizes the Rayleigh quotient

$$S_{\alpha,p} := \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2\, dx}{(\int_\Omega |x|^\alpha |u|^p\, dx)^{2/p}}. \tag{4}$$

Equivalently, a least energy solution is a critical point corresponding to the lowest critical value of the associated energy functional

$$J_{\alpha,p}(u) := \int_\Omega \frac{1}{2}|\nabla u|^2 - |x|^\alpha \frac{|u|^p}{p}\, dx.$$

Our main result in [19] can be summarized as follows :

Theorem. *Let $N \geq 2$. a) Given $2 < p < 2^*$, there exists $\alpha^* > 0$ such that for $\alpha \geq \alpha^*$, no ground state solution of (3) is radially symmetric.*
b) For each $n \in \mathbb{N}$, there exists $\delta_n > 0$ such that the unique ground state solution of (3) is radial provided $\alpha \leq n$ and $p \leq 2 + \delta_n$.
c) Assume $N \geq 3$, then for any $n \in \mathbb{N}$, there exists $\gamma_n > 0$ such that no ground state solution of (3) is radially symmetric if $\alpha \geq 1/n$ and $2^* - \gamma_n < p < 2^*$.

Roughly speaking, this means that for p fixed symmetry breaking always occur for sufficiently large α, but also that α has to be very large if p is close to 2 or can be very small if p is close to 2^*.
Actually, when $p = 2^*$ the minimization problem (4) has no solution, so that symmetry breaking is easily deduced from a Brezis-Nirenberg type argument [5]. A more precise description of the ground state solutions for fixed α and $p \to 2^*$ was subsequently obtained by Cao and Peng [9].
To state their result, remember that the equation $-\Delta U = U^{2^*-1}$ on $\mathbb{R}^N$ has a unique positive solution in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ (up to translations and dilations), which is given by $U(x) := (1+|x|^2)^{(2-N)/2}$. It is standard to denote equation preserving scalings of U by $U_{\varepsilon,y}(x) := \varepsilon^{\frac{2-N}{2}} U(\frac{x-y}{\varepsilon})$.

Theorem (Cao and Peng [9]). *Let $N \geq 3$ and $\alpha > 0$. If (u_p) is a sequence of ground state solutions of (3) with $p \to 2^*$, then up to a subsequence there exists $x_0 \in \partial\Omega$ such that*

- $|\nabla u_p|^2 \to \mu\delta_{x_0}$ *in the sense of measure*

- $|u_p|^{2^*} \to \nu\delta_{x_0}$ *in the sense of measure*
- $\exists\varepsilon > 0$ *such that for* $2^* - p < \varepsilon$, u_p *has a unique maximum point* x_p *such that* $\mathrm{dist}(x_p, \partial\Omega) \to 0$ *as* $p \to 2^*$, $u_p(x_p) \to +\infty$, *and*

$$\lim_{p\to 2^*} \int_\Omega \left|\nabla\left(u_p - U_{\lambda_p,x_p}\right)\right|^2 dx = 0,$$

where $\lambda_p := u_p(x_p)$.

The main reason behind it is that as $p \to 2^*$, the ground state solutions of (3) tend to concentrate on a single point, and because they need to minimize the quotient in (4), this point has to be on the boundary, where the weight is maximal.

The asymptotic of the ground state solutions when p is fixed and $\alpha \to +\infty$ seems less clear, since there is a priori no limit equation. In the remaining of this section, we will give a precise characterization of them.

By the results of Sect. 2, we know that for any ground state solution u_α of (3), there exists some $\xi \in S^{N-1}$ such that the restriction of u_α to any sphere $\partial B(r)$ is symmetric decreasing with respect to the distance to $r\xi$. Without loss of generality, we can assume that ξ is a fixed direction, say $\xi_0 = (1, 0, \cdots, 0)$.

To the solution u_α, we associate v_α defined by

$$v_\alpha(x) := \alpha^{-\frac{2}{p-2}} u_\alpha\left(\frac{x}{\alpha} + \xi_0\right) \qquad \text{on } \Omega_\alpha := \alpha(\Omega - \xi_0).$$

The exponent $\frac{-2}{p-2}$ was chosen as the right scaling so that v_α satisfies the equation

$$\begin{cases} -\Delta v_\alpha = |\dfrac{x}{\alpha} + \xi_0|^\alpha\, v_\alpha^{p-1} & \text{in } \Omega_\alpha \\ \quad v_\alpha = 0 & \text{on } \partial\Omega_\alpha. \end{cases} \tag{5}$$

Lemma 3.1. *There exists* $C > 0$ *independent of* α *such that for all* α *large,*

$$\frac{1}{C} \le \int_{\Omega_\alpha} |\nabla v_\alpha|^2\, dx \le C.$$

First notice that because v_α satisfies Eq. (5), there holds

$$\int_{\Omega_\alpha} |\nabla v_\alpha|^2\, dx = \int_{\Omega_\alpha} \left|\frac{x}{\alpha} + \xi_0\right|^\alpha\, v_\alpha^p\, dx. \tag{6}$$

