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Hölder continuity of harmonic maps from Riemannian polyhedra to spaces of upper bounded curvature

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Abstract. This is an addendum to the recent Cambridge Tract "Harmonic maps between Riemannian polyhedra", by J. Eells and the present author. Hölder continuity of locally energy minimizing maps φ from an admissible Riemannian polyhedron X to a complete geodesic space Y is established here in two cases: (1) Y is simply connected and has curvature $\leqslant 0$ (in the sense of A.D. Alexandrov), or (2) Y is locally compact and has curvature $\leqslant 1$, say, and $\varphi(X)$ is contained in a convex ball in Y satisfying bi-point uniqueness and of radius $R < \pi/2$ (best possible). With Y a Riemannian polyhedron, and $R < \pi/4$ in case (2), this was established in the book mentioned above, though with Hölder continuity taken in a weaker, pointwise sense. For X a Riemannian manifold the stated results are due to N.J. Korevaar and R.M. Schoen, resp. T. Serbinowski.

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1. Introduction and preliminaries

This article is an addendum to the recent book [EF], by J. Eells and the present author, but written in a manner that hopefully permits independent reading. My recent discussions with Professor Eells on the topic of the present work have led to improvements in the presentation.

Certain regularity results for harmonic maps from an admissible Riemannian polyhedron, obtained in [EF], will be established here under weaker, and mostly optimal, hypotheses on the target space and/or the size of the range of the map. Furthermore, Hölder continuity is taken here in the customary locally uniform sense. For this purpose a weak Poincaré inequality for maps is needed in a similar generality (Proposition 2 and Corollary 1). Altogether, Riemannian polyhedra can be replaced as targets by *geodesic spaces* in the regularity and existence theory for harmonic (or locally energy minimizing) maps as developed in [EF, Chaps. 10 and 11]. The domain of such a map shall be an admissible Riemannian polyhedron (X, g).

The main results of the present paper assert that a locally energy minimizing map of (X, g) into a simply connected complete geodesic space of nonpositive Alexandrov curvature is Hölder continuous (Theorem 1); while such a map into a complete locally compact geodesic space of curvature $\leqslant K$ ($K > 0$) is Hölder

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continuous provided that it maps X into a ball of radius $\leqslant \pi/(2\sqrt{K})$ satisfying bipoint uniqueness (Theorem 2).

Alternatively, the target can be a smooth Riemannian *manifold* without boundary (Theorems 3 and 4). The study of maps into a manifold is not quite a particular case of that of a geodesic space target because our concept of energy of maps into a geodesic space, extending that of Korevaar and Schoen [KS1] for maps from a (smooth) Riemannian manifold, requires that the Riemannian metric g on the domain polyhedron X be simplexwise smooth, while bounded measurable components of g suffices for a good concept of energy of maps into a Riemannian manifold. The two concepts of energy are identical (up to a constant factor) when g is simplexwise smooth and when moreover the map is bounded [EF, Theorem 9.2] or the target manifold is simply connected and complete with nonpositive curvature [F3, Proposition 3].

The article [F3], likewise an outgrowth of [EF], deals with existence, uniqueness, and continuity up to the boundary for the solution to the Dirichlet problem for harmonic maps in the same four settings as here.

Recall from [EF, Chapter 4] that a polyhedron X is termed *admissible* if it is dimensionally homogeneous, say of dimension m, and if any two m-simplexes of X with a common face s ($\dim s = 0, 1, \ldots, m-2$) can be joined by a chain of m-simplexes containing s, any two consecutive ones of which have a common $(m-1)$-face containing s (this is trivially fulfilled if $m = 1$).

We denote throughout by X an m-dimensional admissible polyhedron, connected and locally finite, and endowed with a nondegenerate Riemannian metric g whose restriction to each closed m-simplex of X is (at least) bounded and measurable. The associated volume measure on X is denoted by $\mu_g = \mu$, the intrinsic (Riemannian) distance on X by $d_X^g = d_X$, and the closed ball with centre $x \in X$ and radius r by $B_X(x, r)$. The interior of a set $A \subset X$ is denoted by A°.

The metric space (X, d_X) is *locally compact*. Furthermore, (X, d_X) is a *length space*, i.e., for any two points $x_0, x_1 \in X$, $d_X(x_0, x_1)$ is the infimum of the lengths of all rectifiable paths joining x_0 to x_1, [EF, Proposition 4.1]. It follows by the Hopf-Rinow theorem that (X, d_X), if complete, is a *geodesic space*, i.e., the above infimum is a true minimum (cf. e.g. [EF, Chapter 2]).

Based on the work of Korevaar and Schoen [KS1] a concept of *energy* of a map φ of (X, g) into a metric space (Y, d_Y) is developed in [EF, Chapter 9] under the hypothesis that g be *simplexwise smooth* (i.e., g should have a C^∞-smooth restriction to each topdimensional closed simplex of X). The map φ is supposed first of all to have separable range and to be of class $L^2(X, Y)$ in the sense that the distance function $d_X(\varphi(\cdot), y)$ is of class $L^2(X, \mu)$ for some and hence for any point $y \in Y$. The *approximate energy density* $e_\varepsilon(\varphi) \in L^1_{\mathrm{loc}}(X, \mu)$ is then defined for $\varepsilon > 0$ at every point $x \in X$ by

$$e_\varepsilon(\varphi)(x) = \int_{B_X(x,\varepsilon)} \frac{d_Y^2(\varphi(x), \varphi(x'))}{\varepsilon^{m+2}} d\mu(x'). \tag{1.1}$$

Definition 1. The energy of $\varphi : (X, g) \to (Y, d_Y)$ is

$$E(\varphi) = \sup_{f \in C_c(X,[0,1])} \Big(\limsup_{\varepsilon \to 0} \int_X f e_\varepsilon(\varphi)\, d\mu \Big) \quad (\leqslant \infty), \tag{1.2}$$

where C_c stands for continuous functions of compact support. $W^{1,2}_{\mathrm{loc}}(X, Y)$ denotes the space of all maps $X \to Y$ for which $E(\varphi_{|U}) < \infty$ for every relatively compact open set $U \subset X$ (equivalently: the above $\limsup$ is finite for every f).

If X is compact then (1.2) reduces to

$$E(\varphi) = \limsup_{\varepsilon \to 0} \int_X e_\varepsilon(\varphi)\, d\mu.$$

It is shown that, if $\varphi \in W^{1,2}_{\mathrm{loc}}(X, Y)$ (and only then), there exists a nonnegative function $e(\varphi) \in L^1_{\mathrm{loc}}(X, \mu)$, called the *energy density* of φ, such that $e_\varepsilon(\varphi) \to e(\varphi)$ as $\varepsilon \to 0$, in the sense of weak convergence as measures:

$$\lim_{\varepsilon \to 0} \int_X f e_\varepsilon(\varphi)\, d\mu = \int_X f e(\varphi)\, d\mu \tag{1.3}$$

for every $f \in C_c(X)$. In the affirmative case it follows from (1.2), (1.3) that

$$E(\varphi) = \int_X e(\varphi)\, d\mu.$$

For the above assertions, see Steps 2, 3, and 4 of the proof of [EF, Theorem 9.1].[1] These steps are independent of the general requirement in [EF] that also the target of maps $X \to Y$ shall be locally compact. Step 1, however, leading to quasicontinuity of φ, does make use of the local compactness of Y.

A function $u : X \to \mathbb{R}$ is of class $W^{1,2}_{\mathrm{loc}}(X, \mathbb{R})$ in the sense of Definition 1 (with $Y = \mathbb{R}$) if and only if $u \in W^{1,2}_{\mathrm{loc}}(X)$ as defined in [EF, p. 63f.]. [2] If that is the case, the energy density of u equals

$$e(u) = c_m |\nabla u|^2 = c_m g^{ij} \partial_i u\, \partial_j u \qquad \text{a.e. in } X, \tag{1.4}$$

with the usual summation convention. Here $c_m = \omega_m/(m+2)$, ω_m being the volume of the unit ball in $\mathbb{R}^m$. See [EF, Corollary 9.2], which is based on [KS1, Theorem 1.6.2] (where X is a Riemannian domain in a Riemannian manifold), and is also a particular case of [EF, Theorem 9.2].

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[1] In the second paragraph of Step 3 of the proof of [EF, Theorem 9.1] the point is that it makes no difference whether $M = (s_1 \cup s_s)^\circ$ is considered with the given Riemannian metric on X or as the Riemannian domain (M, g^e), g^e denoting a "Euclidean" Riemannian metric on X.

[2] The gradient $\nabla u \in L^2(X)$ defined there is independent of the choice of Cauchy sequence (u_j) in $(\mathrm{Lip}^{1,2}(X), ||\cdot||)$ with $u_j \to u$ in $L^2(X)$. This follows by considering the restrictions of u and the u_j to each open m-simplex of X, the assertion being known to hold for a Riemannian manifold X, see [FHK].

By polarization, (1.3) and (1.4) lead, for any two functions $u, v \in W^{1,2}_{\mathrm{loc}}(X)$, to

$$\int_{B_X(x,\varepsilon)} \frac{(u(x)-u(x'))(v(x)-v(x'))}{\varepsilon^{m+2}}\, d\mu(x') \to c_m \langle \nabla u(x), \nabla v(x)\rangle \tag{1.5}$$

as $\varepsilon \to 0$, in the sense of weak convergence as measures.

Preservation of finite energy of a map under postcomposition with a Lipschitz map [EF, Corollary 9.1] obviously does not require local compactness of the target, nor does the lower semicontinuity of energy [EF, Lemma 9.1].

For any map $\varphi \in W^{1,2}_{\mathrm{loc}}(X, Y)$ and any open set $U \subset X$ it is easily shown that $e(\varphi_{|U}) = e(\varphi)$ μ-a.e. in U, and so $E(\varphi_{|U}) = \int_U e(\varphi)\, d\mu$. It follows that if two maps $\varphi, \psi \in W^{1,2}_{\mathrm{loc}}(X, Y)$ agree μ-a.e. off a closed set $F \subset X$ then $e(\varphi) = e(\psi)$ μ-a.e. in $X \setminus F$, and hence (if $E(\varphi) < \infty$)

$$E(\psi) - E(\varphi) = \int_F (e(\psi) - e(\varphi))\, d\mu. \tag{1.6}$$

A map $\varphi \in W^{1,2}_{\mathrm{loc}}(X, Y)$ is said to be *locally E-minimizing*, or to be a local E-minimizer, if X can be covered by relatively compact subdomains U for each of which $E(\varphi_{|U}) \leqslant E(\psi_{|U})$ for every map $\psi \in W^{1,2}_{\mathrm{loc}}(X, Y)$ such that $\psi = \varphi$ μ-a.e. in $X \setminus U$, [EF, Definition 10.1]. It would clearly amount to the same to require that $\psi = \varphi$ μ-a.e. off some closed set $F \subset U$ (F depending on ψ). If $E(\varphi) < \infty$ (e.g. if X is compact) then the inequality $E(\varphi_{|U}) \leqslant E(\psi_{|U})$ may therefore be replaced equivalently by $E(\varphi) \leqslant E(\psi)$, according to (1.6).

These two comments to the definition of a local E-minimizer apply as well to the following concept of harmonic map, proposed in [EF, Definition 12.1]:

A *harmonic map* $\varphi : X \to Y$ is a *continuous* map of class $W^{1,2}_{\mathrm{loc}}(X, Y)$ which is *bi-locally E-minimizing* in the sense that X can be covered by relatively compact subdomains U for each of which there is an open set $V \supset \varphi(U)$ in Y such that $E(\varphi_{|U}) \leqslant E(\psi_{|U})$ holds for every *continuous* map $\psi \in W^{1,2}_{\mathrm{loc}}(X, Y)$ *with* $\psi(U) \subset V$ and $\psi = \varphi$ μ-a.e. in $X \setminus U$.

Clearly, every continuous local E-minimizer is harmonic (take $V = Y$). In the setting of Theorem 1 or Theorem 2 below, *a continuous local E-minimizer $\varphi : X \to Y$ is the same as a harmonic map*, [EF, Lemma 12.1, Remark 12.1].

2. Formulation of results

We proceed to formulate and comment on the results of the present paper. Proofs are mostly given in subsequent sections. Compactness of the admissible m-dimensional Riemannian polyhedron X is only required when stated. For the concepts of (weakly) harmonic and (weakly) sub/superharmonic functions, see [EF, Definitions 5.1, 5.2, 7.1, and Theorem 7.1].

Some inequalities for harmonic and superharmonic functions. In this subsection the nondegenerate Riemannian metric g on the admissible polyhedron X is merely required to be *bounded* and *measurable* on each m-simplex.

Lemma 1. *Suppose $m > 1$. For any compact $K \subset X$ and any constant $\kappa > 1$ there are constants $\varrho > 0$ and $c > 1$, depending only on X, Λ_X, K, κ, such that for any point $a \in K$ and any harmonic function $u \geqslant 0$ in $B^e_X(a, \kappa r)^\circ \setminus \{a\}$*

$$\max_{\partial B^e_X(a,r)} u \leqslant c \min_{\partial B^e_X(a,r)} u \qquad \text{for} \quad r \leqslant \varrho.$$

Here $\Lambda_X \geqslant 1$ denotes an *ellipticity constant* for g on X (or just on K), i.e., a constant such that, in terms of Euclidean coordinates $x^1, \ldots, x^m$ on each m-simplex s of X,

$$\Lambda_X^{-2} \sum_{i=1}^{m} (\xi^i)^2 \leqslant g_{ij}(x)\xi^i\xi^j \leqslant \Lambda_X^2 \sum_{i=1}^{m} (\xi^i)^2$$

μ-a.e. for $x \in X$, and for every $(\xi^1, \ldots, \xi^m) \in \mathbb{R}^m$. Furthermore, $B^e_X(x, R)^\circ$ denotes the interior of the closed ball $B^e_X(x, R)$ centred at x and of radius R, but relative to the *Euclidean Riemannian metric* g^e on X, given by $g_{ij} = \delta_{ij}$.

