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Cluster lag synchronization of delayed heterogeneous complex dynamical networks via intermittent pinning control

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Abstract

This paper investigates the problem of cluster lag synchronization in the heterogeneous dynamical networks by using an intermittent pinning control strategy. Previous related works mainly focused on the time-varying delays in the self-dynamics, which was not consistent with the real world. The transmission delay in the communication channels is considered in this paper. We present several criteria to guarantee cluster lag synchronization without assuming the coupling matrix being symmetric and irreducible. A decentralized adaptive intermittent pinning control scheme is employed to reduce the control cost. An effective pinned-cluster selection scheme is adopted to guide what kind of clusters should be pinned preferentially. Two simulations are proposed to verify the correctness of the theoretical results.

Keywords Lag synchronization · Cluster synchronization · Adaptive control · Intermittent pinning control · Heterogeneous complex networks

1 Introduction

In the past few decades, the basic properties of complex dynamical networks have been extensively studied [1–3] and have attracted increasing attention, especially on the issues of synchronization [4–6], the typical and interesting collective behaviors. Hitherto, the cluster synchronization and the related control problems have become a hot topic [7–9]. The main reason lies in that it not only can well explain many natural phenomena, but also has potential applications in the areas of secure communication [10],

image processing [11], mechanical engineering [12], etc. Certainly, one has already obtained many important results. Zhang et al. [13] studied the cluster synchronization problem in asymmetric negative coupling networks and found that the couplings of one-node clusters to multi-node clusters have beneficial effects in the cluster synchronization. The authors of [14] investigated the cluster synchronization problem of complex networks with stochastic perturbations and non-identical nodes. Recently, Li et al. [15] studied an attraction region for the frequency synchronization on a class of symmetrically connected power network systems, which was helpful to the insight into the study of the stability of large power grids.

It is worth pointing out that time delay is of great interest to many researchers in recent years [16–18]. For instance, the stability of uncertain delayed and time-varying delayed neural networks was investigated in [16] and [17], respectively. Huang et al. [18] analyzed the synchronization of delayed chaotic systems with parameter mismatches, etc. However, in real applications, there are not only the time-varying delays in the self-dynamics but also the transmission delay between the signal source and destination. Namely, the signal received at the time $t + \sigma$ is always transmitted from the source at the time t. Therefore, the research about cluster lag synchronization has an

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extremely vital significance in theory and practice. However, hitherto, in previous studies, only a few results have been published.

Besides this, in the history of research on the network control, controllers should be added on all nodes. However, in actual applications, researchers find that it is impossible to implement this requirement since there are a large number of nodes in the super-network. With the gradual development of the network theories, an effective control technology, i.e., pinning control technology, is then put forward and applied with great advantage. The controllers only need to be added on a small percentage of nodes. Chen et al. [19] analyzed the feasibility of the pinning control scheme with the pinned nodes as little as possible and proved that the pinning synchronization can be achieved under some mild conditions. This efficient control technology was also used for solving the cluster synchronization problem in [20, 21]. Liu and Chen [22] studied the exponential synchronization problem with the time-varying delays by the pinning control scheme.

In the real world, it is also too expensive to implement the controllers all the time. Therefore, some optimized control technologies were put forward, for instance, the impulse control technology [23], the sampled-data control technology [24] and the event-triggered and self-triggered technology [5, 25]. All these technologies are discontinuous, which have attracted great interests and have been used in practice widely. For the periodical intermittent control technique, the control interval is assumed to be periodic and each cycle consists of work time and rest time. Liu and Chen [26] studied the problem of cluster synchronization by employing this control technology, and several conditions for synchronization were presented. The authors of [27] discussed the finite-time synchronization between two complex dynamical networks under this periodically intermittent control. Cai et al. [28] investigated the cluster synchronization of the complex networks with the time-varying delays in the self-dynamics under the intermittent pinning control scheme.

To the best of our knowledge, adaptive control has also been used in many systems and applications [29–33]. In recent years, there are a lot of works about this control technology. For instance, an adaptive neural network control was proposed in [31], which succeeds in controlling the desired trajectory robustly to a small neighborhood of zero and guarantees the boundedness of all the closed-loop signals at the same time. In [33], the adaptive control was adopted to handle system uncertainties and disturbances. A common feature of these works is that there are fixed pinned nodes or centralized nodes. Obviously, a more reasonable adaptive approach is decentralized (or distributed), which only relies on local information instead of global information of the whole network. A decentralized

adaptive intermittent pinning control scheme was investigated for a class of discrete-time nonlinear hidden leader–follower multi-agent systems in [34]. In [35], this control scheme was adopted for the global synchronization of complex directed dynamical networks. Although some research was done about the synchronization problem in [34–36], few works concerned the decentralized pinning intermittent control about the cluster lag synchronization problem. In this paper, we will solve this problem.

Enlightened by the above discussions, this paper aims at investigating the cluster lag synchronization of the delayed heterogeneous dynamical networks. Considering the delayed dynamical complex networks that involve both the transmission delay in communication channels and the time-varying delays in self-dynamics simultaneously. Several criteria are presented to guarantee a cluster lag synchronization without assuming the coupling matrix being symmetric and irreducible, which is very loose. In addition, a decentralized adaptive intermittent pinning control scheme is proposed to reduce the control cost. Then, an effective pinned-cluster selection scheme is also introduced to guide what kind of clusters should be pinned preferentially. It is helpful to improve the cluster lag synchronization efficiency.

The rest of this paper is organized as follows. Section 2 establishes an intermittent pinning-controlled complex directed dynamical network model, which contains the mixed delays we mentioned as above, and gives some definitions, remarks and lemmas. Several criteria are presented in Sect. 3 to ensure the cluster lag synchronization, and then, some theorems and corollaries are derived. Besides this, a pinned-cluster selection scheme is introduced in this section. In Sect. 4, the final control scheme is proposed. Two simulations are provided in Sect. 5 to verify the correctness of the theoretical results. Section 6 closes the paper with a conclusion.

2 Preliminaries

Consider a standard directed complex network model with N non-identical dynamical nodes, in which each node is an n-dimensional dynamical system with the time-varying delays in the self-dynamics. For $i = 1, 2, \ldots, N$, the network can be described as

$$\dot{x}_i(t) = f_i(t, x_i(t), x_i(t - \tau_i(t))) + c \sum_{j=1, j \neq i}^{N} b_{ij} \Gamma (x_j(t) - x_i(t)),$$

(1)

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^{\mathrm{T}} \in \mathbb{R}^n$ is an n-dimensional state variable of the node i. A continuous vector-valued function $f_i : [0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is used to

control the evolution of the ith node. The time-varying delays $\tau_i(t)$ may be unknown (constant or time varying) but be known by constraints, i.e., $0 \leq \tau_i(t) \leq \tau_i$, $i = 1, 2, \ldots, N$. The positive constant c is the coupling strength. And the coupling matrix $B = (b_{ij})_{N \times N}$ represents the network topology structure, and if there is a directed link from the jth node to the ith node ($j \neq i$), then $b_{ij} > 0$; otherwise, $b_{ij} = 0$. This means that the network is directed and B is an asymmetric coupling matrix. Furthermore, for $i = 1, 2, \ldots, N$, the definition of the diagonal entry of the matrix B is: $b_{ii} = -\sum_{j=1,j \neq i}^{N} b_{ij}$ and thus $\sum_{j=1}^{N} b_{ij} = 0$. The inner connecting matrix $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n) > 0$ is used to describe the internal individual coupling between nodes. The initial conditions of network (1) are obtained from $x_i(s) = \varphi_i(s) \in C([-\tau, 0], \mathbb{R}^n)$, $i = 1, 2, \ldots, N$, where $C([-\tau, 0], \mathbb{R}^n)$ represents the collection of all n-dimensional continuous functions being defined on the range $[-\tau, 0]$ with $\tau = \max_{1 \leq i \leq N} \tau_i$.

Assume that the collection of nodes in the network (1) can be divided into m clusters with $1 \leq m < N$. Without loss of generality, the kth cluster is indicated as C_k, $k \in \Im = \{1, 2, \ldots, m\}$, and let

$C_1 = \{1, 2, \ldots, r_1\}$,
$C_2 = \{r_1 + 1, r_1 + 2, \ldots, r_1 + r_2\}$,
$\ldots,$
$C_m = \{r_1 + r_2 + \cdots + r_{m-1} + 1, r_1 + r_2$
$\qquad + \cdots + r_{m-1} + 2, \ldots, r_1 + r_2 + \cdots + r_{m-1} + r_m\}$,

where $1 \leq r_k \leq N$ and $\sum_{k=1}^{m} r_k = N$. In many real-world systems, such as metabolic, neural, gene regulation or software networks, the independent nodes in each group can be considered as functional units, see [2, 37, 38]. According to their functions, any pair of nodes in different clusters is essentially different. Assume that the nodes from the same cluster have the same dynamical behavior, while the nodes from different clusters show the distinct behaviors. In this case, the dynamical behavior of each node from the kth cluster can be expressed by

$$\dot{x}_i(t) = \tilde{f}_k(t, x_i(t), x_i(t - \tau_k(t))), \quad i \in C_k. \tag{2}$$

Then, the directed delayed dynamical network (1) can be described as

$$\dot{x}_i(t) = \tilde{f}_k(t, x_i(t), x_i(t - \tau_k(t))) + c \sum_{j=1}^{N} b_{ij} \Gamma x_j(t), \quad i \in C_k. \tag{3}$$

In this paper, we pay attention driving network (3) to the cluster lag synchronization by employing the intermittent pinning control scheme. For this purpose, a definition about the cluster lag synchronization of a dynamical network is given.

We first give some notations that will be used. A positive constant σ is the transmission delay, and let $s_k(t - \sigma)$ denote the lag synchronous state for the cluster C_k. Then, the desired cluster lag synchronous state of network (3) denotes by $s(t) = (s_1(t), s_2(t), \ldots s_m(t))$.

Definition 1 We say the cluster lag synchronization of network (3) is realized, if the nodes of network (3) can be divided into m clusters as mentioned above, such that, for any node $i \in C_k$, $k \in \Im$, one has

$$\lim_{t \to +\infty} \|x_i(t) - s_k(t - \sigma)\| = 0 \quad \text{and}$$

$$\lim_{t \to +\infty} \|s_k(t) - s_j(t)\| \neq 0, \quad j \neq k.$$

According to the network (3), we know that each $s_k(t)$ satisfies

$$\dot{s}_k(t) = \tilde{f}_k(t, s_k(t), s_k(t - \tau_k(t)))$$
$$\quad + c \sum_{p=1}^{m} \sum_{j \in C_p} b_{ij} \Gamma s_p(t), \quad i \in C_k, \quad k \in \Im. \tag{4}$$

Apparently, for each $i \in C_k$, the lag synchronous state $s_k(t)$ in the same cluster should be uniform, see [38]. In order to achieve the cluster lag synchronization of network (3), we put forward the following assumption.

Assumption 1 Assume that the coupling matrix $B = (b_{ij}) \in \mathbb{R}^{N \times N}$ of network (3) can be expressed by

$$B = \begin{pmatrix} B_{11} & B_{12} & \ldots & B_{1m} \\ B_{21} & B_{22} & \ldots & B_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \ldots & B_{mm} \end{pmatrix}.$$

Each block $B_{uv} = (b_{ij}) \in \mathbb{R}^{r_u \times r_v} (u, v \in \Im)$ is a matrix that has the same sum of row, which means there exist some constants $\beta_{uv}, u, v \in \Im$, such that $\sum_{j \in C_v} b_{ij} = \beta_{uv}$, $i \in C_u$.

