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# Behaviour of singularities of the rotationally symmetric, volume-preserving mean curvature flow

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**Abstract.** We study developing singularities for surfaces of rotation with free boundaries and evolving under volume-preserving mean curvature flow. We show that singularities form a finite, discrete set along the axis of rotation. We prove a monotonicity formula and conclude that type I singularities are asymtotically cylindrical.

## Introduction

Assume $M^n$ to be a $n$-dimensional manifold and consider a one-parameter family of smooth immersions $x_t : M^n \to \mathbb{R}^{n+1}$. The hypersurfaces $M_t = x_t(M^n)$ are said to move by mean curvature, if $x_t = x(\cdot, t)$ satisfies

$$\frac{d}{dt}x(p,t) = -H(p,t)\nu(p,t), p \in M^n, t > 0. \tag{1}$$

By $\nu(p,t)$ we denote a choice of unit normal of $M_t$ at $x(p,t)$, and by $H(p,t)$ the mean curvature with respect to this normal. The above is a system of quasilinear parabolic equations and if $M^n$ is compact without boundary, or otherwise by imposing suitable growth conditions at infinity or on the boundary, the short-time existence of solutions is guaranteed through standard theory. In addition, the surface area $|M_t|$ of the hypersurfaces is known to decrease and, provided the flow converges, the limit is a minimal surface. Of particular interest is the long-term geometric behaviour of solutions.

Here we are interested in the evolution of compact hypersurfaces $M_t$ enclosing a prescribed constant volume $V$. Specifically, we consider the evolution equation

$$\frac{d}{dt}x(p,t) = -(H(p,t) - h(t))\nu(p,t), p \in M^n, t > 0, \tag{2}$$

where $h(t)$ is the average of the mean curvature.

$$h(t) = \frac{\int_M H dg_t}{\int_M dg_t}$$

and $g_t$ denotes the metric on $M_t$.

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We assume the initial hypersurface $M_0$ to be smoothly embedded in the domain

$$G = \{x \in \mathbb{R}^{n+1} : 0 < x_{n+1} < d\}, d > 0$$

and for its boundary $\emptyset \neq \partial M_0 \subset \partial G$. The vector $\nu(p,t)$ is the outer unit normal.

The surface area $|M_t|$ is again decreasing under the flow defined by (2) and in addition the enclosed volume is constant (see e.g. [17]). In this case the hypersurfaces can be expected to converge to a surface of constant mean curvature which solves the isoperimetric problem.

Mean curvature flow (1) was initially studied by Brakke [9] in a geometric measure theoretical setting. For compact, convex initial surfaces without boundary Huisken [16] proves that they converge asymptotically to round spheres. Entire graphs over $\mathbb{R}^n$ of linear growth "flatten out" as shown by Ecker and Huisken in [13].

The question of the formation of singularities for (1) in the nonconvex case is considered by Huisken [18], Grayson [14], Dziuk and Kawohl [10], and more recently by Altschuler, Angenent and Giga [1], Ecker [12], Huisken and Sinestrari [20].

The main difference and the challenge in the volume-preserving evolution (2), is how to control $h$, which introduces a global character to the problem. Parabolic maximum principles, an important tool in the investigation of evolution equations (see [11]), either fail or become more subtle.

If the initial hypersurface $M_0$ is compact, without boundary and uniformly convex Huisken proves in [17] long-time existence for (2). The uniform convexity is crucial for the proof; using a maximum principle for parabolic systems developed by Hamilton in ([15], Theorem 9.1), Huisken shows that uniform convexity is preserved for $t > 0$. The surfaces $M_t$ are shown to converge to a round sphere enclosing the same volume as $M_0$.

**Results**

In this paper as in [8], except for the volume constraint, we have a free boundary. A convexity assumption would not be natural. Instead, we assume the initial surface to be rotationally symmetric and the hypersurfaces $M_t$ to intersect $\partial G$ orthogonally at the boundary. The motivation is the fact that in solving the isoperimetric problem using methods of the calculus of variations, the minimizers prove to be surfaces of revolution intersecting the obstacle at a right angle [6,7].

Mean curvature flow (without a volume constraint) for complete rotationally symmetric surfaces has been studied by Simon [22]; Dziuk and Kawohl [10], Grayson [14], Huisken [18], Altschuler, Angenent and Giga [1] consider the behaviour of developing singularities. The methods used in [18] and [1] are of essential interest to us, as we deal with rotational symmetry.

In [8] we proved longtime existence for the flow of rotationally symmetric surfaces enclosing a relatively large volume compared to the distance $d$ of the planes and the initial surface area $|M_0|$; the surfaces do not develop singularities and eventually they converge to a cylinder of same enclosed volume. The present paper is complementing the result in [8]: there is no condition on the volume here and we study cases in which singularities do develop at some time $T < \infty$. We investigate the asymptotic behaviour of $M_t$ as $t$ approaches $T$.