The proof of Lemma 1, given in Sect. 3, is an adaptation of the proof of [EF, Proposition 6.2] (in which u was required to be harmonic and $\geqslant 0$ in all of $X \setminus \{a\}$).

The next lemma compares the Green kernels G_U and G^e_U on $U = B^e_X(a, \kappa r)^\circ$ relative to the given Riemannian structure g, resp. the Euclidean Riemannian structure g^e on X, cf. [EF, Theorem 7.3].

Lemma 2. *For any compact $K \subset X$ and any constant $\kappa > 1$ there are constants $\varrho > 0$ and $c > 1$, depending only on X, Λ_X, K, κ, such that for every $a \in K$ and $r \leqslant \varrho$:*

$$c^{-1} G^e_U(x, y) \leqslant G_U(x, y) \leqslant c\, G^e_U(x, y) \qquad \text{for} \ \ x, y \in B^e_X(a, r),$$

whereby $U = B^e_X(a, \kappa r)^\circ$.

Essentially, this lemma can be derived from a result by Biroli and Mosco [BM, Theorem 1.3]. In Sect. 4 below we give an alternative proof by reducing Lemma 2 to [EF, Lemma 7.2] (in which a and r were fixed). This is done by use of an iteration procedure, exploiting the polyhedral structure of the domain space X. The same iteration scheme is employed in the proof of Lemma 1 above.

We proceed to estimate mollified Green functions $G_U \mu_r$, cf. [GH, p. 134]. For a given point $a \in X$ write

$$\mu_r = \mu_{|B^e_X(a,r)}.$$

Lemma 3. *For any compact $K \subset X$ and any constant $\kappa > 1$ there are constants $\varrho > 0$ and $\gamma_1 > \gamma_2 > 0$, depending only on X, Λ_X, K, κ, such that for every $a \in K$ and $r \leqslant \varrho$:*

$$G_U \mu_r \leqslant \gamma_1 r^2 \qquad \text{on} \ \ U = B^e_X(a, \kappa r)^\circ,$$
$$G_U \mu_r \geqslant \gamma_2 r^2 \qquad \text{on} \ \ B^e_X(a, r).$$

This is obtained just like [EF, Sublemma 10.1] (in which a was fixed), with [EF, Lemma 7.2] replaced by Lemma 2 above.

Using Lemmas 1, 2, and 3, we establish in Sect. 5 the following proposition, of which part (a) is essentially contained in [BM, Theorem 1.1]:

Proposition 1. (Harnack inequalities for variable balls.) *For any compact $K \subset X$ and any constant $\kappa > 1$ there are constants $\varrho > 0$ and $c > 1$, depending only on X, Λ_X, K, κ, such that the following holds for every $a \in K$ and $r \leqslant \varrho$:*

(a) *If u is harmonic $\geqslant 0$ in $B_X^e(a, \kappa r)^\circ$ then*

$$\max_{B_X^e(a,r)} u \leqslant c \min_{B_X^e(a,r)} u.$$

(b) *If u is superharmonic $\geqslant 0$ in $B_X^e(a, \kappa r)^\circ$ then*

$$\frac{1}{\mu(B_X^e(a,r))} \int_{B_X^e(a,r)} u \, d\mu \leqslant c \min_{B_X^e(a,r)} u.$$

Maps into metric spaces. In this subsection we suppose that the Riemannian metric g of the admissible polyhedron X be *simplexwise smooth*. We first consider maps of X into an arbitrary metric space (Y, d_Y). In Sect. 6 we establish the following proposition and corollary:

Proposition 2. (Poincaré inequality.) *Suppose that X is compact. Every finite energy map φ of (X, g) into a metric space (Y, d_Y) satisfies the inequality*

$$\inf_{y \in Y} \int_X d_Y^2(\varphi(x), y) \, d\mu(x) \leqslant C \, E(\varphi), \tag{2.1}$$

where C only depends on X as a polyhedron and on the ellipticity constant Λ_X.

This was obtained in [EF, Proposition 9.1] under the extra hypothesis that there exist a (globally) bi-Lipschitz bijection of Y onto a closed subset of a Euclidean space (e.g., Y is a compact Riemannian polyhedron); this allowed for a reduction to the case $Y = \mathbb{R}$.

For the case that (X, g) is a Riemannian manifold, Proposition 2 is contained in a result by Korevaar and Schoen [KS2, §1.4].

The infimum on the left hand side of (2.1) is of course no bigger than the integral mean of $\int d_Y^2(\varphi(x), \varphi(x')) \, d\mu(x)$ as a function of $x' \in X$, taken with respect to $d\mu(x')$; and (2.1) is therefore a consequence of the inequality

$$\frac{1}{\mu(X)} \int_X \int_X d_Y^2(\varphi(x), \varphi(x')) \, d\mu(x) \, d\mu(x') \leqslant C \, E(\varphi). \tag{2.2}$$

Conversely, (2.1) implies that (2.2) holds (with C replaced by $4C$) in the case where Y is a simply connected complete geodesic space of nonpositive curvature in the sense of A.D. Alexandrov [A1], [A2] (cf. [EF, Chapter 2]).[3] And in that case it is well known that the infimum on the left of (2.1) is attained by precisely one point $y = \overline{\varphi}$ of Y, called the *meanvalue* of φ over (X, μ), [KS1, Lemma 2.5.1].

[3] This follows by triangle comparison for the triple $\varphi(x), \varphi(x'), y$ for any $y \in Y$:

$$d_Y^2(\varphi(x), \varphi(x')) \leqslant 2d_Y^2(\varphi(x), y) + 2d_Y^2(\varphi(x'), y) - 4d_Y^2(z, y),$$

where z denotes the midpoint of the geodesic segment $\varphi(x)\varphi(x')$.

Corollary 1. (Weak Poincaré inequality for variable balls.) *There exist constants* $\varrho > 0$, $\kappa > 1$, *and* $C > 0$, *depending only on* X, Λ_X, *such that*

$$\inf_{y \in Y} \int_{B_X(a,r)} d_Y^2(\varphi(x), y)\, d\mu(x) \leqslant Cr^2 \int_{B_X(a,\kappa r)} e(\varphi)\, d\mu \qquad (2.3)$$

for any $a \in X$ *and* $r \leqslant \varrho$.

Compactness of X is not required here. Again, there is a more general version in the spirit of (2.2). In view of [St, Theorem 2.4], Corollary 1 holds even with $\kappa = 1$ (strong Poincaré inequality), but we shall not make use of that fact.

Next we pass to the regularity of local E-minimizers as defined in Sect. 1. We shall say that a map $\varphi : (X, d_X) \to (Y, d_Y)$ between metric spaces is *pointwise Hölder continuous* if, for any point $a \in X$, there are positive constants A, α, δ (allowed to depend on a) such that

$$d_Y(\varphi(x), \varphi(a)) \leqslant A\, d_X(x,a)^\alpha \qquad \text{when } d_X(x,a) < \delta.$$

If this holds for all points $a \in X$ with A, α, δ independent of a, at least locally, we say that φ is locally uniformly Hölder continuous, or simply *Hölder continuous*; it follows then that A, α, δ can be chosen independently of a in any given compact subset of X. For example, every harmonic function on an admissible Riemannian polyhedron is Hölder continuous, by [EF, Theorem 6.3]. [4] In [EF] the distinction between Hölder continuity and pointwise Hölder continuity was not made; and in [EF, Chaps. 10–12] Hölder continuity was understood in the pointwise sense, like in [Jo]. In the present paper, local compactness of the target space Y is only assumed when stated.

Theorem 1. *Let* (Y, d_Y) *be a simply connected complete geodesic space of non-positive curvature. Every local* E*-minimizer* $\varphi : X \to Y$ *is Hölder continuous (after correction on a null set for* μ*).*

In [EF, Theorem 10.1] it was shown that φ is pointwise Hölder continuous, under the extra hypothesis that (Y, d_Y) be (locally) Lipschitz homeomorphic with a closed subset of a Euclidean space (in particular locally compact); this was imposed in order to ensure the validity of the weak Poincaré inequality for balls. The proof was based on a general procedure of Jost [Jo]. The proof of Theorem 1, given in Sect. 7 below (without the indicated extra hypothesis, and with locally uniform Hölder continuity), will proceed by reduction to the proof in [EF] while using Proposition 1 (b), Lemma 3, and Corollary 1 above.

Theorems 11.1 and 11.2 in [EF] likewise hold for geodesic space targets Y (cf. [EF, Remark 11.4]) because the homotopy classification stated in [EF] does not require that X and Y have polyhedral structure; for a proof of that classification see [Sp, Chap. 7, §3, Theorem 8].

For the case where (X, g) is a Riemannian manifold, Theorem 1 was obtained by Korevaar and Schoen [KS1, §2.4]. Earlier this had been established by Gromov and Schoen [GS] when Y is a Riemannian polyhedron. Local E-minimizers φ

[4] In [EF, Theorem 6.3] the constant factor A is missing on the right hand side of eq. (6.30).

from a Riemannian manifold were shown in these two articles to be even Lipschitz continuous; in the more general setting of Theorem 1 above (with polyhedral domain X) φ need not be Lipschitz continuous, not even if $(Y, d_Y) = \mathbb{R}$, cf. [Ch], or [EF, Example 6.1], where X is the Riemann surface of $\sqrt{z}$, including the point 0, and $\varphi(z) = \sqrt{z}$.

Remark (with thanks to James Eells). The notion of Lipschitz manifold in [EF, Example 3.6] has been adjusted by Donaldson and Sullivan [DS] to include quasi-conformal manifolds – described by coordinate atlases whose transition maps are quasiregular homeomorphisms; i.e., homeomorphisms $\theta \in W^{1,n}_{\mathrm{loc}}$ (n being the dimension) whose Jacobian determinant J_θ satisfies $|d\theta(x)| \leqslant K\, J_\theta(x)$ a.e. in the domain U of θ, for some constant $K \geqslant 1$.

Basic properties:

(1) If θ is quasiregular, then there exists $p > n$ such that $\theta \in W^{1,p}_{\mathrm{loc}}$, [BI, Theorem 5.1].

(2) Every quasiregular map is locally Hölder continuous [BI, Theorem 5.2], [HKM, Theorem 14.44].

(3) In the direct methods of variational theory, quasiregular maps appear as quasiminima [GG, §2b], [G, Chapter IX, §1]. However, Hölder continuity for quasiminima is much more problematical [G, Chapter IX, p. 253]; in particular, Harnack's inequality is in doubt [GG, after Definition 1.1].

In the next theorem the complete geodesic target space (Y, d_Y) has instead *upper bounded curvature*, again in the Alexandrov sense. For our proof (in Sect. 8) we need to assume that Y is *locally compact*. Every bounded closed subset of Y is then compact, by the Hopf-Rinow theorem (cf. e.g. [EF, Chap. 2]).

Theorem 2. *Let (Y, d_Y) be a locally compact, complete geodesic space of curvature $\leqslant K$ for some constant $K > 0$. A local E-minimizer $\varphi : X \to Y$ is then Hölder continuous provided that its range $\varphi(X)$ is contained in a closed convex ball $B = B_Y(q, R)$ of radius $R < \pi/(2\sqrt{K})$ satisfying bipoint uniqueness.*

With Y as stated (in fact, even without local compactness), a set $V \subset Y$ of diameter $< \pi/\sqrt{K}$ is said to satisfy *bipoint uniqueness* if every pair of points y_0, y_1 of V can be joined by a unique Y-geodesic in V which is minimizing within V (not necessarily within Y unless V is convex), [5] and if that segment varies continuously with its endpoints (in the uniform topology on curves). Since geodesics in V have no conjugate points [AB1, Theorem 3] this definition agrees with the more restrictive one given by Alexander and Bishop in [AB2], where it is also shown that the continuous dependence is automatic in case Y is locally compact. Bipoint uniqueness of V implies that V is simply connected (two paths in V with the same endpoints are connected even by a geodesic homotopy). As shown by Alexandrov [A1], [A2], a convex set $V \subset Y$ satisfies bipoint uniqueness if and only if every geodesic triangle with vertices in V and perimeter $< 2\pi/\sqrt{K}$ admits *triangle comparison*; then the distance function $d_Y(q, \cdot)$ is convex in V, and hence every ball

[5] See [AB2, §4] for the simple example of a ball V of radius 1 in an ordinary cylinder Y of circumference 3. Thus V need not be convex (in Y), contrary to what was stated in [EF, p. 28]. The proof of [EF, Theorem 11.4] therefore requires that B be convex.

in V is convex. A ball B (open or closed) in Y of radius $R < \pi/(2\sqrt{K})$ satisfies bipoint uniqueness if and only if B satisfies *radial uniqueness* with respect to its centre q; i.e., the particular case of bipoint uniqueness where one of the above two points y_0, y_1 of $V = B$ is q (kept fixed), [AB2, Theorem 4.3 and Proposition 4.2].