Remark 1 Based on Assumption 1, the cluster lag synchronous manifold diagram of the network (3) satisfies

$$M = \left\{ \begin{array}{l} (x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t), \ldots, x_N^{\mathrm{T}}(t))^{\mathrm{T}} \in \mathbb{R}^{nN} : \\ x_1(t) = x_2(t) = \cdots = x_{r_1}(t), \\ x_{r_1+1}(t) = x_{r_1+2}(t) = \cdots = x_{r_1+r_2}(t), \\ \cdots \\ x_{r_1+r_2+\cdots+r_{m-1}+1}(t) = x_{r_1+r_2+\cdots+r_{m-1}+2}(t) = \cdots = x_N(t) \end{array} \right\}.$$

It is an invariant manifold of network (3) and is a precondition for the cluster synchronization [38]. Additionally, the information communication in the same cluster is represented by each diagonal block in the matrix B, and the information communication among different

clusters is expressed by each non-diagonal block, respectively.

Remark 2 In the traditional literature about the cluster lag synchronization of the complex dynamical networks, each block unit in the coupling matrix of the network is supposed to be a zero-row-sum matrix. This condition is introduced to ensure the cluster synchronization of the network, see [39, 40]. This requests that the average of the influence from the other cluster for each node is equal to zero. In fact, it is relatively conservative. In this paper, we assume that each block of the coupling matrix is an equal-row-sum matrix; furthermore, the sum of each block can be different. Therefore, the hypothesis is less conservative than that in [39, 40].

Under Assumption 1, we can reduce formula (4) to

$$\dot{s}_k(t) = \tilde{f}_k(t, s_k(t), s_k(t - \tau_k(t))) + c\sum_{p=1}^m \beta_{kp}\Gamma s_p(t), \quad (5)$$
$$k \in \Im.$$

Formula (5) is referred to as the cluster lag synchronous form of network (3). One can clearly see that the values of the cluster lag synchronous states are sets of un-decoupled trajectories instead of decoupled ones. This is different mostly from previous works [37, 39, 40], where the cluster lag synchronous states were selected as the particular solutions of the decoupled node systems when they worked on the problems of synchronization, i.e.,

$$\dot{s}_k(t) = \tilde{f}_k(t, s_k(t), s_k(t - \tau_k(t))).$$

To achieve the cluster lag synchronization of the delayed dynamical network (3), an intermittent pinning control scheme will be used. As is known, the nodes that in the same cluster usually have the same properties. Thus, while studying the cluster lag synchronization, each cluster can naturally be seen as a whole [37]. According to this, we put each cluster of the network (3) as a whole and control part of them intermittently to achieve the cluster lag synchronization. Let the first $l(1 \le l < m)$ clusters to be pinned, without loss of generality, and then, we have the delayed dynamical network under pinning control as

$$\begin{cases} \dot{x}_i(t) = \tilde{f}_k(t, x_i(t), x_i(t - \tau_k(t))) + c\sum_{p=1}^m \sum_{j \in C_p} b_{ij}\Gamma x_j(t) \\ \quad + u_i(t), \quad i \in C_k, \ 1 \le k \le l, \\ \dot{x}_i(t) = \tilde{f}_k(t, x_i(t), x_i(t - \tau_k(t))) + c\sum_{p=1}^m \sum_{j \in C_p} b_{ij}\Gamma x_j(t), \\ \quad i \in C_k, \ 1 + l \le k \le m. \end{cases} \quad (6)$$

The intermittent controllers $u_i(t) \in \mathbb{R}^n$ are defined as

$$u_i(t) = cd_k(t)\Gamma(s_k(t - \sigma) - x_i(t)), \quad i \in C_k, 1 \le k \le l, \quad (7)$$

where the intermittent feedback control gain $d_k(t)$ of the cluster C_k is described by

$$d_k(t) = \begin{cases} d_k, & \omega T \le t < \omega T + \delta, & i \in C_k, \ 1 \le k \le l, \\ 0, & \omega T + \delta \le t < (\omega + 1)T, & i \in C_k, \ 1 \le k \le l. \end{cases} \quad (8)$$

where $d_k > 0$ is a constant, the control period $T > 0$, the control width $\delta > 0$, and $T - \delta$ is the non-control width, where $\omega = 0, 1, 2, \ldots$. It is found that the control time of controllers (7) is cyclical, and each period T is composed by the work time δ and the rest time $T - \delta$. As a result, the control behavior is intermittent rather than continuous and it can reduce the cost of control.

For $i \in C_k, \ k \in \Im$, define the synchronous errors as $e_{ik}(t) = x_i(t) - s_k(t - \sigma)$. Let $\theta = \delta/T$ be the ratio of the control width δ to the control period T, which is also referred to as the control rate. From the control law (8), the error dynamical system is obtained as

$$\begin{cases} \dot{e}_{ik}(t) = \tilde{f}_k^{\tau_k,\sigma}(t, x_i, s_k) + c\sum_{p=1}^m \sum_{j \in C_p} b_{ij}\Gamma e_{jp}(t) - cd_k\Gamma e_{ik}(t), & \omega T \le t < (\omega+\theta)T, & i \in C_k, \ 1 \le k \le l, \\ \dot{e}_{ik}(t) = \tilde{f}_k^{\tau_k,\sigma}(t, x_i, s_k) + c\sum_{p=1}^m \sum_{j \in C_p} b_{ij}\Gamma e_{jp}(t), & (\omega+\theta)T \le t < (\omega + 1)T, & i \in C_k, \ 1 \le k \le m, \\ \dot{e}_{ik}(t) = \tilde{f}_k^{\tau_k,\sigma}(t, x_i, s_k) + c\sum_{p=1}^m \sum_{j \in C_p} b_{ij}\Gamma e_{jp}(t), & \omega T \le t < (\omega+\theta)T, & i \in C_k, \ l+1 \le k \le m, \end{cases} \quad (9)$$

where $\tilde{f}_k^{\tau_k,\sigma}(t, x_i, s_k) = \tilde{f}_k(t, x_i(t), x_i(t - \tau_k(t))) - \tilde{f}_k((t - \sigma), s_k(t - \sigma), s_k(t - \tau_k(t) - \sigma))$. For the error dynamical system (9), if the zero solution is globally exponentially stable, then obviously, for the pinning-controlled delayed dynamical network (6), the global cluster lag synchronization is achievable.

According to the above, the following assumption is needed.

Assumption 2 For the vector-valued function $\tilde{f}_k(t, x(t), x(t - \tau_k(t)))$, supposing that there exist some constants L_k^0 and $L_k^\tau \ge 0$, such that, for any $x(t), y(t) \in \mathbb{R}^n$ and $k \in \Im$,

$(x(t) - y(t))^{\mathrm{T}} (\tilde{f}_k(t, x(t), x(t - \tau_k(t))) - \tilde{f}_k(t, y(t),$

$\quad y(t - \tau_k(t))))$

$\quad \leq L_k^0 (x(t) - y(t))^{\mathrm{T}} \Gamma (x(t) - y(t))$

$\quad\quad + L_k^\tau (x(t - \tau_k(t)) - y(t - \tau_k(t)))^{\mathrm{T}} \Gamma (x(t - \tau_k(t))$

$\quad\quad - y(t - \tau_k(t))).$

Remark 3 Some requirements about the dynamical change in the nodes in the cluster C_k are given in Assumption 2. If the function $\tilde{f}_k(t, x_i(t), x_i(t - \tau_k(t)))$ of each node in the cluster C_k satisfies the Lipschitz condition [41, 42], then

$$\left\| \left(\tilde{f}_k(t, x(t), x(t - \tau_k(t))) - \tilde{f}_k(t, y(t), y(t - \tau_k(t))) \right) \right\|$$
$$\leq \Lambda_k^0 \| x(t) - y(t) \| + \Lambda_k^\tau \| x(t - \tau_k(t)) - y(t - \tau_k(t)) \|.$$

We can choose $L_k^0 = \frac{2\Lambda_k^0 + \Lambda_k^\tau}{2\lambda_{\min}(\Gamma)}$ and $L_k^\tau = \frac{\Lambda_k^\tau}{2\lambda_{\min}(\Gamma)}$ to satisfy Assumption 1. In addition, almost all the famous chaotic systems with or without delay also satisfy Assumption 1. Such as Lorenz system, Chen system, Rössler system, Chua's circuit, as well as delayed Ikeda equations, delayed cellular neural networks, and so on, see [41, 42].

Lemma 1 (Gershgorin disk theorem [43]) *Let* $A = (a_{ij})_{n \times n}$ *be a complex matrix. For* $i = 1, 2, \ldots, n$, *define* $R_i'(A) = \sum_{j=1, j \neq i} |a_{ij}|$ *as the absolute row sums of A besides the diagonal elements, and consider the n Gershgorin disks as*

$$\{ z \in \mathbb{C} \mid |z - a_{ii}| \leq R_i'(A) \}.$$

All the eigenvalues of A are included in the Gershgorin disks set as

$$G(A) = \bigcup_{i=1}^n \{ z \in \mathbb{C} \mid |z - a_{ii}| \leq R_i'(A) \}.$$

Lemma 2 (Schur complement [41]) *The linear matrix inequality*

$$\begin{pmatrix} Q(x) & U(x) \\ U(x)^{\mathrm{T}} & R(x) \end{pmatrix} < 0,$$

where $Q(x) = Q(x)^{\mathrm{T}}$, $R(x) = R(x)^{\mathrm{T}}$ *and* $U(x)$ *is a matrix with suitable dimensions, can be represented by one of the following situations.*

(1) $Q(x) < 0$, $\quad R(x) - U(x)^{\mathrm{T}} Q(x)^{-1} U(x) < 0$;
(2) $R(x) < 0$, $\quad Q(x) - U(x) R(x)^{-1} U(x)^{\mathrm{T}} < 0.$

3 Intermittent pinning criteria for cluster lag synchronization

We explore the stability condition of the delayed dynamical network (6) under pinning control in this section and introduce some criteria, which can be used to ensure that the globally exponential stability with an intermittent

pinning control scheme is realizable. With Lemma 2, we will show how to set the appropriate control parameters $d_k (k = 1, 2, \ldots, l)$, δ and T in the delayed dynamical network (6) such that the cluster lag synchronization is reachable. For convenience, we will introduce the following notations.

Let $r_0 = 0$, and for each $i \in C_k$, $k \in \Im$, let $a_i^k = \sum_{j \in C_k} (b_{ji} + b_{ij})$, $\quad \xi_i^k = \sum_{j=1}^N b_{ji}$, $\quad A_k = \mathrm{diag}$ $\left(a_{r_0 + r_1 + \cdots + r_{k-1} + 1}^k, \ldots, a_{r_0 + r_1 + \cdots + r_{k-1} + r_k}^k \right)$ and $\quad \Psi_k = \mathrm{diag}$ $\left(\xi_{r_0 + r_1 + \cdots + r_{k-1} + 1}^k, \ldots, \xi_{r_0 + r_1 + \cdots + r_{k-1} + r_k}^k \right).$

Define two block diagonal matrices $Z \in \mathbb{R}^{N \times N}$ and $D \in \mathbb{R}^{N \times N}$ as

$$Z = \begin{pmatrix} Z_{11} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & Z_{22} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & Z_{ll} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & Z_{l+1,l+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & Z_{mm} \end{pmatrix} \text{ and}$$

$$D = \begin{pmatrix} d_1 I_{r_1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & d_2 I_{r_2} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & d_l I_{r_l} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $Z_{kk} = (z_{ij}) \in \mathbb{R}^{r_k \times r_k} = B_{kk}^s + \frac{1}{2} \Psi_k + \left(\frac{L_k^0}{c} \right) I_{r_k} - \frac{1}{2} A_k$, $B_{kk}^s = \frac{1}{2} \left(B_{kk} + B_{kk}^{\mathrm{T}} \right)$, $k \in \Im$ and I_{r_j} represents the r_j-dimensional identity matrix, $j = 1, 2, \ldots, l$.