The main result is the following

**Theorem.** *Assume $V, d \in \mathbb{R}$ to be given, $M_0 \subset G$ to be a smooth, rotationally symmetric, initial hypersurface which intersects $\partial G$ orthogonally at the boundary and encloses the volume $V$ and $M_t$ to evolve under (2), satisfying the same boundary and volume conditions as $M_0$. Then*

(i) *for $0 \leq \beta \leq n-1, \varepsilon > 0$, there are constants $R_0 = R_0(\beta, n, \sup_t |h|)$ and $\gamma < 2\beta$, such that if in addition the upper bound $R$ of the radius of the flow satisfies $R < R_0$, $M_0 \subset \{x \in \mathbb{R}^{n+1} : (n-1-\beta)x_{n+1}^2 \geq (x_1^2 + x_2^2 + \ldots + x_n^2) - \varepsilon\}$ implies that $M_{\frac{\varepsilon}{2\beta-\gamma}}$ has a singularity at the origin;*
(ii) *the singular set is finite and discrete along the axis of rotation;*
(iii) *in case of type I singularities the neighbourhood of blow-up points is asymptotically cylindrical;*
(iv) *if $M_0$ has non-negative mean curvature, then developing singularities are of type I.*

The paper is organised as follows:

The basic estimate, which allows us to use techniques similar to those in mean curvature flow, is established in Sect. 1. By means of the Sturmian Theorem (Theorem 1.1) we can control the number of zeroes of the derivative of the radius function which describes the surfaces of rotation (Corollary 1.2). This helps controlling the number of necks of $M_t$, the lenght of the generating curve (Corollary 1.3), and most importantly the mean value of the mean curvature $h$, even if $M_t$ approach the axis of rotation (Proposition 1.4).

In Sect. 2 we prove the fact that "too thin" necks have to pinch-off, the flow thus developing singularities in finite time (Proposition 2.1 and remark following it). We show that these necks will converge to single points along the axis of rotation (Lemma 2.3, 2.4), so that the singularities form a finite, discrete set on that axis. Gradient and curvature estimates away from the axis can then be obtained as in [8], so that we have convergence of the evolution to a limiting hypersurface.

Section 3 is dedicated to the asymptotic behaviour of singularities. The main ingredient here is the monotonicity formula (Proposition 3.2). In case of Type I singularities and after rescaling, in order to guarantee the second fundamental form of the rescaled surfaces being uniformly bounded, we show that the rescaled flow converges to a cylinder (Theorem 3.9).

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## Notations and preliminaries

Let $G = \{x \in \mathbb{R}^{n+1} : 0 < x_{n+1} < d\}$, for given $d > 0$. We denote by $\Pi_i$, $i = 1, 2$, the two parallel hyperplanes bounding the domain $G$.

The $n$-dimensional manifold $M^n$ is assumed to be smoothly embedded in $\mathbb{R}^{n+1}$, compact, with boundary $\partial M$. The one-parameter family of surfaces obtained by the flow is defined by means of the position vector

$$x : M^n \times [0, T) \to \mathbb{R}^{n+1} ,$$

where $x$ satisfies the evolution equation (2) above.
By $M_t$ we denote the image $M_t = x_t(M^n)$ and $M_0$ will be a given initial surface.
In addition we assume

(i) The hypersurface $M_0$ is rotationally symmetric about an axis which intersects $\Pi_i$ orthogonally. We also use the parametrization

$$\rho_S : [0, d] \to \mathbb{R}$$

for the generating curve of a surface $S$ of revolution. Actually, the flow preserves rotational symmetry (see Fact 1 below).

(ii) The boundary $x_t(\partial M) = \partial M_t$ is contained in $\partial G = \underset{i=1,2}{\to} \bigcup \Pi_i$.

(iii) $M_t$ intersects $\partial G$ orthogonally at the free boundary; i.e. $\dot{\rho}(z) = 0$, for $z = 0$ and $z = d$. Here $\dot{\rho} = \frac{d\rho}{dz}$.

By $g = g_{ij}$ and $A = \{h_{ij}\}$ we denote the metric and the second fundamental form on $M_t$. The mean curvature and the norm of the second fundamental form are given by

$$H = g^{ij} h_{ij}, |A|^2 = g^{ij} g^{kl} h_{ik} h_{jl}.$$

**Facts**

For the proofs compare ([8], Sect. 1).

1. The flow preserves rotational symmetry. This is clear from the evolution equation, since the mean curvature and the normal are symmetric.
2. The surface area $|M_t|$ is decreasing. To see this we need the evolution equation of the metric.
3. The enclosed volume $V$ is preserved.

By standard parabolic theory the flow exists for some short time $0 < t < t_1$. We write also $[0, T_{max})$ to indicate the maximal time interval for which the flow exists.

## 1. The Sturmian theorem and bounds of the mean value $h$ of the mean curvature

In [8] an essential step was to be able to bound $h$ - the mean value of the mean curvature $H$ - assuming that the radius of the rotationally symmetric evolving surfaces is bounded away from zero.

In the present part the estimate on $h$, without any restrictions on the radius, will follow the same steps as ([8], Lemma 2) but we need to guarantee that the generating curve does not begin oscillating as it approaches the axis of rotation.

It turns out that by using the Sturmian theorem we are able to control the zeroes of $\dot{\rho}$, points where the generating curve assumes a horizontal tangent.

Assume $u : \mathbb{R}^2 \to \mathbb{R}$ to be a solution of

$$u_t = a(z,t)u_{zz} + b(z,t)u_z + c(z,t)u \tag{1.1}$$

on $Q = \{(z,t) \in \mathbb{R}^2 : 0 \le z \le 1,\ 0 \le t \le T\}$ with Dirichlet boundary conditions $u(0,t) \equiv 0 \equiv u(1,t)$. The **number of zeroes** of $u(\cdot,t)$ is defined as the supremum of all $k$ such that there exist $0 < z_1 < z_2 < \ldots < z_k < 1$ with

$$u(z_i,t)u(z_{i+1},t) < 0, i = 1,2,\ldots,k-1.$$

For $t \in (0,T)$ let

$$Z_t = \{x \in \mathbb{R} : u(x,t) = 0\}$$

**Theorem 1.1 (Sturmian theorem).** *Assume the coefficients of (1.1) to satisfy*

$$\begin{aligned}
&a > 0, \ \ a, a^{-1}, a_t, a_z, a_{zz} \in L^\infty, \\
&b, b_t, b_z \in L^\infty, \\
&c \in L^\infty
\end{aligned}$$