It is shown that Theorem 2 likewise holds for an *open* convex ball $B = V_Y(q,R) := \{y \in Y : d_Y(y,q) < R\}$ (instead of a closed ball); this is not obvious because bipoint uniqueness may get lost when passing to the closure. It is not known to the author whether convexity of B is needed in Theorem 2 (cf. Theorem 4), and whether local compactness of Y is needed in Theorem 2.

Under the more restrictive hypothesis $R < \pi/(4\sqrt{K})$ (instead of $R < \pi/(2\sqrt{K})$) and with Y for example a Riemannian polyhedron, Theorem 2 was established in [EF, Theorem 10.2] (for a correction, see footnote 6 in Sect. 8 below); our proof for the general case is done by reduction to the case $R < \pi/(4\sqrt{K})$, in which local compactness of Y is not needed. For the case where (X, g) is a Riemannian manifold the theorem was obtained by Serbinowski [Se], even with Lipshitz continuity, and without assuming Y locally compact.

As a consequence of Theorem 2 above, the last assertion of [EF, Theorem 11.4] (about Hölder continuity in $X \setminus bX$ of the variational solution to the Dirichlet problem for maps $X \to Y$ with prescribed restriction to the boundary bX) holds without the stated additional hypothesis that $R < \pi/(4\sqrt{K})$; and Y can now be a locally compact complete geodesic space (instead of a Riemannian polyhedron).

An example by Hildebrandt, Kaul, and Widman [HKW, §6] shows that Theorem 2 is sharp (even when X and Y are Riemannian manifolds), in the sense that a local E-minimizer $\varphi : X \to Y$ with range $\varphi(X)$ in a closed ball of radius $R = \pi/(2\sqrt{K})$ satisfying bipoint uniqueness, need not be continuous (cf. [EF, Example 12.3]).

Especially in the proof of Theorem 2 we shall use "fine" potential theory, i.e., potential theory relative to the *fine topology* of H. Cartan on X – the coarsest topology in which all subharmonic functions are continuous. The fine topology is finer than the metric topology on X (and strictly finer when $\dim X > 1$).

Maps into a Riemannian manifold. In this subsection the Riemannian metric g on each closed m-simplex of the m-dimensional admissible polyhedron X is allowed to be just *bounded* and *measurable* (rather than smooth), and as always nondegenerate, i.e., with elliptic bounds (cf. [EF, Eq. (4.1)]). For maps φ of (X, g) into an n-dimensional separable Riemannian C^1-*manifold* (N, h) (without boundary) there is an immediate concept of energy density $e(\varphi)$ and energy $E(\varphi) = \int e(\varphi)\, d\mu$ in terms of a Riemannian isometric embedding of (N, h) onto a submanifold of some $\mathbb{R}^q$, as in Nash's theorem. If $\varphi^1, \dots, \varphi^q$ denote the components of $\varphi : X \to \mathbb{R}^q$ then φ has locally finite energy if and only if $\varphi^1, \dots, \varphi^q$ are of Sobolev class $W^{1,2}_{\text{loc}}(X)$. In the affirmative case,

$$e(\varphi) = |\nabla\varphi|^2 = \sum_{i=1}^{q} |\nabla\varphi^i|^2.$$

There is an equivalent covariant energy concept (not using Nash's theorem), defined in terms of a countable atlas of charts $\eta : V \to \mathbb{R}^n$, [EF, Definition 9.2, Lemma 9.3]. For simplicity of enunciation we suppose here that X is *compact*.

Definition 2. A map $\varphi : (X, g) \to (N, h)$ has finite energy if and only if

(i) φ can be redefined on a μ-null set so as to become *quasicontinuous*; and
(ii) the components $\varphi^1, \dots, \varphi^n$ of φ in each chart $V \to \mathbb{R}^n$ are of class $W^{1,2}$ on the quasiopen set $\varphi^{-1}(V)$; and
(iii) the *energy density* $e(\varphi)$ of φ, defined μ-a.e. in each of the sets $\varphi^{-1}(V)$ covering X by

$$e(\varphi) = \operatorname{trace}_g \varphi^* h = (h_{\alpha\beta} \circ \varphi)\langle \nabla\varphi^\alpha, \nabla\varphi^\beta \rangle \tag{2.4}$$

(summed over $\alpha, \beta = 1, \dots, n$), is integrable over (X, μ).

The *energy* of φ is then defined by $E(\varphi) = \int e(\varphi)\, d\mu$.

For the terms quasiopen and quasicontinuous, see [EF, Definitions 7.2, 7.3], where the *capacity* is associated with the Dirichlet form $\int_X |\nabla u|^2 d\mu$, $u \in L_0^{1,2}(X)$, see [EF, p. 21 and Proposition 7.3], possibly applied locally, cf. [EF, Remark 7.2].

Note that the above pre-images $\varphi^{-1}(V)$ in (ii) and (iii) are *quasiopen* in view of (i); we refer to Kilpeläinen and Malý [KM] for the Sobolev space $W^{1,2}(U)$ on a quasiopen set $U \subset \mathbb{R}^m$ and the gradient operator ∇ on it, equally applicable in the present setup, where $U \subset X$. Cf. [EF, text following Definition 7.4].

If $\varphi(X)$ has compact closure in N then (iii) is a consequence of (i) and (ii) because only finitely many coordinate patches V meet $\varphi(X)$. If φ is even continuous (or if $\varphi(X)$ is contained in a single coordinate patch V, as in Theorems 3 and 4 below) then $W^{1,2}(\varphi^{-1}(V))$ becomes the usual Sobolev (1,2)-space on $\varphi^{-1}(V)$.

The above definition is independent of the choice of countable atlas on N, [EF, text following Definition 9.2]

If g is simplexwise smooth then the concepts of energy of a map $\varphi : X \to N$ according to Definition 1 and Definition 2 are identical (up to the constant factor c_m from (1.4)) in either of the following cases: (a) $\varphi(X)$ *has compact closure in* N (e.g. φ is continuous), [EF, Theorem 9.2]; or (b) N *is simply connected, complete, and has nonpositive curvature*, [F3, Proposition 3].

In the rest of this subsection we no longer insist that X be compact, nor that g be simplexwise smooth. For any quasiopen set $U \subset X$ define as in [KM, p. 372]

$$W_0^{1,2}(U) = \bigcap \left\{ W_0^{1.2}(G) : G \text{ open in } X, G \supset U \right\}.$$

The well-known concept of weak harmonicity of maps between Riemannian manifolds extends readily to the present setting as follows, [EF, Definition 12.2]:

A *weakly harmonic map* $\varphi : (X, g) \to (N, h)$ is a *quasicontinuous* map of locally finite energy (cf. Definition 2) such that, for any chart $\eta : V \to \mathbb{R}^n$ on N,

$$\int_{\varphi^{-1}(V)} \langle \nabla\lambda, \nabla\varphi^k \rangle d\mu = \int_{\varphi^{-1}(V)} \lambda(\Gamma^k_{\alpha\beta} \circ \varphi)\langle \nabla\varphi^\alpha, \nabla\varphi^\beta \rangle d\mu \tag{2.5}$$

holds for every $k = 1, \dots, n$ and every bounded function λ of class $W_0^{1,2}(\varphi^{-1}(V))$ on the quasiopen set $\varphi^{-1}(V)$.

Weak harmonicity of a map is a *local* property.

A *continuous* map $\varphi : X \to N$ of locally finite energy is weakly harmonic if and only if φ is a harmonic map (as defined in Sect. 1); and this is further equivalent to φ

pulling germs of geodesically convex functions on N back to germs of subharmonic functions on X, [EF, Theorem 12.1]. In the case where also X is a Riemannian manifold the latter equivalence is due to Ishihara [Ish].

Theorem 1 above has a companion for weakly harmonic maps into manifolds:

Theorem 3. *Let (N, h) be a simply connected complete Riemannian manifold of nonpositive sectional curvature. Every weakly harmonic map $\varphi : X \to N$ is Hölder continuous* (*and hence harmonic*).

As in the proof of Theorem 1 in Sect. 7 one may use Lemma 3, Proposition 1 (b), and Corollary 1 above in order to pass from pointwise Hölder continuity to locally uniform Hölder continuity in the proof of [EF, Proposition 12.1]. (As to Corollary 1 in the present setting see [EF, Remark 9.6].)

Theorem 2 above has a similar companion in the present setting (now for targets of upper bounded curvature):

Theorem 4. *Let (N, h) be a complete Riemannian manifold with sectional curvature $\leqslant K$ for some constant $K > 0$. A weakly harmonic map $\varphi : X \to N$ is Hölder continuous provided that its range $\varphi(X)$ is contained in a closed ball $B = B_N(q, R)$ of radius $R < \pi/(2\sqrt{K})$, lying within normal range from each of its points* (*i.e., B shall not meet the cut locus of any of its points*).

Under the remaining hypotheses on N and B the cut locus condition is fulfilled if and only if B satisfies bipoint uniqueness (cf. the paragraph following Theorem 2 above); for the "if" part see e.g. [KN, Theorem 7.1], and for the "only if" part see [K1, §4.4] and [K2, Satz 4].

In the proof of Theorem 4, given in Sect. 9, the ball B is allowed to be closed or open (as in Theorem 2). With pointwise Hölder continuity, B closed, and $R < \pi/(4\sqrt{K})$, the theorem was obtained in [EF, Proposition 12.2] (where convexity of B is not used in the proof).

3. Proof of Lemma 1

We may assume that $\kappa \leqslant 2$ (otherwise replace κ by 2). Because K is compact we may choose $\varrho = \varrho(X, K)$ so that $B^{\mathrm{e}}_X(a, 4\varrho) \Subset \mathrm{st}_X(a)$ (the open star of a in X) for every $a \in K$. Suppose there is no constant c as stated in the lemma. There exists then a sequence of balls $B_j = B^{\mathrm{e}}_X(a_j, r_j)$ with $a_j \in K$, $r_j \leqslant \varrho$; and harmonic functions $u_j \geqslant 0$ in $B^{\mathrm{e}}_X(a_j, \kappa r_j)^\circ$ with respect to some Riemannian metric g_j on X with ellipticity constant $\leqslant \Lambda_X$ such that

$$\max_{\partial B_j} u_j > j \min_{\partial B_j} u_j. \tag{3.1}$$

We may assume that $a_j \to a \in K$ and $r_j \to r \in [0, \varrho]$. Write $\kappa - 1 = 4\delta$ ($\in]0, 1[$).

Step 1. The case $r > 0$. We may further assume that $d^{\mathrm{e}}_X(a_j, a) < \frac{1}{2}\delta r$ and $|r_j - r| < \frac{1}{2}\delta r$ for all j. Then $B^{\mathrm{e}}_X(a, r - \delta r) \subset B_j \subset B^{\mathrm{e}}_X(a, r + \delta r)$, and hence

$$\partial B_j \subset A := B^{\mathrm{e}}_X(a, r + \delta r) \setminus B^{\mathrm{e}}_X(a, r - \delta r)^\circ.$$

Furthermore, u_j is harmonic (with respect to g_j) and $\geqslant 0$ in the open set

$$Y := B^{\mathrm{e}}_X(a, r + 2\delta r)^\circ \setminus \{a, a_1, a_2, \dots\}$$

because $B^{\mathrm{e}}_X(a, r+2\delta r) \subset B^{\mathrm{e}}_X(a_j, \kappa r_j)$ (this uses $\kappa = 1+4\delta \leqslant 3$). Y is connected because X is admissible of dimension $m > 1$, and hence no countable subset of an open ball in Y divides Y. Since $\partial B_j \subset A$, (3.1) contradicts the Harnack inequality [EF, Theorem 6.1] applied to the compact set $A \subset Y$.

Step 2. The remaining case $r_j \to 0$. Here we use an iterative scheme, earlier employed in [EF] (in the proofs of Theorems 5.1 and 7.4 there). Let s denote the carrier of a (i.e., a belongs to the relative interior s° of the simplex s). After embedding the (open) star $S = \mathrm{st}_X(s) = \mathrm{st}_X(a)$ in a Euclidean vector space V with origin at a and norm $|\cdot|$ we denote by $p : V \to \mathbb{R}s$ the orthogonal projection on $\mathbb{R}s$. Because $a_j \to a$ $(= 0)$ and $p(a_j) \to p(a) = a$, we may assume that $a_j \in B^{\mathrm{e}}_X(a, \varrho)$ $(\Subset S)$ and $p(a_j) \in s^\circ \cap B^{\mathrm{e}}_X(a, \varrho)$. Consider for each j the Euclidean homothety ψ_j of V with centre $p(a_j)$ and factor

$$\alpha_j = \frac{1}{\varrho} \max\{|a_j - p(a_j)|, r_j\} > 0. \tag{3.2}$$

Writing $a'_j = \psi_j^{-1}(a_j)$ $(\in V)$, we have

$$a'_j - p(a_j) = \alpha_j^{-1}(a_j - p(a_j)), \tag{3.3}$$

and by (3.2)

$$|a'_j - p(a_j)| = \alpha_j^{-1}|a_j - p(a_j)| \leqslant \varrho. \tag{3.4}$$

Since $|p(a_j)| \leqslant \varrho$ it follows from (3.4) that $|a'_j| \leqslant 2\varrho$, which in turn implies that a'_j lies in $K' := B^{\mathrm{e}}_X(a, 2\varrho)$, a compact subset of S (by the choice of ϱ). Writing $r'_j := \alpha_j^{-1} r_j \leqslant \varrho$, by (3.2), and recalling the notation $B_j = B^{\mathrm{e}}_X(a_j, r_j)$, we infer that

$$B'_j := \psi_j^{-1}(B_j) = B^{\mathrm{e}}_X(a'_j, r'_j) \subset B^{\mathrm{e}}_X(a, 3\varrho) \ \ (\Subset S). \tag{3.5}$$

Use ψ_j^{-1} to transport the Riemannian metric g_j on $B^{\mathrm{e}}_X(a_j, \kappa r_j)^\circ$ into the Riemannian metric $g'_j := \alpha_j^{-2}(\psi_j^{-1})^* g_j$ on $B^{\mathrm{e}}_X(a'_j, \kappa r'_j)^\circ \subset B^{\mathrm{e}}_X(a, 4\varrho) \Subset S$ (because $a'_j \in K'$ and $\kappa r'_j \leqslant 2\varrho$); and g'_j has ellipticity constant $\leqslant \Lambda_X$. Writing $u_j \circ \psi_j = u'_j$ we have from (3.1) its analogue

$$\max_{\partial B'_j} u'_j > j \min_{\partial B'_j} u'_j. \tag{3.6}$$

By (3.5), u'_j is harmonic $\geqslant 0$ with respect to g'_j in $B^{\mathrm{e}}_X(a'_j, \kappa r'_j)^\circ$.

