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ORIGINAL ARTICLE

Correlation measure of hesitant fuzzy soft sets and their application in decision making

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Abstract Hesitant fuzzy soft set (HFSS) allows each element to have different number of parameters and the values of those parameters are represented by multiple possible membership values. HFSS is considered as a powerful tool to represent uncertain information in group decision-making process. In this study, we introduce the concept of correlation coefficient for HFSS and some of its properties. Using correlation coefficient of HFSS, we develop correlation efficiency which shows the significance of the HFSS. We also propose an algorithm to apply correlation coefficient in decision-making problem, where information is presented in hesitant fuzzy environment. In order to extend the application of HFSS, we propose correlation coefficient in the framework of interval-valued hesitant fuzzy soft set (IVHFSS). We also introduce correlation efficiency in the context of IVHFSS. Then the proposed algorithm is extended using IVHFSS for solving decision-making problems. Finally, two examples that are semantically meaningful in real life are illustrated to show the effectiveness of the proposed algorithms.

Keywords HFSS · IVHFSS · Correlation coefficient · Correlation efficiency · GDM

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1 Introduction

In statistical analysis, the correlation coefficient imparts a vital role to measure the strength of the linear relationship between two variables, whereas in fuzzy set theory the correlation measure determines the degree of dependency between two fuzzy sets. It has been proved to be an important measure in data analysis, medical diagnosis, pattern recognition, and especially for decision-making problems [1–6]. An extensive review of decision-making techniques and applications is given in [7]. Hung and Wu [8] first defined the correlation coefficient of fuzzy numbers using the concept of expected value. Gerstenkorn and Manko [9] introduced the correlation coefficients of intuitionistic fuzzy sets (IFSs) which was later analyzed by Szmidt and Kacprzyk [4]. Bustince and Burillo [10] and Hong [11] further extended the concepts of correlation and correlation coefficient for interval-valued intuitionistic fuzzy sets (IVIFSs). Hung and Wu [12] proposed a method to find the correlation coefficients of IFSs by means of centroid. HFS was introduced by Torra and Narukawa [13] and Torra [14]. In order to aggregate hesitant fuzzy information, Yu [15] proposed Heronian mean (HM) aggregation operator and solved a multi-criteria decision-making (MCDM) problem using the proposed HM operator. In [16], Feng et al. studied induced hesitant fuzzy Hamacher correlated geometric (IHFHCG) operator and applied it in real-life problems. Chen et al. [17] derived some correlation coefficient formulas for hesitant fuzzy sets (HFSs) and applied them to clustering analysis in hesitant fuzzy environments. They also developed correlation coefficient formulas for interval-valued HFSs (IVHFSs) and then demonstrated their applications in clustering analysis. For the purpose of defining correlation coefficients of HFSs and IVHFSs, the authors considered that both HFSs and IVHFSs have same length and their values are arranged in ascending order. When the HFSs and IVHFSs do not have same length, the shorter one is extended up to the length of longer one. Meng et al. [18], in their paper, proposed some new correlation coefficients of IVHFSs, where they did not consider the lengths of interval-valued hesitant fuzzy elements (IVHFEs) and the arrangement of their interval values. The authors proposed Shapely weighted correlation coefficient when the elements of a set are correlative. Correlation measures for hesitant fuzzy information and their properties were proposed by Xu and Xia [19]. Tong and Yu [20] emphasized on multiple attribute decision-making (MADM) problems that simultaneously determine attribute weights and decision maker (DM) preferences, and proposed a MADM approach in hesitant fuzzy environment based on extended distance and correlation coefficient measure. The relationship between entropy, similarity measure, and distance measure for hesitant fuzzy sets (HFSs) and interval-valued hesitant fuzzy sets (IVHFSs) was investigated by Farhadinia [21]. In [22], Jun introduced interval neutrosophic hesitant fuzzy set (INHFS) and then defined correlation coefficients of INHFSs. The authors investigated the relationship between the similarity measures and the correlation coefficients of INHFSs. Furthermore, they proposed a MADM method based on the correlation coefficients under interval neutrosophic hesitant fuzzy environment. Liao et al. [23] proposed a novel correlation coefficient formula to measure the relationship between two HFSs. Correlation coefficients of dual hesitant fuzzy sets (DHFSs) was defined by Tyagi [24], where (DHFS) is a generalized form of a hesitant fuzzy set (HFS) which negates the effects of uncertainty inherent in the collected data. A set of aggregation operators under dual hesitant fuzzy environment was developed by Yu [25] and Yu et al. [26]. Wang et al. [27] defined the correlation measures for dual hesitant fuzzy information [28] and discussed their properties. Zhu and Xu [29] focused on the information loss problem of HFSs and developed extended hesitant fuzzy sets (EHFSs) as an extension of HFSs to overcome this problem. The concept of correlation measure for hesitant fuzzy linguistic term set (HFLTS) was investigated by Liao et al. [30]. Motivated by the work found in [30], Das et al. [31] proposed correlation coefficient in the context of hesitant fuzzy linguistic term soft set (HFLTSS) and applied it in decision-making problems.

Recently, Das and Kar [32] have introduced HFSS as a hybridization of HFS and soft set [33]. The authors analyzed some basic operations and applied it in decision-making problems. Since then many researchers [34–37] have contributed on the development on HFSS. In [36], Chen defined generalized HFSS and studied some of their properties. In [37], Ismat and Tabasam used TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) in the context of HFSS. As an extension of HFS, Wei et al. [38] introduced IVHFS. Combining IVHFS with soft set, Zhang et al. [39] proposed interval-valued hesitant fuzzy soft set (IVHFSS) which permits the membership degrees to have a few different intervals. When the real-life decision-making problems cannot be expressed by exact values, then IVHFSS can solve it. Peng and Yong [40] discussed in detail about the various operations and properties of IVHFSS, and finally they presented a decision-making algorithm using TOPSIS method. Recently, Das and Kar [41, 42] and Das et al. [43–46] have introduced some hybridized soft sets and applied them in decision-making problems.

Presently, measuring correlation coefficient of various fuzzy sets has drawn more attention of a number of researchers because correlation measure has proved its potential to decision-making paradigm in uncertain environment. But when some parameters are associated with the set elements, i.e., in case of soft sets, no idea of correlation coefficient is given. As per our knowledge, there is no research work on the application of the correlation coefficient of soft sets to multi-criteria decision-making problems in the existing literature. This is very much essential to derive the degree of cohesiveness among the soft sets, as soft sets have a huge impact on decision-making process. As we know, when the correlation coefficient of a soft set with the other soft sets will be high, importance of that soft set will be more. This idea motivates us to find out the correlation measure of hesitant fuzzy soft sets. HFSs are useful in situations, where each element of a set is permitted to have a few different membership values, which can arise in a GDM problem. Following example narrates the necessity of HFS. Suppose three experts discuss about the membership degree that an alternative satisfies a criterion. Some experts may assign the membership value as 0.3, some other assigns 0.5, while the rest assigns 0.7. No consistency is found among these experts. For such a case, the satisfactory membership degrees are represented by a hesitant fuzzy element {0.3, 0.5, 0.7}, which is different from fuzzy number 0.3 (or 0.5), the interval-valued fuzzy number [0.3, 0.7] and the intuitionistic fuzzy number (0.3, 0.5). Thus the HFS can reflect all possible opinions of the group members. However, HFS is restricted in the sense that when an alternative is defined, it must be defined for all the given criteria/attributes/parameters. But HFSS is soft or more relaxed as an alternative may not be associated with all the given criteria. Decision makers select the criteria as per their own intuitions. Thus importance of using HFSS in decision-making problem is inevitable and it is very much necessary to develop some theories for HFSS. But little has been done about this issue. However, one of the main limitations of HFSS/HFS is that it permits membership to have exact values. But in real-life decision-making problems, exact membership values are often found to be inadequate or insufficient to represent human thoughts properly. Human judgements can be well expressed by fuzzy values which permits an interval instead of exact value. So in this paper, we present IVHFSS to overcome those limitations. To explore the degree of importance of IVHFSS, we introduce correlation coefficients of IVHFSS.

In this paper, we mainly discuss the correlation measures of HFSS and IVHFSS. To do this, the remainder of the paper is organized as follows. Section 2 presents some basic concepts related to HFSs, HFSSs and correlation measures of HFSs. In Sect. 3 we introduce correlation coefficient for HFSSs and obtain some properties. Section 4 firstly defines correlation efficiency. Then we propose a decision-making algorithm based on HFSSs, correlation measures, correlation efficiency, and hesitant fuzzy-ordered weighted averaging (HFOWA) operator. One real case study is performed to demonstrate the need of the proposed algorithm under hesitant fuzzy environments in Sect. 5. The idea of correlation coefficient relevant with IVHFSS is given in Sect. 6 followed by an algorithmic approach in Sect. 7. Section 8 includes a case study relevant with IVHFSS. Comparative study is given in Sect. 9. Finally Sect. 10 summarizes this study and presents future challenges.

2 Preliminaries

This section introduces some concepts related to HFS, HFSS, and HFSM.

2.1 HFS, HFSS, and HFSM

Definition 1 [13, 14] A HFS M on a fixed set X is defined in terms of a function $h_M(x)$ that returns a subset of [0, 1] when applied to X. The HFS M is represented by $M = \{<x, h_M(x)> | x \in X\}$, where $h_M(x)$ is a set of values in [0, 1] which denotes the possible membership degrees of the element $x \in X$ to the set M. Xia and Xu [47] defined $h = h_M(x)$ as hesitant fuzzy element (HFE) and the set of all HFEs by H.

Different HFEs may have different numbers of values. A HFE $h_M(x)$ with k number of values is defined as $h_M^k(x)$, where k is a positive integer. In HFS, the membership degree of an element is represented by several possible values between 0 and 1. HFSs are useful in situations, where the decision makers hesitate in giving their preferences over the elements.