Since u_α is a minimizer for (4), so is v_α for the rescaled version. Let $\varphi \in \mathcal{D}(\mathbb{R}^N)$ positive such that the support of φ is contained in $B(0,1) - \xi_0$, for α sufficiently large

$$\frac{\int_{\Omega_\alpha} |\nabla v_\alpha|^2\, dx}{(\int_{\Omega_\alpha} |\frac{x}{\alpha} + \xi_0|^\alpha\, v_\alpha^p\, dx)^{2/p}} \le \frac{\int_{\mathbb{R}^N} |\nabla\varphi|^2\, dx}{(\int_{\mathbb{R}^N} |\frac{x}{\alpha} + \xi_0|^\alpha\, \varphi^p\, dx)^{2/p}} \le C\frac{\int_{\mathbb{R}^N} |\nabla\varphi|^2\, dx}{(\int_{\mathbb{R}^N} \varphi^p\, dx)^{2/p}},$$

which combined with (6) yields

$$\left(\int_{\Omega_\alpha} |\nabla v_\alpha|^2\, dx\right)^{1-2/p} \le C.$$

Concerning the lower bound, first remark that there exists $C > 0$ such that for sufficiently large α,

$$\left|\frac{x}{\alpha} + \xi_0\right|^{\alpha} \leq C\text{dist}(x, \partial\Omega_\alpha)^{-\gamma}, \ \ \forall x \in \Omega_\alpha,$$

where $\gamma := \frac{N-2}{2}(2^* - p)$ (see Lemma 3.3 for a very similar proof).
Hence,

$$\begin{aligned}\left(\int_{\Omega_\alpha} |\frac{x}{\alpha} + \xi_0|^{\alpha}\, v_\alpha^p dx\right)^{2/p} &\leq C\left(\int_{\Omega_\alpha} \frac{v_\alpha^p}{|\text{dist}\,(x, \partial\Omega_\alpha)\,|^{\gamma}}\, dx\right)^{2/p} \\ &\leq C\left(\int_{\Omega_\alpha} \frac{v_\alpha^2}{|\text{dist}\,(x, \partial\Omega_\alpha)\,|^{2}} dx\right)^{\gamma/p}\left(\int_{\Omega_\alpha} v_\alpha^{2^*}\, dx\right)^{(2-\gamma)/p} \\ &\leq C\int_{\Omega_\alpha} |\nabla v_\alpha|^2 dx,\end{aligned} \tag{7}$$

where C does not depend on α, by the Hardy and Sobolev inequalities. Combining (6) with (7) gives the lower bound and ends the proof. □

A crucial step in the analysis is the following lemma.

Lemma 3.2. *There exist* $\delta > 0$ *and* $R > 0$ *such that for all* α *large,*

$$\int_{\text{dist}(x,\partial\Omega_\alpha)\leq R} v_\alpha^p\, dx \geq \delta.$$

We make the proof by contradiction. Let $w_\alpha(x) := v_\alpha(x - \alpha\xi_0)$, if the thesis is false, there exist sequences $(\delta_i) \to 0$, $(R_i) \to +\infty$ and $(\alpha_i) \to +\infty$ such that $R_i/\alpha_i \to 0$ and

$$\int_{\alpha_i - R_i \leq |x| \leq \alpha_i} w_\alpha^p\, dx < \delta_i. \tag{8}$$

Let φ_i a radial cut-off function such that $\varphi_i \equiv 1$ on $B(0, \alpha_i - R_i)$ and $\varphi_i \equiv 0$ on $B(0, \alpha_i) \setminus B(\alpha_i - R_i/2)$ with $|\nabla\varphi_i| \leq 4/R_i$. There holds:

$$\begin{aligned}&\int_{B(0,\alpha_i)} \left|\frac{x}{\alpha_i}\right|^{\alpha_i} w_{\alpha_i}^p\, dx \\ &\leq \int_{B(0,\alpha_i - R_i/2)} \left|\frac{x}{\alpha_i}\right|^{\alpha_i} w_{\alpha_i}^p\, \varphi_i^p\, dx + \int_{B(0,\alpha_i)\setminus B(0,\alpha_i - R_i)} \left|\frac{x}{\alpha_i}\right|^{\alpha_i} w_{\alpha_i}^p\, dx \qquad (9) \\ &\leq \int_{B(0,\alpha_i - R_i/2)} \left|\frac{x}{\alpha_i}\right|^{\alpha_i} w_{\alpha_i}^p\, \varphi_i^p\, dx + \delta_i.\end{aligned}$$

By Lemma 3.3, there exist $(\varepsilon_i) \to 0$ such that

$$|\frac{x}{\alpha_i}|^{\alpha_i} \leq \varepsilon_i\, \text{dist}(x, \partial B(0, \alpha_i - R_i/2))^{-\gamma}\, \forall\, x \in B(0, \alpha_i - R_i/2),$$

where $\gamma := p - Np/2 + N$ $(0 < \gamma < 2)$ does not depend on i. Using the Hardy-Sobolev inequality as in (7) we thus obtain

$$\int_{B(0,\alpha_i)} \left|\frac{x}{\alpha_i}\right|^{\alpha_i} w_{\alpha_i}^p \, dx \le C\varepsilon_i \left(\int_{B(0,\alpha_i - R_i/2)} |\nabla(w_{\alpha_i}\varphi_i)|^2 \, dx\right)^{p/2} + \delta_i.$$