Because $a'_j \in K'$ and $r'_j \leqslant \varrho$, we may assume that $a'_j \to a' \in K'$ and $r'_j \to r' \in [0, \varrho]$. From Step 1, now with the "dashed" quantities, we infer from (3.6) that $r' = 0$. We may therefore assume that $r'_j < \varrho$, and so r_j $(= \alpha_j r'_j)$ $< \alpha_j \varrho = |a_j - p(a_j)|$, by (3.2). Since $p(a_j) \to 0$ it therefore follows from (3.4) that

$$|a'| = \lim |a'_j - p(a_j)| = \lim \alpha_j^{-1}|a_j - p(a_j)| = \varrho.$$

Moreover, $a' = \lim_j (a'_j - p(a_j)) \in K' \Subset S$, and a' is orthogonal to $\mathbb{R}s$ because $a'_j - p(a_j)$ is so for each j, by (3.3). It follows that $a' \in S \setminus \mathbb{R}s$, and hence the carrier s' of a' in X contains s properly, so that $\dim s' > \dim s$. Iterating the above procedure at most $m - \dim s$ times therefore leads to an absurdity.

4. Proof of Lemma 2

We may assume that $\kappa \leqslant 2$ (otherwise replace κ by 2). Because K is compact we may assume that the (open) star $\mathrm{st}_X(a)$ of each point $a \in K$ satisfies the Poincaré-style inequality [EF, eq. (7.3)], cf. [EF, Remark 7.2]. Every domain U in $\mathrm{st}_X(a)$ has then a symmetric Green kernel G_U, [EF, Theorem 7.3]. We may choose $\varrho = \varrho(X, K) > 0$ so that $B^e_X(a, 4\varrho) \Subset \mathrm{st}_X(a)$ for every $a \in K$. Suppose there is no constant c as stated in the former inequality of the lemma. Then, for any $j = 1, 2, \dots$, there exist $a_j \in K$, $r_j \in]0, \varrho]$, and $(x_j, y_j) \in B_j \times B_j$ such that

$$G_{U_j}(x_j, y_j) < j^{-1} G^e_{U_j}(x_j, y_j), \tag{4.1}$$

where $B_j := B^e_X(a_j, r_j)$ and $U_j := B^e_X(a_j, \kappa r_j)^\circ$. Here G_{U_j} denotes the Green kernel on U_j for some Riemannian metric g_j on U_j with ellipticity constant $\leqslant \Lambda_X$. In particular, $G_{U_j}(x_j, y_j) < \infty$, and so $x_j \neq y_j$ (if $\dim X > 1$). We may assume that $a_j \to a \in K$ and $r_j \to r \in [0, \varrho]$. Write $\kappa - 1 = 4\delta$ $(\in]0, 1])$.

Step 1. The case $r > 0$. We may further assume that $d^e_X(a_j, a) < \frac{1}{2}\delta r$ and $|r_j - r| < \frac{1}{2}\delta r$ for all j. With the abbreviations

$$B = B^e_X(a, r + \delta r), \quad U = B^e_X(a, r + 2\delta r)^\circ, \quad \widetilde{U} = B^e_X(a, r + 6\delta r)^\circ,$$

we have $B_j \subset B \subset U \subset U_j \subset \widetilde{U} \subset B^e_X(a, 4\varrho) \Subset S := \mathrm{st}_X(a)$ for $r \leqslant \varrho$. It follows by [EF, Proposition 7.5 (a)] that

$$G_U \leqslant G_{U_j}, \quad G^e_{U_j} \leqslant G^e_{\widetilde{U}} \qquad \text{on} \quad B \times B. \tag{4.2}$$

According to [EF, Lemma 7.2] there is a constant c_1 such that

$$G^e_{\widetilde{U}} \leqslant c_1 G_{\widetilde{U}} \qquad \text{on} \quad B \times B. \tag{4.3}$$

By further application of [EF, Proposition 7.5 (a)] we have $G_{\widetilde{U}} = G_U + H$, where H is finite and continuous in $U \times U$, while G_U is lower semicontinuous and > 0 there. It follows that H/G_U and hence $G_{\widetilde{U}}/G_U$ are bounded on $B \times B$ off the diagonal, and so

$$G_{\widetilde{U}} \leqslant c_2 G_U \qquad \text{on} \quad B \times B, \tag{4.4}$$

for some constant c_2. Applying successively (4.2) (the latter inequality), (4.3), (4.4), (4.2) (the former inequality), and (4.1), we obtain $0 < G^e_{U_j} < c_1 c_2 j^{-1} G^e_{U_j} < \infty$ at the point $(x_j, y_j) \in B_j \times B_j \subset B \times B$; and this is absurd for $j > c_1 c_2$.

Step 2. The remaining case $r_j \to 0$. Recall the text in the first paragraph of Step 2 in Sect. 3. Write $U'_j = \psi_j^{-1}(U_j)$, and use ψ_j^{-1} to transport the Riemannian metric g_j on U_j into the Riemannian metric $g'_j := \alpha_j^{-2}(\psi_j^{-1})^* g_j$ on U'_j, with ellipticity constant $\leqslant \Lambda_X$. By [EF, Lemma 7.3], G_{U_j} transforms under ψ_j^{-1} into

$$G_{U'_j}(x', y') = \alpha_j^{m-2} G_{U_j}(x, y), \qquad x' = \psi_j^{-1}(x) \in U'_j, \quad y' = \psi_j^{-1}(y) \in U'_j.$$

With $x'_j = \psi_j^{-1}(x_j) \in B'_j$ and $y'_j = \psi_j^{-1}(y_j) \in B'_j$, cf. (3.5), we have $x'_j \neq y'_j$ (if $\dim X > 1$); and (4.1) transforms into

$$G_{U'_j}(x'_j, y'_j) < j^{-1} G^e_{U'_j}(x'_j, y'_j). \tag{4.5}$$

This leads to an absurdity in the same way as described in the last paragraph of Sect. 3. Having thus established the former inequality of the lemma, the latter inequality is proved similarly, by interchanging G and G^e.

5. Proof of Proposition 1

To begin with, choose $\varrho > 0$ so that, for every $a \in K$, $B_X^e(a, \kappa\varrho) \subset \mathrm{st}_X(a)$ (the open star of a in X). Part (a) is proved just like Lemma 1 (now also for $m = 1$) after replacing ∂B_j by B_j and $B_X^e(a_j, \kappa r_j)^\circ \setminus \{a_j\}$ by $B_X^e(a_j, \kappa r_j)^\circ$, etc. Alternatively (for $m > 1$), apply to Lemma 1 the maximum principle for harmonic functions (cf. e.g. [EF, Chapter 2]).

To prove part (b), fix for a while a point $a \in K$ and denote

$$B_r = B_X^e(a, r), \quad U_r = B_X^e(a, \kappa r)^\circ, \quad \mu_r = \mu_{|B_r}, \qquad 0 < r \leqslant \varrho.$$

For ϱ sufficiently small, (U_r, g) has for every $0 < r \leqslant \varrho$ a symmetric Green kernel G_{U_r}, [EF, Theorem 7.3 and Remark 7.2]. In accordance with the F. Riesz theorem (cf. e.g. [EF, Chapter 2]) write

$$u = G_{U_r}\lambda + h,$$

where λ is a positive measure on U_r and $h \geqslant 0$ a harmonic function on U_r. Applying (a) to h in place of u we have

$$\frac{1}{\mu(B_r)} \int_{B_r} h \, d\mu \leqslant \max_{B_r} h \leqslant c \min_{B_r} h.$$

It therefore remains (if $\lambda \neq 0$) to prove that

$$\frac{1}{\mu(B_r)} \int_{B_r} G_{U_r}\lambda \, d\mu \leqslant c \min_{B_r} G_{U_r}\lambda \tag{5.1}$$

(with a possibly larger constant c). For suitable constants $\varrho, \gamma_1 > 0$ we have by Lemma 3 and the symmetry of the Green kernel,

$$\int_{B_r} G_{U_r}\lambda \, d\mu = \int_X G_{U_r}\lambda \, d\mu_r = \int_{U_r} G_{U_r}\mu_r \, d\lambda \leqslant \gamma_1 r^2 \int_{U_r} d\lambda \tag{5.2}$$

for $r \leqslant \varrho$. On the other hand it follows from (5.2) and Lemma 2, and from [EF, Lemma 7.3] with $\alpha = \varrho/r$, that there exist positive constants ϱ, c_1 (depending on X, Λ_X, K, κ only) such that for $r \leqslant \varrho$ and $x, y \in B_r$

$$c_1 G_{U_r}(x, y) \geqslant G_{U_r}^e(x, y) = \left(\frac{\varrho}{r}\right)^{m-2} G_{U_\varrho}^e\left(\frac{\varrho}{r}x, \frac{\varrho}{r}y\right) \geqslant \left(\frac{\varrho}{r}\right)^{m-2} c_2(a), \tag{5.3}$$

$$c_2(a) := \min\{G_{U_\varrho}^e(x, y) : (x, y) \in B_\varrho \times B_\varrho\} > 0, \qquad a \in K. \tag{5.4}$$

Writing $\varepsilon = (\kappa - 1)\varrho/3$, finitely many balls $B_X^e(a_i, \varepsilon)$, $i = 1, \dots, k$, cover K. For each $a \in K \cap B_X^e(a_i, \varepsilon)$ we have $U_\varrho = B_X^e(a, \kappa\varrho)^\circ \supset B_X^e(a_i, \kappa\varrho - \varepsilon)$ and

$B_\varrho = B^e_X(a,\varrho) \subset B^e_X(a_i, \varrho+\varepsilon)$. Because $\varrho+\varepsilon < \kappa\varrho - \varepsilon$ it follows in view of [EF, Proposition 7.5 (a)] that, on $B_\varrho \times B_\varrho$,

$$\begin{aligned} G^e_{U_\varrho} &\geqslant G^e_{B^e_X(a_i,\kappa\varrho-\varepsilon)} \\ &\geqslant c^{(i)} := \min\{G^e_{B^e_X(a_i,\kappa\varrho-\varepsilon)}(x,y) : (x,y) \in B^e_X(a_i,\varrho+\varepsilon)\times B^e_X(a_i,\varrho+\varepsilon)\} \end{aligned}$$

for $i = 1,\ldots,k$ and $a \in K \cap B^e_X(a_i,\varepsilon)$; and hence by (5.4) for every $a \in K$

$$c_2(a) \geqslant c_2 := \min\{c^{(1)},\ldots,c^{(k)}\} > 0, \tag{5.5}$$

where $c_2 = c_2(X,K,\kappa)$. Now insert (5.5) in (5.3), and integrate with respect to $d\lambda(y)$ over U_r. Next, take minimum over $x \in B_r$, and compare the resulting inequality with (5.2); that leads for every $a \in K$ to

$$\int_{B_r} G_{U_r}\lambda\, d\mu \leqslant \frac{c_1}{c_2}\gamma_1\varrho^2\left(\frac{r}{\varrho}\right)^m \min_{B_r} G_{U_r}\lambda.$$

This, in turn, leads to (5.1) after division by $\mu(B_r)$ because $\mu(B_r) \geqslant c_K \Lambda_X^{-2m} r^m$, by the proof of [EF, Lemma 4.4] applied to $X_0 = K$, whereby $c_K > 0$ depends on X and K only.

6. Proof of Proposition 2 and Corollary 1

Step 1. Let the domain be the unit cube $X = [0,1]^m$ in $\mathbb{R}^m$ with the Euclidean structure and Lebesgue measure μ. That case is covered by [KS2, Corollary 1.4.2], but we give an alternative elementary proof in the slightly amplified form (2.2).