Example 1 Let $X = \{x_1, x_2, x_3\}$ be a fixed set. $h_M(x_1) = \{0.5, 0.4, 0.6\}, h_M(x_2) = \{0.7, 0.5\}$, and $h_M(x_3) = \{0.2, 0.3, 0.8, 0.7\}$ be the HFEs of $x_i (i = 1, 2, 3)$ to the set M. Then the HFS M is considered as

$$M = \{(x_1, \{0.5, 0.4, 0.6\}), (x_2, \{0.7, 0.5\}), (x_3, \{0.2, 0.3, 0.8, 0.7\})\}.$$

Torra and Narukawa [13] and Torra [14] defined some operations on HFEs. Let h, h_1, and h_2 are HFEs, then

$$(1)\ h^c = \bigcup_{\gamma \in h} \{1 - \gamma\},\ (2)\ h_1 \cup h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \max\{\gamma_1, \gamma_2\},\ (3)\ h_1 \cap h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \min\{\gamma_1, \gamma_2\}.$$

Xia and Xu [47] defined some operations on the HFEs h, h_1 and h_2 as given below.

$$(1)\ h^\lambda = \cup_{\gamma \in h}\{\gamma^\lambda\}, \lambda > 0,\ (2)\ \lambda h = \cup_{\gamma \in h}\{1 - (1-\gamma)^\lambda\}, \lambda > 0,$$
$$(3)\ h_1 \oplus h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\},$$
$$(4)\ h_1 \otimes h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\gamma_1\gamma_2\}.$$

Definition 2 [47] For a HFE h, $s(h) = \frac{1}{\#h}\sum_{\gamma \in h}\gamma$ is called the score function of h, where $\#h$ is the number of elements in h. For any two HFEs h_1 and h_2, if $s(h_1) > s(h_2)$, then $h_1 > h_2$; if $s(h_1) = s(h_2)$ then $h_1 = h_2$.

One can note that $s(h)$ computes the average value of all elements in h. Thus it expresses the average opinion of decision makes. When the average value is higher, the higher will be the score, resulting in a better HFE. But in some special cases, this comparison method fails to differentiate between two HFEs.

For the purpose of decision making using HFEs, Xia and Xu [47] proposed some aggregation operators for HFEs, two of them are defined below, in Definitions 3 and 4. Let $h_j (j = 1, 2, \ldots, n)$ be a collection of HFEs, $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $h_j (j = 1, 2, \ldots, n)$ with $w_j \in [0, 1]$ and $\sum_{j=i}^n w_j = 1$.

Definition 3 [47] A hesitant fuzzy weighted averaging (HFWA) operator is a mapping $H^n \to H$ such that

$$\mathrm{HFWA}(h_1,h_2,\ldots,h_n)=\sum_{j=1}^{n}w_jh_j$$
$$=\cup_{\gamma_1\in h_1,\gamma_2\in h_2,\ldots,\gamma_n\in h_n}\left\{1-\prod_{j=1}^{n}(1-\gamma_j)^{w_j}\right\}.$$

If $w=(1/n,1/n,\ldots,1/n)^T$, then HFWA operator is reduced to hesitant fuzzy averaging (HFA) operator:

$$\mathrm{HFA}(h_1,h_2,\ldots,h_n)=\sum_{j=1}^{n}\frac{1}{n}h_j=\cup_{\gamma_1\in h_1,\gamma_2\in h_2,\ldots,\gamma_n\in h_n}\left\{1-\prod_{j=1}^{n}(1-\gamma_j)^{\frac{1}{n}}\right\}.$$

Definition 4 [47] Let $h_{\sigma(j)}$ be the jth larget among the collection of HFEs $h_j, j=1,2,\ldots,n$. Here $\sigma(1),\sigma(2),\ldots,\sigma(n)$ is considered as a permutation of $1,2,\ldots,n$, such that $h_{\sigma(j-1)}\geq h_{\sigma(j)}$ for all $j=1,2,\ldots,n$. A hesitant fuzzy-ordered weighted averaging (HFOWA) operator is a mapping HFOWA: $H^n\rightarrow H$ such that

$$\mathrm{HFOWA}(h_1,h_2,\ldots,h_n)=\sum_{j=1}^{n}w_jh_{\sigma(j)}$$
$$=\cup_{\gamma_{\sigma(1)}\in h_1,\gamma_{\sigma(2)}\in h_2,\ldots,\gamma_{\sigma(n)}\in h_n}\left\{1-\prod_{j=1}^{n}\left(1-\gamma_{\sigma(j)}\right)^{w_j}\right\}.$$

The HFOWA operator is based on the concept of ordered weighted averaging (OWA) operator, where the HFEs are initially reordered and then combined with their weights.

Definition 5 [32] Let X be a fixed set, $\widetilde{H}(X)$ be the set of all HFSs of X, E be a set of parameters, and $A\subseteq E$. A HFSS over X is defined by the set of ordered pairs $\left(\widetilde{F}_{\{A\}},E\right)$, where $\widetilde{F}_{\{A\}}$ is a mapping given by, $\widetilde{F}_{\{A\}}:E\rightarrow\widetilde{H}(X)$. For any parameter $e\in A$, $\widetilde{F}_{\{A\}}(e)$ is considered a HFS, which can be expressed as $\widetilde{F}_{\{A\}}(e)=\{<x,h_M(x)>|x\in X\}$, where $h_M(x)$ is a set of values in [0, 1].

In MADM, HFSS is considered to be more flexible for representing the decision makers' opinions as it allows them to select a subset of attributes as among the given set of attributes according their own intuition in an unbiased manner.

Example 2 Let X be set of three shirts, i.e., $X=\{x_1,x_2,x_3\}$. Let $E=\{e_1,e_2,e_3\}=$ {bright, colorful, light}. Suppose $\widetilde{F}_{\{A\}}(e_1)=(\{x_1,(0.3,0.2,0.7)\},\{x_2,(0.5,0.8)\},\{x_3,(0.3,0.4,0.6)\})$, $\widetilde{F}_{\{A\}}(e_2)=(\{x_1,(0.4,0.7)\},\{x_2,(0.4,0.1,0.5)\},\{x_3,(0.4,0.9)\})$, and $\widetilde{F}_{\{A\}}(e_3)=(\{x_1,(0.5,0.4,0.2)\},\{x_2,(0.4,0.1)\},\{x_3,(0.3,0.6)\})$. Thus HFSS is represented as

$$(\widetilde{F}_A,E)=\left\{\begin{array}{l}\langle e_1,(\{x_1,(0.3,0.2,0.7)\},\{x_2,(0.5,0.8)\},\{x_3,(0.3,0.4,0.6)\})\rangle\\ \langle e_2,(\{x_1,(0.4,0.7)\},\{x_2,(0.4,0.1,0.5)\},\{x_3,(0.4,0.9)\})\rangle\\ \langle e_3,(\{x_1,(0.5,0.4,0.2)\},\{x_2,(0.4,0.1)\},\{x_3,(0.3,0.6)\})\rangle\end{array}\right\}.$$

Definition 6 [32] HFSS can be represented well by HFSM. If $X=\{x_1,x_2,\ldots,x_m\}$ and $E=\{e_1,e_2,\ldots,e_n\}$, then HFSM is defined as $\mathcal{F}=(f_{ij})_{m\times n}$ where $f_{ij}=h^k_M(x_i,e_j)$, $i=1,2,\ldots,m$, $j=1,2,\ldots,n$. Here $k>0$ is the number of values which may be different for different HFE. Table 1 presents a HFSM.

Example 3 Let us consider Example 2. According to Definition 6, the corresponding HFSM is given below.

$$\mathcal{F}=(f_{ij})_{m\times n}=\begin{bmatrix}(0.3,0.2,0.7) & (0.4,0.7) & (0.5,0.4,0.2)\\ (0.5,0.8) & (0.4,0.1,0.5) & (0.4,0.1)\\ (0.3,0.4,0.6) & (0.4,0.9) & (0.3,0.6)\end{bmatrix}.$$

2.2 Correlation measure of hesitant fuzzy set

Let $X=\{x_1,x_2,\ldots,x_m\}$ be a discrete universe of discourse, A and B are two HFSs on X denoted as $A=\{(x_i,h_A(x_i))\,|\,x_i\in X,\ i=1,2,\ldots,m\}$ and $B=\{(x_i,h_B(x_i))\,|\,x_i\in X,\ i=1,2,\ldots,m\}$. Membership values of a HFE are not usually given in any specific order. For convenience, Chen et al. [17] arranged them in a decreasing order. For a HFE h, let $\sigma:(1,2,\ldots,n)\rightarrow(1,2,\ldots,n)$ be a permutation satisfying $h_{\sigma(i)}\geq h_{\sigma(i+1)}$, $i=1,2,\ldots,n-1$, such that $h_{\sigma(i)}$ be the ith largest value in h. Since the number of values in different HFEs are normally different, to compute the correlation coefficients between two HFSs, the number of values in the corresponding HFEs are considered to be equal. Let $l_i=\max\{l(h_F(x_i,e_k)),l(h_G(x_i,e_k))\}$, $\forall x_i\in X$, where $l(h_A(x_i))$ and $l(h_B(x_i))$, respectively, represent the number of values in HFEs $h_A(x_i)$ and $h_B(x_i)$. When $l(h_A(x_i))\neq l(h_B(x_i))$, the HFE which has

Table 1 Tabular representation of HFSM

	E_1	E_2	$\cdots$	E_m
x_1	$h^k_E(x_1,e_1)$	$h^k_E(x_1,e_2)$	$\cdots$	$h^k_E(x_1,e_n)$
x_2	$h^k_E(x_2,e_1)$	$h^k_E(x_2,e_2)$	$\cdots$	$h^k_E(x_2,e_n)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_n	$h^k_E(x_m,e_1)$	$h^k_E(x_m,e_2)$	$\cdots$	$h^k_E(x_m,e_n)$

less number of values, some values are added to make it same. In this work, if $l(h_A(x_i)) < l(h_B(x_i))$, then $h_A(x_i)$ is extended by adding its final value, i.e., the minimum value until it has the same length as $h_B(x_i)$.

