



Numerical solutions of fuzzy fractional diffusion equations by an implicit finite difference scheme

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Abstract

Fuzzy fractional diffusion equations are used to model certain phenomena in physics, hydrology biology and amongst others. In this paper, an implicit finite difference scheme is developed, analysed and applied to numerically solve a fuzzy time fractional diffusion equation. For our case, the fuzziness is in the coefficients as well as initial and boundary conditions. The time fractional derivative is defined using the Caputo formula. The stability of the implicit finite difference scheme is analysed by means of the Von Neumann method. A numerical example has been given to check the feasibility of the approach and to examine certain related aspects. It was found that the results obtained are in good agreement with the proposed theory. Hence, the proposed scheme is suitable for solving fuzzy time fractional diffusion equations.

Keywords Fuzzy numbers · Caputo formula · Fuzzy time fractional diffusion equation · Implicit finite difference scheme

1 Introduction

Fractional differential equations have attracted considerable attention for the past 10 years or so. This is evident from the number of publications on such equations in various mathematical and scientific databases. Crisp quantities in the fractional differential equations which are deemed imprecise and uncertain can be replaced by fuzzy quantities to reflect imprecision and uncertainty. This leads to fuzzy fractional differential equations (FFDEs). There have been a number of recent studies on the solutions of FFDEs [1–7]. Agarwal et al. [8] considered the solution of fractional differential equation with uncertainty. The problem in question was an initial value problem involving a fractional ordinary differential equation. The fractional derivative was evaluated using the Riemann–Liouville formula. Later, Allahviranlo et al. [9] introduced the

concept of Riemann–Liouville H-differentiability, which is a direct generalization of the fractional Riemann–Liouville derivative using Hukuhara difference to solve uncertain fractional differential equations (UFDEs) using Mittag-Leffler functions. The obtained explicit solutions of UFDEs were derived by applying the equivalent integral forms of UFDEs. Then, Takaci et al. [10] used the fuzzy Laplace transform to construct exact and approximate solutions of FFDEs in the sense of Caputo Hukuhara differentiability, i.e. the fractional derivative was evaluated using Caputo formula and fuzzy differentiability evaluated using the Hukuhara approach. The obtained results were expressed in the form of fuzzy Mittag-Leffler function. Salahshour et al. [11] then used the fuzzy Laplace transform definition to solve FFDEs under Riemann–Liouville H-differentiability and investigated the efficiency and utility of the Laplace transform method. Later, Ghazanfari and Ebrahimi [12] applied the differential transformation method (DTM) to solve fuzzy fractional diffusion equations. The DTM is an iterative procedure for obtaining analytic series solution of differential equations. It was found that the DTM was a highly effective and simple scheme for obtaining approximate analytical solutions of fuzzy fractional diffusion equations. Chakraverty and Tampaswini [13] later proposed a new computational technique to handle the fuzzy fractional diffusion equations. These approaches convert

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the fuzzy diffusion equation into interval-based *finite difference equations* (FDEs) and then transform the obtained equation into crisp form by using the double parametric form of fuzzy numbers. Finally, this crisp form is solved by the Adomian decomposition method (ADM) to obtain the uncertain bounds of the solution.

Salah et al. [14] developed the homotopy analysis transform method (HATM) to solve fuzzy fractional heat and wave equations. The HATM is a combination of the homotopy analysis method and the Laplace decomposition method. HATM yields approximate analytical solution in the form of a series. It was found that the HATM is efficient, simple and involves less computational work as compared to other analytical methods. To the best of our knowledge, there seems to have been no attempt to solve fuzzy fractional diffusion equations by using finite difference schemes. Most of the papers on the solution of fuzzy fractional diffusion equations involve approximate analytical methods. Our paper will investigate the use of a finite difference scheme for solving fuzzy time fractional diffusion equations. The availability of a reliable and efficient finite difference scheme will facilitate the numerical solution of fuzzy time fractional diffusion equations.

2 Fuzzy time fractional diffusion equation

In this section, we present the general form of time fractional diffusion equation in a fuzzy environment by using the basic concepts of fuzzy properties [15–18]. Consider the one-dimensional fuzzy time fractional diffusion equation with the initial and boundary conditions

$$\begin{aligned} \frac{\partial^\alpha \tilde{u}(x, t, \alpha)}{\partial^\alpha t} &= \tilde{a}(x) \frac{\partial^2 \tilde{u}(x, t)}{\partial x^2} + \tilde{b}(x), \quad 0 < x < l, \quad t > 0 \\ \tilde{u}(x, 0) &= \tilde{f}(x), \quad \tilde{u}(0, t) = \tilde{g}, \quad \tilde{u}(l, 0) = \tilde{z} \end{aligned} \tag{1}$$

where $\tilde{u}(x, t)$ is a fuzzy function [16] of crisp variables t and x , $\frac{\partial^\alpha \tilde{u}(x, t, \alpha)}{\partial^\alpha t}$ is the fuzzy time fractional derivative of order α , $\frac{\partial^2 \tilde{u}(x, t)}{\partial x^2}$ is a fuzzy partial *Hukuhara* derivative [9] with respect to x . $\tilde{a}(x)$ and $\tilde{b}(x)$ are fuzzy functions for the crisp variable x . $\tilde{u}(0, x)$ is the fuzzy initial condition, and $\tilde{u}(0, t)$ as well as $\tilde{u}(l, 0)$ is fuzzy boundary conditions with \tilde{g} , \tilde{z} being fuzzy convex numbers. Finally in Eq. 1, the fuzzy functions $\tilde{a}(x)$, $\tilde{b}(x)$ and $\tilde{f}(x)$ are defined as follows [19]:

$$\begin{cases} \tilde{a}(x) = \tilde{\theta}_1 s_1(x) \\ \tilde{b}(x) = \tilde{\theta}_2 s_2(x) \\ \tilde{f}(x) = \tilde{\theta}_3 s_3(x) \end{cases} \tag{2}$$

where $s_1(x)$, $s_2(x)$ and $s_3(x)$ are the crisp functions of the crisp variable x with $\tilde{\theta}_1$, $\tilde{\theta}_2$ and $\tilde{\theta}_3$ being the fuzzy convex numbers. The fuzzification of Eq. 1 for all $r \in [0, 1]$ is as follows [19]

$$[\tilde{u}(x, t)]_r = \underline{u}(x, t; r), \quad \bar{u}(x, t; r) \tag{3}$$

$$\left[\frac{\partial^\alpha \tilde{u}(x, t, \alpha)}{\partial^\alpha t} \right]_r = \frac{\partial^\alpha \underline{u}(x, t, \alpha; r)}{\partial^\alpha t}, \quad \frac{\partial^\alpha \bar{u}(x, t, \alpha; r)}{\partial^\alpha t} \tag{4}$$