For given k divide the cube X into k^m subcubes σ in the obvious way, each σ having the volume $\mu(\sigma) = 1/k^m$. The integral I on the left hand side of (2.2) then decomposes into the sum $I = \sum_{\sigma,\sigma'} I(\sigma,\sigma')$, where

$$I(\sigma,\sigma') = \int_\sigma\int_{\sigma'} d^2_Y(\varphi(x),\varphi(x'))\, d\mu(x)\, d\mu(x').$$

For a given (ordered) pair of distinct σ,σ' consider a chain of distinct subcubes $\sigma_0,\sigma_1,\ldots,\sigma_n$ of X with $\sigma_0 = \sigma$, $\sigma_n = \sigma'$, and σ_{i-1},σ_i being neighbours, i.e., having a common $(m-1)$-face. For $x \in \sigma$, $x' \in \sigma'$ we then have for any $x_1 \in \sigma_1,\ldots,x_{n-1} \in \sigma_{n-1}$, by the triangle inequality and Cauchy's inequality,

$$d^2_Y(\varphi(x),\varphi(x')) \leqslant n\sum_{i=1}^n d^2_Y(\varphi(x_{i-1}),\varphi(x_i)).$$

Taking meanvalues with respect to $(x_1,\ldots,x_{n-1}) \in \sigma_1\times\ldots\times\sigma_{n-1}$ gives

$$\begin{aligned} \frac{1}{n} d^2_Y(\varphi(x),\varphi(x') &\leqslant k^m\int_{\sigma_1} d^2_Y(\varphi(x),\varphi(x_1))\, d\mu(x_1) \\ &\quad + \sum_{i=2}^{n-1} k^{2m} I(\sigma_{i-1},\sigma_i) + k^m\int_{\sigma_{n-1}} d^2_Y(\varphi(x_{n-1}),\varphi(x'))\, d\mu(x_{n-1}). \end{aligned}$$

By integration with respect to $d\mu(x)\,d\mu(x')$ it follows that

$$I(\sigma,\sigma') \leqslant n\sum_{i=1}^{n} I(\sigma_{i-1},\sigma_i) \tag{6.1}$$

(trivial if $\sigma = \sigma'$ with the chain reducing to σ, σ).

Now choose the chain from σ to $\sigma' \neq \sigma$ specifically as follows: First let σ_0 $(= \sigma), \sigma_1, \dots, \sigma_{n_1}$ be a subchain consisting of successive neighbours, each obtained from σ by translation in the direction of the (positive or negative) x_1-axis. Next let $\sigma_{n_1}, \dots, \sigma_{n_2}$ $(n_2 - n_1 < k)$ be successive neighbours, translates of σ_{n_1} in the x_2-direction, etc., ending with translates $\sigma_{n_{m-1}}, \dots, \sigma_{n_m}$ of $\sigma_{n_{m-1}}$ in the x_m-direction and such that $\sigma_{n_m} = \sigma'$. Then $n = n_m < mk$.

For any pair τ, τ' of neighbouring subcubes of X there are at most k^{m+1} pairs σ, σ' of distinct subcubes of X such that τ, τ' occur as two neighbours σ_{i-1}, σ_i in the chain specified above, joining σ to σ'. Since $n \leqslant mk$ we therefore obtain from (6.1)

$$\sum_{\sigma\neq\sigma'} I(\sigma,\sigma') \leqslant mk^{m+2}{\sum_{\tau,\tau'}}^{*} I(\tau,\tau'),$$

where the star indicates summation over all pairs of neighbouring subcubes τ, τ' of X. Adding to this $\sum_\sigma I(\sigma,\sigma) = \sum_{\tau=\tau'} I(\tau,\tau')$ yields

$$I = \sum_{\sigma,\sigma'} I(\sigma,\sigma') \leqslant mk^{m+2}\sum_{\tau}\Big(\sum_{\tau'} I(\tau,\tau')\Big),$$

where τ, τ' are either neighbours or identical.

For any points $x \in \tau$, $x' \in \tau'$ we have $d_X(x,x') \leqslant \varepsilon := 2\sqrt{m}/k$, and we conclude that

$$\begin{aligned} I &\leqslant mk^{m+2}\sum_{\tau}\int_{\tau} d\mu(x)\int_{B_X(x,\varepsilon)} d_Y^2(\varphi(x),\varphi(x'))\,d\mu(x') \\ &= mk^{m+2}\varepsilon^{m+2}\sum_{\tau}\int_{\tau} e_\varepsilon(\varphi)\,d\mu = C\int_X e_\varepsilon(\varphi)\,d\mu \end{aligned}$$

with $C = m(2\sqrt{m})^{m+2}$, $e_\varepsilon(\varphi)$ being the approximate energy density of φ, see (1.1). For $k \to \infty$, i.e., $\varepsilon \to 0$, this leads to (2.2) for $X = [0,1]^m$ according to (1.2) or (1.3) since $\mu(X) = 1$.

Step 2. Let X be any compact admissible m-dimensional polyhedron endowed with a *Euclidean* Riemannian metric g^e on each m-simplex s (taken as a Euclidean simplex, cf. [EF, Chapter 4]). Let μ_e denote the corresponding volume measure on X. Being compact and covered by the open stars of its vertices, the polyhedron X is finite. Hence there is a constant $\lambda = \lambda(X) > 0$ such that

$$\lambda^{-1} \leqslant \mu_e(s) \leqslant \lambda \qquad \text{for every } m\text{-simplex } s.$$

For any two distinct m-simplexes s, s' of X write

$$I(s,s') = \int_s\int_{s'} d_Y^2(\varphi(x),\varphi(x'))\,d\mu_e(x)\,d\mu_e(x').$$

s can be joined to s' by a chain of distinct m-simplexes $s_0, s_1, \ldots, s_n$ with $s_0 = s$, $s_n = s'$, and s_{i-1}, s_i being neighbours, i.e., having a common $(m-1)$-face, $i = 1, \ldots, n$. (For $s \cap s' \neq \varnothing$ this follows from X being admissible, and for general s, s' it therefore follows from X being connected.) Similarly to (6.1) we obtain

$$\frac{1}{n} I(s, s') \leqslant \sum_{i=1}^{n} \lambda^2 I(s_{i-1}, s_i) \leqslant \lambda^2 \sum_{i=1}^{n} I(s_{i-1} \cup s_i, s_{i-1} \cup s_i) \tag{6.2}$$

with the obvious understanding of the terms in the latter sum. Each $s_{i-1} \cup s_i$ is a "double simplex", bi-Lipschitz homeomorphic to the union of two adjacent unit cubes, cf. [EF, Sublemma 4.3], hence also to a single unit cube $\sigma = [0,1]^m$. Let $\theta : \sigma \to s_{i-1} \cup s_i$ be a bi-Lipschitz bijection with constant $|\theta| \geqslant 1$. Then (2.2), applied to $X = \sigma$ with Lebesgue measure μ (cf. Step 1) and to the map $\varphi \circ \theta : \sigma \to Y$, reads

$$\int_\sigma \int_\sigma d_Y^2\big((\varphi \circ \theta)(\xi), (\varphi \circ \theta)(\xi')\big)\, d\mu(\xi)\, d\mu(\xi') \leqslant C \int_\sigma e(\varphi \circ \theta)\, d\mu. \tag{6.3}$$

For $x = \theta(\xi)$ we have $|\theta|^{-m} d\mu(\xi) \leqslant d\mu_e(x) \leqslant |\theta|^m d\mu(\xi)$ and

$$\theta(B_\sigma(\xi, \varepsilon)) \subset B^e_{s_{i-1} \cup s_i}(x, |\theta|\varepsilon),$$

where B^e refers to a ball relative to the Euclidean Riemannian metric g^e on X. The approximate energy densities (cf. (1.1)) $e_\varepsilon(\varphi \circ \theta)$ on σ and $e'_{|\theta|\varepsilon}(\varphi)$ on $s_{i-1} \cup s_i$, the latter relative to g^e, are therefore related by

$$e_\varepsilon(\varphi \circ \theta)(\xi) \leqslant |\theta|^{2m+2} e'_{|\theta|\varepsilon}(\varphi)(x).$$

Applying (1.2) or (1.3) to the right hand side of (6.3) now leads to

$$I(s_{i-1} \cup s_i, s_{i-1} \cup s_i) \leqslant C|\theta|^{5m+2} \int_{s_{i-1} \cup s_i} e'(\varphi)\, d\mu_e, \tag{6.4}$$

where $e'(\varphi)$ denotes the energy density of φ relative to g^e. Because there are only finitely many pairs of neighbours such as s_{i-1}, s_i, we may replace $C|\theta|^{5m+2}$ by a constant $C_1 = C_1(X)$. When N denotes the number of m-simplexes of X, we have $n + 1 \leqslant N$ in (6.2). Combining (6.2) and (6.4) therefore leads to

$$I(s, s') \leqslant 2C_1 N \lambda^2 E'(\varphi)$$

for any pair of distinct m-simplexes s, s' of X, whereby $E'(\varphi)$ denotes the total energy of φ relative to g^e. For a single m-simplex $s = \theta(\sigma)$, say, we similarly obtain from Step 1: $\sum_s I(s, s) \leqslant C_2 E'(\varphi)$, and we arrive at (2.2) by adding up:

$$\int_X \int_X d_Y^2(\varphi(x), \varphi(x'))\, d\mu_e(x)\, d\mu_e(x') = \sum_{s,s'} I(s, s') \leqslant C' \mu_e(X) E'(\varphi)$$

in terms of a new constant $C' = C'(X)$.

Step 3. (The general case.) We argue as in the proof of (6.4). Let $\Lambda_X = \Lambda$ denote the ellipticity constant for the given Riemannian metric g on the finite polyhedron X, cf. [EF, eq. (4.2)]; then $\Lambda^{-m} d\mu_e \leqslant d\mu_g \leqslant \Lambda^m d\mu_e$, $B_X^e(x, \varepsilon) \subset B_X(x, \Lambda\varepsilon)$, and hence $e'_\varepsilon(\varphi) \leqslant \Lambda^{2m+2} e_{\Lambda\varepsilon}(\varphi)$. From (2.2) relative to g^e (Step 2) with constant C' we therefore obtain the stated inequality (2.2) relative to g, with $\mu = \mu_g$ and the constant $C = \Lambda^{6m+2} C'$.

Proof of Corollary 1. We view X as embedded Lipschitz homemorphically in a Euclidean space V with each simplex embedded as a Euclidean simplex in V, [EF, Lemma 4.1].

Consider first the case of the Euclidean Riemannian metric g^e on X, with volume measure μ_e and balls $B_X^e(a, r)$. Let $S = \mathrm{st}_X(a)$ denote the open star of a point a in X, and fix $R > \varrho > 0$, depending only on X, so that, when a is taken as origin in V, we have $\frac{\varrho}{R}\overline{S} \Subset B_X^e(a, R)^\circ$ and $B_X^e(a, R) \Subset S$, and hence

$$B_X^e(a, \varrho) \Subset \frac{\varrho}{R} S \quad \text{and} \quad \frac{\varrho}{R}\overline{S} \Subset B_X^e(a, R)^\circ .$$

It follows by Proposition 2, with (X, g) replaced by the present compact Riemannian polyhedron $(\frac{\varrho}{R}\overline{S}, g^e)$, that

$$\inf_{y \in Y} \int_{B_X^e(a,\varrho)} d_Y^2(\varphi(\xi), y)\, d\mu_e(\xi) \leqslant C' \varrho^2 \int_{B_X^e(a,R)} e'(\varphi)\, d\mu_e, \tag{6.5}$$

where $e'(\varphi)$ denotes the energy density of φ relative to g^e, and we have written $C'\varrho^2$ in place of $C' = C'(X)$. For given $r \leqslant \varrho$ consider the homothety ψ of V defined by $x = \psi(\xi) = \lambda^{-1}\xi$, where $\lambda = \varrho/r \geqslant 1$. Applying (6.5) to the map $\varphi \circ \psi : B_X^e(a, \varrho) \to Y$ in place of φ, and performing the transformation $\psi : B_X^e(a, \varrho) \to B_X^e(a, r)$, we obtain (via the approximate energy density $e'_\varepsilon(\varphi)(x) = \lambda^2 e_{\lambda\varepsilon}(\varphi \circ \psi)(\xi)$)

$$\inf_{y \in Y} \int_{B_X^e(a,r)} d_Y^2(\varphi(x), y)\, d\mu_e(x) \leqslant C' r^2 \int_{B_X^e(a,\kappa r)} e'(\varphi)\, d\mu_e$$

for $r \leqslant \varrho$, writing $R/\varrho = \kappa$. This is a "Euclidean" version of (2.3). From that we readily derive the general version, noting again that $e'_\varepsilon(\varphi) \leqslant \Lambda^{2m+2} e_{\Lambda\varepsilon}(\varphi)$:

$$\inf_{y \in Y} \int_{B_X(a,r)} d_Y^2(\varphi(x), y)\, d\mu(x) \leqslant C' \Lambda^{4m+4} r^2 \int_{B_X(a,\Lambda^2 \kappa r)} e(\varphi)\, d\mu$$

for $r \leqslant \Lambda^{-2}\varrho$, because $B_X(a, r) \subset B_X^e(a, \Lambda r) \subset B_X(a, \Lambda^2 r)$.

7. Proof of Theorem 1

In the proof of [EF, Theorem 10.1], one now replaces Lemma 10.1 (a) and Sublemma 10.1 by the above Proposition 1 (b) (cf. [EF, Remark 7.4]) and Lemma 3, respectively, in order to allow the point $a \in X$ to vary and to make all estimates locally uniform in a. This leads to locally uniform (rather than just pointwise) Hölder continuity.

