



An optimization study based on Dijkstra algorithm for a network with trapezoidal picture fuzzy numbers

Muhammad Akram¹ · Amna Habib¹ · José Carlos R. Alcantud²

Received: 12 November 2019 / Accepted: 12 May 2020 / Published online: 9 June 2020
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Abstract

Path finding models attempt to provide efficient approaches for finding shortest paths in networks. A well-known shortest path algorithm is the Dijkstra algorithm. This paper redesigns it in order to tackle situations in which the parameters of the networks may be uncertain. To be precise, we allow that the parameters take the form of special picture fuzzy numbers. We use this concept so that it can flexibly fit the vague character of subjective decisions. The main contributions of this article are fourfold: (i) The trapezoidal picture fuzzy number along with its graphical representation and operational laws is defined. (ii) The comparison of trapezoidal picture fuzzy numbers on the basis of their expected values is proposed in terms of their score and accuracy functions. (iii) Based on these elements, we put forward an adapted form of the Dijkstra algorithm that works out a picture fuzzy shortest path problem, where the costs associated with the arcs are captured by trapezoidal picture fuzzy numbers. Also, a pseudocode for the application of our solution is provided. (iv) The proposed algorithm is numerically evaluated on a transmission network to prove its practicality and efficiency. Finally, a comparative analysis of our proposed method with the fuzzy Dijkstra algorithm is presented to support its cogency.

Keywords Picture fuzzy number · Trapezoidal picture fuzzy number · Score function · Shortest path problem · Dijkstra algorithm

1 Introduction

The shortest path problem (SPP) is inspired by a natural question on graphs, which may represent networks of various kinds (e.g., transport networks, inventory systems, or manpower allocation). The optimum paths minimize the sum of ‘weights’ associated with its edges. These ‘weights’ may represent amounts like distances or lengths, and then, the problem becomes a question on finding a shortest way

to travel from one point to another. This interpretation justifies the name given to the general problem.

Some standard algorithms for solving SPPs have been proposed by Dijkstra [16], Bellman [6], Floyd [21], and Warshall [48]. But there are situations that need to implement a vague or uncertain knowledge about the ‘weights’ or ‘distances’ between the nodes of a network. For example, the next day’s transportation scheme does not have accurate a priori information. In an electricity market, the power supply at each departure point, the demand at every destination, and the transmission capacity need to be assessed by experts’ judgments or probabilistic analysis. To handle such type of vague information, Zadeh [51] introduced fuzzy set (FS) theory. The analysis of fuzzy counterparts of the SPP has been a popular topic since the fuzzy SPP was first proposed by Dubois and Prade [19] in 1980.

The Dijkstra algorithm is a milestone in the analysis of the SPP with crisp information. When arc lengths are crisp numbers, it can be easily understood and applied. However, in networks of ambiguous nature, the optimization methods designed for crisp lengths may not be immediately employed

✉ José Carlos R. Alcantud
jcr@usal.es

Muhammad Akram
m.akram@pucit.edu.pk

Amna Habib
amnahabib96@gmail.com

¹ Department of Mathematics, University of the Punjab, New Campus, Lahore, Pakistan

² BORDA Research Unit and IME, University of Salamanca, 37007 Salamanca, Spain

when data are fuzzy numbers. The structure of the problem produces objective values that are fuzzy numbers too. For this reason, Klein [29] explained that ‘fuzzy shortest paths (...) are hampered by the possibility that the fuzzy shortest length may not correspond to an actual path in the network,’ a problem already noted by the pioneering Dubois and Prade [19]. Therefore, some adaptations are required in order to resort to the classical approach. We list but a few solutions in the literature. Liu and Kao [31] use a defuzzification function known as Yager’s ranking index [50]. With the help of this tool, they transform a problem where the arcs are weighted by trapezoidal fuzzy numbers into a crisp formulation. Klein [29] states new models which rely on fuzzy shortest paths and multiple objectives. Peyer et al. [39] generalize the Dijkstra’s SP algorithm and apply it to VLSI routing. Shu-Xi [42] suggests an improved Dijkstra’s SP algorithm with applications. Still other research papers investigate the fuzzy Dijkstra algorithm for SPP, so that it can be applied under several uncertain conditions [15, 26, 31, 38].

In fuzzy set theory, a system of fuzzy numbers (FNs) has been defined in accordance with the classical system of real numbers. They were introduced by Zadeh [52] in order to handle imprecise numerical quantities. Their arithmetic structure was drafted by Dubois and Prade [18], Ma et al. [34], Mizumoto and Tanaka [32], and Nahmias [36], among others.

Atanassov [5] initiated the idea of intuitionistic fuzzy set (IFS) in 1986. This concept turns out to be more beneficial than fuzzy sets in dealing with uncertainties and ambiguities. The notion of FNs is extended to intuitionistic fuzzy numbers (IFNs) by Burillo et al. [7]. The notions of IFNs by various contexts were examined in [10, 25, 43]. Shu et al. [43] gave operational laws for intuitionistic triangular FNs and suggested an algorithm describing fault-tree analysis in intuitionistic fuzzy environment. Zhang and Liu [53] made use of triangular FN to indicate membership functions, which led them to put forward the definition of triangular IFN. Wang [27] proposed intuitionistic trapezoidal fuzzy number (ITFN) and interval ITFN. Wang and Zhang [47] described the expected values for ITFNs. Jianqiang and Zhong [28] defined some aggregation operators on ITFN. In relation with our targets, Rangasamy et al. [41] proposed a method for finding the shortest hyperpath in an intuitionistic fuzzy weighted hypergraph. Mukherjee [35] considered Dijkstra’s algorithm to resolve shortest path problem in networks with intuitionistic fuzzy information. Porchelvi and Banumathy [40] proposed a generalized Dijkstra’s shortest path algorithm with intuitionistic fuzzy information. Akram and Arshad [1] defined bipolar fuzzy number and discussed the trapezoidal bipolar fuzzy TOPSIS method.

Atanassov’s IFS theory has been effectively implemented in various fields. However, there are real situations

when one also wants to emphasize a neutral part between two extreme positions (cf., Alcantud and Laruelle [4] for real motivations in a voting context). It is just natural to add one more level of freedom to the description of IFSs in order to achieve that goal. Such a generalization of IFSs, known as picture fuzzy sets (PFSs), was suggested by Cuong [12, 13], and published in 2013 [14]. Substantially, PFS-based models are useful when one needs to incorporate the opinion of experts that provide vague answers in the way: yes, abstain, no, and refusal. So far, some development has been done in PFS theory. Singh [44] examined the correlation coefficients for PFS and employ them in clustering analysis. Son and Thong [45] gave a new hybrid forecast method based on PF clustering. Thong and Son [46] used PF clustering and the PF rule interpolation technique to discuss multivariable fuzzy forecasting. For additional notations and applications, the readers are referred to [22, 33, 37]. Akram and Zafer [2] presented a remarkable contribution for the interested readers.

In continuation with the prolific lines of research stated above, this paper extends the traditional Dijkstra algorithm so that we can find the minimum cost of the picture fuzzy SPP (PFSP). The PFSP aims at providing decision makers with the PFSP length and the shortest path in a network with picture fuzzy arc lengths. For traveling each arc, the cost parameters are assumed to be TPFNs. A pseudocode for this problem is proposed on the basis of Dijkstra’s methodology. Some operational laws and expected values of TPFNs and the comparison between two TPFNs by score and accuracy functions are explained. Finally, an example is provided to validate the adopted approach and illustrate its usefulness and productiveness. Our results are discussed in comparison with existing models.

The outline of this paper is as follows. Section 2 gives preliminaries and basic definitions. Picture fuzzy numbers are the target of Sect. 3. We review their trapezoidal and triangular formulations, their operations, and the methodologies for comparison. Section 4 explains formally the problem that we investigate. We give the algorithm of solution and explanations about its performance. Section 5 compares our study with the closest papers in the literature. Section 6 gives an application to a transmission network. A comparative analysis is the subject of Sect. 7. Section 8 puts forward some discussion, and Sect. 9 concludes the paper.

2 Basic concepts and notations

A *fuzzy set* on a universal set X is a mapping $A : X \rightarrow [0, 1]$, where $\mu_A(x)$ is assigned to represent the membership degree of x in A and is called the membership function of A [51].