Theorem 1 *Suppose that Assumptions 1 and 2 hold. If there are two positive constants a_1 and a_2 such that*

(i) $a_1 I_N + cZ - cD < 0$,
(ii) $cZ - a_2 I_N \leq 0$,
(iii) $q - p_1 < 0$,
(iv) $\varpi = \lambda - (p_1 + p_2)(1 - \theta) > 0$,

where $p_1 = 2a_1 \lambda_{\min}(\Gamma)$, $\quad p_2 = 2a_2 \lambda_{\max}(\Gamma)$, $q = 2 \left(\max_{1 \leq k \leq m} L_k^\tau \right) \lambda_{\max}(\Gamma)$, *and the unique positive solution of the equation* $\lambda - p_1 + q e^{\lambda \tau} = 0$ *is* $\lambda > 0$, *then the cluster lag synchronization of the delayed dynamical network (6) can be realized.*

Proof See "Appendix 1." $\qquad\qquad\qquad\square$

Remark 4 For a directed complex network with non-identical time-varying delayed dynamical nodes,

Theorem 1 presents a novel cluster lag synchronous condition by combining the intermittent control technique with pinning scheme. As far as we know, there is a little work concerning the cluster lag synchronization of the heterogeneous complex dynamical networks with intermittent pinning control. It should be emphasized that in this paper, we discuss the cluster lag synchronization with both the time-varying delays and transmission delay and treat each cluster as a whole. Moreover, the desired cluster lag synchronous states are chosen as a collection of uncoupled trajectories instead of the particular solutions of the decoupled node systems. Most of the existing researches on the cluster lag synchronization problems are different with the precondition that we set [37, 39, 40]. In addition, from Theorem 1, we can see that the cluster lag synchronous specification relies mostly on the control rate θ rather than the control width δ or the control period T. With regard to the actual problems, the cluster lag synchronization can be realized by randomly selecting the control period T.

Theorem 1 provides a general control standard to ensure that the cluster lag synchronization of the delayed dynamical network (3) can be realized. However, how to choose an appropriate control gain matrix D and a control rate θ is still not clear. Using Lemma 2 and the matrix decomposition method [41, 44], some simplified versions of Theorem 1 are constructed (see below). These simplified versions are more convenient when estimating the control rate and the control gains.

Let $Q = a_1 I_N + cZ$, $a_1 I_N + cZ - cD = Q - cD = \begin{pmatrix} G - c\tilde{D} & 0 \\ 0^{\mathrm{T}} & \tilde{Q} \end{pmatrix}$, $d = \min_{1 \le k \le l} d_k$, where G and $\tilde{D}$ are two block diagonal matrices. The definitions of E and $\tilde{D}$ are

$$G = \begin{pmatrix} cZ_{11} + a_1 I_{r_1} & 0 & \cdots & 0 \\ 0 & cZ_{22} + a_1 I_{r_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & cZ_{ll} + a_1 I_{r_l} \end{pmatrix}$$

and

$$\tilde{D} = \begin{pmatrix} d_1 I_{r_1} & 0 & \cdots & 0 \\ 0 & d_2 I_{r_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_l I_{r_l} \end{pmatrix}.$$

Obviously, the real matrix Q is symmetric, and by removing its first $(r_1 + r_2 + \cdots + r_l)$ row–column pairs, the matrix $\tilde{Q}$ is obtained. In virtue of Lemma 2, we can verify that $\tilde{Q} < 0$ and $d > \frac{a_1}{c} + \max_{1 \le k \le l}(\lambda_{\max}(Z_{kk}))$ is equivalent to $Q - cD < 0$. If d can be large enough (we can

set $d_i > \frac{a_1}{c} + \max_{1 \le k \le l}(\lambda_{\max}(Z_{kk}))$, $i = 1, 2, \ldots, l$, then $Q - cD < 0$ is equivalent to $\tilde{Q} < 0$.

Let $\Theta_{kk} = \left(\vartheta_{ij}^k \right) \in \mathbb{R}^{r_k \times r_k} = B_{kk}^s + \frac{1}{2} \Psi_k - \frac{1}{2} A_k = Z_{kk} - \frac{L_k^0}{c} I_{r_k}$ and $\Upsilon_k = \lambda_{\max}(\Theta_{kk}) + \frac{L_k^0}{c}$, $k \in \Im$.

Noticing that

$$\tilde{Q} = \begin{pmatrix} cZ_{l+1,l+1} + a_1 I_{r_{l+1}} & 0 & \cdots & 0 \\ 0 & cZ_{l+2,l+2} + a_1 I_{r_{l+2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & cZ_{m,m} + a_1 I_{r_m} \end{pmatrix}$$

and $\lambda_{\max}(\Theta_{kk}) \le \Upsilon_k$, $k \in \Im$, the following corollary can be concluded from Theorem 1.

Corollary 1 *Suppose that Assumptions 1 and 2 hold. If the intermittent feedback control gains $d_i(1 \le i \le l)$ are large enough, and there exists a positive constant $a_1 > \frac{q}{2\lambda_{\min}(\Gamma)}$ such that*

(i') $\Upsilon_k < -\frac{a_1}{c}$, $l + 1 \le k \le m$,

(ii') $1 - \frac{\lambda}{p_1 + p_2} < \theta < 1$,

then the cluster lag synchronization of the delayed dynamical network (6) is realizable.

An explanation of the notations in Corollary 1 is given. The definitions of λ, p_1 and q are the same as Theorem 1, and $p_2 = 2c\lambda_{\max}(Z)\lambda_{\max}(\Gamma)$. We set $d_i > \frac{a_1}{c} + \max_{1 \le k \le l}(\lambda_{\max}(Z_{kk}))$.

Furthermore, assume that a_1 is given as $a_1^0 > \frac{q}{2\lambda_{\min}(\Gamma)}$. We substitute $a_1 = a_1^0$ into the equation $\lambda - p_1 + qe^{\lambda \tau} = 0$, where $\lambda = \phi(a_1^0)$. Let $d_i > \frac{a_1^0}{c} + \max_{1 \le k \le l}(\lambda_{\max}(Z_{kk}))$ and $p_1 = 2a_1^0 \lambda_{\min}(\Gamma)$. Besides this, the variables p_2 and q are the same with Corollary 1. Then, we can put the above corollary in the following way.

Corollary 2 *Suppose that Assumptions 1 and 2 hold. If the intermittent feedback control gains $d_i(1 \le i \le l)$ are large enough, and there exists a positive constant $a_1^0 > \frac{q}{2\lambda_{\min}(\Gamma)}$ such that*

(i'') $\Upsilon_k < -\frac{a_1^0}{c}$, $l + 1 \le k \le m$,

(ii'') $1 - \frac{\phi(a_1^0)}{p_1 + p_2} < \theta < 1$,

then the cluster lag synchronization of the delayed dynamical network (6) is realizable.

Remark 5 Corollary 2 allows us to simply determine the control rate θ and the feedback control gains $d_i(1 \le i \le l)$, thus, the intermittent controller (7)–(8) can be conveniently designed.

Remark 6 An undirected delayed dynamical network can be regarded as a special case of the directed delayed dynamical network with symmetrically coupling matrix, i.e., $B = B^{\mathrm{T}}$. Therefore, the theoretical results that we obtained in this paper establish the appropriate conditions for the cluster lag synchronization of the undirected complex dynamical delayed networks under the intermittent pinning control.

In the above, we regard the each cluster as a whole and put forward some low-dimensional specifications to ensure that we can implement the cluster lag synchronization by intermittently controlling a part of clusters of the delayed dynamical network (3). But, how to choose these pinned clusters? We make use of an effective scheme, which was originally proposed in [28]. To facilitate the applications, we give the selection scheme as follows. At first, the following notations are given.

Intra-InDeg(i, k) denotes the sum weights of the directed edges $e(i, j)$ with $j \in C_k$ into $i \in C_k$, i.e., the intra-in-degree of the ith node of the cluster C_k [38]. Extra-InDeg(i, k) denotes the sum weights of the directed edges $e(i, j)$ with $j \notin C_k$ into $i \in C_k$, i.e., the extra-in-degree of the ith node in the cluster C_k [38]. Similarly, Intra - OutDeg(i, k) denotes the sum weights of the directed edges $e(j, i)$ with $j \in C_k$ from $i \in C_k$, i.e., the intra-out-degree of the ith node in the cluster C_k [38]. Extra-OutDeg(i, k) denotes the sum weights of the directed edges $e(j, i)$ with $j \notin C_k$ from $i \in C_k$, i.e., the extra-out-degree of the ith node in the cluster C_k [38]. In addition, InDeg(i, k) and OutDeg(i, k), respectively, denote the in-degree and the out-degree of the ith node of the cluster C_k [38]. According to the definition about the coupling matrix B of network (1), for any $i \in C_k$, one has

Intra-InDeg$(i, k) = \sum_{j \in C_k, j \neq i} b_{ij}$ and Extra-InDeg $(i, k) = \sum_{p=1, p \neq k}^{m} \sum_{j \in C_p} b_{ij}$,
Intra-OutDeg$(i, k) = \sum_{j \in C_k, j \neq i} b_{ji}$ and Extra-OutDeg $(i, k) = \sum_{p=1, p \neq k}^{m} \sum_{j \in C_p} b_{ji}$,
InDeg$(i, k) = $ Intra-InDeg$(i, k) + $ Extra-InDeg$(i, k) = \sum_{j=1, j \neq i}^{N} b_{ij} = -b_{ii}$,
OutDeg$(i, k) = $ Intra-OutDeg$(i, k) + $ Extra-OutDeg$(i, k) = \sum_{j=1, j \neq i}^{N} b_{ji}$.

Furthermore, Intra-InDeg(k) stands for the sum of the intra-in-degree for all nodes of the cluster C_k, i.e., the intra-in-degree of the cluster C_k in network (1) [38]. Extra-InDeg(k) denotes the sum of the extra-in-degree for all nodes of the cluster C_k, i.e., the extra-in-degree of the cluster C_k in network (1) [38]. Similarly, define Intra-OutDeg(k) (Extra-OutDeg(k)) as the intra(extra)-out-degree of the cluster C_k in network (1). This means the sum of the intra(extra)-out-degree for all nodes of the cluster C_k

[38]. Additionally, define InDeg(k) and OutDeg(k) as the in-degree and the out-degree of the cluster C_k in network (1), respectively [38]. Then, we have

$$\mathrm{InDeg}(k) = \sum_{i \in C_k} \mathrm{InDeg}(i, k)$$
$$= \sum_{i \in C_k} (\mathrm{Intra\text{-}InDeg}(i, k) + \mathrm{Extra\text{-}InDeg}(i, k))$$
$$= \mathrm{Intra\text{-}InDeg}(k) + \mathrm{Extra\text{-}InDeg}(k),$$
$$\mathrm{OutDeg}(k) = \sum_{i \in C_k} \mathrm{OutDeg}(i, k)$$
$$= \sum_{i \in C_k} (\mathrm{Intra\text{-}OutDeg}(i, k) + \mathrm{Extra\text{-}OutDeg}(i, k))$$
$$= \mathrm{Intra\text{-}OutDeg}(k) + \mathrm{Extra\text{-}OutDeg}(k)$$

and

$$\mathrm{OutDeg}(k) - \mathrm{InDeg}(k) = \sum_{i \in C_k} (\mathrm{OutDeg}(i, k) - \mathrm{InDeg}(i, k))$$
$$= \sum_{i \in C_k} \left(\sum_{j=1, j \neq i}^{N} b_{ji} - \sum_{j=1, j \neq i}^{N} b_{ij} \right) = \sum_{i \in C_k} \sum_{j=1}^{N} b_{ji}.$$

Obviously, the information about the intra-cluster communication is represented by the intra-in-degree and the intra-out-degree of the cluster, and the information about the inter-clusters communication is represented by the extra-in-degree and the extra-out-degree of the cluster.