$$\left[\frac{\partial^2 \tilde{u}(x, t)}{\partial x^2} \right]_r = \frac{\partial^2 \underline{u}(x, t; r)}{\partial x^2}, \quad \frac{\partial^2 \bar{u}(x, t; r)}{\partial x^2} \tag{5}$$

$$[a(x)]_r = \underline{k}(x; r), \quad \bar{k}(x; r) \tag{6}$$

$$[\tilde{b}(x)]_r = \underline{b}(x; r), \quad \bar{b}(x; r) \tag{7}$$

$$[\tilde{u}(x, 0)]_r = \underline{u}(x, 0; r), \quad \bar{u}(x, 0; r) \tag{8}$$

$$[\tilde{u}(0, t)]_r = \underline{u}(0, t; r), \quad \bar{u}(0, t; r) \tag{9}$$

$$[\tilde{u}(l, t)]_r = \underline{u}(l, t; r), \quad \bar{u}(l, t; r) \tag{10}$$

$$[\tilde{f}(x)]_r = \underline{f}(x; r), \quad \bar{f}(x; r) \tag{11}$$

$$\begin{cases} [\tilde{g}]_r = \underline{g}(r), \quad \bar{g}(r) \\ [\tilde{z}]_r = \underline{z}(r), \quad \bar{z}(r) \end{cases} \tag{12}$$

where

$$\begin{cases} [\tilde{a}(x)]_r = [\underline{\theta}(r)_1, \bar{\theta}_1(r)] s_1(x) \\ [\tilde{b}(x)]_r = [\underline{\theta}(r)_2, \bar{\theta}_2(r)] s_2(x) \\ [\tilde{f}(x)]_r = [\underline{\theta}(r)_3, \bar{\theta}_3(r)] s_3(x) \end{cases} \tag{13}$$

The membership function is defined by using the fuzzy extension principle [19]

$$\begin{cases} \underline{u}(x, t; r) = \min\{\tilde{u}(\tilde{\mu}(r), t) | \tilde{\mu}(r) \in \tilde{u}(x, t; r)\} \\ \bar{u}(x, t; r) = \max\{\tilde{u}(\tilde{\mu}(r), t) | \tilde{\mu}(r) \in \tilde{u}(x, t; r)\} \end{cases} \tag{14}$$

According to [19], by fuzzification of Eq. 1 and defuzzification of Eqs. (2–14), we can rewrite the Eq. 1 in the following new formula. The Lower bound of Eq. 1

$$\begin{cases} \frac{\partial^\alpha \underline{u}(x, t, \alpha)}{\partial^\alpha t} = [\underline{\theta}(r)_1] s_1(x) \frac{\partial^2 \underline{u}(x, t; r)}{\partial x^2} + [\underline{\theta}(r)_2] s_2(x) \\ \underline{u}(x, 0; r) = \underline{\theta}(r)_3 s_3(x) \\ \underline{u}(0, t; r) = \underline{g}(r), \quad \underline{u}(l, t; r) = \underline{z}(r) \end{cases} \tag{15}$$

The upper bound of Eq. 1

$$\begin{cases} \frac{\partial^\alpha \bar{u}(x, t, \alpha)}{\partial^\alpha t} = [\bar{\theta}_1(r)] s_1(x) \frac{\partial^2 \bar{u}(x, t; r)}{\partial x^2} + [\bar{\theta}_2(r)] s_2(x) \\ \bar{u}(x, 0; r) = \bar{\theta}_3(r) s_3(x) \\ \bar{u}(0, t; r) = \bar{g}(r), \quad \bar{u}(l, t; r) = \bar{z}(r) \end{cases} \tag{16}$$

We next consider a discretization of Eqs. 15 and 16.

3 The fuzzy implicit finite difference method

In this section, we present a fuzzy implicit scheme using Caputo formula for time fractional derivative and central difference approximation for second-order space derivative to solve the fuzzy time fractional diffusion equations.

Following the definition of Caputo formula, we discretize the time fractional derivative in Eq. 1 such that [20]:

$$\frac{\partial^\alpha \tilde{u}(x, t, \alpha)}{\partial t^\alpha} = \Delta t^{-\alpha} \sum_{j=0}^n v_j (u_i^{n-j} - u_i^0) + o(\Delta t), \tag{17}$$

where $v_0 = 1$, $v_j = \left(1 - \frac{\alpha+1}{j}\right)v_{j-1}$, $j = 1, 2, \dots$

Also by using the central difference approximation definition, we can discretize the second partial derivatives $\frac{\partial^2 \underline{u}(x,t)}{\partial x^2}$, $\frac{\partial^2 \bar{u}(x,t)}{\partial x^2}$ as follows:

$$\frac{\partial^2 \underline{u}_{i,n}(x, t; r)}{\partial x^2} = \frac{\underline{u}_{i+1,n}(x, t; r) - 2\underline{u}_{i,n}(x, t; r) + \underline{u}_{i-1,n}(x, t; r)}{h^2} \tag{18}$$

$$\frac{\partial^2 \bar{u}_{i,n}(x, t; r)}{\partial x^2} = \frac{\bar{u}_{i+1,n}(x, t; r) - 2\bar{u}_{i,n}(x, t; r) + \bar{u}_{i-1,n}(x, t; r)}{h^2} \tag{19}$$

i indicates a spatial grid point and n a temporal one.

Equations (17, 18, 19) are substituted in Eqs. (14, 15) to obtain:

$$\begin{aligned} &\Delta t^{-\alpha} \sum_{j=0}^n v_j (\underline{u}_i^{n-j} - u_i^0) \\ &= \underline{a}(x, r) \frac{\underline{u}_{i+1,n}(x, t; r) - 2\underline{u}_{i,n}(x, t; r) + \underline{u}_{i-1,n}(x, t; r)}{h^2} + \underline{b}(x, r) \end{aligned} \tag{20}$$

$$\begin{aligned} &\Delta t^{-\alpha} \sum_{j=0}^n v_j (\bar{u}_i^{n-j} - \bar{u}_i^0) \\ &= \bar{a}(x, r) \frac{\bar{u}_{i+1,n}(x, t; r) - 2\bar{u}_{i,n}(x, t; r) + \bar{u}_{i-1,n}(x, t; r)}{h^2} + \bar{b}(x, r) \end{aligned} \tag{21}$$

Now we let $p(r) = \frac{\underline{a}(x,r)\Delta t^\alpha}{h^2}$, and from Eqs. (21, 22), we obtain for all $r \in [0, 1]$

$$\begin{aligned} &- p\underline{u}_{i+1,n}(x, t; r) + (1 + 2p)\underline{u}_{i,n}(x, t; r) - p\underline{u}_{i-1,n}(x, t; r) \\ &= - \sum_{j=1}^{n-1} v_j \underline{u}_{i,n-j} + \left(\sum_{j=0}^{n-1} v_j \right) \underline{u}_{i,0} + \Delta t^\alpha \underline{b}(x, r) \end{aligned} \tag{22}$$

$$\begin{aligned} &- p\bar{u}_{i+1,n}(x, t; r) + (1 + 2p)\bar{u}_{i,n}(x, t; r) - p\bar{u}_{i-1,n}(x, t; r) \\ &= - \sum_{j=1}^{n-1} v_j \bar{u}_{i,n-j} + \left(\sum_{j=0}^{n-1} v_j \right) \bar{u}_{i,0} + \Delta t^\alpha \bar{b}(x, r) \end{aligned} \tag{23}$$

For each spatial grid point, Eqs. (22, 23) are evaluated to yield linear equations. At the end of each time level, a

system of linear equations is obtained. This system is then solved to obtain the values $\tilde{u}(x, t, \alpha)$ for that particular time level.