$|u(z,t)| \le A\exp(Bz^2)$.
*Then for each $t \in (0,T)$ the zero set $Z_t$ of $u$ is a discrete subset of $\mathbb{R}$. Moreover if at $(z_0,t_0)$ both $u$ and $u_z$ vanish then there is a neighbourhood $N = [z_0-\varepsilon, z_0+\varepsilon] \times [t_0-\delta, t_0+\delta]$ of $(z_0,t_0)$ such that*

(i) $u \neq 0$ *on the sides of $N$, i.e.$u(z_0 \pm \varepsilon, t) \neq 0$ for $|t-t_0| \le \delta$,*
(ii) $u(\cdot, t+\delta)$ *has at most one zero in the interval $[z_0-\varepsilon, z_0+\varepsilon]$,*
(iii) $u(\cdot, t-\delta)$ *has at least two zeroes in the interval $[z_0-\varepsilon, z_0+\varepsilon]$.*

The theorem in this form has been proven in [2]; we refer to that paper for additional remarks. The theorem holds exactly as stated above if instead of a Dirichlet boundary condition we impose periodic or Neumann boundary data - in latter case, assume that $a = 1$ and $b = 0$. If $u$ is assumed to be a smooth solution then it also holds for $u(0,t) \neq 0 \neq u(1,t)$ for $0 \le t < T$.
For applications related to the problem we are dealing with see [3–5] and [1].

Let $M_t$ be the evolving surfaces solving (2). As they are rotationally symmetric they can be described by their generating curves, which are given nonparametrically as graphs of the radius function $\rho = \rho(z,t), 0 \le z \le d$. We have the following

**Corollary 1.2.** *Assume $M_t \subset G$ to be smooth surfaces solving (2) described by means of $\rho$ as above. Assume in addition that $\rho(z,t) \ge \varepsilon, \varepsilon > 0$, for $0 \le z \le d, t \in (0,T)$. Then the set $Z_t = \left\{ z \in \mathbb{R} : \dot{\rho}(z,t) = \dfrac{\partial \rho}{\partial z}(z,t) = 0 \right\}$ is a discrete set in $[0,d]$, for every $t \in [0,T)$. Moreover, the number of zeroes of $\dot{\rho}$ is a nonincreasing function of time.*

*Proof.* From the evolution equation (2) we find the equation for $\rho$ to be

$$\frac{\partial \rho}{\partial t} = \frac{\ddot{\rho}}{1+\dot{\rho}^2} - \frac{n-1}{\rho} + h\sqrt{1+\dot{\rho}^2}, \tag{1.2}$$

the mean curvature of a rotationally symmetric $n$-dimensional surface being

$$H = -\frac{\ddot{\rho}}{(1+\dot{\rho}^2)^{3/2}} + \frac{n-1}{\rho\sqrt{1+\dot{\rho}^2}}.$$

Differentiating (1.2) with respect to $z$ we find that $\dot{\rho} = w$ satisfies

$$\frac{\partial w}{\partial t} = \frac{1}{1+\dot{\rho}^2}\ddot{w} + (\frac{n-1}{\rho^2} - \frac{2\ddot{\rho}^2}{(1+\dot{\rho}^2)^2} + \frac{2h\ddot{\rho}}{\sqrt{1+\dot{\rho}^2}})w. \tag{1.3}$$

Using the results of [8] for fixed $\varepsilon > 0$, such that $\rho(z,t) \geq \varepsilon$, we have

$$0 < h \leq c$$

with $c = c(n, \varepsilon)$ (Lemma 2 in above reference), and can consequently obtain uniform bounds on all quantities of interest (gradient, curvature and higher derivatives of the curvature estimates). This means that $w$ solving (1.3) satisfies the conditions of Theorem 1.1, and the result of the corollary follows. □

**Corollary 1.3.** *The length of the generating curves of $M_t$ is bounded for $t \in [0, t_1]$ and any $t_1 < T$. More precisely, we have*

$$\int_0^d \sqrt{1+\dot{\rho}^2}dz \leq c_1$$

*where $c_1 = c_1(k, R)$, with $k$ denoting the number of zeroes of $\dot{\rho}(z, 0)$ in $[0, d]$, and $R$ an upper bound for $\rho$.*

*Proof.* Due to the volume constraint there exists an $R > 0$ such that $\rho(z,t) \leq R$ for $0 \leq z \leq d$, $t \in [0, T]$ (see remark (iii), in 2A. of [8].)