In order to meet the control condition (i″), at least we need to choose some pinned nodes to make $\Upsilon_k < 0$, $l + 1 \leq k \leq m$. To achieve this purpose, based on the degree information of each cluster defined as above, we give the following theorem.

Theorem 2 *For any $k \in \Im$,*

$$\lambda_{\max}(\Theta_{kk}) \leq \frac{\max_{i \in C_k}(\mathrm{OutDeg}(i, k) - \mathrm{InDeg}(i, k))}{2}. \quad (10)$$

In particular, for some k, if $\max_{i \in C_k}(\mathrm{OutDeg}(i, k) - \mathrm{InDeg}(i, k)) < 0$, then $\lambda_{\max}(\Theta_{kk}) < 0$.

Proof Noting $\Theta_{kk} = \left(\vartheta_{ij}^k \right) \in \mathbb{R}^{r_k \times r_k} = B_{kk}^s + \frac{1}{2}\Psi_k - \frac{1}{2}A_k$, then for any $i \in C_k$,

$$\vartheta_{ii}^k + \sum_{j \in C_k, j \neq i} \left| \vartheta_{ij}^k \right| = \left(\sum_{j \in C_k} \frac{b_{ij} + b_{ji}}{2} \right) + \frac{\xi_i^k}{2} - \frac{a_i^k}{2}$$
$$= \frac{1}{2} \left(\sum_{j=1, j \neq i}^{N} b_{ji} \right) + \frac{1}{2} b_{ii} \quad (11)$$
$$= \frac{\mathrm{OutDeg}(i, k) - \mathrm{InDeg}(i, k)}{2}.$$

Let λ^k be an eigenvalue of Θ_{kk}. According to Lemma 1, we get

$$\lambda^k \leq \bigcup_{i \in C_k} \left\{ z \in \mathbb{R} \,\Big|\, |z - \vartheta_{ii}^k| \leq \sum_{j \in C_k, j \neq i} \left| \vartheta_{ij}^k \right| \right\}. \tag{12}$$

With (12), we have

$$\lambda^k \leq \max_{i \in C_k} (\mathrm{OutDeg}(i,k) - \mathrm{InDeg}(i,k))/2, \tag{13}$$

and if $\max_{i \in C_k}(\mathrm{OutDeg}(i,k) - \mathrm{InDeg}(i,k)) < 0$, then $\lambda^k < 0$. The proof is thus completed. □

Remark 7 Theorem 2 provides some useful guidance about how to set up the pinned clusters to achieve the cluster lag synchronization. For $k \in \Im$, define

$$\mathrm{DiffDeg}(k) = \mathrm{OutDeg}(k) - \mathrm{InDeg}(k)$$

as the difference between the out-degree and the in-degree of the cluster C_k. Noticing that for any $k \in \Im$, one has $\max_{i \in C_k}(\mathrm{OutDeg}(i,k) - \mathrm{InDeg}(i,k)) \geq \mathrm{DiffDeg}(k)/r_k$.
Thus, if the out-degrees of some unpinned clusters are bigger than their in-degrees, then according to Theorem 2, for those clusters which are not be pinned, we have $\lambda_{\max}(\Theta_{kk}) > 0$. Namely, condition (i″) may not be met. This means that, to satisfy condition (i″), these clusters whose out-degrees are bigger than their in-degrees should be chosen as pinned candidates preferentially. Based on Theorem 2 and condition (i″), we adopt the following special pinned scheme to choose the pinned clusters for the delayed dynamical networks:

1. Define a cluster information vector

    $$\mathrm{CluInf}(k) = \frac{L_k^0}{c} + \frac{1}{2}\max_{i \in C_k}(\mathrm{OutDeg}(i,k) - \mathrm{InDeg}(i,k)),$$
    $$k \in \Im.$$

2. Select the cluster which has zero-in-degrees as the clusters be pinned firstly.
3. According to the value of $\mathrm{CluInf}(k)$, rearrange the rest of the clusters in descending order. For those clusters having the same $\mathrm{CluInf}(k)$, rearrange them in descending order according to their out-degrees. According to the sequence of the cluster information to increase the number of clusters to be pinned until the restrain condition (i″) holds. In addition, from condition (i″), we know that the number of clusters being selected and pinned to achieve the cluster lag synchronization of the delayed dynamical network (6) is at least equal to l_0 which is bounded by

$$\Upsilon_{l_0 - 1} \geq -\frac{a_1^0}{c} \quad \text{and} \quad \Upsilon_{l_0} < -\frac{a_1^0}{c}.$$

4 Adaptive intermittent pinning control scheme

In order to realize the cluster lag synchronization of the dynamical delayed network (6), we can set the intermittent feedback control gains according to the following condition.

$$d_i > \frac{a_1^0}{c} + \max_{1 \leq k \leq l}(\lambda_{\max}(Z_{kk})), \quad i = 1, 2, \ldots, l. \tag{14}$$

Unfortunately, the theoretical values, i.e., the intermittent feedback control gains given by condition (14), are usually bigger than the actual need. In practice, however, it cannot be realized if the control gains are too large [19]. To make the control gains as small as possible for practical applications, a better way is to adjust the gains by an adaptive intermittent feedback control approach. In this section, by introducing a local adaptive strategy for the intermittent feedback control gains, a decentralized adaptive intermittent pinning control scheme is proposed in order to realize the cluster lag synchronization of the delayed dynamical network (3).

Theorem 3 *Suppose that Assumptions 1 and 2 hold, and there are l clusters being pinned such that condition (i″) satisfies. With the adaptive intermittent pinning control (16), if*

$$1 - \frac{\phi(a_1^0)}{p_1 + p_2} < \theta < 1, \tag{15}$$

then the cluster lag synchronization of the network (6) with transmission delay σ can be realized.

Proof See "Appendix 2." □

The variables a_1^0, p_1 and p_2 are the same as that in Corollary 2, and Eq. (16) is given in the following:

$$u_i(t) = \begin{cases} -cd_i(t)\Gamma e_{ik}(t), & \omega T \leq t < (\omega + \theta)T, \ i \in C_k, \ 1 \leq k \leq l, \\ 0, & \text{otherwise,} \end{cases} \tag{16}$$

where $e_{ik}(t) = x_i(t) - s_k(t - \sigma)$, with the adaptive rules

$$\dot{d}_k(t) = h_k \sum_{i \in C_k} e_{ik}(t)^{\mathrm{T}} \Gamma e_{ik}(t), \quad \omega T \le t < (\omega + \theta)T, \quad (17)$$

where for $k = 1, 2, \ldots, l$, the initial conditions $d_k(0) \ge 0$ and constants $h_k > 0$. Assume that $d_k(t_\omega^-) = \lim_{t \to t_\omega^-} d_k(t)$ exists and $d_k((\omega + 1)T) = d_k(t_\omega^-)$, where $t_\omega = (\omega + \theta)T$.

When the transmission delay $\sigma = 0$, the values of a_1^0, p_1 and p_2 keep the same with Theorem 3. Theorem 3 reduces to the following result.

Theorem 4 *Suppose that Assumptions 1 and 2 hold, and there are l clusters being pinned such that condition* (i″) *satisfies. With the adaptive intermittent pinning control* (19), *if*

$$1 - \frac{\phi(a_1^0)}{p_1 + p_2} < \theta < 1, \quad (18)$$

then the cluster synchronization of the network (6) *can be realized.*

Equation (19) is given in the following:

$$u_i(t) = \begin{cases} -cd_k(t)\Gamma(x_i(t) - s_k(t)), & \omega T \le t < (\omega + \theta)T, i \in C_k, 1 \le k \le l, \\ 0, & \text{otherwise}, \end{cases} \quad (19)$$

with the adaptive rules

$$\dot{d}_k(t) = h_k \sum_{i \in C_k} (x_i(t) - s_k(t))^{\mathrm{T}} \Gamma(x_i(t) - s_k(t)), \quad \omega T \le t < (\omega + \theta)T, \quad (20)$$

where for $k = 1, 2, \ldots, l$, the initial conditions $d_k(0) \ge 0$ and constants $h_k > 0$. Assume that $d_k(t_\omega^-) = \lim_{t \to t_\omega^-} d_k(t)$ exists and $d_k((\omega + 1)T) = d_k(t_\omega^-)$, where $t_\omega = (\omega + \theta)T$.

Remark 8 In [28], the cluster synchronization under the intermittent pinning control has been studied; however, the authors ignored the transmission delay in the process of communication. In this paper, we analyze a more general network model with non-identical dynamical nodes containing both time-varying delays and transmission delay. And an intermittent adaptive control strategy is discussed. The adaptive update rules for the controlled clusters only determined by the state information of the controlled cluster itself and the information of the desired cluster synchronous state of the cluster. Namely, the intermittent adaptive control strategy only depends on some local information rather than the global information of the entire network. Therefore, the results obtained in the paper are more practically applicable.

5 Numerical simulations

In this section, we present two numerical examples to verify the validity of the theoretical results in the previous section.

5.1 Example A: cluster lag synchronization of a directed network with 15 nodes

We design a directed network consisting of 15 non-identical delayed dynamical nodes, and suppose it can be divided into three clusters: $C_1 = \{1, 2, 3, 4, 5\}$, $C_2 = \{6, 7, 8, 9, 10\}$, $C_3 = \{11, 12, 13, 14, 15\}$. The network state equation is

$$\begin{cases} \dot{x}_i(t) = -1.5x_i(t) + 0.75\tanh(x_i(t - \tau_1(t))) \\ \quad + c\sum_{p=1}^{3}\sum_{j \in C_p} b_{ij}\Gamma x_j(t), \quad i \in C_1, \\ \dot{x}_i(t) = -2.5x_i(t) + \tanh(x_i(t - \tau_2(t))) \\ \quad + c\sum_{p=1}^{3}\sum_{j \in C_p} b_{ij}\Gamma x_j(t), \quad i \in C_2, \\ \dot{x}_i(t) = 0.375x_i(t) + \tanh(x_i(t - \tau_3(t))) \\ \quad + c\sum_{p=1}^{3}\sum_{j \in C_p} b_{ij}\Gamma x_j(t), \quad i \in C_3, \end{cases} \quad (21)$$

where $x_i(t) \in \mathbb{R}^3$, $\Gamma = I_3$, $\tau_1(t) = \tau_2(t) = \tau_3(t) = 0.02\frac{e^t}{1+e^t}$, the transmission delay $\sigma = 0.5$, and the coupling strength $c = 8$. It can be verified that $L_1^0 = -1.5$, $L_1^\tau = 0.75$, $L_2^0 = -2.5$, $L_2^\tau = 1$, $L_3^0 = 0.375$, $L_3^\tau = 1$, and thus, Assumption 2 is satisfied. We assume that the characteristics of the coupling matrix $B = (b_{ij})_{15 \times 15} = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}$ are as follows:

$$B_{11} = \begin{pmatrix} -16 & 1 & 0 & 1 & 1 \\ 0 & -16 & 1 & 1 & 1 \\ 1 & 0 & -16 & 1 & 1 \\ 1 & 1 & 1 & -16 & 0 \\ 1 & 1 & 1 & 0 & -16 \end{pmatrix},$$

$$B_{12} = \begin{pmatrix} 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix}, \quad B_{13} = \begin{pmatrix} 0 & 0 & 5 & 0 & 2 \\ 5 & 0 & 0 & 2 & 0 \\ 0 & 5 & 0 & 0 & 2 \\ 5 & 0 & 2 & 0 & 0 \\ 0 & 5 & 0 & 2 & 0 \end{pmatrix},$$

$$B_{21} = \begin{pmatrix} 0 & 0 & 5 & 0 & 2 \\ 5 & 0 & 0 & 2 & 0 \\ 0 & 5 & 0 & 0 & 2 \\ 5 & 0 & 2 & 0 & 0 \\ 0 & 5 & 0 & 2 & 0 \end{pmatrix},$$

$$B_{22} = \begin{pmatrix} -16 & 1 & 0 & 1 & 1 \\ 0 & -16 & 1 & 1 & 1 \\ 1 & 0 & -16 & 1 & 1 \\ 1 & 1 & 1 & -16 & 0 \\ 1 & 1 & 1 & 0 & -16 \end{pmatrix},$$

$$B_{23} = \begin{pmatrix} 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$

$$B_{31} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_{32} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$B_{33} = \begin{pmatrix} -5 & 0 & 1 & 1 & 1 \\ 1 & -5 & 0 & 1 & 1 \\ 1 & 1 & -5 & 0 & 1 \\ 1 & 1 & 1 & -5 & 0 \\ 1 & 1 & 1 & 1 & -6 \end{pmatrix}.$$

Obviously, $\beta_{11} = -13$, $\beta_{12} = 6$, $\beta_{13} = 7$, $\beta_{21} = 7$, $\beta_{22} = -13$, $\beta_{23} = 6$, $\beta_{31} = 1$, $\beta_{32} = 1$, $\beta_{33} = -2$. Therefore, the cluster lag synchronous equations of network (21) can be described as

$$\begin{cases} \dot{s}_1(t) = -105.5s_1(t) + 48s_2(t) + 56s_3(t) + 0.75\tanh(s_1(t - \tau_1(t))), \\ \dot{s}_2(t) = 56s_1(t) - 106.5s_2(t) + 48s_3(t) + \tanh(s_2(t - \tau_2(t))), \\ \dot{s}_3(t) = 8s_1(t) + 8s_2(t) - 15.625s_3(t) + \tanh(s_3(t - \tau_3(t))). \end{cases}$$
(22)

Through calculation, the network (22) has three equilibrium points described by

$$S_1^* = (-5.7456, -5.7007, -5.9245), \quad S_2^* = (0, 0, 0),$$
$$S_3^* = (5.7456, 5.7007, 5.9245).$$

We choose S_3^* as the desired cluster lag synchronous state. By calculation, we get $q = 2$ and $\lambda_{\max}(Z) = 6.0366$. Let $a_1^0 = 12$, from conditions (i″) and (ii″), we have

$$\Upsilon_k < -1.5, \quad l + 1 \le k \le 3, \quad 0.8215 < \theta < 1. \tag{23}$$

Since $\text{CluInf}(1) = -1.1875$, $\text{CluInf}(2) = -3.3125$ and $\text{CluInf}(3) = 7.5469$, according to the descending order of the values, the rearranging order of cluster is C_3, C_1, C_2. Noticing that $\Upsilon_3 = 6.0366$, $\Upsilon_2 = -3.3125$ and $\Upsilon_1 = -2.3095$, then the cluster lag synchronization of the network (21) can be realized by controlling the first rearranged cluster, i.e., C_3. Moreover, from condition (14), we get the lower bound of d_i is 7.5366.

Setting $\theta = 0.85$ and $d_3 = 8$, conditions (14) and (23) are satisfied. Select the control period $T = 0.01$. The network with the intermittent controllers (7)–(8) can realize

the desired cluster lag synchronization by pinning the first rearranged cluster, which uses the parameters chosen as above. Figure 1 shows the time evolutions of $x_i(t)(1 \le i \le 15)$ with different initial values under the intermittent controllers (7)–(8). It can be seen that the cluster lag synchronization is realized with the nodes in the kth cluster converges to the kth component of $S_3^*(k = 1, 2, 3)$.

We apply the adaptive intermittent pinning control schemes (16)–(17) to network (21) based on Theorem 3. Select $\theta = 0.85$, $T = 0.1$, $h_3 = 0.2$ and $\sigma = 0.5$. The time evolutions of $x_i(t)(1 \le i \le 15)$ and $d_3(t)$ are, respectively, illustrated in Figs. 2 and 3. We can see that the cluster lag synchronization of network (21) is realized and the adaptive intermittent feedback gain $d_3(t)$ converges to 0.4741 finally, which is smaller than the fixed intermittent feedback gain we select as above. When the set S_1^* is chosen as the desired cluster lag synchronous state, we can get the similar simulation results, not shown here.

5.2 Example B: cluster lag synchronization of a directed scale-free network

In this case, we consider the cluster lag synchronization of the directed scale-free network which consists of 300 nonidentical delayed dynamical nodes. We suppose that the nodes can be divided into three clusters: $C_1 = \{1, 2, \ldots, 100\}$, $C_2 = \{101, 102, \ldots, 200\}$, $C_3 = \{201, 202, \ldots, 300\}$. The network state equations are as follows:

$$\dot{x}_i(t) = \tilde{f}_k(t, x_i(t), x_i(t - \tau_k(t))) + c\sum_{p=1}^{3}\sum_{j \in C_p} b_{ij}\Gamma x_j(t),$$
$$i \in C_k, \quad k \in \Im = \{1, 2, 3\},$$
(24)

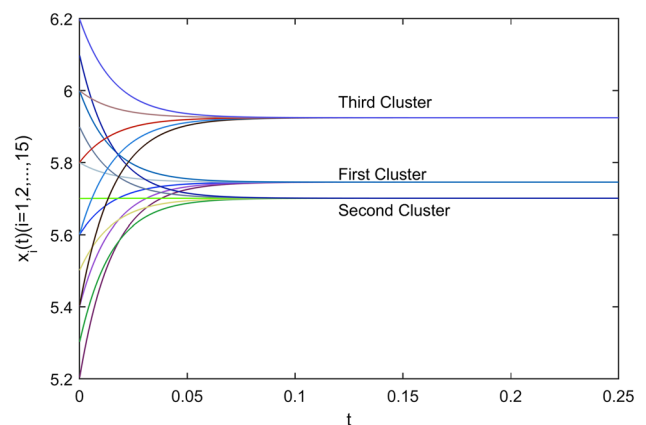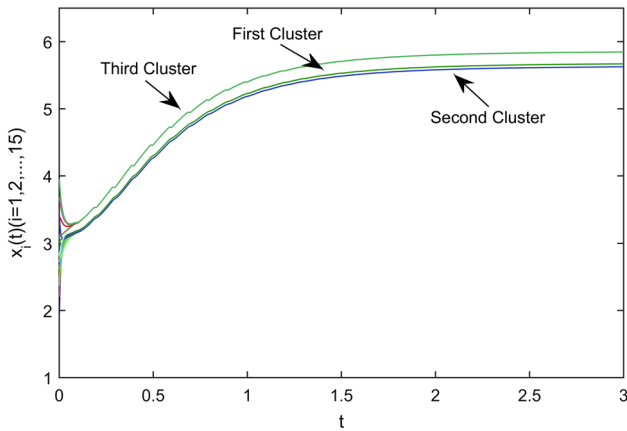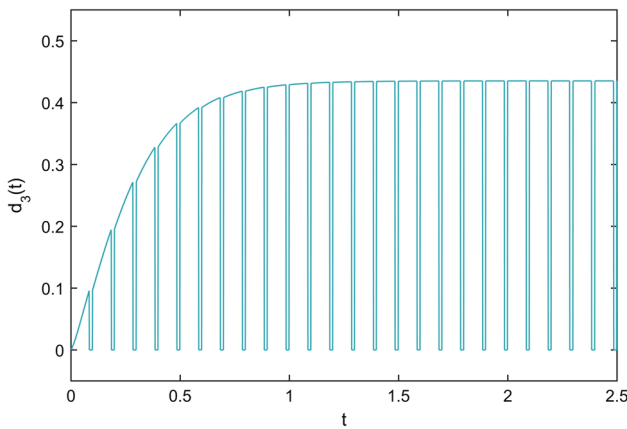where $\Gamma = I_3$, $B = (b_{ij})_{300 \times 300}$ is an asymmetrical coupling

Fig. 1 Time evolutions of $x_i(t)(1 \le i \le 15)$ for network (21) under the intermittent pinning control schemes (7)–(8), where $\sigma = 0.5$, $\theta = 0.85$, $T = 0.01$ and $d_3 = 8$

Fig. 2 Time evolutions of $x_i(t)(1 \leq i \leq 15)$ for network (21) under the adaptive intermittent pinning control schemes (16)–(17), where $\sigma = 0.5$, $\theta = 0.85$, $T = 0.1$ and $h_3 = 0.2$

Fig. 3 Time evolutions of $d_3(t)$ for network (21) under the adaptive intermittent pinning control schemes (16)–(17), where $\sigma = 0.5$, $\theta = 0.85$, $T = 0.1$ and $h_3 = 0.2$

matrix, and the coupling strength $c = 15$. The node dynamics of the three clusters are chosen as the delayed Chua oscillators with different system parameters, i.e.,

$$\dot{x}_i(t) = \tilde{f}_k(t, x_i(t), x_i(t - \tau_k(t)))$$
$$= \Omega_k x_i(t) + \tilde{f}_k^1(x_i(t)) + \tilde{f}_k^2(x_i(t - \tau_k(t))), \quad i \in C_k,$$
$$\tag{25}$$

where

$$x_i(t) = (x_{i1}(t), x_{i2}(t), x_{i3}(t))^T \in \mathbb{R}^3,$$
$$\tilde{f}_k^1(x_i(t)) = \left(-\frac{1}{2}\alpha_k(a - b)\right)((|x_{i1}(t) + 1| - |x_{i1}(t) - 1|), 0, 0)^T \in \mathbb{R}^3,$$
$$\tilde{f}_k^2(x_i(t)) = (0, 0, -\psi_k\varsigma\sin(\upsilon x_{i1}(t - \tau_k(t))))^T \in \mathbb{R}^3,$$
$$\Omega_k = \begin{pmatrix} -\alpha_k(1 + b) & \alpha_k & 0 \\ 1 & -1 & 1 \\ 0 & -\psi_k & -\omega_k \end{pmatrix},$$

