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ORIGINAL ARTICLE

The multi-fuzzy N-soft set and its applications to decision-making

Fatia Fatimah[1] · José Carlos R. Alcantud[2]

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Abstract

The goal of this paper is to introduce a novel hybrid model called multi-fuzzy N-soft set, and to design an adjustable decision-making methodology for solving problems where the inputs appear in this form. The new model enhances the virtues of multi-fuzzy set theory with the benefits of N-soft sets, two models that have been extensively investigated in recent years. The theoretical setting that arises allows us to incorporate data on the occurrence of ratings or grades (the defining characteristic of N-soft sets) in a multi-fuzzy environment. We perform a set-theoretical analysis of multi-fuzzy N-soft sets in order to establish the fundamental properties of their behavior. Then we develop a highly adaptable approach to decision-making in this new setting. This methodology takes advantage of a flexible procedure for the conversion of the original data to a hesitant N-soft setting, where we can resort to scores. Examples illustrate its application and the role of each parameter in the decision-making procedure.

Keywords N-soft sets · Multi-fuzzy sets · Multi-fuzzy soft set · Multi-fuzzy N-soft set

Mathematics Subject Classification 03E05 · 03B52 · 94D05

1 Introduction

Theories that benefit from probability [1,2], fuzzy descriptions [3] or approximations by attributes [4], as well as related models [5], provide useful mathematical approaches to tackle many classes of uncertainties that arise in fields like the social sciences, engineering, or economics. In line with the spirit of fuzzy sets, there exist models that account for margins of error (e.g., intuitionistic fuzzy sets [6,7] or their generalizations [8–12], or some possibility distribution on the possible values (e.g., type-2 fuzzy sets or T-2FSs, and type-n fuzzy sets or T-nFSs [13]). Hesitant fuzzy sets are useful when the membership degree of the options are not uniquely determined, for reasons like group knowledge or hesitation [14,15]. Hesitancy can be merged with other important models (for recent examples, see [16–21]).

✉ Fatia Fatimah
fatia@ecampus.ut.ac.id

José Carlos R. Alcantud
jcr@usal.es

1 Department of Mathematics, Universitas Terbuka, South Tangerang, Indonesia

2 BORDA Research Unit and IME, University of Salamanca, 37007 Salamanca, Spain

The aforementioned references confirm the interest of many scholars in hybrid models. Many possibilities of hybridization of the recent model called N-soft sets (cf., Fatimah et al. [22]) have already been developed. The original contribution of that model is the improvement of the range of applications of soft set theory, which deals with attributes that approximate the universe of discourse. Let us dwell on the advances on that general idea.

Soon after the introduction of soft set theory, Maji et al. [23] and Ali et al. [24] proposed basic algebraic operations on soft sets, thus providing an analytical approach to their theory. Maji et al. [25] favored the use of soft sets in decision-making problems. Maji et al. [26] combined soft sets with other mathematical structures and introduced the hybrid model called fuzzy soft sets, the natural fuzzy generalization of soft sets (see [27] for a recent update on decision making in that setting, and [28] for an application to valuation of assets by valuation fuzzy soft sets). Majumdar and Samanta [29] revised the definition of fuzzy soft sets and proposed the concept of generalized fuzzy soft sets. Alcantud and Muñoz Torrecillas [30] first posed the problem of intertemporal choice of fuzzy soft sets. Peng et al. [31,32] introduced the algorithms for interval-valued fuzzy soft sets in stochastic multi-criteria decision-making and algorithms for neutrosophic soft decision-making. Further, Peng and Garg [33] proposed the novel algorithms for interval-valued fuzzy soft sets in emergency decision-making based on WDBA and CODAS with new information measure. Zhan and Alcantud [34] is an updated summary of the literature on parameter reduction in the setting of soft sets.

The majority of the researchers in soft set theory were bound by one of the following restrictions (see [35] or other updated summaries of hybrid models): (i) the evaluations are either 0 or 1, i.e., the original binary setting; or (ii) the evaluations are real numbers between 0 and 1, like fuzzy soft sets or separable fuzzy soft sets [27]. We find an analysis of incomplete information in both contexts in [36]. However in cases like social judgement systems or the presentation of ranking positions in daily life, we observe information with a discrete structure that however, is not binary. In this regard, Herawan and Deris [37] produced n binary-valued information system in soft sets where each parameter has its own rankings, as compared to rating orders described in Chen et al. [38]. Ali et al. [39] worked with a rating system among the elements of soft sets parameters. Nevertheless N-soft sets are the correct formal representation of the idea of a parameterized description of the universe of objects that depends on a finite number of ordered grades. Fatimah et al. [40,41] referred to other extended soft set models which have been related to social choice [42] . Akram et al. [43] have first combined N-soft sets with fuzzy sets, and links with rough set theory are the subject of Alcantud et al. [44]. Bipolarity, generalized vague soft sets and neutrosophy are also amenable to hybridization with the N-soft set spirit [45–48]. We emphasize that [22,44,45] provide several real examples that speak for the applicability of multinary parameterized descriptions like those appearing in the N-soft set model and its generalizations. Recently, Akram et al. [49] also proposed the concept of parameter reduction in N-soft sets.

In addition, N-soft sets permit to update ideas like hesitancy, which in the framework of soft sets, is simply equivalent to complete ignorance [36]. Now hesitancy can naturally appear in a situation of approximate descriptions, and then it produceds the model called hesitant N-soft sets [50]. A more general approach is hesitant fuzzy N-soft sets [51], that allow us to combine hesitant fuzzy set theory and N-soft sets. These elements model what specific grades are given to objects when parameterization attributes are graded, which can be assigned as partial degrees of membership; and they allow for hesitancy when we describe such membership values.