By Corollary 1.2 the number of oscillations of the radius of $M_t$ for any $t \in [0, t_1], t_1 < T$ ($T$ is first singular time!) is less than that of $M_0$. Let us assume $\dot{\rho}(z, 0)$ to have $k$ zeroes (two of them are at $z = 0$ and $z = d$), i.e. we have $k-1$ intervals in $[0, d]$, such that $\dot{\rho}$ has different sign in adjacent ones. In each of them we can estimate the length of the generating curve by $d + R$ from above, which gives

$$\int_0^d \sqrt{1+\dot{\rho}^2}dz \leq (k-1)(d+R).$$ □

**Proposition 1.4 (Estimates on $h$).** *Assume $\{M_t\}$ to be a family of smooth, rotationally symmetric surfaces, solving (2) for $t \in [0, T)$. Then the mean value $h$ of the mean curvature satisfies*

$$0 < c_2 \leq h \leq c_3,$$

*with $c_2 = c_2(n, d, V)$ and $c_3$ depending on $n, R, \mathcal{A}_V$ and $c_1$, where $\mathcal{A}_V$ is the area of the solution to the given isoperimetric problem.*

*Proof.* As in the proof of Lemma 2 in [8] we have for $H = \kappa_1 + (n-1)\kappa_2$, with $\kappa_1$ and $\kappa_2$ denoting the principal curvatures,

$$h(t) = \frac{1}{|M_t|} \int_M (\kappa_1 + (n-1)\kappa_2) \, \mathrm{dg}_t, t \in [0, T).$$

For the second term we obviously have

$$0 \leq \frac{n-1}{|M_t|} \int_0^d \rho^{n-2}(z,t) dz \leq c(n, \mathcal{A}_V, R), \tag{1.4}$$

since $\rho \leq R$.
For the first term, we remark that $\frac{\ddot{\rho}}{1+\dot{\rho}^2} = \frac{d}{dz}(\arctan \dot{\rho})$, so that after integrating by parts and using $0 \leq (\arctan \dot{\rho})\dot{\rho} \leq \frac{\pi}{2}|\dot{\rho}|$ we obtain

$$\begin{aligned} 0 &\leq \frac{1}{|M_t|} \int_M \kappa_1 dg_t \\ &\leq \frac{n(n-1)\omega_n}{|M_t|} \frac{\pi}{2} \int_0^d \sqrt{1+\dot{\rho}^2} \rho^{n-2} dz \\ &\leq c'(n, \mathcal{A}_V, R) \int_o^d \sqrt{1+\dot{\rho}^2} dz \\ &\leq c''(n, \mathcal{A}_V, R, c_1), \end{aligned} \tag{1.5}$$

by Corollary 1.3.

For the lower bound, we prove that there exist constants $\tilde{d} > 0, \rho_0 > 0$ and an interval $I \subset [0, d]$ with $\mathcal{H}^1(I) \geq \tilde{d}$, such that $\rho > \rho_0$ in $I$.
We remark that the part of any surface $M_t$ lying inside the cylinder $C$ of volume $\frac{V}{2}$ encloses at most that same volume. Note that $C$ has radius $\rho_C = \left(\frac{V}{2\omega_n d}\right)^{1/n}$.

For the parts of $M_t$ not contained in $C$ we want to construct a comparison surface and distribute the remaining volume - which is at least $\frac{V}{2}$ - outside $C$, in a rotationally symmetric way, such that the projection of the new surface onto the $z$-axis corresponds to the least possible width in $z$-direction.
We know that $\rho_{M_t} \leq R$. As comparison object, we choose the annular cylinder with radius $\rho$ satisfying $\rho_C \leq \rho \leq R$, same axis of rotation as $M_t$ and height $\tilde{d}$. We impose the condition $\omega_n (R - \rho_C)^n \, \tilde{d} = \frac{V}{2}$ on the enclosed volume of this annular cylinder.
For the parts of any other surface $M_t$ as above, lying outside $C$, we would have $\rho_{M_t} \geq \rho_C \equiv \rho_0$ in $I$ with $\mathcal{H}^1(I) \geq \tilde{d}$.

This results to the improved lower bound for (1.4)

$$0 < \tilde{c}(n, \mathcal{A}_V, \tilde{d}, V) \leq \frac{n-1}{|M_t|} \int_0^d \rho^{n-2}(z,t) dz,$$

which – combined with (1.5) – completes the proof. □

## 2. Formation of singularities

This section is aimed to understanding the behaviour of singularities. First we show that if the evolving surfaces develop a "neck" close to the axis of rotation, then they eventually pinch-off. We call a neck the part of the surface corresponding to a small neighbourhood of a local minimum of the radius. The proof involves a maximum principle technique similar to one presented in [11] for mean curvature flow. Furthermore, we prove convergence of necks to singularities and the fact that singular points form a discrete set along the axis of rotation. For this part most of the arguments follow the ideas of [1] adjusted to our setting.

**Proposition 2.1 (Pinch-off for thin necks).** *Let* $\{M_t\}$ *be a family of smooth hypersurfaces solving (2). Then for* $0 \leq \beta \leq n-1, t \leq \dfrac{\varepsilon}{2\beta - \gamma}, \varepsilon > 0$ *and* $\gamma \leq 2\beta$*, and if in addition the upper bound of the radius* $R$ *of the flow satisfies* $R < \frac{\beta}{(n-\beta+1)c_3} - d$*,* $M_0 \subset \{x \in \mathbb{R}^{n+1} : (n-1-\beta)x_{n+1}^2 \geq (x_1^2 + x_2^2 + \ldots + x_n^2) - \varepsilon\}$ *implies* $M_t \subset \{x \in \mathbb{R}^{n+1} : (n-1-\beta)x_{n+1}^2 \geq x_1^2 + x_2^2 + \ldots + x_n^2 - \varepsilon + (2\beta - \gamma)t\}$.

(Concerning the conditions of this proposition, see also remarks following the proof.)

*Proof.* Let us first note that in view of Proposition 1.4, $0 < c_2 \leq h(t) \leq c_3 < \infty$, as $h$ is bounded for all $t \in [0, T)$.