$a = -1.4325$, $b = -0.7831$, $\upsilon = 0.5$, $\varsigma = 0.2$, $\tau_k(t) = 0.02$

and the transmission delay $\sigma = 0.2$. In this simulation, we select $\alpha_1 = 10$, $\psi_1 = 19.53$, $\omega_1 = 0.1636$ for the cluster C_1, $\alpha_2 = 10.5$, $\psi_2 = 20.53$, $\omega_2 = 0.2636$ for the cluster C_2, $\alpha_3 = 11$, $\psi_3 = 18.53$, $\omega_3 = 0.3636$ for the cluster C_3. We verify Assumption 2 as

$$(x(t) - y(t))^T (\tilde{f}_k(t, x(t), x(t - \tau_k(t))) - \tilde{f}_k((t), y(t), y(t - \tau_k(t))))$$
$$\leq \frac{1}{2}(x(t) - y(t))^T(\Omega_k + \Omega_k^T)(x(t) - y(t)) + |\alpha_k(a - b)|(x_1(t) - y_1(t))^2$$
$$+ \psi_k\varsigma\upsilon|x_3(t) - y_3(t)||x_1(t - \tau_k(t)) - y_1(t - \tau_k(t))|$$
$$\leq \lambda_{\max}(\tilde{\Omega}_k)(x(t) - y(t))^T(x(t) - y(t))$$
$$+ (\psi_k\varsigma\upsilon)/(2\varpi)(x(t - \tau_k(t)) - y(t - \tau_k(t)))^T(x(t - \tau_k(t))$$
$$-y(t - \tau_k(t)))$$
$$= L_k^0(x(t) - y(t))^T\Gamma(x(t) - y(t)) + L_k^\tau(x(t - \tau_k(t))$$
$$-y(t - \tau_k(t)))^T\Gamma(x(t - \tau_k(t)) - y(t - \tau_k(t))).$$

Here, $\tilde{\Omega}_k = ((\Omega_k + \Omega_k^T)/2 + \text{diag}(|\alpha_k(a - b)|, 0, \varpi(\psi_k\varsigma\upsilon)/2))$, $L_k^0 = \lambda_{\max}(\tilde{\Omega}_k)$ and $L_k^\tau = (\psi_k\varsigma\upsilon)/(2\varpi)$ can be determined by selecting the proper parameter $\varpi > 0$. Therefore, Assumption 2 is satisfied.

For convenience, we take the coupling matrix B as

$$B = (b_{ij})_{300 \times 300}$$
$$= \begin{pmatrix} \Phi_1 - 15I_{100} & 8I_{100} & 7I_{100} \\ 6I_{100} & \Phi_2 - 15I_{100} & 9I_{100} \\ 4I_{100} & 2I_{100} & \Phi_3 - 6I_{100} \end{pmatrix},$$

where Φ_1, Φ_2 and Φ_3 represent the intra-coupling matrices of the three scale-free characteristics clusters. And $\beta_{11} = -15$, $\beta_{12} = 8$, $\beta_{13} = 7$, $\beta_{21} = 6$, $\beta_{22} = -15$, $\beta_{23} = 9$, $\beta_{31} = 4$, $\beta_{32} = 2$, $\beta_{33} = -6$. Hence, the cluster lag synchronous equations of network (24) can be characterized by

$$\dot{s}_k(t) = \tilde{f}_k(t, s_i(t), s_i(t - \tau_k(t))) + c\sum_{p=1}^{3}\beta_{kp}s_p(t),$$
$$k \in \Im. \tag{26}$$

Setting $\varpi = 2$, then we can get that $L_1^0 = 11.3931$, $L_1^\tau = 0.4883$, $L_2^0 = 11.9654$, $L_2^\tau = 0.5133$, $L_3^0 = 11.4442$, $L_3^\tau = 0.4633$, therefore $q = 1.0266$. We have $\text{CluInf}(1) = -1.7405$, $\text{CluInf}(2) = -1.7023$, $\text{CluInf}(3) = 5.7629$ and $\lambda_{\max}(Z) = 5.7629$ through computation. According to the descending order of values, the rearranging order of the cluster is C_3, C_2, C_1. Choosing $a_1^0 = 25$, under conditions (i″) and (ii″), we have

$$\Upsilon_k < -1.6667, \quad l + 1 \leq k \leq 3, \quad \text{and} \quad 0.7802 < \theta < 1. \tag{27}$$

Due to $\Upsilon_3 = 5.7629$, $\Upsilon_2 = -1.7023$, $\Upsilon_1 = -1.7405$, then the cluster lag synchronization of network (24) can be realized by controlling the first rearranged cluster, i.e., C_3.

Considering Theorem 3, we use the adaptive intermittent pinning control schemes (16)–(17) to achieve the cluster lag synchronization of network (24). Setting
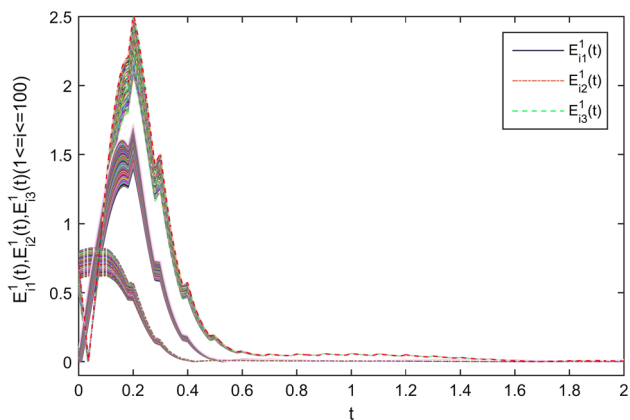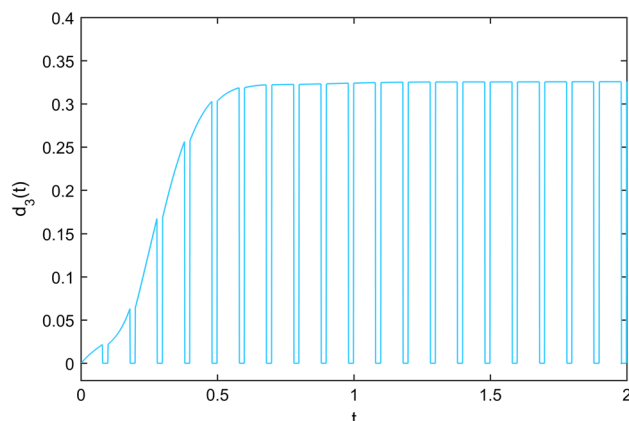
Fig. 4 Time evolutions of the synchronous errors $E_{ij}^1(t) = |x_{ij}(t) - s_{1j}(t - \sigma)|, (i \in C_1, j = 1, 2, 3)$ for the first cluster of network (24) under the adaptive intermittent pinning control schemes (16)–(17), where $\theta = 0.8$, $T = 0.1$, $\sigma = 0.2$ and $h_3 = 0.01$
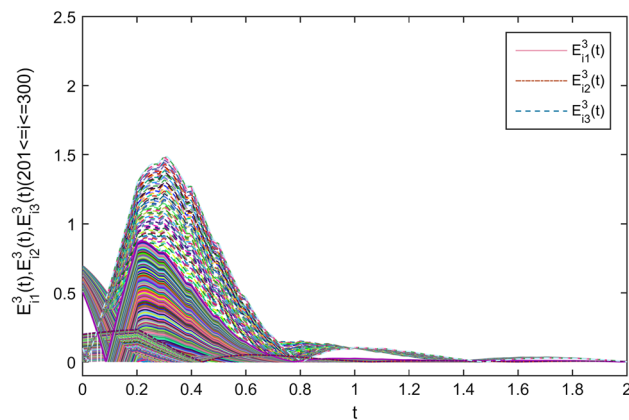
Fig. 6 Time evolutions of the lag synchronous errors $E_{ij}^3(t) = |x_{ij}(t) - s_{3j}(t - \sigma)|, (i \in C_3, j = 1, 2, 3)$ for the third cluster of network (24) under the adaptive intermittent pinning control schemes (16)–(17), where $\theta = 0.8$, $T = 0.1$, $\sigma = 0.2$ and $h_3 = 0.01$
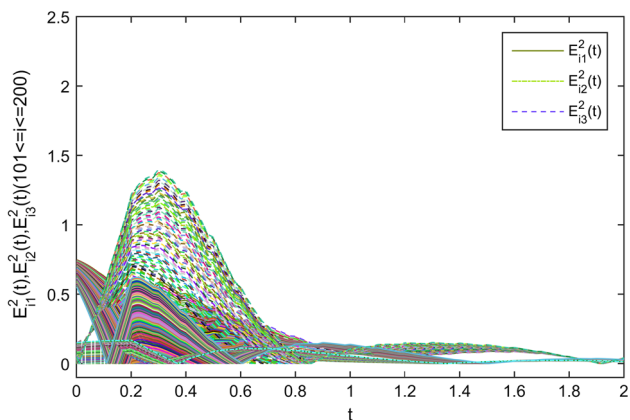
Fig. 5 Time evolutions of the synchronous errors $E_{ij}^2(t) = |x_{ij}(t) - s_{2j}(t - \sigma)|, (i \in C_2, j = 1, 2, 3)$ for the second cluster of network (24) under the adaptive intermittent pinning control schemes (16)–(17), where $\theta = 0.8$, $T = 0.1$, $\sigma = 0.2$ and $h_3 = 0.01$

Fig. 7 Time evolutions of the adaptive intermittent feedback gain $d_3(t)$ for the network (24) under the adaptive intermittent pinning control schemes (16)–(17), where $\theta = 0.8$, $T = 0.1$, $\sigma = 0.2$ and $h_3 = 0.01$

$\theta = 0.8$, conditions (15) and (27) are satisfied. Select a control period $T = 0.1$ and $h_3 = 0.01$. Figures 4, 5, 6 and 7, respectively, show the time evolutions of the synchronous errors $E_{ij}^k(t)(k \in \Im)$ and the adaptive intermittent feedback gain $d_3(t)$. Define the lag synchronous errors as $E^k(t)$, $k \in \Im$, i.e., $E_j^1(t) = \sqrt{\sum_{i=1}^{100} \|x_{ij}(t) - s_{1j}(t - \sigma)\|^2 / 100}$, $E_j^2(t) = \sqrt{\sum_{i=101}^{200} \|x_{ij}(t) - s_{2j}(t - \sigma)\|^2 / 100}$, $E_j^3(t) = \sqrt{\sum_{i=201}^{300} \|x_{ij}(t) - s_{3j}(t - \sigma)\|^2 / 100}$, where $j = 1, 2, 3$. Then, Figs. 8, 9 and 10 show the time evolutions of the lag synchronous errors $E^k(t)$. We can see that the cluster lag

synchronization is realized and the feedback gain $d_3(t)$ converges to 0.3258 at last.
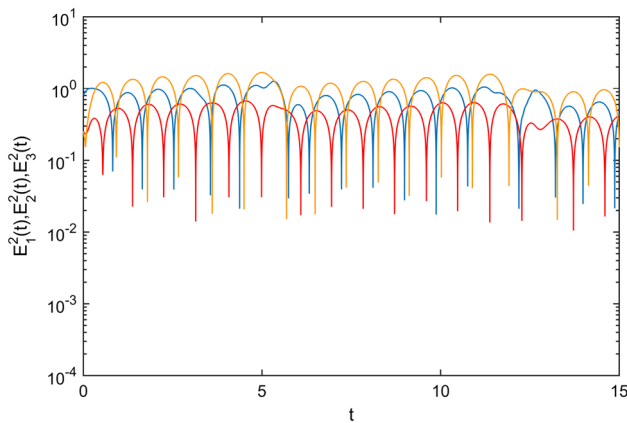
6 Conclusion