On the other hand, using Lemma 3.1

$$\begin{aligned} &\int_{B(0,\alpha_i - R_i/2)} |\nabla(w_{\alpha_i}\varphi_i)|^2 \, dx \\ \le& 2\left[\int_{B(0,\alpha_i - R_i/2)} |\nabla w_{\alpha_i}|^2 \, dx + \int_{\alpha_i - R_i \le |x| \le \alpha_i - R_i/2} w_{\alpha_i}^2 |\nabla \varphi_i|^2 \, dx\right] \\ \le& C\left[1 + R_i^{-2}\left(\int_{\alpha_i - R_i \le |x| \le \alpha_i - R_i/2} w_{\alpha_i}^p \, dx\right)^a \left(\int_{B(0,\alpha_i)} w_{\alpha_i}^{2^*} \, dx\right)^{1-a}\right] \qquad (10) \\ \le& C\left[C + R_i^{-2}\delta_i^a C\left(\frac{1}{\alpha_i}\right)^{2^*(1-a)/2}\right] \\ \le& C, \end{aligned}$$

where $a = (2^* - 2)/(2^* - p)$. Hence, by Lemma 3.1 we get the contradiction

$$0 < \frac{1}{C} \le \int_{B(0,\alpha_i)} |\nabla w_{\alpha_i}|^2 \, dx = \int_{B(0,\alpha_i)} |\frac{x}{\alpha_i}|^{\alpha_i} w_{\alpha_i}^p \, dx \le C\varepsilon_i + \delta_i \to 0.$$

□

Lemma 3.3. *Let* $a > 0$ *be fixed and two sequences* $(R_i) \to +\infty$ *and* $\alpha_i \to +\infty$ *such that* $\frac{R_i}{\alpha_i} \to 0$. *Then there exist* $(\varepsilon_i) \to 0$ *such that*

$$\forall\, r \in [0, \alpha_i - R_i/2], \ \left|\frac{r}{\alpha_i}\right|^{\alpha_i} \le \varepsilon_i \ |\alpha_i - R_i/2 - r|^{-a}.$$

We make the proof by contradiction. Let $s := \frac{\alpha_i - R_i/2 - r}{\alpha_i - R_i/2}$ and suppose that there exist $\varepsilon > 0$ and $s_i \in (0, 1]$ such that

$$\left(\frac{\alpha_i - R_i/2 - s_i(\alpha_i - R_i/2)}{\alpha_i}\right)^{\alpha_i} > \varepsilon\,(s_i(\alpha_i - R_i/2))^{-a}.$$

Taking the log of both sides we get

$$\alpha_i \log\left((1 - s_i)\frac{\alpha_i - R_i/2}{\alpha_i}\right) > \varepsilon - a\log\left(s_i(\alpha_i - R_i/2)\right), \qquad (11)$$

so that necessarily $s_i \to 0$. Hence,

$$\begin{aligned} \alpha_i \log\left((1-s_i)\frac{\alpha_i - R_i/2}{\alpha_i}\right) &= \alpha_i \log\left(1 - s_i - \frac{(1-s_i)R_i}{2\alpha_i}\right) \\ \sim \alpha_i\left(-s_i - \frac{(1-s_i)R_i}{2\alpha_i}\right) &\sim -R_i - \alpha_i s_i \to -\infty. \end{aligned} \tag{12}$$

If (11) is satisfied, then it holds $\log(s_i(\alpha_i - R_i/2)) \to +\infty$, so that $s_i\alpha_i \to +\infty$, but then (12) leads to a contradiction. The lemma is proved. □

We are now in position to state the main result of this section. We denote by $\mathbb{R}^N_-$ the half space $\{x \in \mathbb{R}^N \text{ s.t. } x_1 < 0\}$, where x_1 refers to the first coordinate.

Theorem 3.4. *Let (u_α) be a sequence of least energy solutions of (3) for $\alpha \to +\infty$. Then, up to a subsequence and some rotations, the rescaled functions*

$$v_\alpha(x) := \alpha^{-2/(p-2)} u_\alpha(x/\alpha + \xi_0)$$

satisfy

$$v_\alpha \longrightarrow V^* \text{ in } \mathcal{D}^{1,2}_0(\mathbb{R}^N_-) \text{ and uniformly on any compact subset of } \mathbb{R}^N_-,$$

where V^* is a positive least energy solution of the equation

$$\begin{cases} -\Delta V = \exp(x_1)V^p, & \text{in } \mathbb{R}^N_-, \\ \quad V = 0 & \text{on } \{x_1 = 0\}. \end{cases} \tag{13}$$