This paper takes advantage of the extension of fuzzy sets called multi-fuzzy sets (cf., Sebastian and Ramakrishnan [52]). We describe the formal model that combines them with N-soft sets. The theoretical model that arises is called multi-fuzzy N-soft set, and it incorporates data on the occurrence of ratings or grades in a multi-fuzzy environment. Therefore our novel model generalizes the successful idea of multi-fuzzy soft set (cf., Yang, Tan and Meng [53]). We investigate some useful properties of multi-fuzzy N-soft sets and the main set-theoretic operations on them. Finally we propose a decision-making mechanism that makes full use of the information provided by multi-fuzzy N-soft sets. Examples illustrate its application and the relevance of each parameter.

This paper is organized as follows. In Sect. 2 we review some known models of uncertain knowledge. Section 3 introduces our novel model. We explain their intuitive tabular representation, as well as their algebra and fundamental operators. Brief examples illustrate all the concepts that we produce in this section. Section 4 is concerned with decision-making when the information comes in the form of multi-fuzzy N-soft sets. Section 5 ends the paper with some concluding remarks.

2 Preliminary definitions

This section recaps the concept of N-soft set (cf., Fatimah et al. [22]) as well as multi-fuzzy sets and multi-fuzzy soft sets.

Henceforth we use $\mathcal{P}([0,1])$ to represent the set of all subsets of $[0,1]$ and $\mathcal{P}^*([0,1])$ will represent the set of non-empty subsets of $[0,1]$.

$\mathcal{F}(X)$ is the set of all fuzzy sets on X.

2.1 *N*-soft sets and extensions

The formal representation of universes that are characterized by multinary descriptions in terms of some fixed set of parameters is as follows:

Definition 2.1 [22] Let O denote the universe of objects under consideration and P the set of attributes, $T \subseteq P$. Let $G = \{0, 1, 2, \ldots, N-1\}$ be the set of ordered grades where $N \in \{2, 3, \ldots\}$. A triple (F, T, N) is called an *N-soft set* on O if F is mapping from T to $2^{O\times G}$, with the property that for each $t \in T$ and $o \in O$ there exists a unique $(o, g_t) \in O \times G$ such that $(o, g_t) \in F(t)$, $g_t \in G$.

The interpretation of the pair $(o, g_t) \in F(t)$, is that the element o belongs to the set of t-approximations of the set O with the grade g_t. Its tabular representation is recalled below in Example 2.1. Real examples where data adopt this form appear e.g., in [22,44,45].

If we allow for partial membership degrees in Definition 2.1 then we obtain fuzzy N-soft sets [43] in the following terms:

Definition 2.2 Let O be a universe of objects under consideration and P the set of attributes, $T \subseteq P$. A pair (μ, K) is a *fuzzy N-soft set*, also (F, N)-soft set, when $K = (F, T, N)$ is an N-soft set on O with $N \in \{2, 3, \ldots\}$,

Table 1 Tabular representation of a HNSS (O and T are finite). Each $l(ij)$ gives the length of the subset of grades placed in position (i, j) of this representation

(H, T, N)	t_1	...	t_q
o_1	$\{\eta_{11}^1, \eta_{11}^2, \ldots, \eta_{11}^{l(11)}\}$	...	$\{\eta_{1q}^1, \eta_{1q}^2, \ldots, \eta_{1q}^{l(1q)}\}$
$\vdots$			
o_p	$\{\eta_{p1}^1, \eta_{p1}^2, \ldots, \eta_{p1}^{l(p1)}\}$	...	$\{\eta_{pq}^1, \eta_{pq}^2, \ldots, \eta_{pq}^{l(pq)}\}$

and $\mu : T \to \bigcup_{t \in T} \mathcal{F}(F(t))$ is an application with the property that $\mu(t) \in \mathcal{F}(F(t))$ for each $t \in T$.

Definition 2.2 explains that the function μ associates with each attribute a fuzzy set on the image of this attribute by F. Thus for every $t \in T$ and $o \in O$ there is a unique pair $(o, g_t) \in O \times G$ such that $g_t \in G$ and $\langle (o, g_t), \mu_t(o) \rangle \in \mu(t)$. This means $\mu_t(o) = \mu(t)(o, g_t)$.

There are extensions of the two definitions above that include the possibility of hesitancy. The model that directly allows for hesitancy in the assessment by N-soft sets is the subject of [50], whereas a model that emerges from the combination of fuzzy N-soft sets and hesitancy was defined in [51]. We recall that Torra [54,55] defined hesitant fuzzy set as follows (see also Alcantud and Torra [56]):

Definition 2.3 [54,55] Let O be any set. A *hesitant fuzzy set* (henceforth, HFS) on O is defined in terms of a function h that for each element o of O returns a subset $h(o)$ of [0, 1]. Hence $h(o) \in \mathcal{P}([0, 1])$ for each $o \in O$.

HFS(O) denotes the set of all HFSs on O.

Then we obtain:

Definition 2.4 Let O be a universe of objects and P the set of attributes, $T \subseteq P$. Let $G = \{0, 1, 2, \ldots, N-1\}$ be the set of ordered grades where $N \in \{2, 3, \ldots\}$.

(1) A triple $(\hbar_f, T, N)$ is called *hesitant fuzzy N-soft set* (HFNSS), when $\hbar_f$ is a mapping defined as

$$\hbar_f : O \times T \to G \times \mathcal{P}^*([0, 1]).$$

(2) A triple (H, T, N) is called a *hesitant N-soft set* (henceforth HNSS) on O if H is a mapping $H : T \to 2^{O \times G}$ such that for each $t \in T$ and $o \in O$ there exists at least one pair $(o, g_t) \in O \times G$ such that $(o, g_t) \in H(t)$, $g_t \in G$.