In this paper, we investigated the cluster lag synchronization of the intermittent pinning-controlled directed complex dynamical networks, which have the non-identical dynamical nodes with time-varying delays in the self-dynamics. Each cluster was regarded as a whole in the study of the cluster lag synchronization. The desired cluster lag synchronous states were chosen as a collection of the undecoupled trajectories rather than the particular solutions of the systems with decoupled nodes. We generalized the intermittent control scheme from the cluster
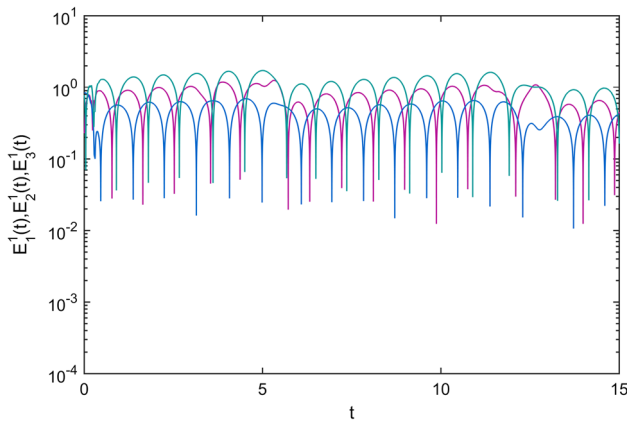
Fig. 8 Time evolutions of the lag synchronous error $E_j^1(t) = \sqrt{\sum_{i=1}^{100} \left\| x_{ij}(t) - s_{1j}(t-\sigma) \right\|^2 \Big/ 100}, (j=1,2,3)$ for the network (24) under the adaptive intermittent pinning control schemes (16)–(17), where $\theta = 0.8$, $T = 0.1$, $\sigma = 0.2$ and $h_3 = 0.01$
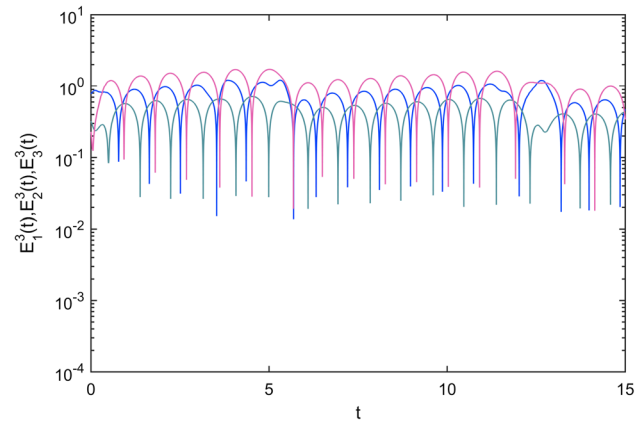
Fig. 10 Time evolutions of the lag synchronous error $E_j^3(t) = \sqrt{\sum_{i=201}^{300} \left\| x_{ij}(t) - s_{3j}(t-\sigma) \right\|^2 \Big/ 100}, (j=1,2,3)$ for the network (24) under the adaptive intermittent pinning control schemes (16)–(17), where $\theta = 0.8$, $T = 0.1$, $\sigma = 0.2$ and $h_3 = 0.01$

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Fig. 9 Time evolutions of the lag synchronous error $E_j^2(t) = \sqrt{\sum_{i=101}^{200} \left\| x_{ij}(t) - s_{2j}(t-\sigma) \right\|^2 \Big/ 100}, (j=1,2,3)$ for the network (24) under the adaptive intermittent pinning control schemes (16)–(17), where $\theta = 0.8$, $T = 0.1$, $\sigma = 0.2$ and $h_3 = 0.01$

synchronization without the transmission delay to the cluster lag synchronization involving both the time-varying delays in the self-dynamics and the transmission delay in the communication channels. By making some mild assumptions, we proposed the main theorems and the criteria for the cluster lag synchronization. Two numerical simulations were given to verify the correctness of the theoretical results.

As a more general case, the cluster lag synchronization of general delayed dynamical networks under the pinning control technology is worth studying in the near future. Topic of the work time of controller being time-varying is also interesting.

Appendix 1

Proof of Theorem 1

Let $E(t) = \left(E_1^T(t), E_2^T(t), \ldots, E_m^T(t) \right)^T$, where $E_1(t) = \left(e_{1,1}^T(t), \ldots, e_{r_1,1}^T(t) \right)^T$, $E_2(t) = \left(e_{r_1+1,2}^T(t), \ldots, e_{r_2,2}^T(t) \right)^T, \ldots E_m(t) = \left(e_{r_1+r_2+\cdots+r_{m-1}+1,m}^T(t), \ldots, e_{N,m}^T(t) \right)^T$.

Construct a Lyapunov candidate as follows, where $\otimes$ is defined as Kronecker product.

$$W(t) = \frac{1}{2} E^T(t)(I_N \otimes I_n)E(t) = \frac{1}{2}\sum_{k=1}^m E_k^T(t)(I_{r_k} \otimes I_n)E_k(t)$$
$$= \frac{1}{2}\sum_{k=1}^m \sum_{i \in C_k} e_{ik}^T(t)e_{ik}(t)$$

When $t \in [\omega T, (\omega + \theta)T)$, $\omega = 0, 1, 2, \ldots$, the time derivative of $W(t)$ along the trajectory of (9) at time t can be given as

$$\dot{W}(t) = \sum_{k=1}^{m} \sum_{i \in C_k} e_{ik}^{\mathrm{T}}(t) \tilde{f}_k^{\tau_k, \sigma}(t, x_i, s_k)$$
$$+ c \sum_{k=1}^{m} \sum_{i \in C_k} \sum_{p=1}^{m} \sum_{j \in C_p} b_{ij} e_{ik}^{\mathrm{T}}(t) \Gamma e_{jp}(t) \qquad (28)$$
$$- c \sum_{k=1}^{l} E_k^{\mathrm{T}}(t)(d_k I_{r_k} \otimes \Gamma) E_k(t).$$

In virtue of Assumption 2, we can get

$$\sum_{k=1}^{m} \sum_{i \in C_k} e_{ik}^{\mathrm{T}}(t) \tilde{f}_k^{\tau_k, \sigma}(t, x_i, s_k)$$
$$\leq \sum_{k=1}^{m} \sum_{i \in C_k} L_k^0 e_{ik}^{\mathrm{T}}(t) \Gamma e_{ik}(t)$$
$$+ \sum_{k=1}^{m} \sum_{i \in C_k} L_k^{\tau} e_{ik}^{\mathrm{T}}(t - \tau_k(t)) \Gamma e_{ik}(t - \tau_k(t)) \qquad (29)$$
$$= \sum_{k=1}^{m} E_k^{\mathrm{T}}(t) \left(L_k^0 I_{r_k} \otimes \Gamma \right) E_k(t)$$
$$+ \sum_{k=1}^{m} E_k^{\mathrm{T}}(t - \tau_k(t)) \left(L_k^{\tau} I_{r_k} \otimes \Gamma \right) E_k(t - \tau_k(t)).$$

Noticing that $\sum_{p=1}^{m} \sum_{i=C_p} b_{ij} = \sum_{j=1}^{N} b_{ij} = 0$, $i = 1, 2, \ldots, N$ with $\tilde{\Gamma} = \mathrm{diag}\left(\sqrt{\gamma_1}, \sqrt{\gamma_2}, \ldots, \sqrt{\gamma_n} \right)$, one has

$$\sum_{k=1}^{m} \sum_{i=C_k} \sum_{p=1}^{m} \sum_{j \in C_p} b_{ij} e_{ik}^{\mathrm{T}}(t) \Gamma e_{jp}(t) = \sum_{k=1}^{m} \sum_{p=1}^{m} E_k^{\mathrm{T}}(t) \left(B_{kp} \otimes \Gamma \right) E_p(t)$$
$$= \sum_{k=1}^{m} E_k^{\mathrm{T}}(t) \left(B_{kk}^s \otimes \Gamma \right) E_k(t) + \sum_{k=1}^{m} \sum_{p=1, p \neq k}^{m} \sum_{i \in C_k} \sum_{j \in C_p} b_{ij} \left(\tilde{\Gamma} e_{ik}(t) \right)^{\mathrm{T}} \left(\tilde{\Gamma} e_{jp}(t) \right)$$
$$\leq \sum_{k=1}^{m} E_k^{\mathrm{T}}(t) \left(B_{kk}^s \otimes \Gamma \right) E_k(t) + \frac{1}{2} \sum_{k=1}^{m} \sum_{p=1, p \neq k}^{m} \sum_{i \in C_k} \sum_{j \in C_p}$$
$$b_{ij} \left(e_{ik}^{\mathrm{T}}(t) \Gamma e_{ik}(t) + e_{jp}^{\mathrm{T}}(t) \Gamma e_{jp}(t) \right)$$
$$= \sum_{k=1}^{m} E_k^{\mathrm{T}}(t) \left(B_{kk}^s \otimes \Gamma \right) E_k(t) - \frac{1}{2} \sum_{k=1}^{m} E_k^{\mathrm{T}}(t)(A_k \otimes \Gamma) E_k(t)$$
$$+ \frac{1}{2} \sum_{k=1}^{m} E_k^{\mathrm{T}}(t)(\Psi_k \otimes \Gamma) E_k(t)$$
$$= \sum_{k=1}^{m} E_k^{\mathrm{T}}(t) \left(\left(B_{kk}^s + \frac{1}{2} \Psi_k - \frac{1}{2} A_k \right) \otimes \Gamma \right) E_k(t) \qquad (30)$$

Substituting (29) and (30) into (28), we get

$$\dot{W}(t) \leq c E^{\mathrm{T}}(t)((Z - D) \otimes \Gamma) E(t)$$
$$+ \frac{q}{2} \left(\sup_{t-\tau \leq s \leq t} \sum_{k=1}^{m} E_k^{\mathrm{T}}(s) E_k(s) \right). \qquad (31)$$

According to condition (i) and the properties of the Kronecker product of matrix [42], from (31), one has

$$\dot{W}(t) \leq E^{\mathrm{T}}(t)((cZ - cD + a_1 I_N) \otimes \Gamma) E(t)$$
$$- a_1 E^{\mathrm{T}}(t)(I_N \otimes \Gamma) E(t) + q \left(\sup_{t-\tau \leq s \leq t} W(s) \right)$$
$$\leq - 2a_1 \lambda_{\min}(\Gamma) W(t) + q \left(\sup_{t-\tau \leq s \leq t} W(s) \right)$$
$$= -p_1 W(t) + q \left(\sup_{t-\tau \leq s \leq t} W(s) \right).$$

Similarly, when $t \in [(\omega + \theta)T, (\omega + 1)T)$, $\omega = 0, 1, 2, \ldots$, using condition (ii), we can deduce

$$\dot{W}(t) \leq E^{\mathrm{T}}(t)((cZ - a_2 I_N) \otimes \Gamma) E(t) + a_2 E^{\mathrm{T}}(t)$$
$$(I_N \otimes \Gamma) E(t) + q \left(\sup_{t-\tau \leq s \leq t} W(s) \right)$$
$$\leq 2a_2 \lambda_{\max}(\Gamma) W(t) + q \left(\sup_{t-\tau \leq s \leq t} W(s) \right)$$
$$= p_2 W(t) + q \left(\sup_{t-\tau \leq s \leq t} W(s) \right).$$

As a result, we have

$$\begin{cases} \dot{W}(t) \leq -p_1 W(t) + q \left(\sup\limits_{t-\tau \leq s \leq t} W(s) \right), & \omega T \leq t < (\omega + \theta)T, \\ \dot{W}(t) \leq p_2 W(t) + q \left(\sup\limits_{t-\tau \leq s \leq t} W(s) \right), & (\omega + \theta)T \leq t < (\omega + 1)T. \end{cases}$$
$$(32)$$

Next, we will prove that conditions (iii)–(iv) imply that, for $t \geq 0$, we have $\dot{W}(t) \leq \left(\sup_{-\tau \leq s \leq 0} W(s) \right) e^{-\varpi t}$.