We make use of the identity

$$\left(\frac{d}{dt} - \Delta^{M_t}\right)|x|^2 = -2n + 2h\langle x, \nu\rangle \tag{2.1}$$

to show that

$$\left(\frac{d}{dt} - \Delta^{M_t}\right)\left(|x|^2 - (n-\beta)x_{n+1}^2 + (2\beta - \gamma)t\right) \leq 0. \tag{2.2}$$

Indeed, we have (using the notations $\Delta = \Delta^{M_t}$ and $\nabla = \nabla^{M_t}$ from now on)

$$\begin{aligned}
&\left(\frac{d}{dt} - \Delta\right)\left(|x|^2 - (n-\beta)x_{n+1}^2 + (2\beta - \gamma)t\right) \\
&\quad = \left(\frac{d}{dt} - \Delta\right)|x|^2 - 2(n-\beta)x_{n+1}\left(\frac{d}{dt} - \Delta\right)x_{n+1} \\
&\quad + 2(n-\beta)|\nabla x_{n+1}|^2 + 2\beta - \gamma \\
&\quad = h(t)\langle x, \nu\rangle - 2n - 2(n-\beta)h(t)x_{n+1}\nu_{n+1} \\
&\quad + 2(n-\beta)|\nabla x_{n+1}|^2 + 2\beta - \gamma \\
&\quad \leq h(t)\langle x, \nu\rangle + 2(n-\beta)h(t)|x_{n+1}\nu_{n+1}| - \gamma,
\end{aligned}$$

where for the last inequality we note that $h(t) \geq 0$ by Proposition 1.4, $|\nabla x_{n+1}|^2 \leq 1, \beta < n$.

The last term above is easily seen to be $2(n-\beta)h(t)|x_{n+1}\nu_{n+1}| \leq 2(n-\beta)h(t)|x||\nu|$, so that for

$$\left(\frac{d}{dt}-\Delta\right)\left(|x|^2-(n-\beta)x_{n+1}^2+2\beta t\right) \\ \leq 2(n-\beta+1)h(t)|x|-\gamma.$$

This is negative for $\gamma > 2(n-\beta+1)c_3(R+d)$. Finally, we need to have $R < \frac{\beta}{(n-\beta+1)c_3} - d$, to make sure that the last condition on $\gamma$ does not contradict $\gamma < 2\beta$, in order for the factor in front of $t$ to be positive.

Having established (2.2) we can use the maximum principle to conclude

$$|x(p,t)|^2-(n-\beta)x_{n+1}^2(p,t)+(2\beta-\gamma)t \\ \leq |x(p,0)|^2-(n-\beta)x_{n+1}^2(p,0)$$

for $p\in M, t\in(0,T)$.
Assuming the initial surface to be such that the right hand side of this inequality is bounded by $\varepsilon$ gives the desired result. □

*Remark.*
1. If $0\leq\beta<n-1$ the surface $M_{\frac{\varepsilon}{2\beta-\gamma}}$ has a singularity at the origin, since it lies in a cone with vertex at $x=0$.
2. The condition on $R$ gives an upper bound on the volume by comparing to a cylinder of radius $R$

$$V \leq \omega_n R^n d < \omega_n\left(\frac{\beta}{(n-\beta+1)c_3}-d\right)^n d.$$

3. Note that although the above result can be interpreted geometrically, as in remark 1, we do not have a comparison principle as in mean curvature flow: disjoint surfaces do not need to remain disjoint, as their average mean curvature varies (see also Lemma 2.2). In particular, the condition on $R$ excludes thin, long cylinders (which are stationary for the flow) as initial surfaces.

**Lemma 2.2 (Intersections of evolving surfaces with equal $h(t)$) .**
*Let $\{M_t^1\}$ and $\{M_t^2\}$ be two families of smooth surfaces of revolution satisfying (2). Assume in addition that for each $t\in[0,T]$ the respective mean values $h^1(t), h^2(t)$ of the mean curvature are equal, $h^1(t)=h^2(t)$.*
*Then either $M_t^1=M_t^2$ or the number of their intersections is finite for $t\in(0,T)$. In the second case this number is non-increasing in time, and decreases whenever $M_t^1$ and $M_t^2$ have a non-transverse intersection.*

*Remarks on the proof.* The proof of this lemma follows exactly the same steps as that of ([5],Theorem 3.2), presented with all details in ([4], Theorem 1.1) - see also Theorem 1.3 in last reference. In our setting of course we describe the surfaces through the generating curve which is given as graph by means of the radius function – this function $\rho_{M_t}$ satisfies the relevant equations.

We turn our attention to the behaviour of necks next. As shown in Sect. 1 of the paper the number of necks of $M_t$ is a finite non-increasing function of time. By Corollary 1.2 and Theorem 1.1 the number of zeroes of $\dot\rho$ decreases each time we have a multiple zero. Therefore we can assume that after some time $t_1$ all zeroes of $\dot\rho$ are simple, and therefore the number of necks remains constant for the rest of the evolution. In the following we consider the evolution for $t_1 < t < T$. We can thus assume $M_t$ to have $m \geq 0$ necks, i.e. $x \to \rho(z,t)$ has $m$ local minima and either $m$ or $m+1$ local maxima, depending on the endpoints. (Still, as the number of zeroes of $\dot\rho$ remains constant, the number of maxima is not varying between $m$ and $m+1$.) This corresponds to the cases

$$0 < \xi_1(t) < \eta_1(t) < \ldots < \xi_m(t) < \eta_m(t) < d, (c_1)$$
$$0 < \eta_1(t) < \xi_1(t) < \ldots < \eta_m(t) < \xi_m(t) < d, (c_2)$$

or

$$0 < \eta_1(t) < \xi_1(t) < \ldots < \eta_m(t) < \xi_m(t) < \eta_{m+1}(t) < d, (c_3)$$

where minima are located at $\xi_j$, maxima at $\eta_j$. By the above considerations minima and maxima are non-degenerate, and by the implicit function theorem $\xi_j(t), \eta_j(t)$ are therefore smooth functions of time.