Because of the results of Sect. 2, we can assume that on each sphere centered at zero, u_α attains its maximum on the segment joining zero and ξ_0. Let R and δ given by Lemma 3.2, and denote v^*_α the restriction of v_α to the set $\{x \text{ s.t. dist}(x, \partial\Omega_\alpha) < R\}$. If

$$\lim_{\alpha\to+\infty} \int_{B(-R\xi_0/2, R/2)} |v_\alpha|^p \, dx \to 0,$$

then also

$$\lim_{\alpha\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,R/2)} |v^*_\alpha|^p \, dx = 0,$$

so that by a P.L. Lions lemma (see [20] Lemma 1.21), $v^*_\alpha \to 0$ in $L^p(\mathbb{R}^N)$, in contradiction with Lemma 3.2. By Lemma 3.1, v_α is bounded in $\mathcal{D}^{1,2}_0(\mathbb{R}^N_-)$, we may thus assume, going if necessary to a subsequence, that $v_\alpha \rightharpoonup V^*$ in $\mathcal{D}^{1,2}_0(\mathbb{R}^N_-)$. Since v_α does not converge to 0 locally in L^p by what precedes, $V^* \neq 0$. Because for any compact subset $\mathcal{K}$ of $\mathbb{R}^N_-$,

$$\left|\frac{x}{\alpha} + \xi_0\right|^\alpha \to \exp(x_1), \text{uniformly on } \mathcal{K} \text{ for } \alpha \to +\infty, \tag{14}$$

it follows by standard elliptic estimates that V^* satisfies (13) and that the convergence even holds in $\mathcal{C}^k_{\text{loc}}$ for any $k \geq 0$. It remains to prove that V^* is a least energy solution of (13) and that the convergence is strong in $\mathcal{D}^{1,2}_0(\mathbb{R}^N_-)$.

Remember that a least energy solution for equation (3),(5) or (13) is also characterized by the fact that it minimizes the energy $E(u) := (\frac{1}{2} - \frac{1}{p}) \int |\nabla u|^2$ on the Nehari constraint, which in the case of (3) is

$$\mathcal{N} := \left\{ u \in H_0^1(\Omega) \text{ s.t. } \langle J'_{\alpha,p}(u), u \rangle = 0 \right\},$$

and is defined analogously for the other equations. Let $V \in \mathcal{D}_0^{1,2}(\mathbb{R}^N_-)$ be any solution of (13), and $(V_n) \in \mathcal{D}(\mathbb{R}^N_-)$ a sequence of smooth functions converging to V in $\mathcal{D}_0^{1,2}(\mathbb{R}^N_-)$. For each n, there is a unique $t_n > 0$ such that $t_n V_n$ belongs to the Nehari manifold of equation (13). Clearly, since $V_n \to V$ in $\mathcal{D}_0^{1,2}(\mathbb{R}^N_-)$, necessarily $t_n \to 1$. For a fixed n, if α is large enough so that the support of V_n is contained in Ω_α, we have $t_n V_n \in H_0^1(\Omega_\alpha)$ and hence $s_{n,\alpha} t_n V_n$ belongs to the Nehari manifold of (3) for some unique $s_{n,\alpha} > 0$. Because the support of V_n is a fixed compact in $\mathbb{R}^N_-$, using (14) we infer that also $s_{n,\alpha} \to 1$ as $\alpha \to +\infty$. Given $\varepsilon > 0$, we first take n large enough so that $t_n < 1 + \varepsilon$, and then choose α so that $s_{n,\alpha} < 1 + \varepsilon$. Since v_α is a least energy solution, we obtain $\limsup E(v_\alpha) \leq (1+\varepsilon)^4 E(V)$, and since ε is arbitrary,

$$\limsup_{\alpha \to +\infty} \int_{\Omega_\alpha} |\nabla v_\alpha|^2 \, dx \leq \int_{\mathbb{R}^N_-} |\nabla V|^2 \, dx. \tag{15}$$

On the other hands, by weak lower semi-continuity

$$\int_{\mathbb{R}^N_-} |\nabla V^*|^2 \, dx \leq \liminf_{\alpha \to +\infty} \int_{\Omega_\alpha} |\nabla v_\alpha|^2 \, dx. \tag{16}$$

As V was arbitrary, it follows that V^* is a least energy solution of (13) and that $\|v_\alpha\| \to \|V^*\|$. The strong convergence follows by uniform convexity. □

Remark 3.5. 1) The existence of a least energy solution for equation (13) can be established using standard concentration-compactness techniques. Actually, our analysis even gives a rather diverted proof of it.
2) Provided one establishes the uniqueness of a least energy solution of (13) up to translations orthogonal to the x_1 direction, then the convergence of the full sequence v_α to V^* holds.
3) It also follows from the convergence, or directly from Sect. 2, that V^* is radially symmetric in the variable $r := |(x_2, \cdots, x_N)|$.

4. Minimizers of the CKN inequalities

The Caffarelli-Kohn-Nirenberg inequality ensures that

$$\left(\int_{\mathbb{R}^N} |\nabla u|^q |x|^{-aq} \, dx \right) \geq S(q,a,b,N) \left(\int_{\mathbb{R}^N} |u|^p |x|^{-bp} \, dx \right)^{\frac{q}{p}} \tag{17}$$

holds for any function $u \in \mathcal{D}(\mathbb{R}^N)$, provided $0 < \frac{1}{q} - \frac{1}{p} = \frac{1+a-b}{N}$, $q > 1$, $p > 1$, and $a \leq b < \frac{N}{q}$.
Some well known particular versions of this inequality are the Sobolev inequality