4 Stability analysis

Ma [21] developed implicit finite difference schemes for the crisp time fractional diffusion equation with source terms. Wang and Qin [22] developed a fuzzy finite difference scheme for heat conduction problem which *did not* involve fractional derivative. In addition, the stability properties of the scheme were also investigated. We shall follow the approaches to analyse stability in [21, 22] in our investigation of the stability of the implicit finite difference schemes proposed in the present for the fuzzy time fractional diffusion equations.

It is first assumed that the discretization of initial condition introduces the fuzzy error $\tilde{\varepsilon}_i^0$.

Let $\tilde{g}_i^0 = \hat{g}_i^0 - \tilde{\varepsilon}_i^0$, \tilde{u}_i^n and \tilde{u}_i^n be the fuzzy numerical solutions of scheme in Eqs. (22, 23) with respect to the initial data's \tilde{g}_i^0 and \hat{g}_i^0 , respectively.

Let $[\tilde{u}_{i+1}^n(x, t; \alpha)]_r = [\underline{u}_{i+1}^n(r), \bar{u}_{i+1}^n(r)]$, where $r \in [0, 1]$.

The error bound is defined as:

$$[\tilde{\varepsilon}_i^n]_r = [\hat{u}_i^n - \tilde{u}_i^n]_r, \tag{24}$$

where

$$[\tilde{\varepsilon}_i^n]_r = \{ \underline{\varepsilon}_i^n(r), \bar{\varepsilon}_i^n(r) \} = \left\{ \begin{aligned} &\underline{u}_i^{n'}(r) - \underline{u}_i^n(r) \\ &\bar{u}_i^{n'}(r) - \bar{u}_i^n(r) \end{aligned} \right. \tag{25}$$

which satisfies the finite difference Eq. 1.

For $n = 1$

$$\begin{cases} -p\underline{\varepsilon}_i^1(r) + (1 + 2p)\underline{\varepsilon}_i^1(r) - s \underline{\varepsilon}_{i-1}^n(r) = \underline{\varepsilon}_i^0(r) \\ -p\bar{\varepsilon}_i^1(r) + (1 + 2p)\bar{\varepsilon}_i^1(r) - s \bar{\varepsilon}_{i-1}^n(r) = \bar{\varepsilon}_i^0(r) \end{cases} \tag{26}$$

For $n \geq 2$

$$\begin{cases} -s \underline{\varepsilon}_{i+1}^n + (1 + 2p)\underline{\varepsilon}_i^n - r \underline{\varepsilon}_{i-1}^n = - \sum_{j=1}^{n-1} v_j \underline{\varepsilon}_i^{n-j} + (\sum_{j=0}^{n-1} v_j) \underline{\varepsilon}_i^0 \\ -s \bar{\varepsilon}_{i+1}^n + (1 + 2p)\bar{\varepsilon}_i^n - r \bar{\varepsilon}_{i-1}^n = - \sum_{j=1}^{n-1} v_j \bar{\varepsilon}_i^{n-j} + (\sum_{j=0}^{n-1} v_j) \bar{\varepsilon}_i^0 \end{cases} \tag{27}$$

Suppose that

$$\begin{cases} \underline{\varepsilon}_i^n = \underline{\lambda}^n(r) e^{\sqrt{-\underline{\theta}}(r)i} \\ \bar{\varepsilon}_i^n = \bar{\lambda}^n(r) e^{\sqrt{-\bar{\theta}}(r)i} \end{cases} \tag{28}$$

Substituting Eq. 28 into Eqs. (26, 27) to obtain:

For $n = 1$

$$\begin{cases} -p \underline{\lambda}(r)e^{\sqrt{-\underline{\theta}(r)(i+1)}} + (1 + 2p)\underline{\lambda}(r)e^{\sqrt{-\underline{\theta}(r)i}} - p \underline{\lambda}(r) e^{\sqrt{-\underline{\theta}(r)(i-1)}} = e^{\sqrt{-\underline{\theta}(r)i}} \\ -p \bar{\lambda}(r)e^{\sqrt{-\bar{\theta}(r)(i+1)}} + (1 + 2p)\bar{\lambda}(r)e^{\sqrt{-\bar{\theta}(r)i}} - p \bar{\lambda}(r) e^{\sqrt{-\bar{\theta}(r)(i-1)}} = e^{\sqrt{-\bar{\theta}(r)i}} \end{cases} \tag{29}$$

For $n \geq 2$

$$\begin{cases} -p \underline{\lambda}^n(r)e^{\sqrt{-\underline{\theta}(r)(i+1)}} + (1 + 2p)\underline{\lambda}^n(r)e^{\sqrt{-\underline{\theta}(r)i}} - p \underline{\lambda}^n(r)e^{\sqrt{-\underline{\theta}(r)(i-1)}} \\ = -\sum_{j=1}^{n-1} v_j \underline{\lambda}^{n-j}(r) e^{\sqrt{-\underline{\theta}(r)i}} + \left(\sum_{j=0}^{n-1} v_j\right) e^{\sqrt{-\underline{\theta}(r)i}} \\ -p \bar{\lambda}^n(r)e^{\sqrt{-\bar{\theta}(r)(i+1)}} + (1 + 2p)\bar{\lambda}^n(r)e^{\sqrt{-\bar{\theta}(r)i}} - p \bar{\lambda}^n(r)e^{\sqrt{-\bar{\theta}(r)(i-1)}} \\ = -\sum_{j=1}^{n-1} v_j \bar{\lambda}^{n-j}(r) e^{\sqrt{-\bar{\theta}(r)i}} + \left(\sum_{j=0}^{n-1} v_j\right) e^{\sqrt{-\bar{\theta}(r)i}} \end{cases} \tag{30}$$