In Definition 2.4 (1), the fact $\hbar_f(o, t) = (g, \hbar_{f_t}(o))$ is interpreted as follows: $\hbar_{f_t}(o)$ is a non-empty set of numbers from [0, 1], denoting the possible membership degrees of the element $o \in O$ to the subset of t-approximations of O (or options approximated by t) with grade g.

According to (2) in Definition 2.4, with each attribute the mapping H assigns a non-empty collection of pairs formed by objects and possible grades. Hence the notion of HNSS extends the following three remarkable models:

(i) When $N = 2$, an HNSS is an incomplete soft set, which is a soft set with missing information.

(ii) If for each $t \in T$ and $o \in O$, H associates *exactly* one pair $(o, g_t) \in O \times G$ such that $(o, g_t) \in H(t)$, $g_t \in G$, is an N-soft set.

(iii) Consequently, when in addition to condition (ii) above we add $N = 2$, we obtain a soft set.

Table 1 recalls the formal representation of the tabular expression of a general HNSS.

Let us clarify these concepts with a brief example:

Example 2.1 In a university, the selection of a lecturer is based on star ratings and gradings awarded by a selection board, which includes vice chancellor, chairman, subject specialist and psychologist. Let $O = \{o_1, o_2, o_3, o_4\}$ be the universe of candidates appearing in an interview of a university and P be the set of attributes "evaluation of candidates by selection board". The subset $T \subseteq P$ such that $T = \{t_1, t_2, t_3\}$ is used. A 4-soft set can be obtained from Table 2.

This information is enough when it is extracted from real data without hesitation or indeterminacy. It pertains to the model in Definition 2.1. But when the assessments are vague and uncertain we may need to use (F, N)-soft sets or even HFNSS, which provide us with more flexible information about how these grades are given to candidates. To this purpose, the following HF4SS (cf., Definition 2.4) is defined:

$$\hbar_f(t_1) = \left\{ \left\langle \frac{(o_1, 2)}{\{0.6, 0.8\}} \right\rangle, \left\langle \frac{(o_2, 0)}{\{0.1\}} \right\rangle, \left\langle \frac{(o_3, 1)}{\{0.3, 0.5\}} \right\rangle, \left\langle \frac{(o_4, 3)}{\{1\}} \right\rangle \right\},$$

$$\hbar_f(t_2) = \left\{ \left\langle \frac{(o_1, 3)}{\{0.8, 0.9\}} \right\rangle, \left\langle \frac{(o_2, 2)}{\{0.3, 0.5, 0.6\}} \right\rangle, \left\langle \frac{(o_3, 1)}{\{0.2, 0.25\}} \right\rangle, \left\langle \frac{(o_4, 2)}{\{0.35, 0.59\}} \right\rangle \right\},$$

$$\hbar_f(t_3) = \left\{ \left\langle \frac{(o_1, 2)}{\{0.4, 0.6\}} \right\rangle, \left\langle \frac{(o_2, 1)}{\{0.1, 0.15, 0.2\}} \right\rangle, \left\langle \frac{(o_3, 2)}{\{0.2, 0.3, 0.55\}} \right\rangle, \left\langle \frac{(o_4, 1)}{\{0.1, 0.12, 0.2\}} \right\rangle \right\}.$$

Table 3 represents this information in tabular form.

An example of a H5SS appears below in Table 6. We emphasize that HNSSs will play an important role in Sect.

Table 2 Tabular representation of the 4-soft set described in Example 2.1

$(F, T, 4)$	t_1	t_2	t_3
o_1	2	3	2
o_2	0	2	1
o_3	1	1	2
o_4	3	2	1

Table 3 Tabular representation of the HF4SS $(\hbar_f, T, 4)$ in Example 2.1

	t_1	t_2	t_3
o_1	$\langle 2, \{0.6, 0.8\}\rangle$	$\langle 3, \{0.8, 0.9\}\rangle$	$\langle 2, \{0.4, 0.6\}\rangle$
o_2	$\langle 0, \{0.1\}\rangle$	$\langle 2, \{0.3, 0.5, 0.6\}\rangle$	$\langle 1, \{0.1, 0.15, 0.2\}\rangle$
o_3	$\langle 1, \{0.3, 0.5\}\rangle$	$\langle 1, \{0.2, 0.25\}\rangle$	$\langle 2, \{0.2, 0.3, 0.55\}\rangle$
o_4	$\langle 3, \{1\}\rangle$	$\langle 2, \{0.35, 0.59\}\rangle$	$\langle 1, \{0.1, 0.12, 0.2\}\rangle$

4, where we put forward a decision-making procedure in the framework initiated by Sect. 3. It is therefore convenient to recall some additional preliminaries about hesitant N-soft sets.

The sets h with the property $\varnothing \neq h \subseteq G = \{0, 1, 2, \ldots, N-1\}$ are called *hesitant N-tuples* (HNT for brevity) in [51]. HNTs lie in $\mathcal{P}^*(G)$. The tabular representation of any HNSS consists of a matrix whose cells are HNTs. For example, all the cells of Table 1 are general HNTs.