Definition 7 [17] For a HFS $A = \{(x_i, h_A(x_i)) \mid x_i \in X,\ i = 1, 2, \ldots, m\}$, the informational energy of the set A is defined as: $I_{\mathrm{HFS}}(A) = \sum_{i=1}^{m}\left(\frac{1}{l_i}\sum_{j=1}^{l_i} h^2_{A\sigma(j)}(x_i)\right)$.

Definition 8 [17] For two HFSs A and B, their correlation is defined by

$$C_{\mathrm{HFS}}(A,B) = \sum_{i=1}^{m}\left(\frac{1}{l_i}\sum_{j=1}^{l_i} h_{A\sigma(j)}(x_i).h_{B\sigma(j)}(x_i)\right).$$

For any two HFSs A and B, the correlation satisfies

(1) $C_{\mathrm{HFS}}(A,A) = I_{\mathrm{HFS}}(A)$,

(2) $C_{\mathrm{HFS}}(A,B) = C_{\mathrm{HFS}}(B,A)$.

Definition 9 [17] Correlation coefficient between two HFSs A and B is given as

$$\rho_{\mathrm{HFS}}(A,B) = \frac{C_{\mathrm{HFS}}(A,B)}{[C_{\mathrm{HFS}}(A,A)]^{\frac{1}{2}}.[C_{\mathrm{HFS}}(B,B)]^{\frac{1}{2}}} = \frac{\sum_{i=1}^{m}\left(\frac{1}{l_i}\sum_{j=1}^{l_i} h_{A\sigma(j)}(x_i)\cdot h_{B\sigma(j)}(x_i)\right)}{\left[\sum_{i=1}^{m}\left(\frac{1}{l_i}\sum_{j=1}^{l_i} h^2_{A\sigma(j)}(x_i)\right)\right]^{\frac{1}{2}}\cdot\left[\sum_{i=1}^{m}\left(\frac{1}{l_i}\sum_{j=1}^{l_i} h^2_{B\sigma(j)}(x_i)\right)\right]^{\frac{1}{2}}}.$$

The correlation coefficient between two HFSs A and B, $\rho_{\mathrm{HFS}}(A,B)$ satisfies the following properties.

(1) $\rho_{\mathrm{HFS}}(A,B) = \rho_{\mathrm{HFS}}(B,A)$

(2) $0 \le \rho_{\mathrm{HFS}}(A,B) \le 1$

(3) $\rho_{\mathrm{HFS}}(A,B) = 1, \quad \text{if } A = B$

Example 4 Let A and B be two hesitant fuzzy sets in $X = \{x_1, x_2, x_3\}$, where $A = \{(x_1, \{0.4, 0.3\}), (x_2, \{0.7, 0.5, 0.2\}), (x_3, \{0.8, 0.6, 0.3\})\}$, and $B = \{(x_1, \{0.7, 0.4\}), (x_2, \{0.6, 0.2, 0.1\}), (x_3, \{0.5, 0.3, 0.2\})\}$.

Now

$$\begin{aligned} C_{\mathrm{HFS}}(A,A) = I_{\mathrm{HFS}}(A) &= \sum_{i=1}^{3}\left(\frac{1}{l_i}\sum_{j=1}^{l_i} h^2_{A\sigma(j)}(x_i)\right) \\ &= \frac{1}{2}\left(0.4^2 + 0.3^2\right) + \frac{1}{3}\left(0.7^2 + 0.5^2 + 0.2^2\right) \\ &\quad + \frac{1}{3}\left(0.8^2 + 0.6^2 + 0.3^2\right) \\ &= 0.75 \end{aligned}$$

Similarly

$$\begin{aligned} C_{\mathrm{HFS}}(B,B) = I_{\mathrm{HFS}}(B) &= \sum_{i=1}^{3}\left(\frac{1}{l_i}\sum_{j=1}^{l_i} h^2_{A\sigma(j)}(x_i)\right) \\ &= \frac{1}{2}\left(0.7^2 + 0.4^2\right) + \frac{1}{3}\left(0.6^2 + 0.2^2 + 0.1^2\right) \\ &\quad + \frac{1}{3}\left(0.5^2 + 0.3^2 + 0.2^2\right) \\ &= 0.59 \end{aligned}$$

Therefore

$$\begin{aligned} C_{\mathrm{HFS}}(A,B) &= \sum_{i=1}^{3}\left(\frac{1}{l_i}\sum_{j=1}^{l_i} h_{A\sigma(j)}(x_i).h_{B\sigma(j)}(x_i)\right) \\ &= \frac{1}{2}(0.4 \times 0.7 + 0.3 \times 0.4) \\ &\quad + \frac{1}{3}(0.7 \times 0.6 + 0.5 \times 0.2 + 0.2 \times 0.1) \\ &\quad + \frac{1}{3}(0.8 \times 0.5 + 0.6 \times 0.3 + 0.3 \times 0.2) \\ &= 0.59 \end{aligned}$$

Hence

$$\rho_{\mathrm{HFS}}(A,B) = \frac{C_{\mathrm{HFS}}(A,B)}{[C_{\mathrm{HFS}}(A,A)]^{\frac{1}{2}}.[C_{\mathrm{HFS}}(B,B)]^{\frac{1}{2}}} = 0.88$$

3 Correlation measure for HFSS

In this section, we introduce the concept of correlation, informational energy, and correlation coefficient in the framework of HFSSs and then discuss their properties. Our proposed definitions are basically extensions of the work done by Chen et al. [17] in the context of HFS. Let (F, A) and (G, B) be two HFSSs defined on the set of elements $\{x_1, x_2, \ldots, x_m\}$ and parameters $\{e_1, e_2, \ldots, e_n\}$. $h_{A\sigma(j)}(x_i, e_k), j = 1, 2, \ldots, l_i$ be the ordered (decreasing) set of membership values in the HFE $h(x_i, e_k)$, $i = 1, 2, \ldots, m$, $k = 1, 2, \ldots, n$. Here $l_i = \max\{l(h_F(x_i, e_k)), l(h_G(x_i, e_k))\}$, where $l(h_F(x_i, e_k))$ and $l(h_G(x_i, e_k))$, respectively, represent the number of values in HFEs $h_F(x_i, e_k)$ and $h_G(x_i, e_k)$.

Definition 10 For two hesitant fuzzy soft sets (F, A) and (G, B), the correlation $C_{\mathrm{HFSS}}\{(F, A), (G, B)\}$ is defined by

$$C_{\mathrm{HFSS}}\{(F,A),(G,B)\} = \sum_{k=1}^{n}\left(\sum_{i=1}^{m}\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(h_{A_{\sigma(j)}}(x_i,e_k)\right)\right].\left[\left(h_{B_{\sigma(j)}}(x_i,e_k)\right)\right]\right)\right).$$

Definition 11 Informational energy $I_{\mathrm{HFSS}}(F, A)$ for HFSS (F, A) is defined as

$$I_{\mathrm{HFSS}}(F,A)=\sum_{k=1}^{n}\left(\sum_{i=1}^{m}\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(h^2_{A_{\sigma(j)}}(x_i,e_k)\right)\right]\right)\right).$$

For any two HFSSs (F,A) and (G,B), the correlation satisfies the following.

(1) $C_{\mathrm{HFSS}}\{(F,A),(F,A)\}=I_{\mathrm{HFSS}}(F,A)$
(2) $C_{\mathrm{HFSS}}\{(F,A),(G,B)\}=C_{\mathrm{HFSS}}\{(G,B),(F,A)\}$

This can be proved easily from Definition 10 and Definition 11.

Definition 12 Correlation coefficient between two HFSSs (F,A) and (G,B) is given as

$$=\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(h_{A_{\sigma(j)}}(x_1,e_1)\right)\right].\left[\left(h_{B_{\sigma(j)}}(x_1,e_1)\right)\right]\right)+\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(h_{A_{\sigma(j)}}(x_1,e_2)\right)\right].\left[\left(h_{B_{\sigma(j)}}(x_1,e_2)\right)\right]\right)+\cdots+\frac{1}{l_i}\left(\sum_{j=1}^{l_i}[(h_{A_{\sigma(j)}}(x_1,e_n))].[(h_{B_{\sigma(j)}}(x_1,e_n))]\right)$$

$$=\sum_{j=1}^{l_i}\frac{\left[\left(h_{A_{\sigma(j)}}(x_1,e_1)\right)\right]}{\sqrt{l_i}}.\frac{\left[\left(h_{B_{\sigma(j)}}(x_1,e_1)\right)\right]}{\sqrt{l_i}}+\sum_{j=1}^{l_i}\frac{\left[\left(h_{A_{\sigma(j)}}(x_1,e_2)\right)\right]}{\sqrt{l_i}}.\frac{\left[\left(h_{B_{\sigma(j)}}(x_1,e_2)\right)\right]}{\sqrt{l_i}}+\cdots+\sum_{j=1}^{l_i}\frac{\left[\left(h_{A_{\sigma(j)}}(x_1,e_3)\right)\right]}{\sqrt{l_i}}.\frac{\left[\left(h_{B_{\sigma(j)}}(x_1,e_3)\right)\right]}{\sqrt{l_i}}$$

$$\rho_{\mathrm{HFSS}}\{(F,A),(G,B)\}=\frac{C_{\mathrm{HFSS}}\{(F,A),(G,B)\}}{\left(C_{\mathrm{HFSS}}\{(F,A),(F,A)\}.C_{\mathrm{HFSS}}\{(G,B),(G,B)\}\right)^{\frac{1}{2}}}$$
$$=\frac{\sum_{k=1}^{n}\left(\sum_{i=1}^{m}\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(h_{A_{\sigma(j)}}(x_i,e_k)\right)\right].\left[\left(h_{B_{\sigma(j)}}(x_i,e_k)\right)\right]\right)\right)}{\left(\sum_{k=1}^{n}\left(\sum_{i=1}^{m}\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(h^2_{A_{\sigma(j)}}(x_i,e_k)\right)\right]\right)\right)\right)^{\frac{1}{2}}.\left(\sum_{k=1}^{n}\left(\sum_{i=1}^{m}\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(h^2_{B_{\sigma(j)}}(x_i,e_k)\right)\right]\right)\right)\right)^{\frac{1}{2}}}$$

The correlation coefficient $\rho_{\mathrm{HFSS}}\{(F,A),(G,B)\}$ between two HFSSs (F,A) and (G,B) satisfies the following properties.