Now let $\begin{cases} \left| \frac{\underline{\lambda}^m}{\bar{\lambda}^m} \right| \leq 1, & m = 1, 2, 3, \dots, n-1 \\ \left| \frac{\bar{\lambda}^m}{\underline{\lambda}^m} \right| \leq 1, & m = 1, 2, 3, \dots, n-1 \end{cases}$

In [17], there is a lemma which states:

The coefficients $v_j = (-1)^j \binom{\alpha}{j}$ ($j = 0, 1, 2, \dots$) satisfy:

Begin with $n = 1$, in Eq. 29 to obtain

$$\begin{cases} -p \underline{\lambda}(r)e^{\sqrt{-\underline{\theta}(r)(i+1)}} + (1 + 2p)\underline{\lambda}(r)e^{\sqrt{-\underline{\theta}(r)i}} - p \underline{\lambda}(r) e^{\sqrt{-\underline{\theta}(r)(i-1)}} = e^{\sqrt{-\underline{\theta}(r)i}} \\ -p \bar{\lambda}(r)e^{\sqrt{-\bar{\theta}(r)(i+1)}} + (1 + 2p)\bar{\lambda}(r)e^{\sqrt{-\bar{\theta}(r)i}} - p \bar{\lambda}(r) e^{\sqrt{-\bar{\theta}(r)(i-1)}} = e^{\sqrt{-\bar{\theta}(r)i}} \end{cases} \tag{31}$$

Divide Eq. 31 on $e^{\sqrt{-\theta i}}$ to obtain:

$$\begin{cases} -p \underline{\lambda}(r)e^{\sqrt{-\underline{\theta}(r)i}} + (1 + 2p)\underline{\lambda}(r) - p \underline{\lambda}(r) e^{-\sqrt{-\underline{\theta}(r)i}} = 1 \\ -p \bar{\lambda}(r)e^{\sqrt{-\bar{\theta}(r)i}} + (1 + 2p)\bar{\lambda}(r) - p \bar{\lambda}(r) e^{-\sqrt{-\bar{\theta}(r)i}} = 1 \end{cases} \tag{32}$$

1. $v_0 = 1, v_j < 0 \quad j = 1, 2, 3, \dots$
2. $\sum_{j=0}^{k-1} v_j > 0 \quad k = 2, 3, \dots$
3. From this lemma and Eq. 30, we obtain
- 4.

$$\begin{cases} \underline{\lambda}(r)[(1 + 2p) - p (e^{\sqrt{-\underline{\theta}(r)i}} + e^{-\sqrt{-\underline{\theta}(r)i})] = 1 \\ \bar{\lambda}(r)[(1 + 2p) - p (e^{\sqrt{-\bar{\theta}(r)i}} + e^{-\sqrt{-\bar{\theta}(r)i})] = 1 \\ \underline{\lambda}(r)[(1 + 2p) - 2p \cos \underline{\theta}(r)] = 1 \\ \bar{\lambda}(r)[(1 + 2p) - 2p \cos \bar{\theta}(r)] = 1 \\ \underline{\lambda}(r)[1 + 2p(1 - \cos \underline{\theta}(r))] = 1 \\ \bar{\lambda}(r)[1 + 2p(1 - \cos \bar{\theta}(r))] = 1 \end{cases} \tag{33}$$

According to the Zadeh extension principle [21], we obtain:

Divide Eq. 33 on $\begin{cases} e^{\sqrt{-\underline{\theta}i}} \\ e^{\sqrt{-\bar{\theta}i}} \end{cases}$ to obtain:

$$\begin{cases} -p \underline{\lambda}^n(r)e^{\sqrt{-\underline{\theta}(r)i}} + (1 + 2p)\underline{\lambda}^n(r) - p \underline{\lambda}^n(r)e^{-\sqrt{-\bar{\theta}i}} = -\sum_{j=1}^{n-1} v_j \underline{\lambda}^{n-j}(r) + \sum_{j=0}^{n-1} v_j \\ -p \bar{\lambda}^n(r)e^{\sqrt{-\bar{\theta}(r)i}} + (1 + 2p)\bar{\lambda}^n(r) - p \bar{\lambda}^n(r)e^{-\sqrt{-\bar{\theta}(r)i}} = -\sum_{j=1}^{n-1} v_j \bar{\lambda}^{n-j}(r) + \sum_{j=0}^{n-1} v_j \end{cases} \tag{34}$$

$$\begin{cases} \underline{\lambda}^n(r)[(1 + 2p) - p (e^{\sqrt{-\underline{\theta}(r)i}} + e^{-\sqrt{-\underline{\theta}(r)i})] = -\sum_{j=1}^{n-1} v_j \underline{\lambda}^{n-j}(r) + \sum_{j=0}^{n-1} v_j \\ \bar{\lambda}^n(r)[(1 + 2p) - p (e^{\sqrt{-\bar{\theta}(r)i}} + e^{-\sqrt{-\bar{\theta}(r)i})] = -\sum_{j=1}^{n-1} v_j \bar{\lambda}^{n-j}(r) + \sum_{j=0}^{n-1} v_j \\ \underline{\lambda}^n(r)[(1 + 2p) - p (2 \cos \underline{\theta}(r))] = -\sum_{j=1}^{n-1} v_j \underline{\lambda}^{n-j}(r) + \sum_{j=0}^{n-1} v_j \\ \bar{\lambda}^n(r)[(1 + 2p) - p (2 \cos \bar{\theta}(r))] = -\sum_{j=1}^{n-1} v_j \bar{\lambda}^{n-j}(r) + \sum_{j=0}^{n-1} v_j \end{cases} \tag{35}$$

$$\begin{cases} \underline{\lambda}^n(r)[1 + 2p(1 - \cos \underline{\theta}(r))] = -\sum_{j=1}^{n-1} v_j \underline{\lambda}^{n-j}(r) + \sum_{j=0}^{n-1} v_j \\ \bar{\lambda}^n(r)[1 + 2p(1 - \cos \bar{\theta}(r))] = -\sum_{j=1}^{n-1} v_j \bar{\lambda}^{n-j}(r) + \sum_{j=0}^{n-1} v_j \end{cases}$$

So,

$$\begin{cases} \underline{\lambda}^n(r) = \frac{1}{1 + 2p(1 - \cos\underline{\theta}(r))} * \left[-\sum_{j=1}^{n-1} v_j \underline{\lambda}^{n-j}(r) + \sum_{j=0}^{n-1} v_j \right] \\ \overline{\lambda}^n(r) = \frac{1}{1 + 2p(1 - \cos\overline{\theta}(r))} * \left[-\sum_{j=1}^{n-1} v_j \overline{\lambda}^{n-j}(r) + \sum_{j=0}^{n-1} v_j \right] \end{cases}$$

Therefore,

$$\begin{cases} |\underline{\lambda}^n(r)| = \frac{1}{1 + 2p(1 - \cos\underline{\theta}(r))} * \left[-\sum_{j=1}^{n-1} v_j \underline{\lambda}^{n-j}(r) + \sum_{j=0}^{n-1} v_j \right] \\ |\overline{\lambda}^n(r)| = \frac{1}{1 + 2p(1 - \cos\overline{\theta}(r))} * \left[-\sum_{j=1}^{n-1} v_j \overline{\lambda}^{n-j}(r) + \sum_{j=0}^{n-1} v_j \right] \end{cases}$$

Thus, $\begin{cases} |\underline{\lambda}^n(r)| \leq 1 \\ |\overline{\lambda}^n(r)| \leq 1 \end{cases}$

Therefore, according to Von Neumann’s criterion for stability, the fuzzy implicit finite difference scheme defined by Eq. 1 is unconditionally stable for all r -level set and for all $0 < \alpha < 1$.