**Lemma 2.3 (Convergence of necks).** *The limits*

$$\lim_{t\to T} \xi_j(t) \textit{and} \lim_{t\to T} \eta_j(t)$$

*exist.*

*Proof.* The proof follows exactly the same arguments as in 5.1 of [1]. We can adapt them to our case by noting that a surface $\tilde{M}_t$ obtained form $M_t$ by reflection about a plane has the same average mean curvature $\tilde{h}(t) = h(t)$, so that the previous result applies.

**Lemma 2.4 (Discrete set of singularities).** *For any compact interval $[a,b]$ not containing any of the points $0, \xi_j(T), \eta_j(T), d, for j = 1, \ldots, m$, there exist $\delta > 0$, $t_1 < T$, such that $\rho(z,t) \geq \delta$ for $x \in [a,b]$, $t \in (t_1, T)$.*

*Proof.* After the following considerations we can argue as in 5.2 of [1]:
Let us work with the interval $[a', b'] \subset (\xi_j(T), \eta_j(T))$ in case $(c_1)$. By the Lemma 2.3, there is a time $t_1 < T$ such that $\dot\rho(z,t) > 0$ for $z \in [a', b']$ and $t \in (t_1, T)$.

We consider the function $\phi = \arctan \dot\rho$, the angle between the tangent to the generating curve and the axis of rotation. By differentiating the evolution equation (1.2) of $\rho$, $\phi$ is found to satisfy the parabolic equation

$$\frac{\partial\phi}{\partial t} - \frac{1}{1+\dot\rho^2}\ddot\phi = \left(\frac{n-1}{\rho^2} + h\sqrt{1+\dot\rho^2}\right)\dot\rho > 0,$$

the right hand side being positive, since $h > 0$ by Proposition 1.4, and $\dot\rho > 0$ on $[a', b']$.

For $\varepsilon > 0$ define

$$f(z,t) = \varepsilon \exp(-\lambda^2 t) \sin(\lambda(z-a')), \text{with} \lambda \equiv \frac{\pi}{b'-a'}.$$

This $f$ solves the heat equation $\frac{\partial}{\partial t} f = \ddot{f}$, with zero boundary data. Also

$$\frac{\partial}{\partial t} f - \frac{1}{1+\dot{\rho}^2} \ddot{f} = \frac{\dot{\rho}^2}{1+\dot{\rho}^2} \ddot{f} < 0,$$

since $\ddot{f} < 0$.

We can now use the maximum principle for parabolic equations and obtain the result by comparing $\phi$ to $f$ as in 5.2 of [1]. □

*Remark.* (Gradient bounds and higher regularity whenever $\rho(x,t) > 0$ for $x \in [a,b], t \in [0,T)$)
The proof of our next lemma relies heavily on the regularity results of [8]. More specifically in that paper we obtained:

(i) Gradient bounds of the form

$$\max_{t>0} \sqrt{1+\dot{\rho}^2} \leq c_4(n, \delta, R, v_0),$$

where $\delta > 0$ denotes a lower bound for the radius, $R$ an upper bound, i.e. $\delta \leq \rho(z,t) \leq R$, and $v_0$ a gradient bound for the initial surface, i.e. $\sqrt{1+\dot{\rho}^2(z,0)} \leq v_0$ ([8], Proposition 4).

(ii) Bounds on the second fundamental form $A$ and all its higher derivatives

$$\max_{t>0} |A|^2 \leq c_\delta(n, \delta, R, v_0) \left( \frac{c_6}{\sqrt{c}} + \frac{1}{ct} \right), t \in [0,T),$$

with $\delta, R, v_0$ as above, $c_6 = c_6(n, \delta, R, v_0, c_3)$, with $h \leq c_3$ as in Proposition 1.4, and $c$ a technical constant ([8], Proposition 5). Also, for each $m \geq 1$ there is a $C_m$ such that

$$|\nabla^m A|^2 \leq C_m$$

uniformly on $M_t$ for $0 < t \leq T$ ([8], Proposition 6).

**Lemma 2.5 (Convergence of the evolving surfaces) .** *The radius $\rho(z,t)$ of the surfaces $M_t$ satisfying (2) converges uniformly to $\rho(z,T)$ as $t \to T$. Furthermore, $\rho(z,t) \in C^\infty(\mathbb{R} \times [0,T])$ provided that $\rho(z,t) > 0$.*

*Proof.* From the above remark we conclude that whenever $\rho$ satisfies $\rho(z,t) \geq \delta$ for all $z \in [a,b]$ we have uniform gradient and higher derivatives bounds on any compact subinterval $[a',b'] \subset (a,b)$. Combining with Lemma 2.4, we obtain that $\rho(z,t)$ converges uniformly in time on any compact subinterval $[a',b'] \subset (a,b)$, with $[a,b]$ in the different categories of Lemma 2.4. Therefore $\rho(z,t)$ converges pointwise for all $z \notin \{0, d, \xi_j(T)\}$.

Furthermore, we can prove as in ([1], Theorem 5.3) that $\rho(z,t)$ is equicontinuous in $z$ for $0 < t < T$, which with the pointwise convergence in a dense subset of $[0,d]$, gives the uniform convergence as $t \to T$. $\square$

Summarising the results of Sect. 2, we know that evolving surfaces with "thin" necks (as in Proposition 2.1) will develop singularities. The singularities form a finite, discrete set along the axis of rotation; their number is bounded from above by the number of necks of the initial surface. As $t \to T$ the surfaces $M_t$ converge away from the singularities uniformly to a limit surface.