For convention, any HNT h is expressed so that $h = \{\eta^{(1)}, \eta^{(2)}, \ldots, \eta^{(l)}\}$ satisfies $\eta^{(1)} < \eta^{(2)} < \ldots < \eta^{(l)}$. The cardinality of h is $l \geq 1$, which also satisfies $l \leq N$. Several scores for h are defined in [51]:

1. Min score: $s_m(h) = \eta^{(1)}$;
2. Max score: $s_M(h) = \eta^{(l)}$;
3. Arithmetic score: $s_a(h) = (\eta^{(1)} + \eta^{(2)} + \cdots + \eta^{(l)})/l$;
4. Geometric score: $s_g(h) = (\eta^{(1)} \cdot \eta^{(2)} \cdots \eta^{(l)})^{1/l}$.

These scores will be one of the ingredients in the decision-making mechanism that we endorse in Sect. 4.

2.2 Multi-fuzzy sets and extensions

We proceed to describe the model that started the branch of literature to which our proposal belongs. Sebastian and Ramakrishnan [52] defined multi-fuzzy sets as follows:

Definition 2.5 A multi-fuzzy set of dimension k on U is a set of k-tuples of numbers from [0, 1] indexed by U. Formally:

$$\{\langle u, \mu_1(u), \ldots, \mu_k(u)\rangle : u \in U\}$$

with $\mu_i : U \longrightarrow [0, 1]$ for each $i = 1, \ldots, k$.

For notational convenience, we can also write the following equivalent notation which is inspired by the practical notation for fuzzy sets:

$$\{u|(\mu_1(u), \ldots, \mu_k(u)) : u \in U\}$$

The set of all multi-fuzzy sets of dimension k on U is denoted by $\mathrm{M}^k\mathrm{FS}(U)$.

Remark 2.1 $\mathrm{M}^k\mathrm{FS}(U)$ has similarities with $\mathrm{HFS}(U)$, since both associate the elements in U with arrays of membership degrees. However there are two key differences: in a multi-fuzzy set, all the arrays have the same number of elements which coincides with its dimension, whereas hesitant fuzzy sets impose no restriction on this number; and in a multi-fuzzy set, the membership degrees associated with a member of U can be repeated, which is not the case of hesitant fuzzy sets.

Other authors took advantage of the idea of multi-fuzzy set and produced more general models. For example, Kar et al. [57] defined type-2 multi-fuzzy sets, and Al-qudah and Hassan [58] defined complex multi-fuzzy sets.

It is also possible to combine the idea of multi-fuzzy set with parameterized descriptions of the universe as a function of a set of attributes. To that purpose, Yang, Tan and Meng [53] defined multi-fuzzy soft sets as follows:

Definition 2.6 A multi-fuzzy soft set of dimension k on U is a pair $(\mathbb{F}, A)$ such that $\mathbb{F} : A \longrightarrow \mathrm{M}^k\mathrm{FS}(U)$.

The set of all multi-fuzzy soft sets of dimension k on U is denoted by $\mathrm{M}^k\mathrm{FSS}(U)$.

Definition 2.6 has soon raised the interest of other scholars too. Al-qudah and Hassan [59] extended the idea of multi-fuzzy soft set and defined complex multi-fuzzy soft set. Dey and Pal [60] gave another extension called generalised multi-fuzzy soft set. In this paper we also generalize multi-fuzzy soft set, but we do that in a different fashion. This is the purpose of the next section.

3 Multi-fuzzy *N*-soft sets

Now we proceed to define our novel model. We are concerned with several elements. We refer our information to U, a universe of objects, and A, a set of attributes (which may be a part of a univeral set of attributes, $A \subseteq P$). We fix two natural numbers k and $N \in \{2, 3, \ldots\}$. The first one will be the dimension of our model. The second discriminates how many degrees of satisfaction of the attributes are allowed, so that we can use $G = \{0, 1, 2, \ldots, N-1\}$ as the set of ordered grades.

Definition 3.1 Let us define $\mathrm{M}^k\mathrm{FS}^N(U)$ as the set of all k-tuples of pairs of elements from $G \times [0, 1]$ indexed by U, i.e., the set of all elements with the form

$\{\langle u, (g_1(u), \mu_1(u)), \ldots, (g_k(u), \mu_k(u))\rangle : u \in U\}$

with $g_i : U \longrightarrow G$ and $\mu_i : U \longrightarrow [0, 1]$, for each $i = 1, \ldots, k$.

For notational convenience, we can also write the following equivalent notation:

$\{u|((g_1(u), \mu_1(u)), \ldots, (g_k(u), \mu_k(u))) : u \in U\}$.

A multi-fuzzy N-soft set of dimension k on U is a pair $(\mathbb{F}, A)$ such that $\mathbb{F}$ is a mapping $\mathbb{F} : A \longrightarrow \mathrm{M}^k\mathrm{FS}^N(U)$. Therefore with each $a \in A$, $\mathbb{F}$ associates a k-tuple of pairs of elements from $G \times [0, 1]$ indexed by U.

Henceforth the set of all multi-fuzzy N-soft sets of dimension k on U will be denoted by $\mathrm{M}^k\mathrm{FSS}^N(U)$.

Remark 3.1 When $N = 1$, the elements in $\mathrm{M}^k\mathrm{FS}^N(U)$ can be identified with elements in $\mathrm{M}^k\mathrm{FS}(U)$. Observe that a generic member of $\mathrm{M}^k\mathrm{FS}^1(U)$ is of the form $\{\langle u, (0, \mu_1(u)), \ldots, (0, \mu_k(u))\rangle : u \in U\}$. We identify it with $\{\langle u, \mu_1(u), \ldots, \mu_k(u)\rangle : u \in U\} \in \mathrm{M}^k\mathrm{FS}(U)$ in a trivial manner.