($q = 2, a = b = 0, p = 2^*$) and the Hardy inequality ($q = p = 2, a = 0, b = 1$). A simple computation, based on the action of the group of dilations, shows that given q, a and b as above, there is just one p for which such an inequality may hold. Because of this invariance under dilations, the associated variational problem

$$S(q, a, b, N) := \inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^q |x|^{-aq} \, dx, \int_{\mathbb{R}^N} |u|^p |x|^{-bp} \, dx = 1 \right\} \tag{18}$$

is critical, and lacks compactness. The existence of minimizers for $S(q, a, b, N)$ as well as their (partial) symmetry has attracted much attention recently. The situation is as follows. Concerning existence, the picture is complete over the whole range of admissible parameters : a minimizer always exists unless $b = a + 1$ or $b = a < 0$. The answer to the question of symmetry properties of minimizers is less complete. The case $a \geq 0$ was essentially treated by Horiuchi [15] by a clever reduction to the case $a = 0$ (where Schwarz symmetrization gives the answer) : the minimizer is radial and unique up to the invariance of the problem.

When $a < 0$, less is known. If $a = b < 0$, while $S(q, a, b, N)$ is never achieved it can be shown that the infimum over the restricted class of radial functions (which we denote by $S_R(q, a, b, N)$) is always achieved. Hence by a lower semi-continuity argument one easily infers that for $a < b$, but sufficiently close one another, no minimizer is radial. This symmetry breaking was first observed by the authors (see [20]) and studied on a more quantitative basis by Catrina and Wang [10] (see also [11]) in the case $p = 2$. More precisely, the following theorem holds.

Theorem (Catrina and Wang [10]).

i) There exists $a_0 \leq 0$ and a function $h(a)$ defined for $a \leq a_0$ satisfying $h(a_0) = a_0$, $a < h(a) < a + 1$ for $a < a_0$ and $a + 1 - h(a) \to 0$ as $a \to -\infty$, such that no minimizer of $S(2, a, b, N)$ is radial if $a < a_0$ and $a < b < h(a)$.
ii) There is an open set H inside the $a-$ negative region containing $\{(a, a), a < 0\}$ such that symmetry breaking occurs in H. (see also [20])

Remark 4.1.

1) When $N = 2$, a_0 is proved to be 0.
2) When $N = 1$, the situation is fully understood, and symmetry breaking occurs everywhere in the a-negative region (when $N = 1$, b is restricted by $b < a + 1/2$).

The proof in [10] is restricted to $q = 2$, because it relies on a change of variable which behaves nicely with respect to the Laplacian but not the q-Laplacian. The usefulness of such kind of transformation, namely

$$u(x) = |x|^{-\frac{N-2-2a}{2}} v \left(-\log(|x|), \frac{x}{|x|} \right), \tag{19}$$

was observed in a different situation by Gidas and Spruck [13].

The purpose of this section is threefold. We first start by giving a simplified proof of the symmetry breaking result above using a different idea as in [19]. It also has the advantage of being valid for general q.

Whether this result is sharp or not was left open in [10] (unless when $N = 1$), i.e. is it true that symmetry breaking occurs only in a subregion of $\{a < 0\}$ or instead do we have complete symmetry breaking like in dimension 1 ? When $q = 2$ and $N \geq 3$, we give a positive answer to this question.

Finally, using the results of Sect. 2, we will prove that in the region where symmetry breaking holds, partial symmetry is preserved and that roughly speaking the minimizers do only depend on two variables : the radial one and one angular one.

The symmetry breaking result is based on an asymptotic analysis of both $S(q, a, b, N)$ and $S_R(q, a, b, N)$ in the next two lemmas.

Lemma 4.2. *The following equality holds :*

$$S_R(q, a, b, N) = \left(\frac{N-q}{N-q-aq}\right)^{1-q-q/p} S_R(q, 0, b-a, N).$$

Let $u \in \mathcal{C}^\infty(\mathbb{R}^N)$ be radial and positive. Define $s := r^\gamma$ and $v(s) := u(r)$ where γ is to be determined later. We have

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla u|^q |x|^{-aq}\, dx &= \omega_{N-1} \int_0^{+\infty} |u'(r)|^q r^{N-1-aq}\, dr \\
&= \omega_{N-1} \int_0^{+\infty} |v'(r^\gamma)|^q \gamma^q r^{\gamma q - q} r^{N-1-aq}\, dr \\
&= \omega_{N-1} \gamma^{q-1} \int_0^{+\infty} s^{\frac{N-1-aq+(q-1)(\gamma-1)}{\gamma}} |v'(s)|^q\, ds \\
&= \gamma^{q-1} \int_{\mathbb{R}^N} |\nabla v|^q\, dx
\end{aligned} \tag{20}$$

provided

$$\frac{N-1-aq+(q-1)(\gamma-1)}{\gamma} = N-1, \quad \text{i.e.} \quad \gamma = \frac{N-q-aq}{N-q}.$$

For this choice of γ, we now have

$$\begin{aligned}
\int_{\mathbb{R}^N} |u|^p |x|^{-bp}\, dx &= \omega_{N-1} \int_0^{+\infty} |u(r)|^p r^{N-1-bp}\, dr \\
&= \omega_{N-1} \int_0^{+\infty} \gamma^{-1} s^{\frac{N-1-bp+1-\gamma}{\gamma}} |v(s)|^p\, ds \\
&= \gamma^{-1} \int_{\mathbb{R}^N} |u|^p |x|^{-(b-a)p}\, dx,
\end{aligned} \tag{21}$$

the last line being just a consequence of the definition of γ and of the equality $\frac{1}{q} - \frac{1}{p} = \frac{1+a-b}{N}$. The lemma follows. □