(1) $\rho_{\mathrm{HFSS}}\{(F,A),(G,B)\}=\rho_{\mathrm{HFSS}}\{(G,B),(F,A)\}$
(2) $0\le\rho_{\mathrm{HFSS}}\{(F,A),(G,B)\}\le 1$
(3) $\rho_{\mathrm{HFSS}}\{(F,A),(G,B)\}=1,\quad \text{if}\ (F,A)=(G,B)$

Proof

(1) This is straightforward.
(2) It is obvious that $\rho_{\mathrm{HFSS}}\{(F,A),(G,B)\}\ge 0$. We will prove only $\rho_{\mathrm{HFSS}}\{(F,A),(G,B)\}\le 1$.

$$C_{\mathrm{HFSS}}\{(F,A),(G,B)\}=\sum_{k=1}^{n}\left(\sum_{i=1}^{m}\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(h_{A_{\sigma(j)}}(x_i,e_k)\right)\right].\left[\left((h_{B_{\sigma(j)}}(x_i,e_k)\right)\right]\right)\right)$$
$$=\sum_{k=1}^{n}\left(\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(h_{A_{\sigma(j)}}(x_1,e_k)\right)\right].\left[\left(h_{B_{\sigma(j)}}(x_1,e_k)\right)\right]\right)\right)+\sum_{k=1}^{n}\left(\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(h_{A_{\sigma(j)}}(x_2,e_k)\right)\right].\left[\left(h_{B_{\sigma(j)}}(x_2,e_k)\right)\right]\right)\right)+\cdots+\sum_{k=1}^{n}\left(\frac{1}{l_i}\left(\sum_{j=1}^{l_i}[(h_{A_{\sigma(j)}}(x_m,e_k))].[(h_{B_{\sigma(j)}}(x_m,e_k))]\right)\right).$$

Now

$$\sum_{k=1}^{n}\left(\frac{1}{l_i}\left(\sum_{j=1}^{l_i}[(h_{A_{\sigma(j)}}(x_1,e_k))].[(h_{B_{\sigma(j)}}(x_1,e_k))]\right)\right)$$

Using the Cauchy–Schwarz inequality,

$$(x_1y_1+x_2y_2+\cdots+x_ny_n)^2\le(x_1^2+x_2^2+\cdots+x_n^2).(y_1^2+y_2^2+\cdots+y_n^2),$$

where $(x_1,x_2,\ldots,x_n)\in R^n,(y_1,y_2,\ldots,y_n)\in R^n$. So, we get

$$\left(\sum_{k=1}^{n}\left(\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(h_{A_{\sigma(j)}}(x_1,e_k)\right)\right].\left[\left(h_{B_{\sigma(j)}}(x_1,e_k)\right)\right]\right)\right)\right)^2$$
$$\le\left[\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(h^2_{A_{\sigma(j)}}(x_1,e_1)\right)\right]\right)+\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(h^2_{A_{\sigma(j)}}(x_1,e_2)\right)\right]\right)+\cdots+\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(h^2_{A_{\sigma(j)}}(x_1,e_n)\right)\right]\right)\right]$$
$$\times\left[\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(h^2_{B_{\sigma(j)}}(x_1,e_1)\right)\right]\right)+\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(h^2_{B_{\sigma(j)}}(x_1,e_2)\right)\right]\right)+\cdots+\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(h^2_{B_{\sigma(j)}}(x_1,e_n)\right)\right]\right)\right]$$
$$=\left[\sum_{k=1}^{n}\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(h^2_{A_{\sigma(j)}}(x_1,e_k)\right)\right]\right)\right].\left[\sum_{k=1}^{n}\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(h^2_{B_{\sigma(j)}}(x_1,e_k)\right)\right]\right)\right]$$

Therefore

$$\left(\sum_{k=1}^{n}\left(\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(h_{A_{\sigma(j)}}(x_1,e_k)\right)\right].\left[\left(h_{B_{\sigma(j)}}(x_1,e_k)\right)\right]\right)\right)\right)$$
$$\leq\left[\sum_{k=1}^{n}\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(h^2_{A_{\sigma(j)}}(x_1,e_k)\right)\right]\right)\right]^{\frac{1}{2}}$$
$$\cdot\left[\sum_{k=1}^{n}\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(h^2_{B_{\sigma(j)}}(x_1,e_k)\right)\right]\right)\right]^{\frac{1}{2}}$$

Hence,

$$C_{\mathrm{HFSS}}\{(F,A),(G,B)\}$$
$$=\sum_{k=1}^{n}\left(\sum_{i=1}^{m}\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(h_{A_{\sigma(j)}}(x_i,e_k)\right)\right].\left[\left(h_{B_{\sigma(j)}}(x_i,e_k)\right)\right]\right)\right)$$
$$\leq\left[\sum_{k=1}^{n}\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(h^2_{A_{\sigma(j)}}(x_1,e_k)\right)\right]\right)\right]^{\frac{1}{2}}$$
$$\cdot\left[\sum_{k=1}^{n}\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(h^2_{B_{\sigma(j)}}(x_1,e_k)\right)\right]\right)\right]^{\frac{1}{2}}$$
$$+\left[\sum_{k=1}^{n}\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(h^2_{A_{\sigma(j)}}(x_2,e_k)\right)\right]\right)\right]^{\frac{1}{2}}$$
$$\cdot\left[\sum_{k=1}^{n}\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(h^2_{B_{\sigma(j)}}(x_2,e_k)\right)\right]\right)\right]^{\frac{1}{2}}$$
$$+\cdots+\left[\sum_{k=1}^{n}\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(h^2_{A_{\sigma(j)}}(x_n,e_k)\right)\right]\right)\right]^{\frac{1}{2}}$$
$$\cdot\left[\sum_{k=1}^{n}\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(h^2_{B_{\sigma(j)}}(x_n,e_k)\right)\right]\right)\right]^{\frac{1}{2}}$$

Thus

$$C_{\mathrm{HFSS}}\{(F,A),(G,B)\}\leq C_{\mathrm{HFSS}}\{(F,A)\}^{\frac{1}{2}}$$
$$.C_{\mathrm{HFSS}}\{(G,B)\}^{\frac{1}{2}}$$

So, $0\leq\rho_{\mathrm{HFSS}}\{(F,A),(G,B)\}\leq 1$.

(3) This is also straightforward.

4 Decision making based on correlation of HFSSs

Based on our proposed formula of correlation coefficient for HFSS, we develop an algorithm under hesitant fuzzy environment. This section also defines correlation efficiency and normalized correlation efficiency, which are required for the algorithm.

Let $A=\{A_1,A_2,\ldots,A_m\}$ be the set of alternatives, $E=\{E_1,E_2,\ldots,E_n\}$ be the set of attributes/criteria, and $\omega=\{\omega_1,\omega_2,\ldots,\omega_n\}$ be the weight vector of the attributes $E_j(j=1,2,\ldots,n)$, where $\omega_j>0$ and $\sum_{j=1}^{n}\omega_j=1$. Let $D=\{D_1,D_2,\ldots,D_t\}$ be the set of decision makers. Decision makers $D_k, k=1,2,\ldots,t$, provide their opinions using HFSS $(F,E)=(\{F_1,E\},\{F_2,E\},\ldots,F_t,E\})$. Let $w=\{w_1,w_2,\ldots,w_t\}$ be the normalized correlation efficiency of HFSS (F_k,E), $k=1,2,\ldots,t$, with $w_k>0$ and $\sum_{k=1}^{t}w_k=1$. The attribute values of the alternatives $A_i(i=1,2,\ldots,m)$ of a decision maker D_k are represented by the hesitant fuzzy soft matrix $R_k=\left(f_{ij}^k\right)_{m\times n}$, where

$f_{ij}^k=h_E^k(x_i,e_j)$, $e\in E$, $x\in A$, $i=1,2,\ldots,m, j=1,2,\ldots,n$.

4.1 Correlation efficiency and normalized correlation efficiency

Definition 13 Correlation efficiency for each HFSS (F_k,E), $k=1,2,\ldots,t$, is defined by $\rho_{\mathrm{HFSS}}(F_k,E)=\frac{\left\{\sum_{k=1}^{t}\rho_{\mathrm{HFSS}}\{(F_k,E),(F_l,E)\}\right\}}{(t-1)}$, where $k\neq l, l=1,2,\ldots,t$.

Definition 14 Normalized correlation efficiency $N\rho_{\mathrm{HFSS}}(F_k,E)$ of HFSS (F_k,E) is defined below.

$$N\rho_{HFSS}(F_k,E)=\frac{\rho_{\mathrm{HFSS}}\{(F_k,E)\}}{\left\{\sum_{k=1}^{t}\rho_{\mathrm{HFSS}}\{(F_k,E)\}\right\}},$$

where $\sum_{k=1}^{t}N\rho_{\mathrm{HFSS}}(F_k,E)=1$.

4.2 Proposed algorithmic approach

Step 1 A group of decision makers $D_k, k=1,2,\ldots,t$, provides their opinions in terms of HFSS (F_k,E), $k=1,2,\ldots,t$.

Step 2 Correlation coefficient $\rho_{\mathrm{HFSS}}\{(F_k,E),(F_l,E)\}$, $k\,/\!=l$, $k,l=1,2,\ldots,t$, for each pair of hesitant fuzzy soft set is computed by Definition 12.

Step 3 Correlation efficiency $\rho_{\mathrm{HFSS}}(F_k,E)$ and normalized correlation efficiency $N\rho_{\mathrm{HFSS}}(F_k,E)$ for each HFSS (F_k,E), $k=1,2,\ldots,t$, are computed as given in Definition 13 and Definition 14.

Step 4 HFSSs (F_k,E), $k=1,2,\ldots,t$, provided by the decision makers D_k, $(k=1,2,\ldots,t)$ are aggregated into a collective decision matrix $R=\left(r_{ij}\right)_{m\times n}$ using the normalized correlation efficiency $N\rho_{\mathrm{HFSS}}(F_k,E), k=1,2,\ldots,t$, which is used as the weight vector $w=\{w_1,w_2,w_3,\ldots,w_t\}$ of decision makers. r_{ij} is computed using HFOWA operator given in Definition 4.