Local compactness of (Y, d_Y) was only used towards the end of the proof of [EF, Theorem 10.1] when a point q was chosen (p. 191) in the intersection of the lower directed family of nonvoid *compact* essential images $\varphi(B^e_X(a,r))$, $r > 0$. In the absence of local compactness, choose instead q_j as any point of $\varphi(B^e_X(a, \beta^j r))$, $j = 1, 2, \ldots$; then eqs. (10.15) and (10.16) in [EF] hold (cf. (10.14) there) with q replaced by q_j, and the rest of the proof remains unchanged. Of course, for the proof of (10.12) in [EF] one now uses the weak Poincaré inequality from Corollary 1 above, in place of [EF, Proposition 9.1] where a restriction was imposed on Y. Because "Euclidean" balls $B^e_X(a,r)$ are involved in (10.12) in [EF], one uses once again the relations $B^e_X(a,r) \subset B_X(a, \Lambda_X r) \subset B^e_X(a, \Lambda_X^2 r)$ (now with r replaced by $5r/6$, cf. eq. (10.11) in [EF]).

8. Proof of Theorem 2

We may assume that X is *compact*. By rescaling the metric d_Y we achieve that $K = 1$, and hence $R < \pi/2$. For brevity, write $B_Y(q, R) = B$. Every geodesic triangle $yy'q$ in B with one vertex at the centre q has perimeter $< 2\pi$ and hence admits a comparison triangle $\tilde{y}\tilde{y}'\tilde{q}$ in the unit sphere S^2 in $\mathbb{R}^3$. Write

$$d_Y(y,q) = v, \quad d_Y(y',q) = v', \quad d_Y(y,y') = d;$$

thus $v, v' \leqslant R$ and $d \leqslant 2R$ $(< \pi)$. For given $0 \leqslant \lambda, \lambda' < 1$ consider the point $y_\lambda \in [y,q]$ (the geodesic segment joining y and q), the point $y'_\lambda \in [y', q]$, and the distance d_λ given by $d_Y(y, y_\lambda) = \lambda v$, $d_Y(y', y'_\lambda) = \lambda' v'$, $d_Y(y_\lambda, y'_\lambda) = d_\lambda$. By abuse of notation we thus have

$$y_\lambda = (1-\lambda)y + \lambda q, \quad y'_\lambda = (1-\lambda')y' + \lambda' q.$$

Similarly write $\tilde{y}_\lambda = (1-\lambda)\tilde{y} + \lambda\tilde{q}$, $\tilde{y}'_\lambda = (1-\lambda')\tilde{y}' + \lambda'\tilde{q}$, and by triangle comparison

$$\tilde{d}_\lambda := d_{S^2}(\tilde{y}_\lambda, \tilde{y}'_\lambda) \geqslant d_\lambda.$$

We shall estimate d_λ from above in terms of d. First a rough estimate: Write $z = (1-\lambda)y' + \lambda q$. The spherical cosine relation applied to $\tilde{y}\tilde{y}'\tilde{q}$ with angle θ at $\tilde{q}$ may be written as

$$\cos d_{S^2}(\tilde{y}, \tilde{y}') = \cos(v - v') - (1 - \cos\theta)\sin v \sin v'.$$

Replacing $\tilde{y}, \tilde{y}'$ by $\tilde{y}_\lambda, \tilde{z}$ amounts to replacing v, v' by $(1-\lambda)v, (1-\lambda)v'$ whilst keeping θ, and so $\cos d_{S^2}(\tilde{y}_\lambda, \tilde{z}) \geqslant \cos d_{S^2}(\tilde{y}, \tilde{y}')$. Hence, by triangle comparison,

$$d_Y(y_\lambda, z) \leqslant d_{S^2}(\tilde{y}_\lambda, \tilde{z}) \leqslant d_{S^2}(\tilde{y}, \tilde{y}') = d_Y(y, y') = d.$$

Since $d_Y(z, y'_\lambda) = |\lambda' - \lambda| v' \leqslant \frac{1}{2}\pi|\lambda' - \lambda|$, it follows by the triangle inequality that

$$d_\lambda = d_Y(y_\lambda, y'_\lambda) \leqslant d + \tfrac{1}{2}\pi|\lambda' - \lambda|. \tag{8.1}$$

To obtain a more precise upper estimate of d_λ in terms of d we estimate the compared distance $\tilde{d}_\lambda \geqslant d_\lambda$. We thus obtain by spherical trigonometry

$$\begin{aligned}\sin v \sin v' \cos d_\lambda \geqslant\ & \sin(v-\lambda v)\sin(v'-\lambda' v')\cos d \\ & + \sin(v-\lambda v)\sin(\lambda' v')\cos v \\ & + \sin v' \sin(\lambda v)\cos(v'-\lambda' v'),\end{aligned}$$

cf. [EF, p. 193]. After division by $\sin v \sin v'$, and manipulations serving to render (1.2) through (1.5) applicable, this leads to the latter inequality (10.17) in [EF] (now with the convex set V of diameter $D < \pi/2$ replaced by the ball B of radius $R < \pi/2$):

$$\cos d - \cos d_\lambda = 2\sin^2(\tfrac{1}{2}\, d_\lambda) - 2\sin^2(\tfrac{1}{2} d) \leqslant \sum_{j=1}^{4} R^{(j)}, \tag{8.2}$$

where

$$\begin{aligned} R^{(1)} :&= -2\sin^2(\tfrac{1}{2}\, d)\Big(1 - \frac{\sin(v-\lambda v)}{\sin v}\,\frac{\sin(v'-\lambda' v')}{\sin v'}\Big) \\ &= -2\sin^2(\tfrac{1}{2}\, d)[\lambda v \cot v + \lambda' v' \cot v' + O(\lambda^2 + {\lambda'}^2)] \\ &\leqslant -\frac{\sin^2 R}{2R^2}\, d^2 R \cot R[2\lambda - (\lambda - \lambda') + O(\lambda^2 + {\lambda'}^2)], \end{aligned}$$

the functions $t^{-2}\sin^2 t$ and $t \cot t$ being decreasing and > 0 for $0 < t \leqslant R$ $(< \pi/2)$;

$$\begin{aligned} R^{(2)} :&= \cos(\lambda v)\cos(\lambda' v')(\cos v - \cos v')\Big(\frac{\tan(\lambda v)}{\sin v} - \frac{\tan(\lambda' v')}{\sin v'}\Big) \\ &= (\cos v - \cos v')\Big(\frac{\tan(\lambda v)}{\sin v} - \frac{\tan(\lambda' v')}{\sin v'}\Big)(1 + O(\lambda^2 + {\lambda'}^2)); \\ R^{(3)} :&= (\cos v - \cos v')^2 \frac{\sin(\lambda v)}{\sin v}\,\frac{\sin(\lambda' v')}{\sin v'} \\ &= (\cos v - \cos v')^2 O(\lambda^2 + {\lambda'}^2); \\ R^{(4)} :&= 2\sin^2\frac{\lambda v - \lambda' v'}{2} - 2\sin^2\frac{v - v'}{2}\,\frac{\sin(\lambda v)}{\sin v}\,\frac{\sin(\lambda' v')}{\sin v'} \\ &\leqslant \tfrac{1}{2}(\lambda v - \lambda' v')^2. \end{aligned}$$

With U as in the definition of a local E-minimizer (Sect. 1), consider a function $\lambda \in \mathrm{Lip}_c(U)$, $0 \leqslant \lambda < 1$, and apply the above to

$$y = \varphi(x), \quad y' = \varphi(x'), \quad \lambda = \lambda(x), \quad \lambda' = \lambda(x')$$

for $x \in U$ and $x' \in B_U(x,\varepsilon)$; and write $y_\lambda = \varphi_\lambda(x)$, $y'_\lambda = \varphi_\lambda(x')$. Then

$$\begin{aligned} v = v(x) = d_Y(\varphi(x), q)\ , \quad & v' = v(x') = d_Y(\varphi(x'), q), \\ d = d_Y(\varphi(x), \varphi(x'))\ , \quad & d_\lambda = d_Y(\varphi_\lambda(x), \varphi_\lambda(x')). \end{aligned}$$

By [EF, Corollaries 9.1 and 9.2], $v \in W^{1,2}(U)$ because $d_Y(\cdot, q) \in \mathrm{Lip}(Y)$. The product of two bounded $W^{1,2}$-functions being a bounded $W^{1,2}$-function, it follows by [EF, Lemma 5.2] that the functions $v \cot v$, $\cos v$, λv, $\tan(\lambda v)/\sin v$ $(= \lambda[\tan(\lambda v)/(\lambda v)]\,[v/\sin v])$ and similarly $\sin(\lambda v)/\sin v$ are of class $W^{1,2}(U)$. Furthermore, $\varphi_\lambda = \varphi$ in $U \setminus \operatorname{supp}\lambda$. Denoting by $\|\lambda\|_{\mathrm{Lip}}$ the Lipschitz constant of λ we have $|\lambda(x) - \lambda(x')| = \|\lambda\|_{\mathrm{Lip}} O(\varepsilon)$ because $x' \in B_X(x, \varepsilon)$. From (8.1) we therefore obtain for $\varepsilon \to 0$

$$d_\lambda \leqslant d + \|\lambda\|_{\mathrm{Lip}} O(\varepsilon).$$

We proceed to compare $\frac{1}{2}(d_\lambda^2 - d^2)$ with $\cos d - \cos d_\lambda$. At points x such that $d_\lambda(x) \geqslant d(x)$ we have by Taylor expansion

$$\frac{1}{2!}(d_\lambda^2 - d^2) - \frac{1}{4!}(d_\lambda^4 - d^4) \leqslant \cos d - \cos d_\lambda$$

with $d_\lambda^4 - d^4 = d^3 \|\lambda\|_{\mathrm{Lip}} O(\varepsilon)$ since $d \leqslant \pi$. In particular,

$$\tfrac{1}{2}(d_\lambda^2 - d^2) \leqslant (\cos d - \cos d_\lambda) + d^2 \|\lambda\|_{\mathrm{Lip}} O(\varepsilon), \tag{8.3}$$

also when $d_\lambda \leqslant d$ (and then even without the remainder term).[6] Inserting all this in (8.2) and noting that $\varphi_\lambda = \varphi$ off $F := \operatorname{supp}\lambda$, we obtain by (1.2) through (1.6)

$$\begin{aligned}
0 &\leqslant \frac{1}{2}E(\varphi_\lambda) - \frac{1}{2}E(\varphi) = \frac{1}{2}\int_F (e(\varphi_\lambda) - e(\varphi))d\mu = \frac{1}{2}E((\varphi_\lambda)_{|U}) - \frac{1}{2}E(\varphi_{|U}) \\
&\leqslant \limsup_{f \in C_c(U,[0,1])} \limsup_{\varepsilon \to 0} \int_U f(x)\, d\mu(x) \int_{B_U(x,\varepsilon)} \frac{1}{\varepsilon^{m+2}} \sum_{j=1}^{4} R^{(j)}\, d\mu(x') \\
&\leqslant -\frac{\sin 2R}{2R} \int_U \lambda e(\varphi)\, d\mu + E(\varphi) O(\|\lambda\|_{L^\infty}^2) \\
&\qquad + c_m(1 + O(\|\lambda\|_{L^\infty}^2)) \int_U \Big\langle \nabla \cos v, \nabla \frac{\tan(\lambda v)}{\sin v} \Big\rangle d\mu \\
&\qquad + c_m O(\|\lambda\|_{L^\infty}^2) \int_U |\nabla \cos v|^2 d\mu \\
&\qquad + c_m \frac{1}{2} \int_U |\nabla(\lambda v)|^2 d\mu.
\end{aligned}$$

For the second inequality here we use the interpretation of the supremum in (1.2) (now with U in place of X) as the limit of the corresponding increasing net with the upper directed index set $C_c(U, [0,1])$ (ordered pointwise).

For given $\lambda \in \mathrm{Lip}_c^+(U)$, $\lambda \not\equiv 0$, apply this with $t\lambda$ in place of λ above, $0 < t < 1/\|\lambda\|_{L^\infty}$; divide by t, and let $t \to 0$, noting that

$$\nabla \frac{\tan(t\lambda v)}{t \sin v} \to \nabla \frac{\lambda v}{\sin v} \qquad \text{in} \quad L^2(U) \text{ as } \quad t \to 0.$$

[6] The inequality (8.3) replaces the former inequality (10.17) in [EF], valid only for $d_\lambda \geqslant d$.

This leads to

$$\int_U \Big\langle \nabla \cos v, \nabla \frac{\lambda v}{\sin v} \Big\rangle d\mu \geqslant \frac{c'}{c_m} \int_U \lambda e(\varphi)\, d\mu, \qquad \lambda \in \mathrm{Lip}_c^+(U),$$

where $c' = (2R)^{-1} \sin 2R$. Here $\lambda v / \sin v$ can be replaced by any $\lambda \in \mathrm{Lip}_c^+(U)$, cf. [EF, p. 195], and we conclude that, for $\lambda \in \mathrm{Lip}_c^+(U)$,

$$\int_U \langle \nabla \cos v, \nabla \lambda \rangle d\mu \geqslant \frac{c'}{c_m} \int_U \lambda \frac{\sin v}{v} e(\varphi)\, d\mu \geqslant \frac{c}{c_m} \int_U \lambda e(\varphi)\, d\mu, \tag{8.4}$$

where $c = 2c'/\pi$. By application of a Lipschitz partition of unity (cf. [EF, p. 41 f.]) U can be replaced here by all of X. The result is expressed by saying that

$$\varDelta \cos v \leqslant -\frac{c}{c_m} e(\varphi) \tag{8.5}$$

holds in X in the weak sense. In particular, $\cos v$ is *weakly superharmonic* in X in the sense of [EF, Definition 5.2].