Let us recall the notion of a fuzzy number. A *fuzzy number* \tilde{n} , is a normal and convex fuzzy subset of X with

piecewise continuous membership functions [19]. ‘Normality’ implies that there exists $x \in \mathbb{R}$ such that $\bigvee_x \mu_{\tilde{n}}(x) = 1$ and ‘convexity’ means that for all $x_1, x_2 \in X$ and $\alpha \in [0, 1]$,

$$\mu_{\tilde{n}}(\alpha x_1 + (1 - \alpha)x_2) \geq \min(\mu_{\tilde{n}}(x_1), \mu_{\tilde{n}}(x_2)).$$

A *triangular fuzzy number* \tilde{n} [9] can be characterized by three real numbers $a \leq b \leq c$, denoted by a triplet (a, b, c) , where the membership can be determined as follows:

$$\mu_{\tilde{n}}(x) = \begin{cases} \frac{x - a}{b - a}, & a \leq x < b \\ \frac{c - x}{c - b}, & b \leq x \leq c \\ 0, & \text{otherwise.} \end{cases}$$

A *trapezoidal fuzzy number* \tilde{n} [9] can be completely characterized by four real numbers $a \leq b \leq c \leq d$, and for this reason it is often denoted as $\tilde{n} = (a, b, c, d)$ for brevity— where the membership can be defined as follows:

$$\mu_{\tilde{n}}(x) = \begin{cases} \frac{x - a}{b - a}, & a \leq x < b \\ 1, & b \leq x \leq c \\ \frac{d - x}{d - c}, & c < x \leq d \\ 0, & \text{otherwise.} \end{cases}$$

For two arbitrary trapezoidal fuzzy numbers $\tilde{n}_1 = (a_1, b_1, c_1, d_1)$ and $\tilde{n}_2 = (a_2, b_2, c_2, d_2)$, the addition [9] of \tilde{n}_1 and \tilde{n}_2 is defined as

$$\tilde{n}_1 + \tilde{n}_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2).$$

To simplify the representation of FNs, some basic features of FNs might be seen as a point that corresponds to the typical value of the magnitude that the FN \tilde{n} represents. A parameter utilized for such purpose is the *expected value* of the fuzzy number $\tilde{n} = (a, b, c, d)$, given by (see [20, 24])

$$EV(\tilde{n}) = \frac{1}{2} \int_0^1 (f_{\tilde{n}}^{-1}(\alpha) + g_{\tilde{n}}^{-1}(\alpha))d\alpha,$$

where $f_{\tilde{n}}$ and $g_{\tilde{n}}$ are the sides of continuous fuzzy number. Sometimes its generalization, called *weighted expected value*, might be interesting. It is defined as

$$EV_q(\tilde{n}) = (1 - q) \int_0^1 f_{\tilde{n}}^{-1}(\alpha)d\alpha + q \int_0^1 g_{\tilde{n}}^{-1}(\alpha)d\alpha,$$

where $q \in [0, 1]$ (see [23]).

3 Picture fuzzy numbers

Fuzzy sets have been extended in many ways in order to gain flexibility and applicability. The next interesting concept will be the generalization that inspires our analysis.

Definition 3.1 [14] A *picture fuzzy set* A on universe X is defined as

$$A = \{ \langle z, \mu_A(z), \eta_A(z), \nu_A(z) \rangle | z \in X \},$$

where $\mu_A(z), \eta_A(z), \nu_A(z) \in [0, 1]$ are positive, neutral, and negative membership functions, respectively, of element z in A such that

$$0 \leq \mu_A(z) + \eta_A(z) + \nu_A(z) \leq 1, \quad \text{for every } z \in X.$$

Moreover, $\pi_A(z) = 1 - \mu_A(z) - \eta_A(z) - \nu_A(z)$ is called refusal membership degree of z to the set A .

Here, we define trapezoidal picture fuzzy numbers (TPFNs) which capture more flexible information than intuitionistic trapezoidal FNs. We also discuss their geometric interpretations. First, we recall the concept of picture fuzzy number.

Definition 3.2 A *picture fuzzy number* \tilde{n} in the set of real numbers \mathbb{R} can be defined as a PFS $\tilde{n} = \{ \langle z, \mu_{\tilde{n}}(z), \eta_{\tilde{n}}(z), \nu_{\tilde{n}}(z) \rangle : z \in \mathbb{R} \}$, where

$$\mu_{\tilde{n}}(z) = \begin{cases} f_{\tilde{n}}^L(z), & p_1 \leq z < q \\ \alpha_{\tilde{n}}, & q \leq z \leq r \\ f_{\tilde{n}}^R(z), & r < z \leq s_1 \\ 0, & \text{otherwise,} \end{cases}$$

$$\eta_{\tilde{n}}(z) = \begin{cases} g_{\tilde{n}}^L(z), & p_2 \leq z < q \\ \beta_{\tilde{n}}, & q \leq z \leq r \\ g_{\tilde{n}}^R(z), & r < z \leq s_2 \\ 0, & \text{otherwise,} \end{cases}$$

$$\nu_{\tilde{n}}(z) = \begin{cases} h_{\tilde{n}}^L(z), & p_3 \leq z < q \\ \gamma_{\tilde{n}}, & q \leq z \leq r \\ h_{\tilde{n}}^R(z), & r < z \leq s_3 \\ 0, & \text{otherwise.} \end{cases}$$

For each $a \in \mathbb{R}$, these elements are its positive, neutral, and negative membership degrees, respectively.

In addition to the above definition, the following restrictions are worth mentioning:

1. $f_{\tilde{n}}^L(z), g_{\tilde{n}}^R(z)$, and $h_{\tilde{n}}^R(z)$ are continuous monotone increasing functions.
2. $f_{\tilde{n}}^R(z), g_{\tilde{n}}^L(z)$, and $h_{\tilde{n}}^L(z)$ are continuous monotone decreasing functions.
3. $f_{\tilde{n}}^L(z), f_{\tilde{n}}^R(z); g_{\tilde{n}}^L(z), g_{\tilde{n}}^R(z);$ and $h_{\tilde{n}}^L(z), h_{\tilde{n}}^R(z)$ are left, right basis functions of positive, neutral, and negative membership functions, respectively.

Thus, a PFN has three parameters: a positive membership function $\mu_{\tilde{n}}(z)$ which shows the extent by which an expert guesses that the element belongs to (p_1, q, r, s_1) , a neutral membership function $\eta_{\tilde{n}}(z)$ which expresses the extent to which an element abstains to be in (p_2, q, r, s_2) , and a

negative membership function $\eta_{\tilde{n}}(z)$ which indicates the extent of element to be not in (p_3, q, r, s_3) , where $p_1, p_2, p_3, q, r, s_1, s_2, s_3 \in \mathbb{R}$. Also,

$$\pi_{\tilde{n}}(z) = 1 - \mu_{\tilde{n}}(z) - \eta_{\tilde{n}}(z) - \nu_{\tilde{n}}(z)$$

implies the extent of refusal of PFN. The smaller the $\pi_{\tilde{n}}(z)$, the more certain is the PFN. A PFN is represented graphically in Fig. 1.

3.1 Trapezoidal picture fuzzy numbers

A PFN is of trapezoidal type if for each element z , the shape of its three associated membership functions is trapezoidal. Figure 2 displays a trapezoidal picture fuzzy number. In formal terms, when

$$\begin{aligned} f_{\tilde{n}}^L(z) &= \frac{z - p_1}{q - p_1} \alpha_{\tilde{n}}, & f_{\tilde{n}}^R(z) &= \frac{s_1 - z}{s_1 - r} \alpha_{\tilde{n}}, \\ g_{\tilde{n}}^L(z) &= \frac{q - z + \beta_{\tilde{n}}(z - p_2)}{q - p_2}, & g_{\tilde{n}}^R(z) &= \frac{z - r + \beta_{\tilde{n}}(s_2 - z)}{s_2 - r}, \\ h_{\tilde{n}}^L(z) &= \frac{q - z + \gamma_{\tilde{n}}(z - p_3)}{q - p_3}, & h_{\tilde{n}}^R(z) &= \frac{z - r + \gamma_{\tilde{n}}(s_3 - z)}{s_3 - r}, \end{aligned}$$

for $0 \leq \alpha_{\tilde{n}}, \beta_{\tilde{n}}, \gamma_{\tilde{n}} \leq 1$ satisfying $\alpha_{\tilde{n}} + \beta_{\tilde{n}} + \gamma_{\tilde{n}} \leq 1$, the PFN is called *trapezoidal picture fuzzy number*, denoted as,

$$\tilde{n} = \langle ([p_1, q, r, s_1]; \alpha_{\tilde{n}}), ([p_2, q, r, s_2]; \beta_{\tilde{n}}), ([p_3, q, r, s_3]; \gamma_{\tilde{n}}) \rangle.$$

Remark 3.1.1 The trapezoidal picture FNs generalize picture fuzzy intervals. In a TPFN \tilde{n} , sometimes we observe

$$[p_1, q, r, s_1] = [p_2, q, r, s_2] = [p_3, q, r, s_3].$$

Under such circumstances, it can be characterized as $\tilde{n} = \langle [p, q, r, s]; \alpha_{\tilde{n}}, \beta_{\tilde{n}}, \gamma_{\tilde{n}} \rangle$. Therefore, \tilde{n} is in fact a picture fuzzy interval $[q, r]$, and p and s are the left and right extremes of its spread, respectively. Observe also that the increase and decrease in its membership functions are linear from p to q and from r to s .