Step 5 In collective decision matrix, hesitant fuzzy weighted averaging (HFWA) operator, given in Definition 3, is used to obtain the HFEs $h_i(i=1,2,\ldots,m)$ for the alternatives A_i, $i=1,2,\ldots,m$.

Step 6 Compute the score values $s(h_i)$ of h_i, $i=1,2,\ldots,m$ by Definition 2.

Table 2 HFSMs for $(F_i, E),\ i = 1,2,3,4$

	E_1 (F_1, E)	E_2	E_3	E_4	E_1 (F_2, E)	E_2	E_3	E_4
A_1	(0.2, 0.5, 0.3)	(0.3, 0.1, 0.5, 0.7)	(0.7, 0.2, 0.6)	(0.4, 0.2)	(0.4, 0.3, 0.1, 0.2)	(0.5, 0.6, 0.2)	(0.4, 0.2, 0.7, 0.2)	(0.4, 0.3, 0.1)
A_2	(0.5, 0.3, 0.7, 0.9)	(0.4, 0.1, 0.6)	(0.4, 0.1, 0.4)	(0.6, 0.3, 0.7, 0.1)	(0.4, 0.5)	(0.3, 0.5, 0.6, 0.1)	(0.4, 0.6, 0.1)	(0.3, 0.6)
A_3	(0.4, 0.2, 0.1, 0.6)	(0.8, 0.3, 0.5)	(0.5, 0.6, 0.4, 0.7)	(0.5, 0.6, 0.2)	(0.8, 0.4, 0.8)	(0.4, 0.2, 0.3)	(0.7, 0.2, 0.4)	(0.3, 0.5, 0.2, 0.7)
A_4	(0.5, 0.4)	(0.1, 0.6, 0.3, 0.9)	(0.5, 0.7)	(0.4, 0.7, 0.3)	(0.2, 0.5, 0.1, 0.6)	(0.5, 0.7, 0.1, 0.1)	(0.4, 0.2, 0.7)	(0.3, 0.1, 0.5)
	E_1 (F_3, E)	E_2	E_3	E_4	E_1 (F_4, E)	E_2	E_3	E_4
A_1	(0.3, 0.4)	(0.4, 0.1, 0.5)	(0.3, 0.5, 0.1, 0.1)	(0.5, 0.2)	(0.5, 0.2, 0.1, 0.5)	(0.3, 0.2, 0.1)	(0.4, 0.5, 0.3)	(0.8, 0.9)
A_2	(0.4, 0.2, 0.5, 0.9)	(0.3, 0.7)	(0.4, 0.2, 0.5)	(0.6, 0.5, 0.2)	(0.8, 0.7)	(0.3, 0.5, 0.6, 0.2)	(0.4, 0.3, 0.5, 0.1)	(0.4, 0.6, 0.9, 0.3)
A_3	(0.5, 0.7, 0.7)	(0.3, 0.5, 0.6, 0.2)	(0.4, 0.2)	(0.5, 0.6, 0.2, 0.1)	(0.3, 0.2, 0.1)	(0.7, 0.6)	(0.5, 0.3)	(0.1, 0.4, 0.2)
A_4	(0.5, 0.3, 0.2)	(0.3, 0.6)	(0.5, 0.2, 0.5, 0.1)	(0.9, 0.8, 0.7)	(0.4, 0.5, 0.2, 0.5)	(0.3, 0.2, 0.1)	(0.9, 0.7, 0.9)	(0.6, 0.3)

Step 7 By ranking $s(h_i)$, we get the priorities of the alternatives $A_i,\ i = 1,2,\ldots,m$ and select the best one.

5 Case study I

In this section, we discuss a problem concerning an investment company, which wants to invest an amount of money in the best possible option. Initially the company decides a group of four different sectors to invest the money, which are different companies of car, food, computer, and arms company. From these sectors, the company selects, respectively, Suzuki (A_1), McDonald's (A_2), IBM (A_3), and Alexander Arms (A_4) as a set of four alternatives for investment purpose. The investment company will take the decision after analyzing the following four criteria/attributes: (1) risk (E_1), (2) growth (E_2), (3) environmental impact (E_3), and (4) company capital (E_4). The criteria weight is given as $\omega = (0.4, 0.3, 0.2, 0.1)$. The investment company forms a group of four experts $D_k,\ (k = 1,2,3,4)$ for taking the investment decision. The experts $D_k,\ (k = 1,2,3,4)$ provide the potential information of the alternatives $A = \{A_1, A_2, A_3, A_4\}$ with respect to the attributes $E = \{E_1, E_2, E_3, E_4\}$ by the HFSMs $(F, E) = (\{F_1, E\}, \{F_2, E\}, \{F_3, E\}, \{F_4, E\})$ listed in Table 2. Correlation measures for every pair of HFSSs, i.e., $C_{\text{HFSS}}\{(F_k, E), (F_l, E)\},\quad k = 1,2,\ldots,t, l = 1,2,\ldots,t,$ are given in Table 3. We calculate correlation coefficients for each pair of HFSSs $\rho_{\text{HFSS}}\{(F_k, E), (F_l, E)\},\quad k = 1,\ 2,3,4,\ l = 1,2,3,4$, which is shown in Table 4. Correlation efficiency $\rho_{\text{HFSS}}(F_k, E)$ and normalize correlation efficiency $N\rho_{\text{HFSS}}(F_k, E)$ are also computed as given in Table 4. Next we compute the collective decision matrix, given in Table 5, using HFOWA operator and the weight vector $w = \{w_1, w_2, w_3, w_4\} = \{0.25665, 0.252997, 0.248528, 0.241825\}$. These weights are derived from the normalize correlation efficiency of individual HFSS. Each of the alternatives, given in Table 5, are aggregated using HFWA operator with the attribute weight $\omega = \{\omega_1, \omega_2, \omega_3, \omega_4\} = \{0.4, 0.3, 0.2, 0.1\}$ to obtain the aggregated HFEs for each of the alternatives $A = \{A_1, A_2, A_3, A_4\}$, which are $h_{A_1} = (0.6, 0.4, 0.2, 0.2)$, $h_{A_2} = (0.7, 0.5, 0.4, 0.3)$, $h_{A_3} = (0.6, 0.5, 0.3, 0.3)$, and $h_{A_4} = (0.6, 0.5, 0.3, 0.2)$ as shown in Table 6. Then we calculate the score values $s(h_{A_i})\,(i = 1,2,3,4)$ of those HFEs, which are $S(h_{A_1}) = 0.35$, $S(h_{A_2}) = 0.475$, $S(h_{A_3}) = 0.425$, $S(h_{A_4}) = 0.4$, and shown in Table 6. Since $S(h_{A_2}) > S(h_{A_3}) > S(h_{A_4}) > S(h_{A_1})$, so $A_2 > A_3 > A_4 > A_1$, thus the most desirable alternative for investment is A_2, i.e., McDonald's as per opinion of the group of experts.

Table 3 Correlation measures of HFSS pairs

	(F_1, E)	(F_2, E)	(F_3, E)	(F_4, E)
(F_1, E)	3.8075	2.9358	3.0775	3.365
(F_2, E)	2.9358	3.0708	2.805	2.8475
(F_3, E)	3.0775	2.805	3.4674	2.9
(F_4, E)	3.365	2.8475	2.9	4.17

Table 4 Correlation coefficients of HFSS pairs and correlation efficiency

	(F_1, E)	(F_2, E)	(F_3, E)	(F_4, E)	Correlation efficiency	Normalized Correlation efficiency
(F_1, E)	1	0.8585	0.8470	0.8445	0.85	0.25665
(F_2, E)	0.8585	1	0.8596	0.795738	0.8379	0.252997
(F_3, E)	0.8470	0.8596	1	0.7626	0.8231	0.248528
(F_4, E)	0.8445	0.795738	0.7626	1	0.8009	0.241825

Table 5 Collective decision matrix

	E_1	E_2	E_3	E_4
A_1	(0.5, 0.4, 0.2, 0.1)	(0.6, 0.4, 0.2, 0.1)	(0.6, 0.4, 0.2, 0.2)	(0.6, 0.4, 0.4)
A_2	(0.8, 0.6, 0.5, 0.4)	(0.6, 0.4, 0.3, 0.2)	(0.5, 0.4, 0.2, 0.1)	(0.7, 0.5, 0.3, 0.2)
A_3	(0.6, 0.6, 0.3, 0.3)	(0.7, 0.5, 0.4, 0.3)	(0.6, 0.4, 0.3, 0.3)	(0.6, 0.4, 0.2, 0.2)
A_4	(0.5, 0.4, 0.3, 0.2)	(0.7, 0.4, 0.2, 0.2)	(0.7, 0.6, 0.4, 0.4)	(0.7, 0.5, 0.4, 0.2)

Table 6 Aggregated alternatives and score values

Alternatives	Aggregated HFE	Score value
A_1	(0.6, 0.4, 0.2, 0.2)	0.35
A_2	(0.7, 0.5, 0.4, 0.3)	0.475
A_3	(0.6, 0.5, 0.3, 0.3)	0.425
A_4	(0.6, 0.5, 0.3, 0.2)	0.4

6 Correlation of interval-valued hesitant fuzzy soft sets

In real-group decision-making problems, sometimes it becomes very much difficult for the experts to assign specific values for the membership degrees for certain elements, where a range of values in [0, 1] might be more realistic. This leads the necessity to introduce the concept of interval-valued hesitant fuzzy soft set (IVHFSS).

Definition 15 [48] Let X be a fixed set, an interval-valued hesitant fuzzy set (IVHFS) on X is in terms of a function that when applied to each x in X and returns a subset of interval values in [0, 1]. IVHFS is defined as

$$M = \left\{ <x, \widehat{h}_M(x)> | x \in X \right\},$$

where $\widehat{h}_M(x)$ is a set of some interval values in [0, 1] denoting the possible membership degrees of the element $x \in X$ to the set E. For convenience, Wei et al. [48] defined $\widehat{h}_M(x) = \widehat{h} = [\gamma^L, \gamma^R]$ as an interval-valued hesitant fuzzy element (IVHFE) and $\widehat{H}$ the set of all IVHFEs.