We shall prove that φ (after correction on a nullset) is Hölder continuous in a neighbourhood of any point of X. If $R < \pi/4$ this follows from the proof of [EF, Theorem 10.2] (slightly modified as in the beginning of the proof of Theorem 1 in Sect. 7 above, and therefore valid even if Y were not locally compact), applied to the convex compact ball $V = B_Y(q, R)$ of diameter $D \leqslant 2R < \pi/2$. We may therefore *assume that* $R \geqslant \pi/4$.

According to [EF, Theorem 9.1] we may define $\varphi : X \to Y$ on a nullset so that φ becomes *quasicontinuous*, Y being locally compact, by hypothesis. By [EF, Proposition 7.8 (c)] this means that φ is *finely continuous* off a certain *polar set* $P \subset X$, cf. [EF, pp. 18, 109]. It follows that

$$\varphi(X \setminus P) \subset B = B_Y(q, R), \tag{8.6}$$

the nullset of points $x \in X \setminus P$ with $\varphi(x) \notin B$ being finely open and hence void, by Lemma 4 below.

Fix $\varrho > 0$ with $\varrho < \pi/2 - R$ $(\leqslant \pi/4)$. For any point $z \in B_Y(q, \varrho)$ consider the distance function

$$v(x, z) = d_Y(\varphi(x), z), \qquad x \in X. \tag{8.7}$$

Because $d_Y(\cdot, z)$ is Lipschitz, $v(\cdot, z)$ and $\cos v(\cdot, z)$ are of class $W^{1,2}(X)$ by [EF, Corollaries 9.1 and 9.2]. Since $v(x, z) \leqslant R + \varrho < \pi/2$ (in particular, any triangle $\varphi(x)\varphi(x')z$ has perimeter $< 2\pi$), it follows from (8.5), applied with q and R replaced by z and $R + \varrho$, respectively, that

$$\varDelta \cos v(\cdot, z) \leqslant -\frac{c}{c_m} e(\varphi)$$

in the weak sense, where c only depends on m and $R + \varrho$. In particular, $\cos v(\cdot, z)$ is *weakly superharmonic*. Furthermore, the distance function $v(\cdot, z)$ from (8.7) is finely continuous in $X \setminus P$ because φ is so. The weakly superharmonic function $\cos v(\cdot, z) \in W^{1,2}(X)$ is therefore *finely superharmonic* in $X \setminus P$. Indeed, X is a countable union of domains $\Omega \subset X$ satisfying the Poincaré-style inequality [EF, eq. (7.3)] and hence admitting a symmetric Green kernel G, as in [EF, Theorem 7.3]. And [F2, Theorem 11] applies with $\mathbb{R}^m$ replaced in our setting by any such domain Ω, showing that $\cos v(\cdot, z)$ is finely superharmonic q.e. in X; hence everywhere in $X \setminus P$, by fine continuity in view of the removable singularity theorem for finely superharmonic functions [F1, Theorem 9.14].

By the same removable singularity theorem the bounded finely superharmonic function $\cos v(\cdot, z) : X \setminus P \to [\cos(R + \varrho), 1]$ agrees on $X \setminus P$ with a unique finely superharmonic function $f_z : X \to [\cos(R + \varrho), 1]$, and f_z is even a *usual* superharmonic function, [F1, Theorem 9.8]. In particular, f_z is lower semicontinuous (relative to the metric topology on X) and *finely continuous*. Consequently, the function $v^*(\cdot, z) := \operatorname{Arc}\cos f_z : X \to [0, R + \varrho]$ is *upper semicontinuous* and *finely continuous*; and $v^*(\cdot, z) = v(\cdot, z)$ in $X \setminus P$.

Let $a \in X$ be given. We denote by $C(\varphi, a) \subset B$ the *cluster set* for $\varphi(x)$ as $x \to a$ (in the metric topology) through points $x \in X \setminus P$, $x \neq a$. Note that $C(\varphi, a)$ is compact and nonvoid, B being compact by the Hopf-Rinow theorem (cf. e.g. [EF, Chapter 2]).

Replacing the metric topology on X by the fine topology we are led to the *fine cluster set* $C_f(\varphi, a)$ for φ at a, now with $x \to a$ in the *fine* topology (through points $x \in X \setminus P$, $x \neq a$). Again, $C_f(\varphi, a)$ is compact and nonvoid. Furthermore, $C_f(\varphi, a) \subset C(\varphi, a)$, the fine topology on X being finer than the metric topology.

Fix for a while a point $z \in B_Y(q, \varrho)$, as just before (8.7). For any fine cluster point $y \in C_f(\varphi, a)$, $d_Y(y, z)$ is a fine cluster value of the distance function $v(x, z)$ from (8.7) as $x \to a$, $x \in X \setminus P \setminus \{a\}$, i.e., $d_Y(y, z) \in C_f(v(\cdot, z), a)$. Hence $d_Y(y, z)$ equals the actual fine limit $v^*(a, z)$ of $v^*(x, z) = v(x, z)$ as $x \to a$ finely through $X \setminus P$. Thus,

$$d_Y(y, z) = v^*(a, z), \qquad y \in C_f(\varphi, a), \tag{8.8}$$

for any $z \in B_Y(q, \varrho)$. In other words, *the fine cluster set $C_f(\varphi, a)$ lies on each sphere $S_Y(z, v^*(a, z)) := \{y \in Y : d_Y(y, z) = v^*(a, z)\}$*. It follows that *the metric cluster set $C(\varphi, a)$ lies on each ball $B_Y(z, v^*(a, z))$*. Indeed, since $v^*(\cdot, z)$ is upper semicontinuous, $v^*(a, z)$ majorizes the metric cluster values $d_Y(y, z)$, $y \in C(\varphi, a)$, of $v^*(x, z) = v(x, z)$ as $x \to a$, $x \in X \setminus P \setminus \{a\}$:

$$d_Y(y, z) \leqslant v^*(a, z), \qquad y \in C(\varphi, a). \tag{8.9}$$

In order to proceed by induction define the integer k by

$$2^{k-1}\varrho \leqslant R + \varrho - \pi/4 < 2^k \varrho; \tag{8.10}$$

this is possible, and $k > 0$, because $R \geqslant \pi/4$. For $j \in \{0, 1, \dots, k\}$ write $\varrho_j = 2^j \varrho$, and choose R_j with $R_0 = R$ so that $R_j + \varrho_j$ increases strictly with j from $R + \varrho$ $(< \pi/2)$ up to

$$R_k + \varrho_k < \min\{\pi/2, \pi/4 + \varrho_k\}; \tag{8.11}$$

this is possible because $R + \varrho < \pi/4 + \varrho_k$ by (8.10). It follows that

$$\varrho_j < \pi/4 \leqslant R_j \quad \text{for } j < k; \qquad 0 < R_k < \pi/4. \tag{8.12}$$

Indeed, $R_j \geqslant R+\varrho-\varrho_j \geqslant R+\varrho-\varrho_{k-1} \geqslant \pi/4$ by (8.10); hence $\varrho_j < \pi/2-R_j \leqslant \pi/4$; next, $R_k + \varrho_k < \pi/4 + \varrho_k$ by (8.11), and $R_k > R_{k-1} + \varrho_{k-1} - \varrho_k = R_{k-1}-\varrho_{k-1} > 0$ because $R_{k-1} \geqslant \pi/4 > \varrho_{k-1}$, as just established with $j = k-1$.

Consider for $0 < \delta \leqslant 1$ the image $S(a,\delta)$ of the closed star of a in X under the Euclidean homothety with centre a and factor δ; then $S(a,\delta)$ is a compact admissible Riemannian polyhedron. Suppose for some $j \in \{0,\dots,k\}$ that $X_j := S(a,\delta_j)$ (with $0 < \delta_j \leqslant 1$), $Y_j \subset Y$, and $q_j \in Y_j$ have been defined so that $d_Y(q,q_j) \leqslant \varrho_j - \varrho$, that Y_j is a closed convex subset of $B \cap B_Y(q_j,R_j)$ (hence satisfies bipoint uniqueness, along with B), and that

$$\varphi(X_j \setminus P) \subset Y_j \;\; (\subset B_Y(q_j,R_j)). \tag{8.13}$$

For $j = 0$ this is fulfilled with $\delta_0 = 1$, $Y_0 = B$, and $q_0 = q$, cf. (8.6). We proceed to show that either φ is Hölder continuous in a neighbourhood of a, or else $j < k$ and there exist δ_{j+1}, Y_{j+1}, q_{j+1} conforming with the above requirements. In the latter case, repeat the argument with j replaced by $j+1$, etc.; that will complete the proof that φ is Hölder continuous in a neighbourhood of the given point a.

For the given index $j \in \{0,\dots,k\}$ let X_j, Y_j, q_j, R_j, and ϱ_j take the place of X, Y, q, R, and ϱ, recalling that $R_j + \varrho_j < \pi/2$. If $j = k$ then $R_j < \pi/4$ by (8.12), and it follows from (8.13) by the previous argument for the case $R < \pi/4$ that φ is Hölder continuous in a neighbourhood of a (after correction on a nullset).

We may therefore assume that $j < k$. Because $d_Y(q,q_j) \leqslant \varrho_j - \varrho$ we may apply (8.7), (8.8), and (8.9) to points $z \in B_Y(q_j,\varrho_j)$ ($\subset B_Y(q,\varrho)$), in particular to $z = q_j$. Consider a number r such that $\varrho_j \leqslant r < \pi/4$, cf. (8.12). Since $d_Y(q,q_j) \leqslant \varrho_j - \varrho \leqslant R - \pi/4 < R - r$ by (8.10), we have $B_Y(q_j,r) \subset B$, and so $B_Y(q_j,r)$ is convex (hence satisfies bipoint uniqueness along with B).

If $v^*(a,q_j) \leqslant \varrho_j$ then $C(\varphi,a) \subset B_Y(q_j,\varrho_j) \subset B_Y(q_j,r)^\circ$ for $\varrho_j < r < \pi/4$, by (8.9). By a standard compactness argument it follows that there exists $0 < \delta \leqslant 1$ such that $\varphi(x) \in B_Y(q_j,r)^\circ$ for $x \in S(a,\delta) \setminus P$. Because $r < \pi/4$ we have therefore found above that φ is Hölder continuous on $S(a,\delta)$ (a neighbourhood of a in X) after correction on a nullset (a subset of $P \cup \{a\}$, by the way).

In the remaining case where $v^*(a,q_j) > \varrho_j$, $B_Y(q_j,\varrho_j)$ does not meet the sphere $S_Y(q_j,v^*(a,q_j))$ containing $C_f(\varphi,a)$, by (8.8) applied to $z = q_j$. For any two distinct fine cluster points $y_1, y_2 \in C_f(\varphi,a)$ ($\subset B$) let z_i ($i = 1,2$) denote the nearest point projection of y_i on $B_Y(q_j,\varrho_j)$ (convex compact in B, as shown above, now for $r = \varrho_j$), i.e., the point of the geodesic segment $[y_i,q_j]$ at distance ϱ_j from q_j. If $z_1 \neq z_2$ then (again by (8.8) for $z = q_j$)

$$d_Y(y_1,q_j) = d_Y(y_2,q_j) < d_Y(y_2,z_1) + d_Y(z_1,q_j)$$

since z_1 is not on $[y_2,q_j]$. Subtracting $d_Y(z_1,q_j)$ we infer that $d_Y(y_1,z_1) < d_Y(y_2,z_1)$, which contradicts (8.8) for $z = z_1 \in B_Y(q_j,\varrho_j)$, because $y_1, y_2 \in C_f(\varphi,a)$.

We have thus found that *all fine cluster points* $y \in C_f(\varphi,a)$ *have the same nearest point projection* q_{j+1} *on* $B_Y(q_j,\varrho_j)$, and by (8.8) (applied to $z = q_{j+1}$) the same distance from q_{j+1}. That distance equals $d_Y(y,q_{j+1}) = d_Y(y,q_j) - \varrho_j$.

Because $Y_j \subset B_Y(q_j, R_j)$ we therefore obtain from (8.8) (now also with $z = q_j$) and (8.13)

$$v^*(a, q_{j+1}) = v^*(a, q_j) - \varrho_j \leqslant R_j - \varrho_j = R_j + \varrho_j - \varrho_{j+1} < R_{j+1}. \quad (8.14)$$

It therefore follows by (8.9) (with $z = q_{j+1}$) and by (8.13), (8.14) that the usual cluster set $C(\varphi, a) \subset Y_j$ does satisfy

$$C(\varphi, a) \subset B_Y(q_{j+1}, v^*(a, q_{j+1})) \subset Y_j \cap B_Y(q_{j+1}, R_{j+1})^\circ \subset B_{Y_j}(q_{j+1}, R_{j+1}).$$

As in (8.6) there exists accordingly $0 < \delta_{j+1} \leqslant \delta_j$ ($\leqslant 1$) such that $X_{j+1} := S(a, \delta_{j+1})$ satisfies

$$\varphi(X_{j+1} \setminus P) \subset Y_{j+1} := B_{Y_j}(q_{j+1}, R_{j+1}),$$

cf. (8.13); and the ball Y_{j+1} in Y_j of radius $R_{j+1} < \pi/2 - \varrho_{j+1} < \pi/2$ is a closed convex subset of Y_j and hence of B. Finally, $d_Y(q, q_{j+1}) \leqslant d_Y(q, q_j) + \varrho_j \leqslant \varrho_{j+1} - \varrho$.