5 Numerical example

In this section, we implement the implicit finite difference approximations to solve fuzzy time fractional diffusion equations for different orders of α to investigate the implicit finite difference method. The Wolfram

$$\begin{cases} \underline{\lambda}^n(r) = \frac{1}{1 + 2p(1 - \cos\underline{\theta}(r))} * \left[\sum_{j=1}^{n-1} (-v_j) \underline{\lambda}^{n-j}(r) + \sum_{j=0}^{n-1} v_j \right] \\ \overline{\lambda}^n(r) = \frac{1}{1 + 2p(1 - \cos\overline{\theta}(r))} * \left[\sum_{j=1}^{n-1} (-v_j) \overline{\lambda}^{n-j}(r) + \sum_{j=0}^{n-1} v_j \right] \\ \underline{\lambda}^n(r) \leq \frac{1}{1 + 2p(1 - \cos\underline{\theta}(r))} * \left[\sum_{j=1}^{n-1} (-v_j) |\underline{\lambda}^{n-j}(r)| + \sum_{j=0}^{n-1} v_j \right] \\ \overline{\lambda}^n(r) \leq \frac{1}{1 + 2p(1 - \cos\overline{\theta}(r))} * \left[\sum_{j=1}^{n-1} (-v_j) |\overline{\lambda}^{n-j}(r)| + \sum_{j=0}^{n-1} v_j \right] \\ \underline{\lambda}^n(r) \leq \left[\sum_{j=1}^{n-1} (-v_j) + \sum_{j=0}^{n-1} v_j \right] * \frac{1}{1 + 2p(1 - \cos\underline{\theta}(r))} \\ \overline{\lambda}^n(r) \leq \left[\sum_{j=1}^{n-1} (-v_j) + \sum_{j=0}^{n-1} v_j \right] * \frac{1}{1 + 2p(1 - \cos\overline{\theta}(r))} \\ \underline{\lambda}^n(r) = \frac{1}{1 + 2p(1 - \cos\underline{\theta}(r))} [(-v_1) + (-v_2) + \dots + (-v_{n-1}) + v_0 + v_1 + v_2 + \dots + v_{n-1}] \\ \overline{\lambda}^n(r) = \frac{1}{1 + 2p(1 - \cos\overline{\theta}(r))} [(-v_1) + (-v_2) + \dots + (-v_{n-1}) + v_0 + v_1 + v_2 + \dots + v_{n-1}] \\ \underline{\lambda}^n(r) = \frac{1}{1 + 2p(1 - \cos\underline{\theta}(r))} * v_0 \\ \overline{\lambda}^n(r) = \frac{1}{1 + 2p(1 - \cos\overline{\theta}(r))} * v_0 \\ \underline{\lambda}^n(r) = \text{Min} \left[\frac{1}{1 + 2p(1 - \cos\underline{\theta}(r))} \right] \leq 1 \\ \overline{\lambda}^n(r) = \text{Max} \left[\frac{1}{1 + 2p(1 - \cos\overline{\theta}(r))} \right] \leq 1 \end{cases}$$

Mathematica10 software was used to conduct the numerical experiment for the proposed method.

Example 1 Consider fuzzy time fractional diffusion equations [10]

$$\frac{\partial^\alpha \tilde{u}(x, t)}{\partial t^\alpha} = \frac{\partial^2 \tilde{u}(x, t)}{\partial x^2}, \quad 0 < x < l, \quad t > 0 \tag{36}$$

subject to the boundary conditions $\tilde{u}(0, t) = \tilde{u}(1, t) = 0$ and initial condition

$$\tilde{u}(x, 0) = \tilde{k} \sin(\pi x), \quad 0 < x < 1, \tag{37}$$

where $\tilde{\alpha}(r) = [0.1r - 0.1, 0.1 - 0.1r]$ for all $r \in [0, 1]$. The exact solution of Eq. 36 was given in [10]:

$$\tilde{u}(x, t, \alpha; r) = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n} t^{n\alpha}}{\Gamma(n\alpha + 1)} \tilde{k}(r) \sin(\pi x) \tag{38}$$

The absolute error of the solution of Eq. 36 can be defined as:

$$[\tilde{E}]_r = |\tilde{u}(t, x; r) - \tilde{u}(t, x; r)| = \begin{cases} [\underline{E}]_r = |\underline{u}(t, x; r) - \underline{u}(t, x; r)| \\ [\overline{E}]_r = |\overline{u}(t, x; r) - \overline{u}(t, x; r)| \end{cases} \tag{39}$$

According to Eqs. (20, 21) in Sect. 3 the implicit finite difference method formula for solving Eq. 36 is as follows:

$$\begin{aligned} \Delta t^{-\alpha} \sum_{j=0}^n v_j (\underline{u}_i^{n-j} - \underline{u}_i^0) \\ = \frac{\underline{u}_{i+1,n}(x, t; r) - 2\underline{u}_{i,n}(x, t; r) + \underline{u}_{i-1,n}(x, t; r)}{h^2} \end{aligned} \tag{40}$$

$$\begin{aligned} \Delta t^{-\alpha} \sum_{j=0}^n v_j (\overline{u}_i^{n-j} - \overline{u}_i^0) \\ = \frac{\overline{u}_{i+1,n}(x, t; r) - 2\overline{u}_{i,n}(x, t; r) + \overline{u}_{i-1,n}(x, t; r)}{h^2} \end{aligned} \tag{41}$$

As explained in Sect. 3 we obtain

$$\begin{aligned} -p\underline{u}_{i+1,n}(x, t; r) + (1 + 2p)\underline{u}_{i,n}(x, t; r) - p\underline{u}_{i-1,n}(x, t; r) \\ = -\sum_{j=1}^{n-1} v_j \underline{u}_{i,n-j} + \left(\sum_{j=0}^{n-1} v_j \right) \underline{u}_{i,0} \end{aligned} \tag{42}$$

$$\begin{aligned} -p\overline{u}_{i+1,n}(x, t; r) + (1 + 2p)\overline{u}_{i,n}(x, t; r) - p\overline{u}_{i-1,n}(x, t; r) \\ = -\sum_{j=1}^{n-1} v_j \overline{u}_{i,n-j} + \left(\sum_{j=0}^{n-1} v_j \right) \overline{u}_{i,0} \end{aligned} \tag{43}$$