Concerning the asymptotic of $S(q, a, b, N)$ we have the following :

Lemma 4.3. *For a sufficiently negative, $S(q,a,b,N) \leq C|a|^{q-N+qN/p}$, where C only depends on q and p.*

Let $u \in \mathcal{C}_0^\infty(B(0,1))$, define $e := (1,0,\cdots,0)$ and $u_a(x) := u\left(|a|(x-e)\right).$ Hence,

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u_a|^q |x|^{-aq}\,dx &= \int_{\mathbb{R}^N} |a|^q |\nabla u(|a|(x-e))|^q |x|^{-aq}\,dx \\ &\leq \left(1+\frac{1}{|a|}\right)^{q|a|} |a|^{q-N} \int_{B(0,1)} |\nabla u|^q\,dx.\end{aligned} \tag{22}$$

And on the other hand,

$$\left(\int_{\mathbb{R}^N} |u_a|^p |x|^{-bp}\,dx\right)^{q/p} \geq \left(1-\frac{1}{|a|}\right)^{q|b|} |a|^{-Nq/p} \left(\int_{B(0,1)} |u|^p\,dx\right)^{q/p}, \tag{23}$$

so that, for a sufficiently negative,

$$S(q,a,b,N) \leq (\exp(2q)S(q,p,B(0,1)) + o(1))\,|a|^{q-N+\frac{Nq}{p}}, \tag{24}$$

where $S(q,p,B(0,1))$ is the best constant for the embedding of $W_0^{1,q}(B(0,1))$ into $L^p(B(0,1))$. This ends the proof. □

Hence, the following symmetry breaking result holds for large negative a :

Theorem 4.4. *Given q and p, there exist $a_0(q,p,N)$ such that for $a < a_0(q,p,N)$ and b satisfying $\frac{1}{q}-\frac{1}{p}=\frac{1+a-b}{N}$, no minimizer for $S(q,a,b,N)$ is radial.*

It follows from Lemma 4.2 that for a sufficiently negative,

$$S_R(q,a,b,N) \geq (S_R(q,0,b-a,N)+o(1))|a|^{-1+q+q/p}.$$

But $-1+q+q/p > q-N+Nq/p$ as soon as $N>1$. The result is then a direct consequence of the preceding lemma. □

Remark 4.5.

1) One can deduce from the previous analysis an asymptotic bound for the region where symmetry breaking occurs, more precisely $(1+a-b) \leq C(\log(|a|))^{-1}$ for $|a|$ large. This bound is not sharp.
2) It also follows from the proof that a_0 can be chosen uniformly as long as p,q remain bounded and the difference $|p-q|$ bounded away form zero.
3) After this work was completed we received [7] where the case $q \neq 2$ is also treated, but the method is much less direct.

We now come to the persistence of a symmetry zone in the region $\{a<0\}$. Remember [15] that the minimization problem

$$S_R(q,a,b,N) := \inf\left\{\int_{\mathbb{R}^N} |\nabla u|^q |x|^{-aq}\,dx, u \text{ radial}, \int_{\mathbb{R}^N} |u|^p |x|^{-bp}\,dx = 1\right\} \tag{25}$$

has, for $N \geq 3$, a unique minimizer up to invariance given by

$$U_{\alpha,\beta}(r) = \left(\alpha + \beta r^{\frac{(p-q)(N-q-aq)}{q(q-1)}}\right)^{-\frac{q}{p-q}}, \tag{26}$$

where α and β are positive constants. Notice also that the appropriate functional spaces in which (18) or (25) are treatable are respectively $\mathcal{D}_a^{1,q} := \{u \in L^1_{\rm loc}$ $s.t.\ |x|^{-a}|\nabla u| \in L^q(\mathbb{R}^N)\}$ and its restriction to radial functions, and hence are depending on a. It seems then difficult to obtain a symmetry result for a negative but small by a perturbative argument from the case $a = 0$ (where Schwarz symmetrization works). Instead, we will take advantage of the $a-$depending transformation (19) to work in a fixed space, but then restrict ourselves to the case $q = 2$.

The Euler-Lagrange equation of (18), after elimination of the Lagrange multiplier using the homogeneity, is given by

$$-\mathrm{div}(|x|^{-2a}\nabla u) = |x|^{-bp}u^{p-1}, \tag{27}$$

and is transformed after the transformation (19) into the Eq. (see [10])

$$-\Delta_g v + (\frac{N-2-2a}{2})^2 v = v^p, \tag{28}$$

where Δ_g refers to the Laplace-Beltrami operator (analysts sign) on the infinite cylinder $\mathbb{R} \times S^{N-1}$. It is convenient to write any point in $\mathbb{R} \times S^{N-1}$ by (t, θ), and to denote by dg its canonical volume form.