Example 3.1.1 Consider a trapezoidal picture fuzzy number

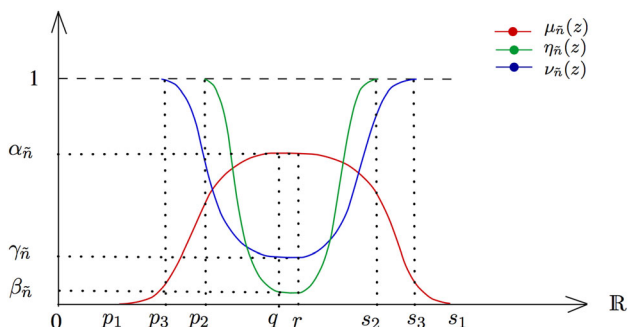


Fig. 1 Picture fuzzy number $\tilde{n} = \{(z, \mu_{\tilde{n}}(z), \eta_{\tilde{n}}(z), \nu_{\tilde{n}}(z)) : z \in \mathbb{R}\}$

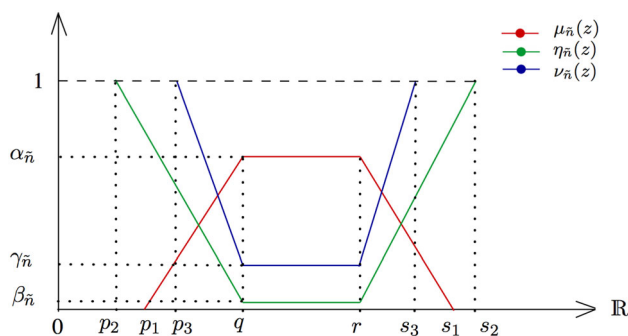


Fig. 2 Trapezoidal picture fuzzy number $\tilde{n} = \{(z, \mu_{\tilde{n}}(z), \eta_{\tilde{n}}(z), \nu_{\tilde{n}}(z)) : z \in \mathbb{R}\}$ in its compact representation $\tilde{n} = \langle ([p_1, q, r, s_1]; \alpha_{\tilde{n}}), ([p_2, q, r, s_2]; \beta_{\tilde{n}}), ([p_3, q, r, s_3]; \gamma_{\tilde{n}}) \rangle$.

$$\tilde{3} = \langle [-1, 2, 4, 6]; 0.5, 0.1, 0.2 \rangle.$$

When $z = 2$, its positive membership degree to be the fuzzy number $\tilde{3}$ is 0.5, its neutral membership to be fuzzy number $\tilde{3}$ is 0.1, its negative membership to be the fuzzy number $\tilde{3}$ is 0.2, and its refusal membership to be the fuzzy number $\tilde{3}$ is 0.2.

3.2 Triangular picture fuzzy numbers

A trapezoidal picture FN where $q = r$ reduces to a triangular picture FN. Thus, the membership functions in a triangular picture fuzzy number

$$\tilde{n} = \langle ([p_1, q, r_1]; \alpha_{\tilde{n}}), ([p_2, q, r_2]; \beta_{\tilde{n}}), ([p_3, q, r_3]; \gamma_{\tilde{n}}) \rangle$$

can be defined as:

$$\begin{aligned} \mu_{\tilde{n}}(z) &= \begin{cases} \frac{\alpha_{\tilde{n}}(z - p_1)}{q - p_1}, & p_1 \leq z \leq q \\ \frac{\alpha_{\tilde{n}}(r_1 - z)}{r_1 - q}, & q \leq z \leq r_1 \\ 0, & \text{otherwise,} \end{cases} \\ \eta_{\tilde{n}}(z) &= \begin{cases} \frac{q - z + \beta_{\tilde{n}}(z - p_2)}{q - p_2}, & p_2 \leq z \leq q \\ \frac{z - q + \beta_{\tilde{n}}(r_2 - z)}{r_2 - q}, & q \leq z \leq r_2 \\ 0, & \text{otherwise,} \end{cases} \\ \nu_{\tilde{n}}(z) &= \begin{cases} \frac{q - z + \gamma_{\tilde{n}}(z - p_3)}{q - p_3}, & p_3 \leq z \leq q \\ \frac{z - q + \gamma_{\tilde{n}}(r_3 - z)}{r_3 - q}, & q \leq z \leq r_3 \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

for each z , where $0 \leq \alpha_{\tilde{n}}, \beta_{\tilde{n}}, \gamma_{\tilde{n}} \leq 1$ such that $\alpha_{\tilde{n}} + \beta_{\tilde{n}} + \gamma_{\tilde{n}} \leq 1$.

Remark 3.2.1 The triangular picture fuzzy numbers can be denoted by $\tilde{n} = \langle [p, q, r]; \alpha_{\tilde{n}}, \beta_{\tilde{n}}, \gamma_{\tilde{n}} \rangle$, whenever

$$[p_1, q, r_1] = [p_2, q, r_2] = [p_3, q, r_3],$$

where q is the mean value and $p = p_1 = p_2 = p_3$ and $r = r_1 = r_2 = r_3$ are the left and right extremes of its spread, respectively. Similar to trapezoidal picture fuzzy numbers, the increase and decrease in the membership functions are linear from p to q and from q to r .

Intuitively, a PFN is of triangular type if the shape of each of its membership function is triangular. Figure 3 displays a triangular picture fuzzy number.

Note 3.1 For understanding the concept of PFNs, the following points are important.

1. In the strict sense of the definition of triangular picture FN, a PFS having triplet $(\mu_{\tilde{n}}, \eta_{\tilde{n}}, \nu_{\tilde{n}})$ at more than one point does not qualify for a triangular picture fuzzy number. However, a trapezoidal picture fuzzy number can accept this triplet at more than one point.
2. Membership functions having triangular and trapezoidal shapes are most often utilized to depict fuzzy numbers, but other shapes may be desirable in certain applications. Membership functions taking bell shapes are quite typical; their form is displayed in Fig. 1.
3. The membership functions of PFNs are not necessarily symmetric.
4. The PFN \tilde{n} is a normal or traditional PFN, if $\alpha_{\tilde{n}} = 1$, $\beta_{\tilde{n}} = 0$ and $\gamma_{\tilde{n}} = 0$.

3.3 Operations on trapezoidal picture fuzzy numbers

In this subsection, we discuss some arithmetic operations on trapezoidal picture fuzzy numbers (TPFNs).