Most real situations are finite, both in the set of alternatives and the attributes that we are interested in. Suppose that $U = \{u_1, \ldots u_n\}$, then a multi-fuzzy N-soft set of dimension k on U is a pair $(\mathbb{F}, A)$ such that for each $a \in A$,

$$\mathbb{F}(a) = \{\langle u_1, (g_1(u_1), \mu_1(u_1)), \ldots, (g_k(u_1), \mu_k(u_1))\rangle, \ldots, \langle u_n, (g_1(u_n), \mu_1(u_n)), \ldots, (g_k(u_n), \mu_k(u_n))\rangle\}$$

or alternatively,

$$\mathbb{F}(a) = \{u_1|((g_1(u_1), \mu_1(u_1)), \ldots, (g_k(u_1), \mu_k(u_1))), \ldots, u_n|((g_1(u_n), \mu_1(u_n)), \ldots, (g_k(u_n), \mu_k(u_n)))\}$$

3.1 An illustrative example

Let us clarify the concepts and notations above with a brief example. In particular, this example shows how we can refer to a tabular form in order to display the information contained in a multi-fuzzy N-soft set of any dimension. This is the practical input that we refer to in our application (cf. Example 4.1 below).

Example 3.1 Let $U = \{u_1, u_2, u_3\}$ be the universe of candidates and $A \subseteq P$ is such that $A = \{t_1, t_2\}$ is the set of attributes.

A multi-fuzzy 4-soft set of dimension 2 on U is defined by the assignments

$$\mathbb{F}(t_1) = \{\langle u_1, (2, 0.45), (3, 0.15)\rangle, \langle u_2, (3, 0.55), (1, 0.35)\rangle, \langle u_3, (1, 0.30), (2, 0.65)\rangle\},$$

Table 4 Tabular representation of the multi-fuzzy 4-soft set of dimension 2 described in Example 3.1

	t_1	t_2
u_1	((2, 0.45), (3, 0.15))	((1, 0.15), (2, 0.75))
u_2	((3, 0.55), (1, 0.35))	((3, 0.25), (1, 0.40))
u_3	((1, 0.30), (2, 0.65))	((2, 0.20), (1, 0.70))

and

$$\mathbb{F}(t_2) = \{\langle u_1, (1, 0.15), (2, 0.75)\rangle, \langle u_2, (3, 0.25), (1, 0.40)\rangle, \langle u_3, (2, 0.20), (1, 0.70)\rangle\}.$$

We can also display this object with the related notation

$$\mathbb{F}(t_1) = \left\{\frac{u_1}{(2, 0.45), (3, 0.15)}, \frac{u_2}{(3, 0.55), (1, 0.35)}, \frac{u_3}{(1, 0.30), (2, 0.65)}\right\},$$

and

$$\mathbb{F}(t_2) = \left\{\frac{u_1}{(1, 0.15), (2, 0.75)}, \frac{u_2}{(3, 0.25), (1, 0.40)}, \frac{u_3}{(2, 0.20), (1, 0.70)}\right\}.$$

Alternatively, this information can be retrieved from Table 4.

3.2 A set-theoretical analysis of multi-fuzzy *N*-soft sets

We now investigate the fundamental set-theoretical operations in the setting that we have defined. First we busy ourselves with inclusion. Our approach is twofold, although both notions of inclusion will ultimately produce the same concept of equality.

Let us fix the grade sets $G = \{0, 1, 2, \ldots, N-1\}$ and $G' = \{0, 1, 2, \ldots, N'-1\}$. Suppose that we have $\mathcal{A} \in \mathrm{M}^k\mathrm{FS}^N(U)$ and $\mathcal{A}' \in \mathrm{M}^k\mathrm{FS}^{N'}(U)$, which are defined as

$$\mathcal{A} = \{\langle u, (g_1(u), \mu_1(u)), \ldots, (g_k(u), \mu_k(u))\rangle : u \in U\}$$

and

$$\mathcal{A}' = \{\langle u, (g'_1(u), \mu'_1(u)), \ldots, (g'_k(u), \mu'_k(u))\rangle : u \in U\}$$

with $g_i : U \longrightarrow G$, $g'_i : U \longrightarrow G'$, $\mu_i, \mu'_i : U \longrightarrow [0, 1]$, for each $i = 1, \ldots, k$. We define two successively weaker forms of inclusion between $\mathcal{A}$ and $\mathcal{A}'$:

(i) $\mathcal{A} \sqsubseteq \mathcal{A}'$ if and only if $g_i(u) = g'_i(u)$ and $\mu_i(u) \leq \mu'_i(u)$, for each $i = 1, \ldots, k$ and $u \in U$.
(ii) $\mathcal{A} \Subset \mathcal{A}'$ if and only if $g_i(u) \leq g'_i(u)$ and $\mu_i(u) \leq \mu'_i(u)$, for each $i = 1, \ldots, k$ and $u \in U$.

Obviously, $\mathcal{A} \sqsubseteq \mathcal{A}'$ implies $\mathcal{A} \Subset \mathcal{A}'$. Example 3.2 will show some situations where these inclusions are different, a fact that is quite easy to spot.

By inspiration of the previous concepts, we are now ready to define two types of inclusions for multi-fuzzy N-soft sets. Then we explore their mutual relationships by a negative example.

Definition 3.2 Let us fix $(\mathbb{F}, A) \in \mathrm{M}^k\mathrm{FSS}^N(U)$ and $(\mathbb{F}', A') \in \mathrm{M}^k\mathrm{FSS}^{N'}(U)$.

- (i) We denote $(\mathbb{F}, A) \sqsubseteq (\mathbb{F}', A')$ if and only if $A \subseteq A'$ and $\mathbb{F}(a) \sqsubseteq \mathbb{F}'(a)$, for each $a \in A$.
- (ii) We denote $(\mathbb{F}, A) \Subset (\mathbb{F}', A')$ if and only if $A \subseteq A'$ and $\mathbb{F}(a) \Subset \mathbb{F}'(a)$, for each $a \in A$.