Equation (27) is invariant under the group action $U \mapsto U_\varepsilon(.) := \varepsilon^{\frac{2-N}{2}} U(./\varepsilon)$, which translates in the variable V into the invariance under the group action $V \mapsto V_t(.) := V(. - t)$. The following proposition states that the first two eigenspaces of the linearized problem are full-filed by these invariances.

Proposition 4.6. *Let $a = 0$, $b \neq 0$ and U of the form (26) appropriately scaled so that*

$$-\Delta U = |x|^{-bp}U^{p-1}.$$

Then if V corresponds to U under the transformation (19), the spectrum of the operator

$$v \longmapsto \left(-\Delta_g + \left(\frac{N-2}{2}\right)^2 I\right)^{-1} (V^{p-2}v)$$

on $L^2(V^{p-2}dg)$, is of the form

$$1 = \lambda_1 > \lambda_2 = \frac{1}{p-1} > \lambda_3 \cdots \to 0,$$

where $\langle\lambda_1\rangle$ is spanned by V and $\langle\lambda_2\rangle$ is spanned by $\partial_t V$.

With the help of the inverse transform of (19), it suffices to prove that the first two eigenvalues of

$$\begin{cases} -\Delta u = \lambda |x|^{-bp} U^{p-2} u \\ \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \end{cases}$$

are 1 and $p-1$, spanned respectively by U and $\partial/\partial_\varepsilon U_{|\varepsilon=1}$.
We claim that an eigenfunction corresponding to an eigenvalue $\lambda \leq (p-1)$ has to be radial. This being proved, the reduced ODE is then easily seen to have eigenfunctions U and $\partial/\partial_\varepsilon U_{|\varepsilon=1}$ corresponding to $\lambda = 1$ and $(p-1)$. Since U is positive and $\partial/\partial_\varepsilon U_{|\varepsilon=1}$ has only one zero, the classical Sturm-Liouville theory ensures that 1 and $(p-1)$ are the two first eigenvalues, with multiplicity 1.

Let us denote by $\Psi_i,\ i = 0, 1, \cdots$ the sequence of spherical harmonics in dimension N, which form a Hilbert basis of $L^2(S^{N-1})$ and are eigenfunctions of the Laplace-Beltrami operator on S^{N-1} :

$$\begin{cases} -\Delta_\theta \Psi_i = \sigma_i \Psi_i \\ \lambda_0 = 0,\ \sigma_1 = \cdots = \sigma_N = N-1,\ \sigma_{N+1} > \sigma_N. \end{cases}$$

To prove the claim, it suffices to show that for all $i \geq 1$,

$$\varphi_i(r) := \int_{S^{N-1}} u(r,\theta)\Psi_i(\theta)\, d\theta \equiv 0,$$

if u is an eigenfunction corresponding to $\lambda \leq (p-1)$. We have,

$$\begin{aligned} \Delta_r \varphi_i &= \int_{S^{N-1}} \Delta_r u(r,\theta)\Psi_i(\theta)\, d\theta \\ &= \int_{S^{N-1}} -\frac{1}{r^2}\Delta_\theta u(r,\theta)\Psi_i(\theta)\, d\theta - \int_{S^{N-1}} \lambda r^{-bp} U^{p-2} u \Psi_i\, d\theta \\ &= \int_{S^{N-1}} \frac{\sigma_i}{r^2} u(r,\theta)\Psi_i(\theta)\, d\theta - \int_{S^{N-1}} \lambda r^{-bp} U^{p-2}(r) u(r,\theta)\Psi_i\, d\theta \\ &= \left(\frac{\sigma_i}{r^2} - \lambda r^{-bp} U^{p-2}(r)\right) \varphi_i(r). \end{aligned} \tag{29}$$

On the other hand, we have

$$\frac{\partial}{\partial r}(\Delta U) = \frac{\partial}{\partial r}\left(\frac{\partial^2 U}{\partial r^2} + \frac{N-1}{r}\frac{\partial U}{\partial r}\right) = \Delta\left(\frac{\partial U}{\partial r}\right) - \frac{N-1}{r^2}\frac{\partial U}{\partial r},$$

so that multiplying (29) by $\frac{\partial U}{\partial r}$ and integrating on some fixed ball $B(0,R)$ yields

$$
\begin{aligned}
0 &= \int_{B(R)} \Delta\varphi_i \frac{\partial U}{\partial r} + \left(\lambda r^{-bp} U^{p-2} - \frac{\sigma_i}{r^2}\right)\varphi_i \frac{\partial U}{\partial r} \\
&= \int_{B(R)} \varphi_i \Delta\left(\frac{\partial U}{\partial r}\right) + \left(\lambda r^{-bp} U^{p-2} - \frac{\sigma_i}{r^2}\right)\varphi_i \frac{\partial U}{\partial r} \\
&+ \int_{\partial B(R)} \left(\frac{\partial U}{\partial r}\frac{\partial \varphi_i}{\partial r} - \varphi_i \frac{\partial^2 U}{\partial r^2}\right) \\
&= \int_{B(R)} \frac{N-1}{r^2}\varphi_i \frac{\partial U}{\partial r} + \frac{\partial}{\partial r}\left(-r^{-bp}U^{p-1}\right)\varphi_i + \left(\lambda r^{-bp} U^{p-2} - \frac{\sigma_i}{r^2}\right)\varphi_i \frac{\partial U}{\partial r} \\
&+ \int_{\partial B(R)} \left(\frac{\partial U}{\partial r}\frac{\partial \varphi_i}{\partial r} - \varphi_i \frac{\partial^2 U}{\partial r^2}\right) \\
&= \int_{B(R)} \frac{(N-1)-\sigma_i}{2}\varphi_i \frac{\partial U}{\partial r} + \int_{B(R)} \left(\lambda - (p-1)\right) U^{p-2}\frac{\partial U}{\partial r}\varphi_i \\
&+ \int_{B(R)} bpr^{-bp-1}U^{p-1}\varphi_i + \int_{\partial B(R)} \left(\frac{\partial U}{\partial r}\frac{\partial \varphi_i}{\partial r} - \varphi_i \frac{\partial^2 U}{\partial r^2}\right).
\end{aligned} \tag{30}
$$