Case of an open ball. The above proof, up to and including (8.5), applies unaltered when B is instead an *open* ball $V_Y(q, R)$ (with the properties stated in the theorem). However, deducing from (8.5) that φ is Hölder continuous, cf. the proof of [EF, Theorem 10.1, p. 189], seems to require that B be closed, in order for $\varphi : X \to B$ to possess a meanvalue over certain subsets of X. According to [EF, Remark 7.4] the weakly superharmonic function $\cos v > \cos R$ differs only on a μ-nullset from a superharmonic function $f = f_q$ on X. Actually, $f \geqslant \cos R$ holds everywhere in X because the nullset Z where $f < \cos R$ is finely open, and hence void by Lemma 4 below. From $f > \cos R$ μ-a.e. and $f \geqslant \cos R$ everywhere it follows that $f > \cos R$ everywhere in X, by the minimum principle for superharmonic functions, cf. e.g. [EF, p. 18]. Define $v^* : X \to [0, \pi/2]$ by $\cos v^* = f$; then $v^* = v$ a.e. and $v^* < R$ everywhere in X. Since X is compact, and $-f$ and hence v^* are upper semicontinuous, we conclude that indeed $R^* := \max v^* < R < \pi/2$, i.e., $\varphi(X) \subset B_Y(q, R^*)$ – a closed ball contained in $V_Y(q, R)$, and therefore convex and satisfying bipoint uniqueness.

The following potential theoretic lemma is essentially known:

Lemma 4. (a) *Every polar set $e \subset X$ is a μ-nullset.* (b) *Every finely open μ-nullset $Z \subset X$ is void.*

Proof. In view of [EF, Theorem 7.2 and Remark 7.2] we have the Dirichlet space $L_0^{1,2}(X)$, hypoelliptic in the sense of Feyel and de La Pradelle [FP]. In particular, X is a $\mathcal{P}$-harmonic space [EF, pp. 19, 108].

Ad (a). There exists an E-potential $p \in L_0^{1,2}(X)$ ($\subset W_{\mathrm{loc}}^{1,2}(X)$) such that $p = \infty$ on e, [FP, 32°], see also [EF, p. 109]; it follows that $\mu(e) = 0$.

Ad (b). Exploiting the polyhedral structure of X this can be reduced to the known case $X = \mathbb{R}^m$. We give a direct, potential theoretic proof for the present, more general case:

For any set $A \subset X$ the *base* of A is defined as the set $b(A)$ of points of X at which A is not *thin* in the sense of Brelot, cf. e.g. [CC, §6.3]; in particular, $b(A) \subset A$ if (and only if) A is finely closed [CC, Proposition 6.3.3 (b)].

According to Lassoued [L] the harmonic space X satisfies the axiom of domination and therefore also the axiom of polarity, see e.g. [CC, Corollary 9.2.3]. The latter axiom means that (for any set $A \subset X$) the set $e = A \setminus b(A)$ of points of A at which A is thin, is *polar* [CC, Theorem 9.1.1], and hence a μ-nullset, by (a).

For the finely closed set $A = X \setminus Z$ there are two finite potentials u and $v \leqslant u$ (e.g. $v = \hat{R}_u^A$) such that

$$b(A) = \{x \in X : u(x) = v(x)\},$$

[CC, Propositions 7.2.1 and 7.2.2]. By hypothesis, $X \setminus A = Z$ is a μ-nullset, and so is therefore $X \setminus b(A) = Z \cup e$. Thus $u = v$ holds μ-a.e. in X, and indeed everywhere by [FP, 33°], according to which

$$u(x) = \operatorname{ess}\liminf_{y\to x}\, u(y) = \operatorname{ess}\liminf_{y\to x}\, v(y) = v(x)$$

for every $x \in X$. Consequently, $X = b(A) \subset A$, i.e., $Z = \varnothing$.

9. Proof of Theorem 4

We may assume that X is *compact* and that $K = 1$, hence $R < \pi/2$. Choose normal coordinates $y^1, \dots, y^n$ centred at q and defined in some relatively compact domain $V = V_q$ in N containing $B = B_N(q, R)$. Write

$$w(y) = u(y)^2 = d_N^2(y, q) = \sum_{k=1}^{n} (y^k)^2, \qquad y \in V, \tag{9.1}$$

and denote $H(w)$ the Hessian of w, formed by the covariant second derivatives

$$w_{\alpha\beta} = \partial_\alpha \partial_\beta w - \Gamma_{\alpha\beta}^k\, \partial_k w \tag{9.2}$$

of w. Then $H(w) - 2ch$ is positive semidefinite for $c = R \cot R$:

$$w_{\alpha\beta}(y)\xi^\alpha\xi^\beta \geqslant 2c\, h_{\alpha\beta}(y)\xi^\alpha\xi^\beta, \qquad y \in B,\ (\xi^1, \dots, \xi^n) \in \mathbb{R}^n. \tag{9.3}$$

In [HKW, eq. (2.2)] this is derived (with a similar constant c) from Rauch's comparison theorem. We reproduce from [EF, p. 231 f.] an alternative proof of (9.3):

Because N has sectional curvature $\leqslant 1$, hence also Alexandrov curvature $\leqslant 1$, and because B satisfies bi-point uniqueness, every triangle $yy'q$ in B admits triangle comparison, its perimeter being $\leqslant 4R < 2\pi$. By continuity it suffices to consider the case where $y \neq q$ and $y \in B^\circ$. Consider a geodesic segment $\gamma : [-\varepsilon, \varepsilon] \to B^\circ$, parametrized by path length and satisfying $\gamma(0) = y \neq q$. Triangle comparison of $\gamma(t)\gamma(-t)q$ with median $\gamma(0)q$ gives, writing for a while $u(t)$ for $(u \circ \gamma)(t)$, and later $w(t)$ for $(w \circ \gamma)(t)$, cf. (9.1),

$$\cos u(t) + \cos u(-t) - 2\cos u(0) \cos t \leqslant 0.$$

Subtracting $2\cos u(0)(1-\cos t)$ on both sides, and dividing by t^2, we obtain for $t \to 0$

$$\frac{d^2}{dt^2}\cos\sqrt{w(t)}\,\Big|_{t=0} \leqslant -\cos u(0),$$

and so

$$\ddot{w}(0) \geqslant \frac{2u(0)}{\tan u(0)} + \frac{1}{2u(0)}\left(\frac{1}{u(0)} - \frac{1}{\tan u(0)}\right)\dot{w}(0)^2 \geqslant 2c$$

with $c = R\cot R$, as stated above. Thus $(w\circ\gamma)^{\cdot\cdot}(0) \geqslant 2c$. By (9.2) and the differential equation $\ddot{\gamma}_k + \Gamma^k_{\alpha\beta}\dot{\gamma}^\alpha\dot{\gamma}^\beta = 0$ for the geodesic γ of speed 1, that is, $(h_{\alpha\beta}\circ\gamma)\dot{\gamma}^\alpha\dot{\gamma}^\beta = 1$, we obtain

$$(w_{\alpha\beta}\circ\gamma)\dot{\gamma}^\alpha\dot{\gamma}^\beta \geqslant 2c(h_{\alpha\beta}\circ\gamma)\dot{\gamma}^\alpha\dot{\gamma}^\beta \qquad \text{at} \quad t=0.$$

Inserting $\gamma(0) = y$ this establishes (9.3) because we may choose the geodesic γ through y with initial velocity $\dot{\gamma}(0) = \xi$ when $|\xi| = 1$, i.e., $h_{\alpha\beta}(y)\xi^\alpha\xi^\beta = 1$.

Because $\varphi \in W^{1,2}(X,N)$, the components $\varphi^k = y^k\circ\varphi$ of φ are of class $W^{1,2}(X)$ (Definition 2 in Sect. 2). They are moreover bounded since $\varphi(X)\subset B$, and so $\sum_{k=1}^n(\varphi^k)^2 = w\circ\varphi = d_N^2(\varphi,q) \leqslant R^2$, cf. (9.1). Consequently, $w\circ\varphi \in W^{1,2}(X)$, and $\nabla(w\circ\varphi) = 2\sum_{k=1}^n \varphi^k\nabla\varphi^k = [(\partial_k w)\circ\varphi]\nabla\varphi^k$; alternatively, this follows from the chain rule [EF, Lemma 5.2].[7] For $\lambda \in \mathrm{Lip}(X)$ we therefore obtain

$$\int_X \langle\nabla\lambda, \nabla(w\circ\varphi)\rangle d\mu = \int_X \langle[(\partial_\alpha w)\circ\varphi]\nabla\lambda, \nabla\varphi^\alpha\rangle d\mu \tag{9.4}$$
$$= \int_X \langle\nabla(\lambda\,[(\partial_\alpha w)\circ\varphi]), \nabla\varphi^\alpha\rangle d\mu - \int_X \lambda\,[(\partial_\alpha\partial_\beta w)\circ\varphi]\langle\nabla\varphi^\alpha, \nabla\varphi^\beta\rangle d\mu.$$

Note that, since $(\partial_k w)\circ\varphi = 2\varphi^k$ is bounded and of class $W^{1,2}(X)$, so is $\lambda[(\partial_k w)\circ\varphi]$, by [EF, Remark 5.1 (a)]. In the definition (2.5) of weak harmonicity of φ we may accordingly replace λ by $\lambda[(\partial_k w)\circ\varphi]$. In the resulting equation insert (9.2) (after composition with φ), and compare with (9.4). That leads to

$$-\int_X \langle\nabla\lambda, \nabla(w\circ\varphi)\rangle d\mu = \int_X \lambda(w_{\alpha\beta}\circ\varphi)\langle\nabla\varphi^\alpha, \nabla\varphi^\beta\rangle d\mu \tag{9.5}$$

for every Lipschitz function λ on X. From (9.3) it follows that

$$(w_{\alpha\beta}\circ\varphi)\langle\nabla\varphi^\alpha, \nabla\varphi^\beta\rangle \geqslant 2c(h_{\alpha\beta}\circ\varphi)\langle\nabla\varphi^\alpha, \nabla\varphi^\beta\rangle = 2c\,e(\varphi), \tag{9.6}$$

by the present definition (2.4) of the energy density $e(\varphi)$. Indeed, for each $y \in B$, the positive semidefinite $n\times n$ matrix with entries $a_{\alpha\beta} = w_{\alpha\beta}(y) - 2c\,h_{\alpha\beta}(y)$,

[7] In the chain rule for *maps* $\varphi : X \to N$ in [EF, Remark 9.7] (not used in the present paper) a hypothesis should be added in order to ensure the *global* finiteness of the Sobolev norm of $v\circ\varphi$ on the quasiopen set $U \Subset X$ with $\varphi(U)\subset V$ ($V\subset N$ being the domain of the given C^1-function v). It suffices to add the requirement that $\varphi(U)\Subset V$ (for then $\varphi(U)$ meets only finitely many coordinate patches W of N). In particular, it suffices to suppose that the map φ be *continuous* (as it is in all applications of the chain rule for maps in [EF]). – Also note that, in the version of the chain rule in [EF, eq. (7.26)], $U\subset X$ should have been $U\Subset X$.

cf. (9.3), has a symmetric square root $(b_{\alpha\beta})$, and for $y = \varphi(x)$ it follows that (at almost every point $x \in X$)

$$\big((w_{\alpha\beta} \circ \varphi) - 2c(h_{\alpha\beta} \circ \varphi)\big)\langle \nabla\varphi^\alpha, \nabla\varphi^\beta \rangle = (a_{\alpha\beta} \circ \varphi)\langle \nabla\varphi^\alpha, \nabla\varphi^\beta \rangle$$
$$= \sum_{\alpha,k=1}^{n} |b_{\alpha k}\nabla\varphi^\alpha|^2 \geqslant 0.$$

Inserting (9.6) in (9.5) gives

$$-\int_X \langle \nabla\lambda, \nabla(w \circ \varphi)\rangle d\mu \geqslant 2c \int_X \lambda e(\varphi) d\mu \geqslant 0 \tag{9.7}$$

for every Lipschitz function $\lambda \geqslant 0$ on X. In particular, the function $w \circ \varphi = v^2(\cdot, q)$ (cf. (8.7)) is *weakly subharmonic*, [EF, Definition 5.2]. Furthermore, φ is *quasicontinuous* according to (i) in Definition 2.

For the proof that φ is Hölder continuous (after correction on a nullset) suppose first that B is *closed*, i.e., $B = B_N(q, R)$. If $R < \pi/4$ then B has diameter $\leqslant 2R < \pi/2$, and the proof of (9.7) carries over with q replaced by any point $z \in B$, i.e., with $w \circ \varphi = d_N^2(\varphi(\cdot), z)$. This plays the role of [EF, Lemma 10.2]; but note that we do not require in the present setting that B be convex. The rest of the proof of [EF, Theorem 10.2] (slightly modified as in the proofs of Theorems 1 and 2 above) therefore carries over and shows that φ in the present setting is indeed Hölder continuous when $R < \pi/4$. The case $R \geqslant \pi/4$ reduces to the case $R < \pi/4$ exactly as in the second part of the proof of Theorem 2 above, beginning with the paragraph containing (8.6), while replacing Y by N and the weakly subharmonic function $-\cos v$, $v = d_Y(\varphi(\cdot), q)$, by the present weakly subharmonic function $w \circ \varphi = v^2$.

As at the end of the proof of Theorem 2 this allows us to reduce the case of an *open* ball B in Theorem 4 to that of a closed ball.

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