## 3. Monotonicity formula and asymptotic behaviour of singularities

In this part we follow the methods used in [18,19]. We derive a monotonicity formula for the volume-flow, which is used to show that type I singularities become asymptotically self-similar. After rescaling the evolving surfaces converge to a homothetic solution of the same type as in the mean curvature flow. Those are described and fully classified in [19]. In a more general setting for mean curvature flow one can investigate the formation of singularities of any type, recently done in [20,23].

The fact that the average mean curvature is bounded suggests that the behaviour of singularities after parabolic rescaling is the same as by mean curvature flow. This issue will be pursued in a subsequent paper.

We define $k$ to be the backward heat kernel on $\mathbb{R}^{n+1}$

$$k(x,t) = \frac{1}{(2\pi(T-t))^{\frac{n}{2}}} \exp\left(-\frac{|x|^2}{4(T-t)}\right),$$

$x \in \mathbb{R}^{n+1}, 0 < t < T$.

A straightforward calculation leads to the following

**Lemma 3.1 .** *The backward heat kernel $k$ satisfies the equations*

$$\frac{dk}{dt} = \left[\frac{n}{2(T-t)} - \frac{|x|^2}{4(T-t)^2} + \frac{H-h}{2(T-t)}\langle x,\nu\rangle\right] k,$$

*and*

$$\left(\frac{d}{dt} + \Delta^{M_t}\right) k = \left\{\frac{H}{(T-t)}\langle x,\nu\rangle - \frac{\langle x,\nu\rangle^2}{4(T-t)^2} - \frac{h}{2(T-t)^2}\right\} k.$$

**Proposition 3.2 (Monotonicity formula) .** *For the backward heat kernel $k$ we have*

$$\frac{d}{dt}\left(f(t)\int_M k(x,t)dg_t\right) \le -\frac{1}{2}f(t)\int_M \left|H - \frac{\langle x,\nu\rangle}{2(T-t)}\right|^2 k(x,t)dg_t,$$

*where* $f(t) = \dfrac{1}{(T-t)^{\frac{n}{2}}} \exp\left(-\dfrac{1}{2}\displaystyle\int_0^t h^2(s)ds\right)$.

*Proof.* Note that $f(t)$ is a well-defined smooth function for $0 \leq t < T$, as we are dealing with a smooth flow (compare regularity remark in Sect. 2 and Lemma 2.5) and $0 < c_2 \leq h \leq c_3$. We have

$$\frac{d}{dt}\int_M k(x,t)dg_t = \int_M \frac{d}{dt}k(x,t)dg_t + \int_M k(x,t)\frac{d}{dt}(dg_t)$$
$$= \int_M \left(\frac{n}{2(T-t)} - \frac{|x|^2}{4(T-t)^2} + \frac{H-h}{2(T-t)}\langle x,\nu\rangle - H(H-h)\right)k(x,t)dg_t,$$

from the above and since $\frac{d}{dt}(dg_t) = -H(H-h)dg_t$.
Therefore, and with $f$ defined as above, we find

$$\frac{d}{dt}\left(f(t)\int_M k(x,t)dg_t\right) =$$

$$= f(t)\int_M \left\{-\frac{|x|^2}{4(T-t)^2} + \frac{(H-h)}{2(T-t)}\langle x,\nu\rangle - H^2 + hH - \frac{1}{2}h^2\right\}kdg_t$$
$$= f(t)\int_M \left\{\frac{1}{2}\left[-\frac{|x|^2}{4(T-t)^2} + \frac{H}{(T-t)}\langle x,\nu\rangle - H^2\right] - \frac{1}{2}\frac{|x|^2}{4(T-t)^2}\right.$$
$$\left.-\frac{h}{2(T-t)}\langle x,\nu\rangle - \frac{1}{2}H^2 + hH - \frac{1}{2}h^2\right\}kdg_t$$
$$\leq f(t)\int_M \left\{-\frac{1}{2}\left[H - \frac{\langle x,\nu\rangle}{2(T-t)}\right]^2 - \frac{1}{2}\frac{|x|^2}{4(T-t)^2}\right.$$
$$\left.+\left(\frac{1}{2}h^2 + \frac{1}{2}\frac{|\langle x,\nu\rangle|^2}{4(T-t)^2}\right) - \frac{1}{2}H^2 + hH - \frac{1}{2}h^2\right\}kdg_t$$
$$\leq f(t)\int_M \left\{-\frac{1}{2}\left[H - \frac{\langle x,\nu\rangle}{2(T-t)}\right]^2 - \frac{1}{2}(H-h)^2\right\}kdg_t,$$

as $|\langle x,\nu\rangle|^2 \leq |x|^2$. □

We proceed as in [18] and rescale the evolving surface $M_t$ near a singular point. First, we note that at a singular point the curvature will become unbounded.

**Definition.** We call $x_0 \in \mathbb{R}^{n+1}$ a **blow-up point** if there is a $p \in M^n$ such that $x(p,t) \to x_0$ and $|A|(p,t) \to \infty$ as $t \to T$. From Sect. 2 we know that the flow will develop a blow-up point along the axis of rotation if $M_t$ have thin necks. Assume $0 \in \mathbb{R}^{n+1}$ to be a blow-up point. Define the rescaled surfaces $\tilde{M}_s$ by

$$\tilde{x}(p,s) = (2(T-t))^{-\frac{1}{2}}x(p,t),$$

where $s(t) = -\frac{1}{2}\log(T-t)$.