Definition 3.3 Let $\tilde{n}_1 = \langle [p_1, q_1, r_1, s_1]; \alpha_{\tilde{n}_1}, \beta_{\tilde{n}_1}, \gamma_{\tilde{n}_1} \rangle$, and $\tilde{n}_2 = \langle [p_2, q_2, r_2, s_2]; \alpha_{\tilde{n}_2}, \beta_{\tilde{n}_2}, \gamma_{\tilde{n}_2} \rangle$ be two TPFNs, then

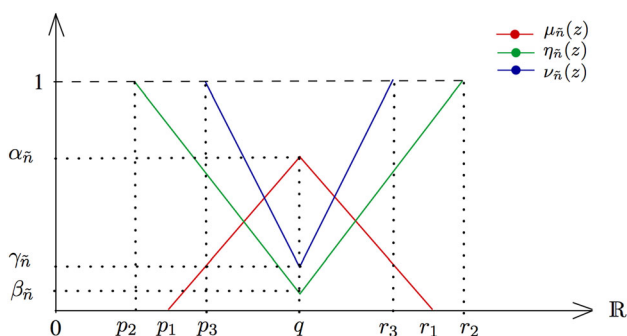


Fig. 3 Triangular picture fuzzy number $\tilde{n} = \{ (z, \mu_{\tilde{n}}(z), \eta_{\tilde{n}}(z), \nu_{\tilde{n}}(z)) : z \in \mathbb{R} \}$ in its compact representation $\tilde{n} = \langle [p, q, r]; \alpha_{\tilde{n}}, \beta_{\tilde{n}}, \gamma_{\tilde{n}} \rangle$ with $[p_1, q, r_1] = [p_2, q, r_2] = [p_3, q, r_3]$

1. $\tilde{n}_1 + \tilde{n}_2 = \langle [p_1 + p_2, q_1 + q_2, r_1 + r_2, s_1 + s_2]; \alpha_{\tilde{n}_1} + \alpha_{\tilde{n}_2} - \alpha_{\tilde{n}_1}\alpha_{\tilde{n}_2}, \beta_{\tilde{n}_1}\beta_{\tilde{n}_2}, \gamma_{\tilde{n}_1}\gamma_{\tilde{n}_2} \rangle;$
2. $\tilde{n}_1 \cdot \tilde{n}_2 = \langle [p_1p_2, q_1q_2, r_1r_2, s_1s_2]; \alpha_{\tilde{n}_1}\alpha_{\tilde{n}_2}, \beta_{\tilde{n}_1} + \beta_{\tilde{n}_2} - \beta_{\tilde{n}_1}\beta_{\tilde{n}_2}, \gamma_{\tilde{n}_1} + \gamma_{\tilde{n}_2} - \gamma_{\tilde{n}_1}\gamma_{\tilde{n}_2} \rangle;$
3. $\omega\tilde{n}_1 = \langle [\omega p_1, \omega q_1, \omega r_1, \omega s_1]; 1 - (1 - \alpha_{\tilde{n}_1})^\omega, (\beta_{\tilde{n}_1})^\omega, (\gamma_{\tilde{n}_1})^\omega \rangle, \omega \geq 0;$
4. $\tilde{n}_1^\omega = \langle [p_1^\omega, q_1^\omega, r_1^\omega, s_1^\omega]; (\alpha_{\tilde{n}_1})^\omega, 1 - (1 - \beta_{\tilde{n}_1})^\omega, 1 - (1 - \gamma_{\tilde{n}_1})^\omega \rangle, \omega \geq 0.$

We state a theorem which illustrates the calculation rules between two PFNs. The proof is straightforward and therefore omitted.

Theorem 3.4 Let $\tilde{n}_1 = \langle [p_1, q_1, r_1, s_1]; \alpha_{\tilde{n}_1}, \beta_{\tilde{n}_1}, \gamma_{\tilde{n}_1} \rangle$, and $\tilde{n}_2 = \langle [p_2, q_2, r_2, s_2]; \alpha_{\tilde{n}_2}, \beta_{\tilde{n}_2}, \gamma_{\tilde{n}_2} \rangle$ be two TPFNs. Then, the following properties must hold:

1. $\tilde{n}_1 + \tilde{n}_2 = \tilde{n}_2 + \tilde{n}_1;$
2. $\tilde{n}_1 \cdot \tilde{n}_2 = \tilde{n}_2 \cdot \tilde{n}_1;$
3. $\omega(\tilde{n}_1 + \tilde{n}_2) = \omega\tilde{n}_1 + \omega\tilde{n}_2, \omega \geq 0;$
4. $\omega_1\tilde{n}_1 + \omega_2\tilde{n}_1 = (\omega_1 + \omega_2)\tilde{n}_1, \omega_1, \omega_2 \geq 0;$
5. $\tilde{n}_1^\omega \cdot \tilde{n}_2^\omega = (\tilde{n}_1 \cdot \tilde{n}_2)^\omega, \omega \geq 0;$
6. $\tilde{n}_1^{\omega_1} \cdot \tilde{n}_1^{\omega_2} = \tilde{n}_1^{\omega_1 + \omega_2}, \omega_1, \omega_2 \geq 0.$

3.4 Comparison of trapezoidal picture fuzzy numbers based on expected values

Consider the functions

$$f_n^L(z) = \frac{z - p_1}{q - p_1} \alpha_{\tilde{n}}, \quad z \in [p_1, q), \quad f_n^R(z) = \frac{s_1 - z}{s_1 - r} \alpha_{\tilde{n}}, \quad z \in (r, s_1]$$

in a positive membership function $\mu_{\tilde{n}}(z)$ of PFN \tilde{n} , which are strictly linear monotone increasing and strictly linear monotone decreasing, respectively. Their inverse functions $f_n^{L-1}(h)$, and $f_n^{R-1}(h)$ can be defined, respectively, as,

$$f_n^{L-1}(h) = p_1 + \frac{h}{\alpha_{\tilde{n}}}(q - p_1), \quad h \in [0, \alpha_{\tilde{n}}];$$

$$f_n^{R-1}(h) = s_1 + \frac{h}{\alpha_{\tilde{n}}}(r - s_1), \quad h \in [0, \alpha_{\tilde{n}}].$$

The positive membership degree of TPFN \tilde{n} lies in $[\alpha_{\tilde{n}}, 1 - \beta_{\tilde{n}} - \gamma_{\tilde{n}}]$.

Definition 3.5 The expected value of TPFN $\tilde{n} = \langle [p, q, r, s]; \alpha_{\tilde{n}}, \beta_{\tilde{n}}, \gamma_{\tilde{n}} \rangle$ is defined as

$$I_\omega(\tilde{n}) = \frac{1}{2} \left[\int_0^{\alpha_{\tilde{n}}} [(1 - \omega)f_n^{L-1}(h) + \omega f_n^{R-1}(h)] dh + \int_0^{1 - \beta_{\tilde{n}} - \gamma_{\tilde{n}}} [(1 - \omega)f_n^{L-1}(h) + \omega f_n^{R-1}(h)] dh \right],$$

where $0 \leq \omega \leq 1$. The value of ω interprets the expert's risk preference; $\omega > 0.5$ indicates the expert's love risk;

$\omega < 0.5$ tells the hate risk of expert; and $\omega = 0.5$ expresses the indifferent risk reference.

Next, we give a theorem which will be used subsequently in defining score function and accuracy function of TPFNs.

Theorem 3.6 For TPFN $\tilde{n} = \langle [p, q, r, s]; \alpha_{\tilde{n}}, \beta_{\tilde{n}}, \gamma_{\tilde{n}} \rangle$, the expected value is $I_{\tilde{n}} = \frac{1}{8}[(p + q + r + s) \times (1 + \alpha_{\tilde{n}} - \beta_{\tilde{n}} - \gamma_{\tilde{n}})]$.

Proof Let $\tilde{n} = \langle [p, q, r, s]; \alpha_{\tilde{n}}, \beta_{\tilde{n}}, \gamma_{\tilde{n}} \rangle$ be a TPFN. The standard expected value of \tilde{n} can be computed as follows.

$$\begin{aligned} I(\tilde{n}) &= \frac{1}{4} \left[\int_0^{\alpha_{\tilde{n}}} [f_{\tilde{n}}^{L-1}(h) + f_{\tilde{n}}^{R-1}(h)] \, dh \right. \\ &\quad \left. + \int_0^{1-\beta_{\tilde{n}}-\gamma_{\tilde{n}}} [f_{\tilde{n}}^{L-1}(h) + f_{\tilde{n}}^{R-1}(h)] \, dh \right] \\ &= \frac{1}{4\alpha_{\tilde{n}}} \left[\int_0^{\alpha_{\tilde{n}}} [p_1 + h(q - p_1) + s_1 + h(r - s_1)] \, dh \right. \\ &\quad \left. + \frac{1}{1 - \beta_{\tilde{n}} - \gamma_{\tilde{n}}} \int_0^{1-\beta_{\tilde{n}}-\gamma_{\tilde{n}}} [p_1 + h(q - p_1) \right. \\ &\quad \left. + s_1 + h(r - s_1)] \, dh \right] \\ &= \frac{1}{4} \left[\left(p_1 + s_1 + \frac{q - p_1 + r - s_1}{2} \right) \alpha_{\tilde{n}} \right. \\ &\quad \left. + \left(p_1 + s_1 + \frac{q - p_1 + r - s_1}{2} \right) (1 - \beta_{\tilde{n}} - \gamma_{\tilde{n}}) \right] \\ &= \frac{1}{8} [(p + q + r + s) \times (1 + \alpha_{\tilde{n}} - \beta_{\tilde{n}} - \gamma_{\tilde{n}})]. \end{aligned}$$

□

A convenient way for comparing PFNs is by using score function and accuracy function.