Define $\zeta(\lambda) = \lambda - p_1 + q e^{\lambda \tau}$, where $p_1 > q > 0$ such that $\zeta(0) < 0$, $\zeta(+\infty) > 0$, $\zeta'(\lambda) > 0$. Utilizing the monotonicity and continuity of $\zeta(\lambda)$, the equation $\lambda - p_1 + q e^{\lambda \tau} = 0$ has a unique positive solution $\lambda > 0$. Let $M_0 = \sup_{-\tau \leq s \leq 0} W(s)$, $V(t) = e^{\lambda t} W(t)$, $t \geq 0$. Let $S(t) = V(t) - h M_0$, where $h > 1$ is a constant. Apparently, for any $t \in [-\tau, 0]$, one has

$$S(t) < 0. \qquad (33)$$

Then, we indicate for all $t \in [0, \theta T]$, that

$$S(t) < 0. \qquad (34)$$

From (33), there exist a $t_0 \in [0, \theta T]$ such that

$$S(t_0) = 0, \quad \dot{S}(t_0) \geq 0, \qquad (35)$$
$$S(t) < 0, \quad -\tau \leq t < t_0. \qquad (36)$$

By (32), (35) and (36), we have

$$\dot{S}(t_0) = \lambda V(t_0) + e^{\lambda t_0} \dot{W}(t_0) \leq (\lambda - p_1) V(t_0)$$
$$+ q e^{\lambda t_0} \left(\sup_{t_0 - \tau \leq s \leq t_0} W(s) \right). \qquad (37)$$

On the other hand, from (35) and (36), we have $V(t) < h M_0$, $-\tau \leq t < t_0$ and $V(t_0) = h M_0$. As a result, we get $W(t) < h M_0 e^{-\lambda t}$, $-\tau \leq t < t_0$, it leads to

$\left(\sup_{t_0-\tau\leq s\leq t_0} W(s)\right)<e^{\lambda\tau}hM_0e^{-\lambda t_0}$. This means $e^{\lambda t_0}\left(\sup_{t_0-\tau\leq s\leq t_0} W(s)\right)<e^{\lambda\tau}hM_0=e^{\lambda\tau}V(t_0)$.

From (37), it can be deduced that $\dot{S}(t_0)<\left(\lambda-p_1+qe^{\lambda\tau}\right)V(t_0)=0$. This is a contradiction with inequality (35), thus, (34) is established. With (33), we get that

$$W(t)<hM_0e^{-\lambda t}, \quad \forall t\in[-\tau,\theta T] \tag{38}$$

Let $\vartheta=p_1+p_2$, for $t\in[\theta T,T)$, we have $Q(t)=V(t)-hM_0e^{\vartheta(t-\theta T)}<0$. Otherwise, there exist a $t_1\in[\theta T,T)$, such that

$$Q(t_1)=0, \quad \dot{Q}(t_1)\geq 0, \tag{39}$$

$$Q(t)<0, \quad \theta T\leq t<t_1. \tag{40}$$

For $\tau>0$, according to (38)–(40), we always have

$$\sup_{t_1-\tau\leq s\leq t_1} W(s)<e^{\lambda\tau}W(t_1).$$

$$W(t)=\begin{cases}\dfrac{1}{2}\displaystyle\sum_{k=1}^m\sum_{i\in C_k}e_{ik}^{\mathrm{T}}(t)e_{ik}(t)+\dfrac{1}{2}e^{-p_1(t-\omega T)}\displaystyle\sum_{k=1}^l\dfrac{c}{h_k}\left(d_k(t)-d_k^*\right)^2, & \omega T\leq t<(\omega+\theta)T,\\[2mm]\dfrac{1}{2}\displaystyle\sum_{k=1}^m\sum_{i\in C_k}e_{ik}^{\mathrm{T}}(t)e_{ik}(t)+\dfrac{1}{2}e^{-p_1(t-\omega T)}\displaystyle\sum_{k=1}^l\dfrac{c}{h_k}d_k^{*2}, & (\omega+\theta)T\leq t<(\omega+1)T.\end{cases}$$

Then, one has $\dot{Q}(t_1)<0$, which is contradicts with (39). Thus, for $t\in[\theta T,T)$,

$$V(t)<hM_0e^{\vartheta(1-\theta)T}.$$

With (33) and (34), we have

$$V(t)<hM_0e^{\vartheta(1-\theta)T}, \quad \text{for} \quad t\in[-\tau,T).$$

Similarly, we can prove

$$V(t)<hM_0e^{\vartheta(1-\theta)T}, \quad \text{for} \quad t\in[T,(1+\theta)T) \quad \text{and}$$

$$V(t)<hM_0e^{\vartheta(t-2\theta)T}, \quad \text{for} \quad t\in[(1+\theta)T,2T).$$

Through a mathematical induction, for any integer $\omega=0,1,2,\ldots$, we can derive the estimates of $V(t)$ as

$$\begin{cases}V(t)<hM_0e^{\omega\vartheta(1-\theta)T}\leq hM_0e^{\vartheta(1-\theta)t}, & \omega T\leq t<(\omega+\theta)T,\\V(t)<hM_0e^{\vartheta[t-(m+1)\theta T]}\leq hM_0e^{\vartheta(1-\theta)t}, & (\omega+\theta)T\leq t<(\omega+1)T.\end{cases}$$

Let $h\to 1$, from the definition of $V(t)$, we have

$$W(t)\leq M_0e^{-[\lambda-\vartheta(1-\theta)]t}=\left(\sup_{-\tau\leq s\leq 0} W(s)\right)e^{-\varpi t}, \quad t\geq 0.$$

Therefore, for the error dynamical system (9) the zero solution is globally exponentially stable. The proof of Theorem 1 is thus completed.

Appendix 2

Proof of Theorem 3

Construct a piecewise Lyapunov candidate function

$$W(t)=\frac{1}{2}\sum_{k=1}^m\sum_{i\in C_k}e_{ik}^{\mathrm{T}}(t)e_{ik}(t)+\frac{1}{2}\Pi(t)\sum_{k=1}^l ce^{-p_1t}\frac{\left(d_k(t)-d_k^*\right)^2}{h_k},$$

where $d_k^*>0$, $k=1,2,\ldots,l$ are constants, and $\Pi(\cdot)$ is a piecewise function defined as

$$\Pi(s)=\begin{cases}1, & -\tau\leq s\leq 0,\\e^{p_1\omega T}, & \omega T\leq s<(\omega+1)T, \omega=0,1,2,\ldots\end{cases}$$

Obviously, while $\omega=0,1,2,\ldots$, we have

With Assumptions 1 and 2, the time derivative of $W(t)$ along the trajectory of (9) can be calculated as follows. When $t\in[\omega T,(\omega+\theta)T)$, $\omega=0,1,2,\ldots$, we have

$$\begin{aligned}\dot{W}(t)\leq{} & E^{\mathrm{T}}(t)\left(\left(cZ-cD^*+a_1^0I_N\right)\otimes\Gamma\right)E(t)\\&+\frac{q}{2}\left(\sup_{t-\tau\leq s\leq t}\sum_{k=1}^m E_k^{\mathrm{T}}(s)E_k(s)\right)\\&-a_1^0E^{\mathrm{T}}(t)(I_N\otimes\Gamma)E(t)-\frac{p_1}{2}e^{-p_1(t-\omega T)}\\&\sum_{k=1}^l\frac{c}{h_k}\left(d_k(t)-d_k^*\right)^2,\end{aligned} \tag{41}$$

where D^* is a modified block diagonal matrix of D through replacing the principal-diagonal sub-matrices $d_kI_{r_k}$ by $\left(e^{-p_1\theta T}d_k^*\right)I_{r_k}, k=1,2,\ldots,l$.

Set $Q^*=a_1^0I_N+cZ$, $a_1^0I_N+cZ-cD^*=Q^*-cD^*=\begin{pmatrix}G^*-c\tilde{D}^* & 0\\0^{\mathrm{T}} & Q_r^*\end{pmatrix}$, where $G^*=\begin{pmatrix}cZ_{11}+a_1^0I_{r_1} & 0 & \cdots & 0\\0 & cZ_{22}+a_1^0I_{r_2} & \cdots & 0\\\vdots & \vdots & \ddots & \vdots\\0 & 0 & \cdots & cZ_{ll}+a_1^0I_{r_l}\end{pmatrix}$, $\tilde{D}^*=e^{-p_1\theta T}\begin{pmatrix}d_1^*I_{r_1} & 0 & \cdots & 0\\0 & d_2^*I_{r_2} & \cdots & 0\\\vdots & \vdots & \ddots & \vdots\\0 & 0 & \cdots & d_l^*I_{r_l}\end{pmatrix}$, and Q_r^* is the minor

matrix of Q^* by removing its first $(r_1 + r_2 + \cdots + r_l)$ row–column pairs. It is obviously that Q^* is a real symmetric matrix. From the pinning condition (i''), we conclude that

$$\lambda_{\max}\left(cZ_{kk} + a_1^0 I_{r_k}\right) = \lambda_{\max}\left(c\Theta_{kk} + L_k^0 I_{r_k} + a_1^0 I_{r_k}\right)$$
$$< c\lambda_{\max}(\Theta_{kk}) + L_k^0 + a_1^0 < 0,$$
$$l+1 \le k \le m.$$

This means $Q_r^* < 0$. As a result, for $k = 1, 2, \cdots, l$, when $d_k^* > 0$ are large enough such that $d_k^* > \frac{e^{p_1 \theta T} \lambda_{\max}(G^*)}{c}$. It is easy to get $Q^* - cD^* < 0$. This follows Lemma 2 directly.

With (41), we can deduce that

$$\dot{W}(t) \le -\frac{p_1}{2} \sum_{k=1}^{m} \sum_{i \in C_k} e_{ik}^T(t) e_{ik}(t) - \frac{p_1}{2} e^{-p_1(t-\omega T)}$$
$$\sum_{k=1}^{l} \frac{c}{h_k} \left(d_k(t) - d_k^*\right)^2$$
$$+ \frac{q}{2} \left(\sup_{t-\tau \le s \le t} \sum_{k=1}^{m} \sum_{i \in C_k} e_{ik}^T(s) e_{ik}(s) \right)$$
$$\le -p_1 W(t) + q \left(\sup_{t-\tau \le s \le t} W(s) \right).$$

In the same way, when $t \in [(\omega + \theta)T, (\omega + 1)T)$, $\omega = 0, 1, 2, \ldots$, we obtain that

$$\dot{W}(t) \le p_2 W(t) + q \left(\sup_{t-\tau \le s \le t} W(s) \right).$$

Namely,

$$\begin{cases} \dot{W}(t) \le -p_1 W(t) + q \left(\sup_{t-\tau \le s \le t} W(s) \right), & \omega T \le t < (\omega + \theta)T, \\ \dot{W}(t) \le p_2 W(t) + q \left(\sup_{t-\tau \le s \le t} W(s) \right), & (\omega + \theta)T \le t < (\omega + 1)T. \end{cases}$$

Then, by a similar proof of Theorem 1, we can prove that condition (15) means

$$\dot{W}(t) \le \left(\sup_{-\tau \le s \le 0} W(s) \right) e^{-\varpi t}, \quad t \ge 0.$$

Therefore, the global cluster lag synchronization of the delayed dynamical network (6) under the adaptive intermittent pinned controllers (16)–(17) is realizable. The proof of Theorem 3 is thus completed.

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