We remark that the surfaces $\tilde{M}_s$ are defined for $-\frac{1}{2}\log T \leq s < \infty$. The scaling factor is chosen in such a manner that the curvature of $\tilde{M}_s$ remains uniformly bounded and we are able to prove convergence to a limit surface $\tilde{M}_\infty$.

Any quantities defined on $\tilde{M}_s$ will be denoted by a tilde.

**Lemma 3.3 .** *The rescaled surfaces $\tilde{M}_s$ defined by $\tilde{x}(s)$ satisfy the evolution equation*

$$\frac{\partial}{\partial s}\tilde{x} = -\left(\tilde{H} - \tilde{h}\right)\tilde{\nu} + \tilde{x}$$

*Proof.* Using $\tilde{H}(p,s) = \sqrt{2(T-t)}H(p,t)$, $\tilde{h}(p,s) = \sqrt{2(T-t)}h(p,s)$ and $\tilde{\nu} = \nu$. □

**Lemma 3.4 (Rescaled monotonicity formula).**
*Define $\tilde{k}(\tilde{x}, s) = \exp\left(-\frac{1}{2}|\tilde{x}|^2\right)$ . Then*

$$\frac{d}{ds}\left(f(t(s))\int_{\tilde{M}_s}\tilde{k}(\tilde{x},s)d\mathcal{H}^n(\tilde{x})\right)$$
$$\leq -\frac{1}{2}f(t(s))\int_{\tilde{M}_s}\tilde{k}(\tilde{x},s)\left|\tilde{H} - \langle\tilde{x},\tilde{\nu}\rangle\right|^2 d\mathcal{H}^n(\tilde{x}).$$

*Proof.* This is Lemma 3.2 adjusted to the rescaled quantities, and with $\frac{dt}{ds} = 2(T-t)$. Here, $d\mathcal{H}^n$ denotes integration with respect to the $n$-dimensional Hausdorff measure. □

We need the following

**Definition.** *The singularity is* **type I** if

$$\max_{M_t}|A|^2 \leq \frac{C_0}{T-t}$$

for some constant $C_0$; otherwise it is called **type II** .

In mean curvature flow without volume constraints it is known that spheres, convex surfaces, cylinders, rotationally symmetric shrinking necks in $\mathbb{R}^3$ develop singularities of type I (see [18,19]); a one-dimensional shrinking loop which forms a cusp is a type II singularity (see [5]).

We consider the case of type I singularities and use Lemma 3.4 to control their asymptotic behaviour. After homothetically rescaling (as above) the curvature of $\tilde{M}_s$ is uniformly bounded, i.e.

$$|\tilde{A}|^2 \leq C_0.$$

Exactly as in ([18], Proposition 2.3) or ([17], Theorem 4.1), we obtain the following

**Proposition 3.5.** *For each $m \geq 0$ there is $C(m) < \infty$ such that $|\tilde{\nabla}^m\tilde{A}| \leq C(m)$ holds on $\tilde{M}_s$ uniformly in s; here, $C(m)$ depends on $n, m, C_0$ and $M_0$.*

In order to investigate the behaviour of $\tilde{M}_s$ as $s \to \infty$, we need first to guarantee that $\tilde{M}_s$ does not disappear at infinity. This is as in Lemma 3.3 of [18]:

**Lemma 3.6.** *There exists $p \in M^n$ such that $\tilde{x}(p,s)$ remains bounded for $s \to \infty$.*

**Proposition 3.7 (Convergence of the rescaled surfaces to $\tilde{M}_\infty$).** *Assume the blow-up point to be a type I singularity. Then for each sequence $s_j$ there is a subsequece $s_{j_k}$ such that $\tilde{M}_{s_{j_k}} \cap B_\delta(0)$ converges smoothly to an immersed non-empty limiting surface $\tilde{M}_\infty$.*

*Proof.* This is Proposition 3.4 of [18] - see also [21]. □

We are now in the position of using Lemma 3.4. to obtain

**Proposition 3.8.** *Each limiting hypersurface $\tilde{M}_\infty$, as obtained in Proposition 3.7, satisfies the equation*

$$\tilde{H} = \langle \tilde{x}, \tilde{\nu} \rangle \,.$$

*Proof.* From the monotonicity formula we obtain

$$\begin{aligned} \int_0^\infty f(t(s)) \int_{\tilde{M}_s \cap B_\delta(0)} \tilde{k} \left| \tilde{H} - \langle \tilde{x}, \tilde{\nu} \rangle \right|^2 d\mathcal{H}^n(\tilde{x}) ds &\leq 2 \int_{\tilde{M}_s \cap B_\delta(0)} \tilde{k} d\mathcal{H}^n(\tilde{x}) \\ &\leq 2 \int_M k dg_0 < \infty. \end{aligned}$$

Observing that we have uniform estimates on the curvature and its higher derivatives (Proposition 3.5), we obtain the result from this inequality. □

This leads to the following.

**Theorem 3.9.** *If $\tilde{M}_\infty$ is a smooth limiting hypersurface in $\mathbb{R}^{n+1}$ as obtained above, satisfying in particular $\tilde{H} = \langle \tilde{x}, \tilde{\nu} \rangle$ and $\tilde{H} \geq 0$, then $\tilde{M}_\infty$ is a cylinder.*

*Proof.* This follows from Theorem 5.1 in [19], or Proposition 5.4 in [18]. □

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