Definition 3.7 Let $\tilde{n} = \langle [p, q, r, s]; \alpha_{\tilde{n}}, \beta_{\tilde{n}}, \gamma_{\tilde{n}} \rangle$ be a TPFN, then

$$S(\tilde{n}) = I_{\tilde{n}} \times (\alpha_{\tilde{n}} - \beta_{\tilde{n}} - \gamma_{\tilde{n}})$$

is named as *score function* of \tilde{n} , where $I_{\tilde{n}}$ is the expected value of \tilde{n} , and $S(\tilde{n}) \in [-1, 1]$.

Example 3.4.1 It is simple to observe that score function defined above is imperfect. As an example, consider two TPFNs $\tilde{n}_1 = \langle [p, q, r, s]; 0.4, 0.3, 0.2 \rangle$, and $\tilde{n}_2 = \langle [p, q, r, s]; 0.4, 0.2, 0.3 \rangle$. Although they have different neutral and negative membership values, their score value coincides. Thus, they are not comparable by means of the function defined above.

The aforementioned fact consequently leads to define accuracy function to improve the performance of the score function:

Definition 3.8 Let $\tilde{n} = \langle [p, q, r, s]; \alpha_{\tilde{n}}, \beta_{\tilde{n}}, \gamma_{\tilde{n}} \rangle$ be a TPFN, then

$$H(\tilde{n}) = I_{\tilde{n}} \times (\alpha_{\tilde{n}} + \beta_{\tilde{n}} + \gamma_{\tilde{n}})$$

is named as *accuracy function* of \tilde{n} , where $I_{\tilde{n}}$ is the expected value of \tilde{n} , and $H(\tilde{n}) \in [0, 1]$.

Based on the score and accuracy functions, we next propose a comparison law for TPFNs. This procedure is common to other models of uncertain knowledge [3].

Definition 3.9 Let $\tilde{n}_1 = \langle [p_1, q_1, r_1, s_1]; \alpha_{\tilde{n}_1}, \beta_{\tilde{n}_1}, \gamma_{\tilde{n}_1} \rangle$, $\tilde{n}_2 = \langle [p_2, q_2, r_2, s_2]; \alpha_{\tilde{n}_2}, \beta_{\tilde{n}_2}, \gamma_{\tilde{n}_2} \rangle$ be two TPFNs with score functions $S(\tilde{n}_1)$, and $S(\tilde{n}_2)$; and the accuracy functions $H(\tilde{n}_1)$, and $H(\tilde{n}_2)$, respectively, then

1. If $S(\tilde{n}_1) > S(\tilde{n}_2)$, then $\tilde{n}_1 > \tilde{n}_2$;
2. If $S(\tilde{n}_1) = S(\tilde{n}_2)$, then
 - (a) if $H(\tilde{n}_1) > H(\tilde{n}_2)$, then $\tilde{n}_1 > \tilde{n}_2$;
 - (b) if $H(\tilde{n}_1) = H(\tilde{n}_2)$, then $\tilde{n}_1 = \tilde{n}_2$.

Voting context The above definitions can be explained by taking ‘voting’ as an example. The score function $S(\tilde{n}) = I_{\tilde{n}} \times (\alpha_{\tilde{n}} - \beta_{\tilde{n}} - \gamma_{\tilde{n}})$ can represent the goal difference, while the accuracy function $H(\tilde{n}) = I_{\tilde{n}} \times (\alpha_{\tilde{n}} + \beta_{\tilde{n}} + \gamma_{\tilde{n}})$ can be translated as effectual degree of voting. If $S(\tilde{n})$ increases, it can be deduced that majority people will vote for $\alpha_{\tilde{n}}$. On the other side, if $H(\tilde{n})$ increases, it can be guessed that fewer people will refuse to vote. Thus, $H(\tilde{n})$ indicates the effectual degree of voting.

4 Dijkstra algorithm for the picture fuzzy shortest path problem

Dijkstra algorithm was first proposed in 1956 by Edsger Dijkstra and published in 1959 [16]. The problem is to determine the SP in weighted network from a source node s to the rest of the nodes, where the weights associated with links (i, j) are often mentioned as ‘costs.’ To deal with uncertain circumstances, generalized Dijkstra algorithms have been discussed by many researchers. In the picture fuzzy environment, there is a need to deal with two important issues.

1. The addition of two edges with picture fuzzy arc lengths.
2. The comparison of distance values related to two distinct paths having edge lengths depicted by PFNs:

Based on their expected values, the fuzzy Dijkstra algorithm can be extended to a picture fuzzy Dijkstra algorithm. We do this in our next subsection. Afterward, we illustrate our algorithm with a fully developed example.

4.1 Our extension of the Dijkstra algorithm

The algorithm identifies all nodes with their own states, where the state of a node consists of two specificities, namely distance value and status label. The ‘distance value’ of node represents a measure of its distance from source s , and the ‘status label’ is a feature specify whether distance value of a node is equal to the shortest distance. If yes, status label is *permanent*; otherwise, it is *temporary*. The algorithm retains and upgrades the nodes incrementally. At every step, a single node is assigned as *current*. The pseudocode for the proposed method is presented below in Algorithm 4.1.

The set of notations used in Algorithm 4.1 are explained in Table 1.

4.1.1 Description and complexity of Algorithm 4.1

At very initial stage, the algorithm specifies the distance values of all nodes in Lines 2–5; therefore, the time complexity of Lines 2–5 is $O(n)$, where n is total number of nodes in G . Line 6 assigns labels for starting/source node, indicating that the node has no predecessor. The temporary labeled nodes are printed in Line 7 in a set T . The block of Lines 8–22 is the main nested loop of Algorithm 1. Lines 8–12 compute the temporary labels for each node j ,

Algorithm 4.1. Pseudocode for the proposed picture fuzzy Dijkstra algorithm

```

1. function Picture Fuzzy Dijkstra ( $G, s$ )
2.   for each node  $j$  in  $G$                                 //Initialization
3.     status label [ $j$ ]  $\leftarrow (\infty, t)$ ;           //An attribute specifying distance value of node  $j$ 
4.     previous [ $j$ ]  $\leftarrow$  not defined;             //Former node in optimized path from the source node
5.   end for
6.   status label [ $s$ ]  $\leftarrow (0, p)$ ;                //Distance of source node to itself
7.    $T \leftarrow$  set of all possible nodes with temporary labels in  $G$ ;
8.   while  $T$  is non void                                //The main loop
9.      $i \leftarrow$  node in  $T$  with lowest distance value  $d_i$ ;
10.    if status label [ $i$ ]  $\leftarrow (\infty, p)$  then
11.      stop;                                           //All other nodes are impenetrable from source node
12.    end if
13.    delete  $i$  from  $T$ ;
14.    for every  $j$  such that there exist link  $(i, j)$ 
15.       $alt \leftarrow d_i + c_{ij}$ ;                       //Defuzzification of picture fuzzy number
16.      if  $alt < d_j$  then                               //Comparison of distance values to obtain smallest distance
17.         $d_j \leftarrow alt$ ;
18.        previous [ $j$ ]  $\leftarrow [i]$ ;
19.      end if
20.      status label [ $j$ ]  $\leftarrow (alt, p)$ ;           //Updated distance label
21.    end for
22.  end while
23.  return status label [];
24. end function

```

provided that it is not yet permanently labeled. This stage involves the defuzzification of TPFNs based on their expected values. The for loop on Line 14 runs n times; therefore, its complexity is $O(n)$. Ignoring the constant, the running time of if conditional in Lines 10–12 is $O(n)$. Line 15 provides a node with the lowest distance value alt . The comparison of distance values is calculated in Lines 16–19. Line 20 finally updates the status label of respective nodes. The running time of block 8–22, which is nested between two loops of sizes n , is $O(n^2)$. If we still reach any temporary labeled node, then return according to Line 23.