At $\Delta x = h = 0.1$ and $\Delta t^\alpha = (0.01)^{0.5} = 0.1$ to get $p(r) = \frac{\Delta t^\alpha}{h^2} = \frac{0.1}{0.1^2}$ we have the following results:

Tables 1 and 2 and Figs. 1, 2, 3, 4 and 5 show that both the implicit finite difference and exact solution at $t = 0.05$,

Table 1 Lower solution of Eq. 36 by implicit FDM at $t = 0.05$, $\alpha = 0.5$ and for all $r \in [0, 1]$

r -level	$\underline{u}(0.9, 0.05; r)$	$[\underline{E}]_r$
0	-0.007854034999317416	0.0005933599789698339
0.2	-0.006283227999453932	0.0004746879831758668
0.4	-0.004712420999590449	0.0003560159873818988
0.6	-0.0031416139997269667	0.00023734399158793383
0.8	-0.0015708069998634825	0.00011867199579396691
1	0	0

Table 2 Upper solution of Eq. 36 by implicit FDM at $t = 0.05$, $\alpha = 0.5$ and for all $r \in [0, 1]$

r -level	$\overline{u}(0.9, 0.05; r)$	$[\overline{E}]_r$
0	0.007854034999317416	0.0005933599789698339
0.2	0.006283227999453932	0.0004746879831758668
0.4	0.004712420999590449	0.0003560159873818988
0.6	0.0031416139997269667	0.00023734399158793383
0.8	0.0015708069998634825	0.00011867199579396691
1	0	0

$\alpha = 0.5$ and for all $r \in [0, 1]$ attain the triangular fuzzy number shape and thus satisfy the fuzzy number properties as explained in [21].

Now we compare between the numerical and exact solutions of Eq. 36, for different orders of α .

Figures 6, 7 and 8 shows that both the implicit finite difference and exact solutions satisfy the fuzzy number properties by attaining the triangular fuzzy number shape. Also, the exact solution agrees with the implicit finite difference solutions for different values of α . The comparison of numerical and exact solutions when $\alpha =$

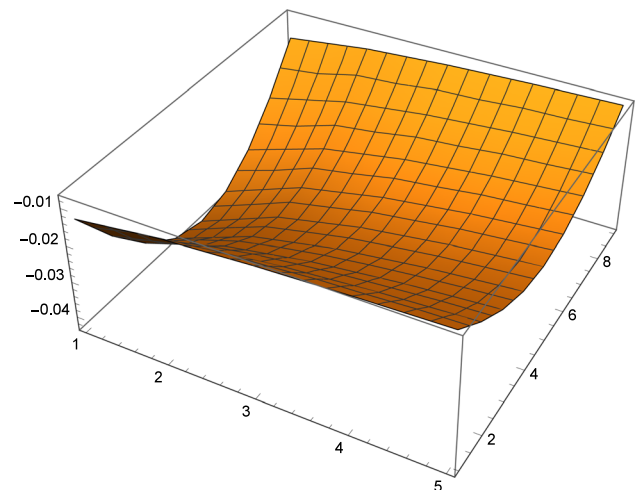


Fig. 1 The exact solution for the lower bound of Eq. 36 at $\Delta t = 0.01$, $h = 0.1$ and $r = 0$

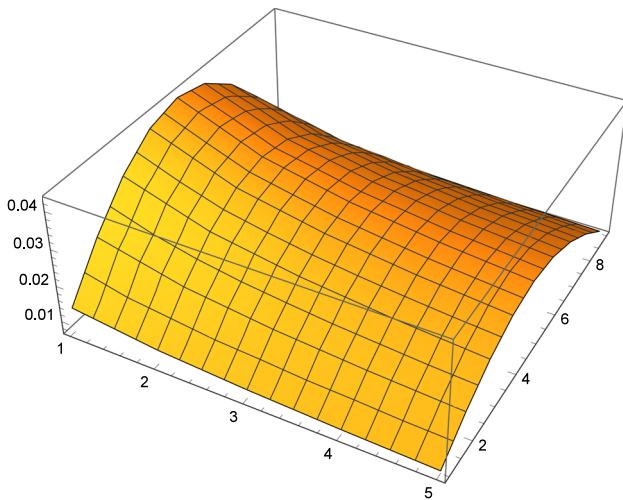


Fig. 2 The exact solution for the upper bound of Eq. 36 at $\Delta t = 0.01$, $h = 0.1$ and $r = 0$

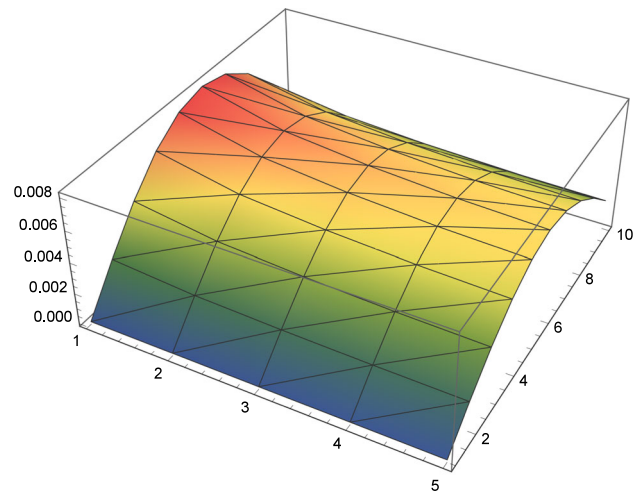


Fig. 5 the lower implicit FDM of the solution of Eq. 36 at $\alpha = 0.2$ for all $r \in [0, 1]$

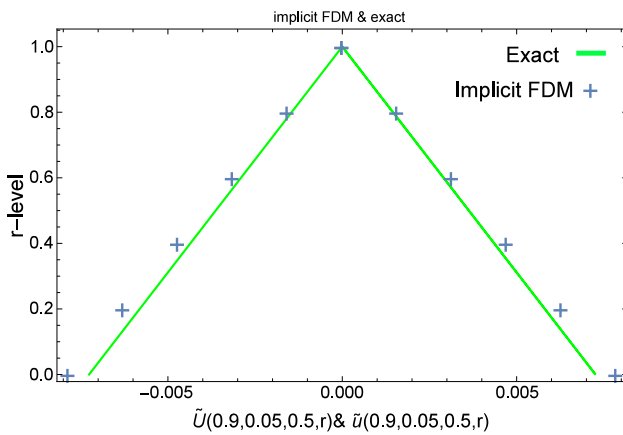


Fig. 3 Exact and implicit FDM of the solution of Eq. 36 at $\alpha = 0.5$, $x = 0.9$, $t = 0.05$ for all $r \in [0, 1]$