Given three IVHFEs $\widehat{h} = [\gamma^L, \gamma^R]$, $\widehat{h}_1 = [\gamma_1^L, \gamma_1^R]$, and $\widehat{h}_2 = [\gamma_2^L, \gamma_2^R]$, $\lambda > 0$. Wei et al. [48] defined their operations as follows:

$$(1)\ \widehat{h}^{\lambda} = \cup_{\gamma \in \widehat{h}} \left\{ \left(\gamma^L\right)^{\lambda}, \left(\gamma^R\right)^{\lambda} \right\}, \lambda > 0,$$

$$(2)\ \lambda\widehat{h} = \cup_{\gamma \in \widehat{h}} \left\{ 1 - \left(1 - \gamma^L\right)^{\lambda}, 1 - \left(1 - \gamma^R\right)^{\lambda} \right\}, \lambda > 0,$$

$$(3)\ \widehat{h}_1 \oplus \widehat{h}_2 = \cup_{\gamma_1 \in \widehat{h}_1, \gamma_2 \in \widehat{h}_2} \left\{ \gamma_1^L + \gamma_2^L - \gamma_1^L\gamma_2^L, \gamma_1^R + \gamma_2^R - \gamma_1^R\gamma_2^R \right\},$$

$$(4)\ \widehat{h}_1 \otimes \widehat{h}_2 = \cup_{\gamma_1 \in \widehat{h}_1, \gamma_2 \in \widehat{h}_2} \left\{ \gamma_1^L\gamma_2^L, \gamma_1^R\gamma_2^R \right\}.$$

Here $\gamma^L, \gamma^R, \gamma_1^L, \gamma_2^L, \gamma_1^R, \gamma_2^R \in [0, 1]$.

For the purpose of finding correlation coefficient of IVHFS, we arrange the intervals in $\widehat{h}_M(x_i)$ in decreasing order. This is achieved based on a transitive order relation between two intervals defined by Moore [49] and Moore et al. [50]. Let $\sigma : (1, 2, \ldots, n) \rightarrow (1, 2, \ldots, n)$ be a permutation satisfying $h_{M\sigma(i)}(x_i) \geq h_{M\sigma(i)}(x_{i+1})$, $i = 1, 2, \ldots, n-1$, and $\widehat{h}_{M\sigma(j)}(x_i)$ be the jth largest interval in $\widehat{h}_M(x_i)$, where

$$\widehat{h}_{M\sigma(j)}(x_i) = \left[\widehat{h}^L_{M\sigma(j)}(x_i), \widehat{h}^R_{M\sigma(j)}(x_i), \right] \subset [0, 1], j = 1, 2, \ldots, l_i,$$

are intervals and

$$\widehat{h}^L_{M\sigma(j)}(x_i) = \inf \widehat{h}_{M\sigma(j)}(x_i),\ \widehat{h}^R_{M\sigma(j)}(x_i) = \sup \widehat{h}_{M\sigma(j)}(x_i).$$

Definition 16 [48] For an IVHFE $\widehat{h}$, $s(\widehat{h}) = \frac{1}{\#h}\sum_{\gamma \in h}\gamma$ is called the score function of $\widehat{h}$, where $\#h$ is the number of the interval fuzzy values in $\widehat{h}$, and $s(\widehat{h})$ is an interval fuzzy values belonging to [0, 1]. For two IVHFEs $\widehat{h}_1$ and $\widehat{h}_2$, if $s(\widehat{h}_1) \geq s(\widehat{h}_2)$, then $\widehat{h}_1 \geq \widehat{h}_2$.

Definition 17 [17] Informational energy for of an IVHFS $\widetilde{A} = \left\{ <x_i, \widehat{h}_M(x_i)> | x_i \in X, i = 1, 2, \ldots, m \right\}$ is defined by

$$E_{\text{IVHFS}}(\widetilde{A}) = \sum_{i=1}^{m}\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(\gamma^L_{\widetilde{A}_{\sigma(j)}}(x_i)\right)^2 + \left(\gamma^R_{\widetilde{A}_{\sigma(j)}}(x_i)\right)^2\right]\right).$$

Definition 18 [17] For two IVHFSs $\widetilde{A}$ and $\widetilde{B}$, their correlation is defined by

$$C_{\text{IVHFS}}(\widetilde{A}, \widetilde{B}) = \left(\sum_{i=1}^{m}\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\gamma^L_{\widetilde{A}_{\sigma(j)}}(x_i).\gamma^L_{\widetilde{B}_{\sigma(j)}}(x_i) + \gamma^R_{\widetilde{A}_{\sigma(j)}}(x_i).\gamma^R_{\widetilde{B}_{\sigma(j)}}(x_i)\right]\right)\right).$$

For any two IVHFSs $\widetilde{A}$ and $\widetilde{B}$, the correlation satisfies the following:

(1) $C_{\text{IVHFS}}(\widetilde{A}, \widetilde{A}) = E_{\text{IVHFS}}(\widetilde{A})$

(2) $C_{\text{IVHFS}}(\widetilde{A}, \widetilde{B}) = C_{\text{IVHFS}}(\widetilde{B}, \widetilde{A})$

Definition 19 [17] Correlation coefficient between two IVHFS $\widetilde{A}$ and $\widetilde{B}$ is

$$\rho_{\text{IVHFS}}(\widetilde{A}, \widetilde{B}) = \frac{C_{\text{IVHFS}}(\widetilde{A}, \widetilde{B})}{\left(C_{\text{IFHFS}}(\widetilde{A}, \widetilde{A}).C_{\text{IVHFS}}(\widetilde{B}, \widetilde{B})\right)^{\frac{1}{2}}} = \frac{\sum_{i=1}^{m}\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\gamma^L_{\widetilde{A}_{\sigma(j)}}(x_i).\gamma^L_{\widetilde{B}_{\sigma(j)}}(x_i) + \gamma^R_{\widetilde{A}_{\sigma(j)}}(x_i).\gamma^R_{\widetilde{B}_{\sigma(j)}}(x_i)\right]\right)}{\left(\sum_{i=1}^{m}\frac{1}{l_i}\left(\sum_{j=1}^{l_i}[(\gamma^L_{\widetilde{A}_{\sigma(j)}}(x_i))^2 + (\gamma^R_{\widetilde{A}_{\sigma(j)}}(x_i))^2]\right)\right)^{\frac{1}{2}}.\left(\sum_{i=1}^{m}\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(\gamma^L_{\widetilde{B}_{\sigma(j)}}(x_i)\right)^2 + \left(\gamma^R_{\widetilde{B}_{\sigma(j)}}(x_i)\right)^2\right]\right)\right)^{\frac{1}{2}}}.$$

The correlation coefficient between two IVHFSs $\widetilde{A}$ and $\widetilde{B}$, $\rho_{\text{IVHFS}}(\widetilde{A}, \widetilde{B})$ satisfies the following properties.

(1) $\rho_{\text{IVHFS}}(\widetilde{A}, \widetilde{B}) = \rho_{\text{IVHFS}}(\widetilde{B}, \widetilde{A})$

(2) $0 \leq \rho_{\text{IVHFS}}(\widetilde{A}, \widetilde{B}) \leq 1$

(3) $\rho_{\text{IVHFS}}(\widetilde{A}, \widetilde{B}) = 1, \quad \text{if } \widetilde{A} = \widetilde{B}$

IVHFSS was introduced by Zhang et al. [51] which is defined below.

Definition 20 [51] Let U be an initial universe and E be a set of parameters. *IVHFS* (U) denotes the set of all interval-valued hesitant fuzzy sets of U. Let $A \subseteq E$. A pair $(\widehat{F}, A)$ is an interval-valued hesitant fuzzy soft set over U, where $\widehat{F}$ is a mapping given by $\widehat{F} : A \rightarrow IVHFS(U)$.

In other words, an interval-valued hesitant fuzzy soft set is a parameterized family of interval-valued hesitant fuzzy subsets of U. Thus, its universe is the set of all interval-valued hesitant fuzzy sets of U. An interval-valued hesitant fuzzy soft set is also a special case of a hesitant fuzzy soft set because it is still a mapping from parameters to interval-valued hesitant fuzzy sets of U.

$\forall \varepsilon \in A, F(\varepsilon)$ is an interval-valued hesitant fuzzy set of U for parameter ε. $F(\varepsilon)$ can be written as: $F(\varepsilon) = \left\{x, \widehat{h}_{F(\varepsilon)}(x)\right\}$. Here $\widehat{h}_{F(\varepsilon)}(x)$ is the set of interval-valued fuzzy membership degrees that object x holds on parameter ε.

Example 5 Consider an interval-valued hesitant fuzzy soft set $<\widehat{F}, A>$, where U is the set of six stages of heart diseases under the consideration of a decision maker to prescribe, which is denoted by $U = \{h_1, h_2, \ldots, h_6\} =$ {Stage 'I', Stage 'II', Stage 'III', Stage 'IV', Stage 'V'} and A is a parameter set, where $A = \{e_1, e_2, \ldots, e_5\} =$ {Chest pain, Palpitation, Dizziness, Fainting, Fatigue}. The interval-valued hesitant fuzzy soft set $<\widehat{F}, A>$ describes the "possibilities of various stages of heart disease" to the decision maker.