4.1.2 Presentation of the algorithm

The algorithm steps are as follows:

1. Allocate distance value of zero to node s and mark as permanent node, i.e., $(0, p)$, and to every other node, assign ∞ as a distance value and mark them as temporary, i.e., (∞, t) . Nominate node s as current node.
2. Consider a set T consisting of nodes having temporary labels which can be attained from current node through a link (i, j) . Upgrade distance values, which is based on defuzzification of PFNs associated with links. For any $j \in T$, the node j with d_j as distance value is overwritten as $\min\{d_j, d_i + c_{ij}\}$. Find out a node j having lowest distance value within all nodes in T , name it alt . Upgrade the node alt with permanent label and designate it as a current node.
3. If we cannot reach any temporary labeled node, then all of the temporary labels turn into permanent labels and we are done. Otherwise, we return to step 2.

Thus, using this algorithm, one can find out the cost of the SP from a sole node to a unique destination node. As soon as the SP to the target node is evaluated, the algorithm will be stopped.

Table 1 Set of notations used in Algorithm 4.1

Notation	Description
T	Set of temporary labeled nodes
P	Set of permanently labeled nodes
i	A node which has lowest distance value from source node s
j	A variant
d_j	Updated distance label from s
c_{ij}	Cost of the link (i, j)
$P_{\text{previous}[j]}$	Closest node to current j along the shortest path from s
alt	Alternative variant of distance employed by way of comparison

5 Significance of the proposed method

This section is devoted to describe a comparison of our proposed method with other well-established techniques. In passing, we overview previous approaches to successful formulations of the shortest path problem in uncertain environments, a literature that we contribute to with our study.

5.1 Fuzzy shortest path problem defined by Okada [38]

Dubois and Prade [17] defined the possibility indices of comparison. Okada [38] used this concept to provide a modified definition by means of possibility variables. Then, he first solved SPP in an uncertain environment. His approach took interactivity into consideration to compare fuzzy path lengths. He then introduced the concept of degree of possibility for each arc on network. Okada's algorithm used α -level sets of FNs combined with the technique of the multiple labeling method.

5.2 Fuzzy shortest path problem defined by Deng et al. [15]

A canonical representation of operations on triangular fuzzy numbers was introduced by Chou [11], which is based on graded mean integration representation. This technique allowed Deng et al. [15] to give a fresh insight to the SPP in an uncertain environment. They revisit the fuzzy Dijkstra algorithm. Their approach determines the addition of two edges by canonical representation, whose result is a crisp number. Their way to compare the distance between two different paths has an advantage in the manner that the shortest path is obtained without ranking FNs.

5.3 Intuitionistic fuzzy shortest path problem defined by Mukherjee [35]

The concept of IF value was given by Xu [49]. Mukherjee [35] considered networks whose cost parameters are IF

values. For solving the SPP that arises, the methodology of Mukherjee follows the definition of score function of an IF value defined by Chen and Tan [10] and its accuracy function as given by Hong and Choi [25]. Based on these two functions, Xu and Yager [49] defined an order relation to rank IF values. Then, Mukherjee utilized this ranking mechanism and used the IF hybrid geometric operator introduced by Xu [49] to aggregate the IF values. He used the above implementations to search out shortest paths in the IF environment by following the spirit of the Dijkstra algorithm.

5.4 Intuitionistic fuzzy shortest path problem defined by Geetharamani and Jayagowri [22]

Hyung et al. [30] defined a similarity measure between fuzzy sets and elements. With the avail of this tool, Geetharamani and Jayagowri [22] utilized the highest similarity degree to identify the shortest path in a network with intuitionistic trapezoidal FNs. They introduced the IFSP length formula on two path length by using half-inverse membership and nonmembership function. The rankings given to the paths are based on the highest similarity degree, which helps the decision makers to identify the preferable path alternatives.

5.5 A view of the proposed Dijkstra algorithm

In contrast with the aforementioned approaches, the technique proposed in this paper is more efficient to find out a shortest path. The main advantage of using expected values of FNs is that they produce just a single value. Decision-making can be done easily without the process of ranking FNs. This is computationally beneficial for solving SPPs in an environment with highly uncertain parameters. Table 2 summarizes the features of four types of models that may be used in the treatment of SPPs.

We argue that picture fuzzy sets have advantages over intuitionistic fuzzy sets and ordinary fuzzy sets, as they provide a more realistic picture of practical situation. Consequently, our technique deals with SPP from a source

node to destination node for a network with trapezoidal picture fuzzy arc lengths.

The picture fuzzy shortest path analysis proceeds in the following way. The literature provides a solution to the problem of generating a single value of a fuzzy variable *V* defined by evidence ‘*V in fuzzy number*’ (Chanas and Nawakost [8]). Subsequently, Heilpern [24] proved that the generative expectation of the fuzzy variable *V* induced by evidence ‘*V in fuzzy number*’ is equal to the expected value of FN. Since then, the concept of expected value of FN has been studied widely and used in a number of decision-making approaches. Jianqiang and Zhang [28] evaluated the expected values of IFNs and applied them to decision-making.

- Our method first modifies the concept of expected values to TPFNs. We establish the novel results for expected values of TPFNs.
- We use this way of implementations to solve a well-known shortest path algorithm, the so-called Dijkstra algorithm, in which the process of defuzzification of TPFNs assigned to arcs of network is done by computing their expected values.
- In order to determine the shortest distance value, the comparison of TPFNs is worked out in terms of score function, based on expected values of TPFNs, which leads directly to a crisp number.

Hence, our proposed implementation is logically sound, more efficient and easy to compute when compared with other fuzzy shortest path approaches.

6 Transmission network application

In this section, we apply the proposed solution to an example with synthetic data. Consider a small-sized transmission network whose edges are assigned to TPFNs. It is displayed in Fig. 4, and the TPFNs with the membership values corresponding to each arc are given in Table 3. The shortest path starting with node (1) and whose

Table 2 Comparison with other fuzzy or crisp models

Shortest path problem under models	Edges\links	Positive membership degree	Negative membership degree	Neutral membership degree
Crisp set	Crisp number	Unable to deal	Unable to deal	Unable to deal
Fuzzy set	Fuzzy number	Able to handle	Unable to deal	Unable to deal
Intuitionistic fuzzy set	Intuitionistic	Able to handle	Able to handle	Unable to deal
Fuzzy set	Fuzzy number			
Picture fuzzy set	Picture fuzzy number	Able to handle	Able to handle	Able to handle

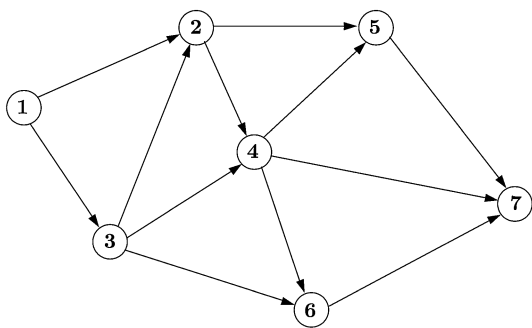


Fig. 4 Transmission network. See Table 3 for the arc lengths

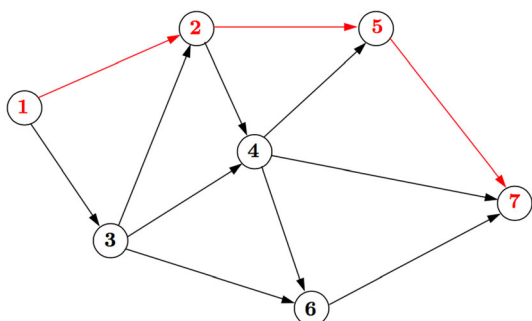


Fig. 5 Shortest path in the transmission network described in Fig. 4

terminal node is (7) will be determined by the application of our Algorithm 4.1.