It is pretty easy to observe that $(\mathbb{F}, A) \sqsubseteq (\mathbb{F}', A')$ implies $(\mathbb{F}, A) \Subset (\mathbb{F}', A')$. However the two notions are different. The next example illustrates the concepts above as well as their differences:

Example 3.2 Consider the multi-fuzzy 4-soft set of dimension 2 on U in Example 3.1. Let $A = \{t_1, t_2\}$.

Additionally, we define a multi-fuzzy 3-soft set of dimension 2 on U by the assignments

$$\begin{aligned}\mathbb{F}'(t_1) = \{&\langle u_1, (2, 0.25), (1, 0.1)\rangle, \langle u_2, (1, 0.3), (1, 0.25)\rangle,\\ &\langle u_3, (0, 0.10), (2, 0.65)\rangle\},\\ \mathbb{F}'(t_2) = \{&\langle u_1, (1, 0.15), (1, 0.5)\rangle, \langle u_2, (1, 0.20), (0, 0.40)\rangle,\\ &\langle u_3, (1, 0.15), (1, 0.6)\rangle\}.\end{aligned}$$

Then $(\mathbb{F}', A) \Subset (\mathbb{F}, A)$ although $(\mathbb{F}', A) \sqsubseteq (\mathbb{F}, A)$ is false. Thus the concepts in Definition 3.2 are different.

In passing, observe that one has $\mathbb{F}'(t_1) \Subset \mathbb{F}(t_1)$ although $\mathbb{F}'(t_1) \sqsubseteq \mathbb{F}(t_1)$ is false, and also $\mathbb{F}'(t_2) \Subset \mathbb{F}(t_2)$ holds true despite the fact that $\mathbb{F}'(t_2) \sqsubseteq \mathbb{F}(t_2)$ is false.

Notice that one can define equality of multi-fuzzy N-soft sets by reference to any of these two notions of inclusion, which happen to be equivalent. The latter fact is easy to prove. Thus we propose:

Definition 3.3 Let us fix $(\mathbb{F}, A) \in \mathrm{M}^k\mathrm{FSS}^N(U)$ and $(\mathbb{F}', A') \in \mathrm{M}^k\mathrm{FSS}^{N'}(U)$. Then $(\mathbb{F}, A) = (\mathbb{F}', A')$ if and only if $(\mathbb{F}, A) \sqsubseteq (\mathbb{F}', A') \sqsubseteq (\mathbb{F}, A)$

Equivalently, $(\mathbb{F}, A) = (\mathbb{F}', A')$ if and only if $(\mathbb{F}, A) \Subset (\mathbb{F}', A') \Subset (\mathbb{F}, A)$.

To conclude this section, we define the complement of a multi-fuzzy N-soft set.

Select $\mathcal{A} \in \mathrm{M}^k\mathrm{FS}^N(U)$ defined as

$$\mathcal{A} = \{\langle u, (g_1(u), \mu_1(u)), \ldots, (g_k(u), \mu_k(u))\rangle : u \in U\}$$

with $g_i : U \longrightarrow G$ and $\mu_i : U \longrightarrow [0, 1]$, for each $i = 1, \ldots, k$. Recall that the respective complements of the fuzzy sets μ_i are the fuzzy sets $\mu_i^c : U \longrightarrow [0, 1]$ such that $\mu_i^c(u) = 1 - \mu_i(u)$ for every $u \in U$. Then,

$$\mathcal{A}^c = \{\langle u, (g_1(u), \mu_1^c(u)), \ldots, (g_k(u), \mu_k^c(u))\rangle : u \in U\}$$

is the complement of $\mathcal{A}$. This concept allows us to express the idea of complement of multi-fuzzy N-soft sets in the following terms:

Definition 3.4 Let us fix $(\mathbb{F}, A) \in \mathrm{M}^k\mathrm{FSS}^N(U)$. Its complement is the pair $(\mathbb{F}^c, A)$ defined by the expression

$$\mathbb{F}^c : A \longrightarrow \mathrm{M}^k\mathrm{FS}^N(U) \text{ with } \mathbb{F}^c(a) = (\mathbb{F}(a))^c.$$

Remark 3.2 We had observed in Remark 3.1 that when $N = 1$, multi-fuzzy N-soft sets are multi-fuzzy soft sets (with a trivial identification). It is easy to check that with this simple identification, Definition 3.4 extends the definition of complement of a multi-fuzzy soft set in Yang, Tan and Meng [53, Definition 11].

4 Decision-making based on multi-fuzzy *N*-soft sets

Suppose a practical multi-attribute decision-making problem with a finite set of alternatives $U = \{u_1, \ldots u_n\}$, and a finite set of attributes $A = \{a_1, \ldots a_m\}$. We need to select an alternative from U. The new situation that we want to investigate is the case where our input comes in the form a multi-fuzzy N-soft set of dimension k on U. Therefore for each $a_j \in A$, we know

$$\begin{aligned}\mathbb{F}(a_j) = \{ &\langle u_1, (g_1^j(u_1), \mu_1^j(u_1)), \ldots, (g_k^j(u_1), \mu_k^j(u_1))\rangle, \ldots ,\\ &\langle u_n, (g_1^j(u_n), \mu_1^j(u_n)), \ldots, (g_k^j(u_n), \mu_k^j(u_n))\rangle \}.\end{aligned} \tag{4.1}$$

Our strategy consists of two fundamental steps:

(1) First we induce a hesitant N-soft set from the input data.
(2) Secondly, we apply the algorithm of weighted choice values in Akram et al. [50] to this induced hesitant N-soft set.