Suppose

$$F(e_1) = \left\{ \begin{array}{l} <h_1, [(0.5, 0.7), (0.2, 0.4), (0.6, 0.8), (0.1, 0.3)] > , <h_2, [(0.6, 0.8), (0.75, 0.86)] > , <h_3, [(0.4, 0.6), (0.5, 0.7), (0.2, 0.4)] > \\ <h_4[(0.2, 0.4), (0.6, 0.8), (0.4, 0.6), (0.5, 0.7)] > , <h_5, [(0.3, 0.5), (0.2, 0.4)] > , <h_6, [(0.75, 0.85), (0.4, 0.6), (0.8, 0.9)] > \end{array} \right\},$$

$$F(e_2) = \left\{ \begin{array}{l} <h_1, [(0.7, 0.9), (0.6, 0.8)] > , <h_2[(0.2, 0.4), (0.6, 0.8), (0.4, 0.6), (0.55, 0.75)] > , <h_3, [(0.3, 0.5), (0.2, 0.4)] > \\ <h_4, [(0.5, 0.7), (0.2, 0.4), (0.6, 0.8)] > , <h_5, [(0.6, 0.8), (0.75, 0.86), (0.6, 0.8)] > , <h_6, [(0.7, 0.8), (0.45, 0.65), (0.82, 0.93)] > \end{array} \right\},$$

$$F(e_3) = \left\{ \begin{array}{l} <h_1, [(0.2, 0.4), (0.4, 0.6), (0.5, 0.8)] > , <h_2, [(0.85, 0.95), (0.2, 0.4), (0.4, 0.6)], <h_3, [(0.08, 0.16), (0.3, 0.5)] > \\ <h_4, [(0.4, 0.6), (0.5, 0.7), (0.2, 0.4), (0.7, 0.9)], <h_5, [(0.7, 0.9), (0.6, 0.8)] > , <h_6, [(0.35, 0.55), (0.43, 0.66), (0.8, 0.9)] > \end{array} \right\},$$

$$F(e_4) = \left\{ \begin{array}{l} <h_1, [(0.2, 0.4), (0.4, 0.6)] > , <h_2, [(0.85, 0.95), (0.2, 0.4), (0.4, 0.6)], <h_3, [(0.08, 0.16), (0.3, 0.5)] > \\ <h_4, [(0.4, 0.6), (0.5, 0.7), (0.2, 0.4), (0.7, 0.9)], <h_5, [(0.7, 0.9), (0.6, 0.8), (0.1, 0.3)] > , <h_6, [(0.35, 0.55), (0.43, 0.66), (0.8, 0.9)] > \end{array} \right\},$$

$$F(e_5) = \left\{ \begin{array}{l} <h_1, [(0.2, 0.4), (0.5, 0.8), (0.5, 0.7)] > , <h_2, [(0.85, 0.95), (0.2, 0.4), (0.4, 0.6)], <h_3, [(0.08, 0.16), (0.3, 0.5), (0.4, 0.6)] > \\ <h_4, [(0.4, 0.6), (0.5, 0.7), (0.2, 0.4), (0.7, 0.9)], <h_5, [(0.7, 0.9), (0.6, 0.8), (0.24, 0.36)] > , <h_6, [(0.8, 0.9), (0.43, 0.66), (0.8, 0.9)] > \end{array} \right\}.$$

Hence the IVIFSS $<\widehat{F}, A>$ is defined as

$$\left\langle \widehat{F}, A \right\rangle = \left\{ \begin{array}{c} \left(e_1, \left\{ \begin{array}{l} <h_1, [(0.5, 0.7), (0.2, 0.4), (0.6, 0.8), (0.1, 0.3)] > , <h_2, [(0.6, 0.8), (0.75, 0.86)] > , <h_3, [(0.4, 0.6), (0.5, 0.7), (0.2, 0.4)] > \\ <h_4[(0.2, 0.4), (0.6, 0.8), (0.4, 0.6), (0.5, 0.7)] > , <h_5, [(0.3, 0.5), (0.2, 0.4)] > , <h_6, [(0.75, 0.85), (0.4, 0.6), (0.8, 0.9)] > \end{array} \right\} \right) \\ \left(e_2, \left\{ \begin{array}{l} <h_1, [(0.7, 0.9), (0.6, 0.8)] > , <h_2[(0.2, 0.4), (0.6, 0.8), (0.4, 0.6), (0.55, 0.75)] > , <h_3, [(0.3, 0.5), (0.2, 0.4)] > \\ <h_4, [(0.5, 0.7), (0.2, 0.4), (0.6, 0.8)] > , <h_5, [(0.6, 0.8), (0.75, 0.86), (0.6, 0.8)] > , <h_6, [(0.7, 0.8), (0.45, 0.65), (0.82, 0.93)] > \end{array} \right\} \right) \\ \left(e_3, \left\{ \begin{array}{l} <h_1, [(0.2, 0.4), (0.4, 0.6), (0.5, 0.8)] > , <h_2, [(0.85, 0.95), (0.2, 0.4), (0.4, 0.6)], <h_3, [(0.08, 0.16), (0.3, 0.5)] > \\ <h_4, [(0.4, 0.6), (0.5, 0.7), (0.2, 0.4), (0.7, 0.9)], <h_5, [(0.7, 0.9), (0.6, 0.8)] > , <h_6, [(0.35, 0.55), (0.43, 0.66), (0.8, 0.9)] > \end{array} \right\} \right) \\ \left(e_4, \left\{ \begin{array}{l} <h_1, [(0.2, 0.4), (0.4, 0.6)] > , <h_2, [(0.85, 0.95), (0.2, 0.4), (0.4, 0.6)], <h_3, [(0.08, 0.16), (0.3, 0.5)] > \\ <h_4, [(0.4, 0.6), (0.5, 0.7), (0.2, 0.4), (0.7, 0.9)], <h_5, [(0.7, 0.9), (0.6, 0.8), (0.1, 0.3)] > , <h_6, [(0.35, 0.55), (0.43, 0.66), (0.8, 0.9)] > \end{array} \right\} \right) \\ \left(e_5, \left\{ \begin{array}{l} <h_1, [(0.2, 0.4), (0.5, 0.8), (0.5, 0.7)] > , <h_2, [(0.85, 0.95), (0.2, 0.4), (0.4, 0.6)], <h_3, [(0.08, 0.16), (0.3, 0.5), (0.4, 0.6)] > \\ <h_4, [(0.4, 0.6), (0.5, 0.7), (0.2, 0.4), (0.7, 0.9)], <h_5, [(0.7, 0.9), (0.6, 0.8), (0.24, 0.36)] > , <h_6, [(0.8, 0.9), (0.43, 0.66), (0.8, 0.9)] > \end{array} \right\} \right) \end{array} \right\}$$

In this example, we observe that the precise value of an object corresponding to a parameter is unknown while the lower and upper limits are given. Here we cannot present the precise membership degree of how chest pain is involved with stage 'I' of heart disease in successive observations, however, stage 'I' has the membership degree of at least 0.5 and at most 0.7 for the chest pain parameter at the first observation. At the second observation, stage 'I' has the membership degree of at least 0.2 and at most 0.4 for the same parameter and so on.

Below we define informational energy, correlation, and correlation coefficient of IVHFSS. These definitions can be considered as the extensions of the ideas given in Definitions 10, 11, and 12, and Definitions 17, 18, and 19.

Definition 21 Correlation $C_{\text{IVHFSS}}\left\{(\widehat{F}, A), (\widehat{G}, B)\right\}$ for two interval-valued hesitant fuzzy soft set $(\widehat{F}, A)$ and $(\widehat{G}, B)$ is given below.

$$C_{\text{IVHFSS}}\left\{(\widehat{F}, A), (\widehat{G}, B)\right\} = \sum_{k=1}^{n} \left(\sum_{i=1}^{m} \frac{1}{l_i} \left(\sum_{j=1}^{l_i} \left[\gamma^L_{A_{\sigma(j)}}(x_i, e_k).\gamma^L_{B_{\sigma(j)}}(x_i, e_k) + \gamma^R_{A_{\sigma(j)}}(x_i, e_k).\gamma^R_{B_{\sigma(j)}}(x_i, e_k) \right] \right) \right)$$

Definition 22 Informational energy for interval-valued hesitant fuzzy soft set $(\widehat{F},A)$ is

$$E_{\text{IVHFSS}}(\widehat{F},A) = \sum_{k=1}^{n}\left(\sum_{i=1}^{m}\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(\gamma^{L}_{A_{\sigma(j)}}(x_i,e_k)\right)^2+\left(\gamma^{R}_{A_{\sigma(j)}}(x_i,e_k)\right)^2\right]\right)\right).$$

For any two IVHFSSs $(\widehat{F},A)$ and $(\widehat{G},B)$, the correlation satisfies the following.

(1) $C_{\text{IVHFSS}}\left\{(\widehat{F},A),(\widehat{F},A)\right\} = E_{\text{IVHFSS}}(\widehat{F},A)$

(2) $C_{\text{IVHFSS}}\left\{(\widehat{F},A),(\widehat{G},B)\right\} = C_{\text{IVHFSS}}\left\{(\widehat{G},B),(\widehat{F},A)\right\}$

Definition 23 Correlation coefficient between two IVHFSS $(\widehat{F},A)$ and $(\widehat{G},B)$

$$\rho_{\text{IVHFSS}}\left\{(\widehat{F},A),(\widehat{G},B)\right\} = \frac{C_{\text{IVHFSS}}\left\{(\widehat{F},A),(\widehat{G},B)\right\}}{\left(C_{\text{IVHFSS}}\left\{(\widehat{F},A),(\widehat{F},A)\right\}.C_{\text{IVHFSS}}\left\{(\widehat{G},B),(\widehat{G},B)\right\}\right)^{\frac{1}{2}}}$$

$$= \frac{\sum_{k=1}^{n}\left(\sum_{i=1}^{m}\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\gamma^{L}_{A_{\sigma(j)}}(x_i,e_k).\gamma^{L}_{B_{\sigma(j)}}(x_i,e_k)+\gamma^{R}_{A_{\sigma(j)}}(x_i,e_k).\gamma^{R}_{B_{\sigma(j)}}(x_i,e_k)\right]\right)\right)}{\sum_{k=1}^{n}\left(\sum_{i=1}^{m}\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(\gamma^{L}_{A_{\sigma(j)}}(x_i,e_k)\right)^2+\left(\gamma^{R}_{A_{\sigma(j)}}(x_i,e_k)\right)^2\right]\right)\right)^{\frac{1}{2}}.\sum_{k=1}^{n}\left(\sum_{i=1}^{m}\frac{1}{l_i}\left(\sum_{j=1}^{l_i}\left[\left(\gamma^{L}_{B_{\sigma(j)}}(x_i,e_k)\right)^2+\left(\gamma^{R}_{B_{\sigma(j)}}(x_i,e_k)\right)^2\right]\right)\right)^{\frac{1}{2}}}.$$

The correlation coefficient $\rho_{\text{IVHFSS}}\left\{(\widehat{F},A),(\widehat{G},B)\right\}$ between two IVHFSSs $(\widehat{F},A)$ and $(\widehat{G},B)$ satisfies the following properties.