Let T be the set of temporary labeled nodes, and let P represent the set of permanent labeled nodes. At the initial stage, the source node (1) is transferred from set T to set P , because it has zero distance from (1) to (1) which is the shortest. The steps to identify the shortest path in the network described by Fig. 4 and to work out the shortest distance value among all paths are explained as follows:

Step 1: We calculate the distance from the source node (1) to its accessible neighbors (2) and (3) (see Fig. 4). So we have two routes, one is $(1) \rightarrow (2)$ and other is $(1) \rightarrow (3)$. The use of expected values of TPFNs in the shortest path finding problem is explained as follows:

For the first route $(1) \rightarrow (2) : \langle [2, 3, 4, 6]; 0.6, 0.1, 0.2 \rangle = \tilde{n}_1$, say (see Table 3), the distance value can be obtained as

$$I_{\tilde{n}_1} = \frac{1}{8} [(2 + 3 + 4 + 6) \times (1 + 0.6 - 0.1 - 0.2)]$$

$$= \frac{1}{8} (15 \times 1.3) = 2.4375.$$

$$S(\tilde{n}_1) = I_{\tilde{n}_1} \times (0.6 - 0.1 - 0.2)$$

$$= 2.4375 \times 0.3 = 0.73125.$$

For the second route $(1) \rightarrow (3) : \langle [17, 10, 13, 14]; 0.3, 0.15, 0.1 \rangle = \tilde{n}_2$, say (see Table 3), the distance value can be obtained as

$$I_{\tilde{n}_2} = \frac{1}{8} [(17 + 10 + 13 + 14) \times (1 + 0.3 - 0.15 - 0.1)]$$

$$= \frac{1}{8} (44 \times 1.05) = 5.775.$$

$$S(\tilde{n}_2) = I_{\tilde{n}_2} \times (0.3 - 0.15 - 0.1)$$

$$= 5.775 \times 0.05 = 0.28875.$$

Since the comparison law for TPFNs is based on the score function, we conclude that $\tilde{n}_2 < \tilde{n}_1$ as $S(\tilde{n}_2) < S(\tilde{n}_1)$, i.e., $0.28875 < 0.73125$. Hence, the shortest route in this step is $(1) \rightarrow (3)$. Thus, node (3) becomes permanently labeled and moved to the set P .

Step 2: We consider $(1) \rightarrow (3)$ as current path. From $(1) \rightarrow (3)$ to its neighbors (2), (4), and (6), the shortest distance is computed to each route. The route $(1) \rightarrow (3) \rightarrow (2)$ has length 1.00875, while in step 1, $(1) \rightarrow (2)$ has length 0.73125, which is smaller. So the status label of the node (2) is temporary, i.e., $(0.73125, t)$. We notice that neither of the distance values in this step is smaller than that of the path $(1) \rightarrow (2)$, calculated in the previous step. Thus, the node (2) is labeled permanent and moves from T to P .

Step 3: The path $(1) \rightarrow (2)$ is considered as current. In this step, we have two possible routes $(1) \rightarrow (2) \rightarrow (4)$ and $(1) \rightarrow (2) \rightarrow (5)$. The status label of node (4) remains fixed, and the node (5) is assigned by $(1.03375, t)$. Till now, node (6) has the smallest distance value, so its status label is permanent, i.e., $(0.9486, p)$.

Step 4: The current path is $(1) \rightarrow (3) \rightarrow (6)$. We obtain single route $(1) \rightarrow (3) \rightarrow (6) \rightarrow (7)$ with length 2.285475. In this step, node (5) moves from T to P with status label $(1.03375, p)$.

Table 3 Arc lengths of the network in Fig. 4

Arc	Trapezoidal picture fuzzy number	Arc	Trapezoidal picture fuzzy number
(1, 2)	$\langle [2, 3, 4, 6]; 0.60, 0.10, 0.20 \rangle$	(3, 6)	$\langle [17, 20, 22, 24]; 0.40, 0.25, 0.10 \rangle$
(1, 3)	$\langle [7, 10, 13, 14]; 0.30, 0.15, 0.10 \rangle$	(4, 5)	$\langle [7, 8, 9, 10]; 0.55, 0.15, 0.25 \rangle$
(2, 4)	$\langle [10, 11, 14, 15]; 0.60, 0.10, 0.20 \rangle$	(4, 6)	$\langle [3, 5, 7, 9]; 0.45, 0.00, 0.30 \rangle$
(2, 5)	$\langle [4, 5, 6, 7]; 0.50, 0.40, 0.00 \rangle$	(4, 7)	$\langle [12, 13, 15, 17]; 0.60, 0.10, 0.20 \rangle$
(3, 2)	$\langle [1, 5, 7, 11]; 0.50, 0.20, 0.10 \rangle$	(5, 7)	$\langle [15, 19, 20, 21]; 0.40, 0.15, 0.20 \rangle$
(3, 4)	$\langle [6, 9, 11, 13]; 0.60, 0.00, 0.40 \rangle$	(6, 7)	$\langle [12, 15, 17, 18]; 0.55, 0.25, 0.15 \rangle$

Step 5: $(1) \rightarrow (2) \rightarrow (5)$ is being considered as current path. The node (7) is allocated with temporary status label $(1.525937, t)$, and (4) is labeled permanently as $(1.45875, p)$.

Step 6: The current path is $(1) \rightarrow (3) \rightarrow (4)$. Proceeding in same manner, we conclude that none of the distances from the current path to its neighbors is shorter than $(1) \rightarrow (2) \rightarrow (5) \rightarrow (7)$, which has previously calculated with distance value 1.5259375. Finally, the node (7) is dragged to the set P .

At the final stage, the set T turns into empty set. Our search is complete. We display all the computations in detail in Table 4.

The shortest path $(1) \rightarrow (2) \rightarrow (5) \rightarrow (7)$ found according to the proposed picture fuzzy Dijkstra algorithm

is highlighted in Fig. . Its length is 1.5259375, which is smallest among the lengths of all other possible paths.

7 Comparative analysis

In this section, a comparative analysis of our approach with the shortest path technique suggested by Deng et al. [15] is presented. We apply our method to the numerical example of transportation network illustrated in [15, 26]. We compare the results to verify the validity of the proposed implementation.

Consider a transportation network as displayed in Fig. 6 with 23 nodes and 40 arcs. It is assumed that the arc lengths of the Network 6 are trapezoidal fuzzy numbers with membership functions as shown in Table 5. The shortest

Table 4 Steps of the picture fuzzy Dijkstra algorithm 4.1 for the network in Fig. 4

Current path	Distance value	Temporary status	Permanent status
Initial stage	$T = \{(2), (3), (4), (5), (6), (7)\}$		$P = \{(1)\}$
(1)	$(1) \rightarrow (2) = 0.73125$ $(1) \rightarrow (3) = 0.28875$ $(1) \rightarrow \text{others} = \infty$	$(2) : (0.73125, t)$ $(3) : (0.28875, t)$	$(1) : (0, p)$ $(3) : (0.28875, p)$
$(1) \rightarrow (3)$	$(1) \rightarrow (3) \rightarrow (2) = 1.00875 > 0.73125 = (1) \rightarrow (2)$ $(1) \rightarrow (3) \rightarrow (4) = 1.45875$ $(1) \rightarrow (3) \rightarrow (6) = 0.9486$ $(1) \rightarrow (3) \rightarrow \text{others} = \infty$	$(2) : (0.73125, t)$ $(4) : (1.45875, t)$ $(6) : (0.9486, t)$	$(1) : (0, p)$ $(3) : (0.28875, p)$ $(2) : (0.73125, p)$
$(1) \rightarrow (2)$	$(1) \rightarrow (2) \rightarrow (4) = 3.16875 > 1.45875 = (1) \rightarrow (3) \rightarrow (4)$ $(1) \rightarrow (2) \rightarrow (5) = 1.03375$ $(1) \rightarrow (2) \rightarrow \text{others} = \infty$	$(4) : (1.45875, t)$ $(6) : (0.9486, t)$ $(5) : (1.03375, t)$	$(1) : (0, p)$ $(3) : (0.28875, p)$ $(2) : (0.73125, p)$ $(6) : (0.9486, p)$
$(1) \rightarrow (3) \rightarrow (6)$	$(1) \rightarrow (3) \rightarrow (6) \rightarrow (7) = 2.285475$ $(1) \rightarrow (3) \rightarrow (6) \rightarrow \text{others} = \infty$	$(4) : (1.45875, t)$ $(5) : (1.03375, t)$ $(7) : (2.285475, t)$	$(1) : (0, p)$ $(3) : (0.28875, p)$ $(2) : (0.73125, p)$ $(6) : (0.9486, p)$ $(5) : (1.03375, p)$
$(1) \rightarrow (2) \rightarrow (5)$	$(1) \rightarrow (2) \rightarrow (5) \rightarrow (7) = 1.5259375 < 2.285475 = (1) \rightarrow (3) \rightarrow (6) \rightarrow (7)$ $(1) \rightarrow (2) \rightarrow (5) \rightarrow \text{others} = \infty$	$(4) : (1.45875, t)$ $(7) : (1.5259375, t)$	$(1) : (0, p)$ $(3) : (0.28875, p)$ $(2) : (0.73125, p)$ $(6) : (0.9486, p)$ $(5) : (1.03375, p)$ $(4) : (1.45875, p)$
$(1) \rightarrow (3) \rightarrow (4)$	$(1) \rightarrow (3) \rightarrow (4) \rightarrow (5) = 2.191875 > 1.03375 = (1) \rightarrow (2) \rightarrow (5)$ $(1) \rightarrow (3) \rightarrow (4) \rightarrow (6) = 1.97625 > 0.9486 = (1) \rightarrow (3) \rightarrow (6)$ $(1) \rightarrow (3) \rightarrow (4) \rightarrow (7) = 2.1 > 1.5259375 = (1) \rightarrow (2) \rightarrow (5) \rightarrow (7)$ $(1) \rightarrow (3) \rightarrow (4) \rightarrow \text{others} = \infty$	$(7) : (1.5259375, t)$	$(1) : (0, p)$ $(3) : (0.28875, p)$ $(2) : (0.73125, p)$ $(6) : (0.9486, p)$ $(5) : (1.03375, p)$ $(4) : (1.45875, p)$ $(7) : (1.5259375, p)$
Final stage	$T = \{\}$		$P = \{(1), (2), (3), (4), (5), (6), (7)\}$