The first step requires one of the general constructions in Definition 4.1 or 4.2:

Definition 4.1 Let us fix $(\mathbb{F}, A) \in \mathrm{M}^k\mathrm{FSS}^N(U)$. Its induced HNSS is the triple $(H_{\mathbb{F}}, A, N)$ such that cell (i, j) of its tabular form is $\{g_1^j(u_i), \ldots, g_k^j(u_i)\}$, where $\mathbb{F}(a_j)$ is given as in Eq. (4.1).

Table 5 Tabular representation of the multi-fuzzy 5-soft set of dimension 3 in Example 4.1

$(\mathbb{F}, A)$	a_1	a_2	a_3
u_1	((1, 0.2), (2, 0.4), (0, 0.5))	((1, 0.5), (1, 0.6), (2, 0.6))	((0, 0.4), (0, 0.4), (1, 0.5))
u_2	((4, 0.2), (2, 0.3), (3, 0.5))	((3, 0.5), (2, 0.4), (4, 0.4))	((2, 0.5), (1, 0.3), (2, 0.1))
u_3	((1, 0.3), (2, 0.6), (4, 0.1))	((2, 0.5), (2, 0.4), (1, 0.6))	((2, 0.4), (1, 0.1), (0, 0.4))
u_4	((1, 0.7), (1, 0.3), (2, 0.6))	((2, 0.4), (1, 0.7), (1, 0.4))	((4, 0.5), (0, 0.2), (2, 0.4))

Observe that the transformation in Definition 4.1 may be too crude since it is totally insensitive to the value of the respective membership degrees. In other words, the values of the $\mu_t^j(u_i)$'s are irrelevant for the purpose of producing $(H_{\mathbb{F}}, A, N)$. A more refined procedure establishes a minimum admissible threshold for each attribute, so that only grades whose membership degrees are higher than the corresponding threshold are selected:

Definition 4.2 Let us fix $(\mathbb{F}, A) \in \mathrm{M}^k\mathrm{FSS}^N(U)$ and $\tau = (\tau_1, \ldots, \tau_m)$, a vector of thresholds with $\tau_i \in [0, 1]$ for each i. The HNSS τ-induced by $(\mathbb{F}, A)$ is the triple $(H_{\mathbb{F}}^{\tau}, A, N)$ such that cell (i, j) of its tabular form is $\{g_t^j(u_i) | \mu_t^j(u_i) \geq \tau_j,\ t = 1, \ldots, k\}$, where $\mathbb{F}(a_j)$ is given as in Equation (4.1).

Of course, Definition 4.1 gives the HNSS τ-induced by $(\mathbb{F}, A)$ with $\tau = (0, \ldots, 0)$. Thus Definition 4.2 generalizes Definition 4.1.

The next example illustrates the procedure that we have put forward. It also gives intuitive explanations about its application.

Example 4.1 Let $U = \{u_1, u_2, u_3, u_4\}$ be the universe of options and $A = \{a_1, a_2, a_3\}$ be the set of relevant attributes.

We suppose that the input information about the performance of the options with respect to each attribute is given by $(\mathbb{F}, A)$, the multi-fuzzy 5-soft set of dimension 3 whose tabular information is displayed in Table 5.

Let us fix $\tau = (0.4, 0.5, 0.3)$. This vector establishes the thresholds associated with the respective attributes. At any cell, grades above the corresponding threshold (no matter at which position) are selected. The H5SS that we obtain is $(H_{\mathbb{F}}^{\tau}, A, 5)$ whose tabular representation is in Table 6.

In operational terms, the production of Table 6 proceeds as follows. At the cells of the first column of Table 5, we select the grades that have been awarded a membership degree above $\tau_1 = 0.4$. At the cells of its second column, we select the grades awarded with a membership degree above $\tau_2 = 0.5$. And at the cells of its third column, we select the grades awarded with a membership degree above $\tau_3 = 0.3$.

Once this induced hesitant N-soft set has been computed, the second step in our decision-making procedure consists of the application of Algorithm 2 (the algorithm of weighted choice values) in Akram et al. [50]. It acts on the induced hesitant N-soft set as a post-processed modelization of the inputs.

Table 6 Tabular representation of the H5SS τ-induced by $(\mathbb{F}, A)$ in Table 5, when the thresholds associated with the respective attributes are $\tau = (0.4, 0.5, 0.3)$

$(H_{\mathbb{F}}^{\tau}, A, 5)$	a_1	a_2	a_3
u_1	{0, 2}	{1, 2}	{0, 1}
u_2	{3}	{3}	{1, 2}
u_3	{2}	{1, 2}	{0, 2}
u_4	{1, 2}	{1}	{2, 4}

In conclusion, we can endorse Algorithm 1 below. Notice that in this algorithm we have specified the list of elements that the practitioner can select. The decision-making mechanism is greatly adaptable since the practitioner can single out the most suitable score, fix the number of grades, and adjust the dimension of admissible evaluations. In addition, one can optionally fix acceptable thresholds for admissibility of the grades (one for each attribute); and the weights of each attribute.

Taken together, these discretionary elements ensure a high level of flexibility and adaptability to the needs of the practitioner.

Algorithm 1 - Decision-making by weighted choice values of induced HNSSs.

Constituents of the method: (i) a score s for HNTs (e.g., geometric or arithmetic score), (ii) the admissible level for grades, i.e., $N \in \{2, 3, \cdots\}$, and (iii) the dimension k for the admissible N-ary fuzzy evaluations.