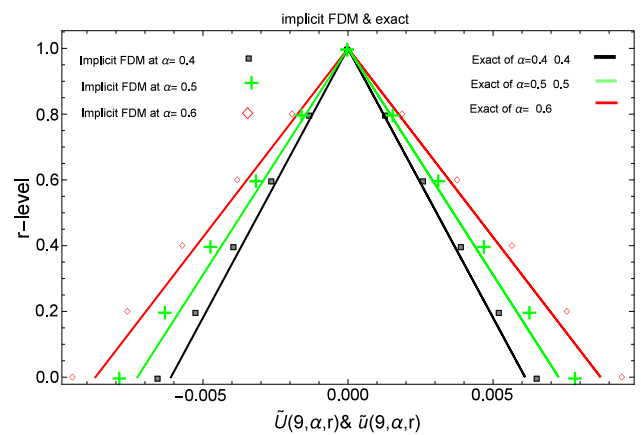


Fig. 6 the lower implicit FDM of the solution of Eq. 36 at $\alpha = 0.5$ for all $r \in [0, 1]$

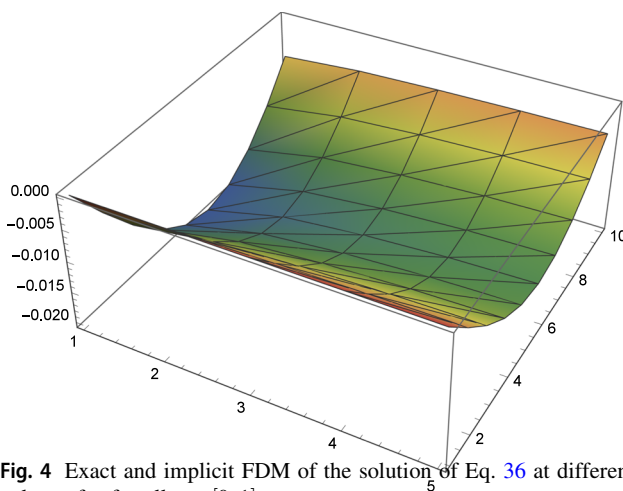


Fig. 4 Exact and implicit FDM of the solution of Eq. 36 at different values of α for all $r \in [0, 1]$

0.4, 0.5, 0.6 shows that the scheme is accurate and the results confirm our theoretical analysis.

Example 2 Consider fuzzy time fractional diffusion equations [14]

$$\frac{\partial^\alpha \tilde{u}(x, t)}{\partial t^\alpha} = \frac{1}{2} \cdot x^2 \frac{\partial^2 \tilde{u}(x, t)}{\partial x^2}, \quad 0 < x < l, t > 0 \tag{44}$$

subject to the boundary conditions $\tilde{u}(0, t) = \tilde{u}(1, t) = 0$ and initial condition

$$\tilde{u}(x, 0) = \tilde{\alpha}x^2, \quad 0 < x < 1, \tag{45}$$

where $\tilde{\alpha}(r) = [0.1r - 0.01, 0.01 - 0.01r]$ for all $r \in [0, 1]$. The exact solution of Eq. 44 was given in [10] :

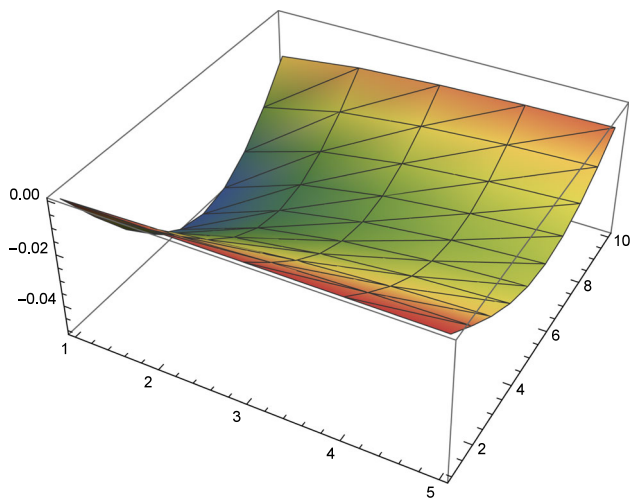


Fig. 7 the upper implicit FDM of the solution of Eq. 36 at $\alpha = 0.2$ for all $r \in [0, 1]$

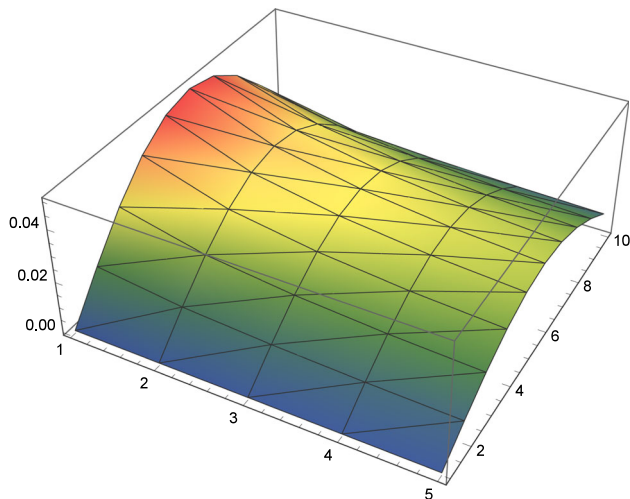


Fig. 8 the upper implicit FDM of the solution of Eq. 36 at $\alpha = 0.5$ for all $r \in [0, 1]$

$$\tilde{u}(x, t, \alpha; r) = \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} \tilde{k}(r)x^2 \tag{46}$$

The absolute error of the solution of Eq. 44 can be defined as:

Table 3 Lower solution of Eq. 44 by implicit FDM at $t = 0.05, \alpha = 0.7$ and for all $r \in [0, 1]$

r -level	$\bar{u}(0.9, 0.05; r)$	$[E]_r$
0	$-32217126986791893 \times 10^{-5}$	$8924974815617505 \times 10^{-6}$
0.2	$-25773701589535144 \times 10^{-5}$	$7139979852492011 \times 10^{-6}$
0.4	$-19330276192151353 \times 10^{-5}$	$5354984889369801 \times 10^{-6}$
0.6	$-12886850794767572 \times 10^{-5}$	$3569989926246054 \times 10^{-6}$
0.8	$-64434253978383786 \times 10^{-6}$	$1784994963130027 \times 10^{-7}$
1	0	0

$$[\tilde{E}]_r = |\tilde{u}(t, x; r) - \tilde{u}(t, x; r)| = \begin{cases} [E]_r = |\underline{u}(t, x; r) - \underline{u}(t, x; r)| \\ [E]_r = |\overline{u}(t, x; r) - \overline{u}(t, x; r)| \end{cases} \tag{47}$$