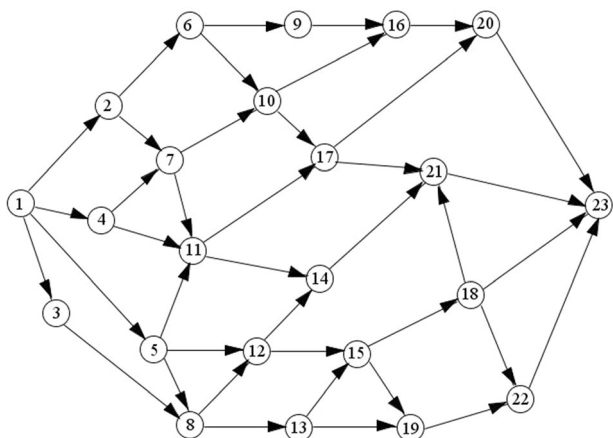


Fig. 6 Transportation network. See Table 5 for arc lengths

path starting with node (1) and whose terminal node is (23) is determined by application of our Algorithm 1. The computation of distance values requires defuzzification of the FNs assigned to each arc. This procedure is done by computing the expected values of FNs.

The comparative study provides the same shortest path, with approximately equal shortest distance value, than the solution in [15]. The results are shown in Table 6.

Hence, the case analysis assures that the decision results of Deng et al. [15] are consistent with our proposed approach, which endorses the authenticity of our method.

8 Discussion

Picture fuzzy sets deal with one more level of freedom than the levels in the description of IFSs. This is done with the inclusion of a neutral part (η) between two extreme positions (μ) and (ν) with the constraint $\mu + \eta + \nu \leq 1$. To cope with several physical quantities, we use a number system in PFS theory, known as PFNs, which can be viewed as a set of real numbers near a given real number or interval. For the reasons stated above, PFNs can express uncertainty and vagueness in parameters more practically than FNs and IFNs. They can be effectively used for computations too.

The investigation on shortest paths stands high among the classic problems in graph theory. Many algorithms have been proposed in the literature [6, 16, 21, 48] in order to offer a solution for the SPP in a network. A prevailing algorithm is Dijkstra algorithm [16]. It has been examined in fuzzy and intuitionistic fuzzy models, where the costs assigned to the arcs were FNs and IFNs, respectively. The PF Dijkstra algorithm that we investigate deals with real situations where the arc lengths are TPFNs. This method differs from previous approaches not only because of this reason, but also because it takes into account a neutral membership grade in addition to the positive and negative membership values in the evaluation of shortest paths.

Methods that compare FNs and IFNs include the application of score functions [18, 28, 35, 36], graded mean

Table 5 Arc lengths of the network in Fig. 6

Arc	Membership function	Arc	Membership function	Arc	Membership function
(1, 2)	(12, 13, 15, 17)	(7, 10)	(9, 10, 12, 13)	(15, 18)	(8, 9, 11, 13)
(1, 3)	(9, 11, 13, 15)	(7, 11)	(6, 7, 8, 9)	(15, 19)	(5, 7, 10, 12)
(1, 4)	(8, 10, 12, 13)	(8, 12)	(5, 8, 9, 10)	(16, 20)	(9, 12, 14, 16)
(1, 5)	(7, 8, 9, 10)	(8, 13)	(3, 5, 8, 10)	(17, 20)	(7, 10, 11, 12)
(2, 6)	(5, 10, 15, 16)	(9, 16)	(6, 7, 9, 10)	(17, 21)	(6, 7, 8, 10)
(2, 7)	(6, 11, 11, 13)	(10, 16)	(12, 13, 16, 17)	(18, 21)	(15, 17, 18, 19)
(3, 8)	(10, 11, 16, 17)	(10, 17)	(15, 19, 20, 21)	(18, 22)	(3, 5, 7, 9)
(4, 7)	(17, 20, 22, 24)	(11, 14)	(8, 9, 11, 13)	(18, 23)	(5, 7, 9, 11)
(4, 11)	(6, 10, 13, 14)	(11, 17)	(6, 9, 11, 13)	(19, 22)	(15, 16, 17, 19)
(5, 8)	(6, 9, 11, 13)	(12, 14)	(13, 14, 15, 18)	(20, 23)	(13, 14, 16, 17)
(5, 11)	(7, 10, 13, 14)	(12, 15)	(12, 14, 15, 16)	(21, 23)	(12, 15, 17, 18)
(5, 12)	(10, 13, 15, 17)	(13, 15)	(10, 12, 14, 15)	(22, 23)	(4, 5, 6, 8)
(6, 9)	(6, 8, 10, 11)	(13, 19)	(17, 18, 19, 20)		
(6, 10)	(10, 11, 14, 15)	(14, 21)	(11, 12, 13, 14)		

Table 6 Comparative analysis of the decision results

Methods	Shortest path	Shortest distance value
FDA by Deng et al. [15]	(1) → (5) → (11) → (17) → (21) → (23)	52.5001
Proposed method	(1) → (5) → (11) → (17) → (21) → (23)	52.5

integration representations based on canonical form [15], order relations [38], specific comparison indices [50], etc. In this study, the comparison of TPFNs on the basis of their expected values is investigated by defining score and accuracy functions. This advantageous way to compare PFNs deems the PF Dijkstra algorithm more powerful to perform qualitative and quantitative studies of different networks.

Concerning applicability, it should be mentioned that depending on weather and other unanticipated factors, the travel time between two cities may be expressed by PF variables even if their geometric distance remains fixed. Hence, our approach is more flexible and general, as different PFNs can be chosen by decision makers according to the different costs associated with arcs of networks.

9 Conclusions and future research

Picture fuzzy sets possess many advantages over IFSs because their membership functions are inherently ambiguous, and they can be modeled to minimize the consequences of uncertainty in intuitionistic fuzzy logic systems. Dijkstra's algorithm is widely known and studied in various disciplines. The introduction described several research papers that extend its applicability to fuzzy values [15, 26, 31, 38] and intuitionistic fuzzy values [22, 35, 41]. This paper has enlarged the scope of application of the Dijkstra algorithm so that it can solve SPPs for networks with picture fuzzy arc lengths. It can be dutifully employed for both triangular picture fuzzy numbers and TPFNs. The operational laws and expected values of TPFNs and the comparison of two TPFNs by score and accuracy functions have been introduced. This most general type of FNs has been used to capture the unreliable costs of traveling across every arc in networks. TPFNs can accommodate both the optimistic and pessimistic beliefs of the experts. To illustrate and validate the proposed algorithm, we have carried out a step-by-step practical application to a small-sized fictitious network and discussed our results by comparing with existing methods.

Future developments of this study should include the investigation of more effective solution methods for PFSPPs and additional work into the extensions and applications of the proposed method to other domains.

Compliance with ethical standards

Conflict of interest The authors declare no conflict of interest.

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