Optional parameters of the method: (i) a vector τ of thresholds, i.e., a threshold $\tau_j \in [0, 1]$ associated with each attribute a_j, and (ii) a vector w of weights, i.e., a number $w_j \in (0, 1]$ associated with each attribute a_j.

Input: $U = \{u_1, \ldots, u_n\}$, a universe of objects, and $A = \{a_1, \ldots, a_m\}$, a set of attributes. A multi-fuzzy N-soft set of dimension k, $(\mathbb{F}, A)$.

Steps:

1: Compute the HNSS $(H_{\mathbb{F}}^{\tau}, A, N)$ in its tabular form.

2: Compute scores $s(h_j(u_i))$ of all the HNTs in this table, $\forall\ j = 1, \ldots, m, \forall\ u_i \in U$.

3: Compute $\upsilon_i^w = \sum_{j=1}^{m} w_j \times s(h_j(u_i))$, for all $u_i \in U$.

4: Compute all the indices l that satisfy $\upsilon_l^w = \max_{i=1,\ldots,n} \upsilon_i^w$.

Output: Any of the alternatives for which $\upsilon_l^w = \max_{i=1,\ldots,n} \upsilon_i^w$ can be chosen.

Table 7 Computations at Steps 2 and 3: Case 1 of Example 4.2

	a_1	a_2	a_3	υ_i^w
u_1	1	1.5	0.5	1
u_2	3	3	1.5	2.4
u_3	2	1.5	1	1.4
u_4	1.5	1	3	1.9

Table 8 Computations at Steps 2 and 3: Case 2 of Example 4.2

	a_1	a_2	a_3	υ_i^w
u_1	0	1.41	0	0.282
u_2	3	3	1.41	2.205
u_3	2	1.41	0	0.882
u_4	1.41	1	2.83	2.038

For illustration, we apply this decision-making mechanism to the case of Example 4.1. We accomplish this goal in Example 4.2 which illustrates two different strategies. Both utilize the same vector of threshold, but each uses their own scores on HNTs and vectors of attributes.

Example 4.2 Suppose that we need to select an optimal alternative in a situation that has been assessed by $(\mathbb{F}, A)$, the multi-fuzzy 5-soft set of dimension 3 described by Table 5

When $\tau = (0.4, 0.5, 0.3)$, the H5SS that we obtain at Step 1 of Algorithm 1 is $(H^{\tau}_{\mathbb{F}}, A, 5)$ whose tabular representation is in Table 6. For illustration, we distinguish two cases with different specifications of the constituent items. They help us to show the flexibility and adaptability of Algorithm 1.

Case 1: we use the arithmetic score on HNTs, and the vector of weights of the attributes is $w = (0.2, 0.4, 0.4)$. Then the computations at Steps 2 and 3 are in Table 7. Attending at its last column, Step 4 recommends that option u_2 should be selected.

Case 2: we use the geometric score on HNTs, and the vector of weights of the attributes is $w = (0.3, 0.2, 0.5)$. Then the computations at Steps 2 and 3 are in Table 8. Attending at its last column, Step 4 recommends that option u_2 should be selected.

5 Concluding remarks

This work stands at the crossroads between two different perceptions of vagueness in knowledge acquisition. In a sense, it is in continuation of the thriving line of research inaugurated by Fatimah et al. [22] who defined the concept of N-soft set. This model constitutes a practical extension of soft sets because many real examples taken from real-world situations have insisted on their applicability [22,44,45]. Besides it has also proven its flexibility from a theoretical perspective: The model is amenable to hybridization with other views of vagueness and uncertainty. Research works like Akram et al. [8,43,50,51,61], Kamacı and Petchimuthu [45], Chen et al. [46], Riaz et al. [47] or Liu et al. [48] have developed hybrid models that assimilate other noteworthy features of frameworks for approximate knowledge. In a different line of inspection, Alcantud et al. [44] has established strong links with rough set theory. And even topological studies exist in an N-soft framework (cf., Riaz et al. [62]) as a natural extension of soft topology [63–65].

In this paper we have continued exploring the scope of multinary descriptions of the universe of alternatives with additional traits. We have designed a novel hybrid model that improves the performance of N-soft sets with the additional advantages of multi-fuzzy sets [52]. The model called multi-fuzzy N-soft set is therefore a blend of multi-fuzzy sets with N-soft sets. With respect to multi-fuzzy soft sets [53] and more particular structures (like fuzzy soft sets [26] or soft sets [4]), the new model guarantees a more accurate, flexible and reliable framework to approach decision-making under approximate information. An advantage with respect to hesitant fuzzy N-soft sets is that a multi-fuzzy N-soft set stores the information about who submitted each opinion, and in particular, repetitions are allowed. Neither of these features are satified by hesitant fuzzy N-soft sets. On the contrary, a comparative disadvantage of multi-fuzzy N-soft sets is that the number of evaluations that they capture is fixed. Therefore if we resort to a multi-fuzzy N-soft set to represent the opinions of various agents, they can neither abstain nor hesitate (although they can attach partial memberships to their unique gradual assessment, as in hesitant fuzzy N-soft sets or fuzzy N-soft sets).

The formal definition of multi-fuzzy N-soft sets is accompanied with a practical description (in finite environments) for the purpose of implementability. We have illustrated its main features with examples, and we have investigated its basic properties and fundamental operations, plus its relationships with existing models.

Our model can be effectively used for decision-making purposes. A flexible algorithm allows us to select several parameters and then proceed with various steps that produce well-defined selections of the set of alternatives.

In this fashion, we pave the way to the proposal of additional models that combine other successful concepts from the literature.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest regarding the publication of this research article.

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