According to Eqs. (20, 21) in Sect. 3 the implicit finite difference method formula for solving Eq. 44 is as follows:

$$\begin{aligned} \Delta t^{-\alpha} \sum_{j=0}^n v_j (\underline{u}_i^{n-j} - \underline{u}_i^0) \\ = \frac{1}{2} x^2 \frac{\underline{u}_{i+1,n}(x, t; r) - 2\underline{u}_{i,n}(x, t; r) + \underline{u}_{i-1,n}(x, t; r)}{h^2} \end{aligned} \tag{48}$$

$$\begin{aligned} \Delta t^{-\alpha} \sum_{j=0}^n v_j (\overline{u}_i^{n-j} - \overline{u}_i^0) \\ = \frac{1}{2} x^2 \frac{\overline{u}_{i+1,n}(x, t; r) - 2\overline{u}_{i,n}(x, t; r) + \overline{u}_{i-1,n}(x, t; r)}{h^2} \end{aligned} \tag{49}$$

As explained in Sect. 3 we obtain

$$\begin{aligned} -p\underline{u}_{i+1,n}(x, t; r) + (1 + 2p)\underline{u}_{i,n}(x, t; r) - p\underline{u}_{i-1,n}(x, t; r) \\ = -\sum_{j=1}^{n-1} v_j \underline{u}_{i,n-j} + \left(\sum_{j=0}^{n-1} v_j \right) \underline{u}_{i,0} \end{aligned} \tag{50}$$

$$\begin{aligned} -p\overline{u}_{i+1,n}(x, t; r) + (1 + 2p)\overline{u}_{i,n}(x, t; r) - p\overline{u}_{i-1,n}(x, t; r) \\ = -\sum_{j=1}^{n-1} v_j \overline{u}_{i,n-j} + \left(\sum_{j=0}^{n-1} v_j \right) \overline{u}_{i,0} \end{aligned} \tag{51}$$

At $\Delta x = h = 0.1$ and $\Delta t^\alpha = (0.001)^{0.5} = 0.01$ to get $p(r) = \frac{1}{2} x^2 \frac{\Delta t^\alpha}{h^2}$ we have the following results:

Tables 3 and 4 and Fig. 9 show that both the implicit finite difference and exact solution at $t = 0.05, \alpha = 0.7$ and for all $r \in [0, 1]$ attain the triangular fuzzy number shape and thus satisfy the fuzzy number properties as explained in [21].

Now we compare between the numerical and exact solutions of Eq. 44, for different orders of α .

Figure 10 shows that both the implicit finite difference and exact solutions satisfy the fuzzy number properties by attaining the triangular fuzzy number shape. Also, the exact solution agrees with the implicit finite difference solutions

Table 4 Upper solution of Eq. 44 by implicit FDM at $t = 0.05$, $\alpha = 0.7$ and for all $r \in [0, 1]$

r -level	$\bar{u}(0.9, 0.05; r)$	$[E]_r$
0	$32217126986791893 \times 10^{-5}$	$8924974815617505 \times 10^{-6}$
0.2	$25773701589535144 \times 10^{-5}$	$7139979852492011 \times 10^{-6}$
0.4	$19330276192151353 \times 10^{-5}$	$5354984889369801 \times 10^{-6}$
0.6	$12886850794767572 \times 10^{-5}$	$3569989926246054 \times 10^{-6}$
0.8	$64434253978383786 \times 10^{-6}$	$1784994963130027 \times 10^{-7}$
1	0	0

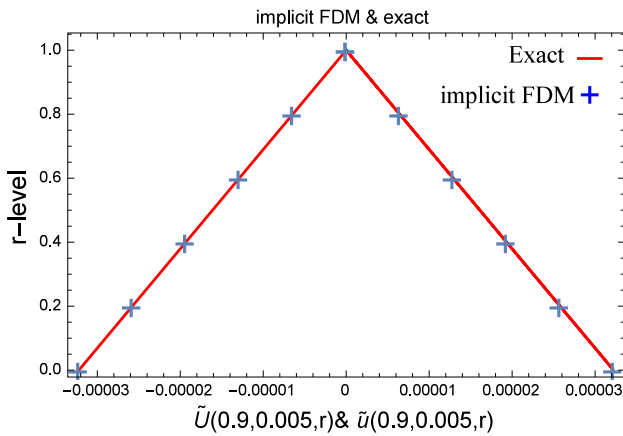


Fig. 9 Exact and implicit FDM of the solution of Eq. 44 at $\alpha = 0.7$, $x = 0.9$, $t = 0.05$ and for all $r \in [0, 1]$

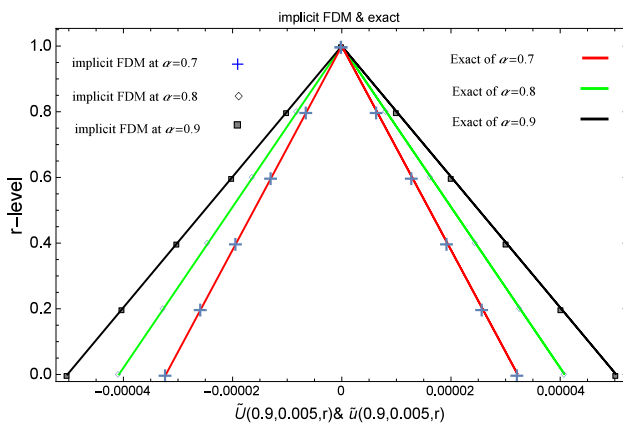


Fig. 10 Exact and implicit FDM of the solution of Eq. 44 at different values of α for all $r \in [0, 1]$

for different values of α . The comparison of numerical and exact solutions when $\alpha = 0.7, 0.8, 0.9$ shows that the scheme is accurate and the results confirm our theoretical analysis.

6 Conclusions

In this paper, an implicit finite difference scheme has been implemented to obtain the numerical solution for a fuzzy time fractional diffusion equations. The Caputo formula was used for the time fractional derivative. The obtained results by the implicit finite difference scheme satisfy the fuzzy number properties by taking the triangular fuzzy number shape. We have also shown that the implicit finite difference scheme is unconditionally stable. A comparative study of the numerical and exact solution at different values of α indicates that the scheme is feasible and accurate. The proposed implicit scheme can be used to obtain accurate numerical solutions of fuzzy time fractional diffusion equations. The presented scheme may be extended to nonlinear fuzzy fractional diffusion equations, and this will be investigated in detail at a later stage.

Compliance with ethical standards

Conflict of interest All authors declare they have no conflict of interest.

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