Let R be the first zero of φ_i different from zero, with the convention that $R = +\infty$ if φ_i is of constant sign. Without loss of generality, we can assume that φ_i is non negative on $(0,R)$. Hence, $\frac{\partial \varphi_i}{\partial r}_{|r=R} \le 0$ and the four integrals in the last line of (30) are all non negative, with the third one being positive unless $\varphi_i \equiv 0$. This ends the claim and the proof. □

Proposition 4.7. *There exist $\varepsilon > 0$, a neighborhood $\mathcal{V}$ of V in $H^1(\mathbb{R}\times S^{N-1})$ and a function*

$$
\begin{aligned}
\mathcal{U} : [0,\varepsilon) &\longrightarrow \mathcal{V} \\
\lambda &\longmapsto \mathcal{U}(\lambda)
\end{aligned}
$$

such that for all $\lambda \in [0,\varepsilon)$,

$$
-\Delta u + \left(\frac{N-2}{2}\right)^2 u = u^{p-1} - \lambda u \Longleftrightarrow u = \mathcal{U}(\lambda)(\cdot - t_0, \cdot)
$$

for some $t_0 \in \mathbb{R}$.

Let K denote the operator $-\Delta_g + (\frac{N-2}{2})^2 I$ and

$$
T(\lambda, u) := \left(u - K^{-1}(u^{p-1} - \lambda u) + \langle u, \partial_t V\rangle \partial_t V\right)
$$

defined on $L^2(V^{p-2}dg)$. We have,

$$
\partial T(0,V)(v) = v - K^{-1}((p-1)V^{p-1}v) + \langle v, \partial_t V\rangle \partial_t V.
$$

If $\partial T(0,V)(v) = 0$, then taking the scalar product of the preceding equality with $\partial_t V$ we get

$$0 = \|\partial V\|^2_{L^2(V^{p-2}dg)} \langle v, \partial_t V\rangle.$$

Hence,

$$K^{-1}(V^{p-1}v) = \frac{1}{p-1}v \qquad \text{and} \qquad \langle v, \partial_t V\rangle = 0,$$

so that $v = 0$ by Proposition 4.6. Conversely, if $w \in L^2(V^{p-2}dg)$, we write

$$w_0 := w - \langle w, \partial_t V\rangle \frac{\partial_t V}{\|\partial_t V\|^2},$$

so that $w_0 \in (\partial_t V)^{\perp}$. By the Fredholm alternative, there exists $v_0 \in (\partial_t V)^{\perp}$ such that

$$-(p-1)K^{-1}(V^{p-1}v_0) + v_0 = w_0.$$

Then, if $v := v_0 + \langle w, \partial V\rangle \partial_t V \|\partial_t V\|^{-2}$, $\partial T(0,V)(v) = w$. Being continuous and bijective, $\partial T(0,V)$ is a homeomorphism; the implicit function theorem ends the proof. □

Theorem 4.8. *Let $N \geq 3$. Given $2 < p < 2^*$ there exists $\varepsilon > 0$ such that for any $0 \leq -a \leq \varepsilon$ and b satisfying $1/2 - 1/p = (1 + a - b)/N$, any minimizer $u \in \mathcal{D}^{1,2}_a(\mathbb{R}^N)$ of $S(2,a,b,N)$ is radially symmetric.*

Let u_a be a minimizer for $S(2,a,b,N)$, after applying the transformation (19) to u_a and a possible scaling $u_a \to \alpha_a u_a$, we get a solution v_a of

$$-\Delta_g v + \left(\frac{N-2}{2}\right)^2 v = v^{p-1} - \lambda_a v$$

where $\lambda_a = (N-2-2a)^2/4 - (N-2)^2/4$. Up to a translation, we can assume that $\langle v_a, \partial_t V\rangle = 0$, so that by standard elliptic estimates and uniqueness we have $v_a \to V$. By the preceding proposition, v_a is unique and hence constant in the θ variable. Coming back to the original variable, we deduce that u_a was radially symmetric, which ends the proof. □

Remark 4.9. When $N = 2$, the conditions for the Caffarelli-Kohn-Nirenberg inequalities to hold imply that $a < 0$. The above method to deduce preserved symmetry by a continuation argument from the situation $a = 0$ hence can no longer be used. It is an interesting open question in that case to decide between preserved or broken symmetry for small negative a.

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