(1) $\rho_{\text{IVHFSS}}\left\{(\widehat{F},A),(\widehat{G},B)\right\} = \rho_{\text{IVHFSS}}\left\{(\widehat{G},B),(\widehat{F},A)\right\}$

(2) $0 \le \rho_{\text{IVHFSS}}\left\{(\widehat{F},A),(\widehat{G},B)\right\} \le 1$

(3) $\rho_{\text{IVHFSS}}\left\{(\widehat{F},A),(\widehat{G},B)\right\} = 1, \quad \text{if } (\widehat{F},A) = (\widehat{G},B)$

7 Decision making based on correlation of IVHFSS

In this section, we revise the algorithm, as proposed earlier in Sect. 4, in the context of IVHFSS. This section also presents IVHFOWA operator, correlation efficiency, and normalize correlation efficiency of IVHFSS.

Here $A=\{A_1,A_2,\ldots,A_m\}$, $E=\{E_1,E_2,\ \ldots,E_n\}$, $\omega=\{\omega_1,\omega_2,\ldots,\omega_n\}$, $D=\{D_1,D_2,\ldots,D_t\}$, m, n, and t are similar as mentioned in Sect. 4. Decision makers $D_k, k=1,2,\ldots,t$, provide their opinions using IVHFSS $(\widehat{F},E)=\left(\{\widehat{F}_1,E\},\{\widehat{F}_2,E\},\ \ldots,\widehat{F}_t,E\}\right)$. Let $w=\{w_1,w_2,\ldots,w_t\}$ be the normalized correlation efficiency of IVHFSS $(\widehat{F}_k,E)$, $k=1,2,\ldots,t$, with $w_k>0$ and $\sum_{k=1}^{t}w_k=1$. The attribute values of the alternatives $A_i(i=1,2,\ldots,m)$ of a decision maker D_k are represented by the interval-valued hesitant fuzzy soft matrix $\widehat{R}_k=\left(\widehat{f}^{k}_{ij}\right)_{m\times n}$, where $\widehat{f}_{ij}=h^{k}_{E}(x_i,e_j)$, $\quad e\in E,\ x\in A,\ i=1,2,\ldots,m, j=1,2,\ldots,n.$

7.1 IVHFOWA operator, correlation efficiency, and normalized correlation efficiency

For the purpose of decision making, IVHFEs are aggregated. Wei et al. [48] proposed some aggregation operators for IVHFEs, two of them are defined below. Let $\widehat{h}_j(j=1,2,\ldots,n)$ be a collection of IVHFEs, $w=(w_1,w_2,\ldots,w_n)^T$ is the weight vector of $\widehat{h}_j(j=1,2,\ldots,n)$ with $w_j\in[0,1]$ and $\sum_{j=i}^{n}w_j=1$.

Definition 24 [48] An interval-valued hesitant fuzzy weighted averaging (IVHFWA) operator is a mapping $\widehat{H}^{n}\rightarrow\widehat{H}$ such that

$$\text{IVHFWA}\left(\widehat{h}_1,\widehat{h}_2,\ldots,\widehat{h}_n\right)=\sum_{j=1}^{n}w_j\widehat{h}_j=\cup_{\gamma_1\in\widehat{h}_1,\gamma_2\in\widehat{h}_2,\ldots,\gamma_n\in\widehat{h}_n}\left\{1-\prod_{j=1}^{n}\left(1-\gamma^{L}_j\right)^{w_j},1-\prod_{j=1}^{n}\left(1-\gamma^{R}_j\right)^{w_j}\right\}.$$

Definition 25 [47] Let $\widehat{h}_{\sigma(j)}$ be the jth larget among the collection of IVHFEs $\widehat{h}_j, j=1,2,\ldots,n$. Here $\sigma(1),\sigma(2),\ldots,\sigma(n)$ is considered as a permutation of $1,2,\ldots,n$, such that $\widehat{h}_{\sigma(j-1)}\ge\widehat{h}_{\sigma(j)}$ for all $j=1,2,\ldots,n$. An interval-valued hesitant fuzzy-ordered weighted averaging (IVHFOWA) operator is a mapping IVHFOWA: $\widehat{H}^{n}\rightarrow\widehat{H}$ such that

$$\text{IVHFOWA}\left(\widehat{h}_1, \widehat{h}_2, \ldots, \widehat{h}_n\right) = \sum_{j=1}^{n} w_j \widehat{h}_{\sigma(j)} = \cup_{\gamma_{\sigma(1)} \in \widehat{h}_1, \gamma_{\sigma(2)} \in \widehat{h}_2, \ldots, \gamma_{\sigma(n)} \in \widehat{h}_n} \left\{1 - \prod_{j=1}^{n}\left(1 - \gamma^L_{\sigma(j)}\right)^{w_j}, 1 - \prod_{j=1}^{n}\left(1 - \gamma^R_{\sigma(j)}\right)^{w_j}\right\}.$$

Definition 26 Correlation efficiency for each IVHFSS $(\widehat{F}_k, E)$, $k = 1, 2, \ldots, t$, is defined by

$$\rho_{\text{IVHFSS}}(\widehat{F}_k, E) = \frac{\left\{\sum_{k=1}^{t} \rho_{\text{IVHFSS}}\left\{(\widehat{F}_k, E), (\widehat{F}_l, E)\right\}\right\}}{(t-1)},$$

where $k \neq l, l = 1, 2, \ldots, t$.

Definition 27 Normalized correlation efficiency $N\rho_{\text{IVHFSS}}(\widehat{F}_k, E)$ of IVHFSS $(\widehat{F}_k, E)$ is defined below.

$$N\rho_{\text{IVHFSS}}(\widehat{F}_k, E) = \frac{\rho_{\text{IVHFSS}}\left\{(\widehat{F}_k, E)\right\}}{\left\{\sum_{k=1}^{t} \rho_{\text{IVHFSS}}\left\{(\widehat{F}_k, E)\right\}\right\}},$$

where $\sum_{k=1}^{t} N\rho_{\text{IVHFSS}}(\widehat{F}_k, E) = 1$.

7.2 Revised algorithmic approach

Step 1 A group of decision makers $D_k, k = 1, 2, \ldots, t$, provides their opinions in terms of IVHFSS $(\widehat{F}_k, E)$, $k = 1, 2, \ldots, t$.

Step 2 Correlation coefficient $\rho_{\text{IVHFSS}}\left\{(\widehat{F}_k, E), (\widehat{F}_l, E)\right\}$, $k \neq l$, $k, l = 1, 2, \ldots, t$, for each pair of IVHFSS is computed by Definition 23.

Step 3 Correlation efficiency $\rho_{\text{IVHFSS}}(\widehat{F}_k, E)$ and normalized correlation efficiency $N\rho_{\text{IVHFSS}}(\widehat{F}_k, E)$ for each IVHFSS $(\widehat{F}_k, E)$, $k = 1, 2, \ldots, t$, is computed as given in Definition 26 and Definition 27.

Step 4 IVHFSSs $(\widehat{F}_k, E)$, $k = 1, 2, \ldots, t$, provided by the decision makers D_k, $(k = 1, 2, \ldots, t)$ are aggregated into a collective decision matrix $\widehat{R} = (\widehat{r}_{ij})_{m \times n}$ using the normalized correlation efficiency $N\rho_{\text{IVHFSS}}\left(\widehat{F}_k, E\right)$, $k = 1, 2, \ldots, t$, which is used as the weight vector $w = \{w_1, w_2, \ldots, w_t\}$ of decision makers. $\widehat{r}_{ij}$ is computed using IVHFOWA operator by Definition 25.

Step 5 In collective decision matrix, interval-valued hesitant fuzzy weighted averaging (IVHFWA) operator [48] (given in Definition 24) is used to obtain the IVHFEs $\widehat{h}_i (i = 1, 2, \ldots, m)$ for the alternatives $A_i (i = 1, 2, \ldots, m)$.

Step 6 Compute the score values $s(\widehat{h}_i)$ of $\widehat{h}_i (i = 1, 2, \ldots, m)$ by Definition 16.

Step 7 By ranking $s(h_i)$, we get the priorities of the alternatives $A_i (i = 1, 2, \ldots, m)$ and select the best one.

8 Case study II

This is continuation of case study I by considering the interval-valued representation of each HFE. Here opinions of each expert D_k, $(k = 1, 2, 3, 4)$ are shown using interval-valued hesitant fuzzy soft matrices (IVFSMs) $\left(\widehat{F}, E\right) = \left(\left\{\widehat{F}_1, E\right\}, \left\{\widehat{F}_2, E\right\}, \left\{\widehat{F}_3, E\right\}, \left\{\widehat{F}_4, E\right\}\right)$ listed in Table 7. These IVFSMs are normalized (in descending order of interval values) for evaluating the subsequent correlation measurements as followed. Correlation measure for every pair of IVHFSS, i.e., $C_{\text{IVHFSS}}\left\{\left(\widehat{F}_k, E\right), \left(\widehat{F}_l, E\right)\right\}$, $k = 1, 2, \ldots, t, l = 1, 2, \ldots, t$, is given in Table 8. In Table 9, we calculate correlation coefficient for each pair of IVHFSS $\rho_{\text{IVHFSS}}\left\{\left(\widehat{F}_k, E\right), \left(\widehat{F}_l, E\right)\right\}$, $k = 1, 2, 3, 4$, $l = 1, 2, 3, 4$. We also calculate correlation efficiency $\rho_{\text{IVHFSS}}\left(\widehat{F}_k, E\right)$ and normalized correlation efficiency $N\rho_{\text{IVHFSS}}\left(\widehat{F}_k, E\right)$ in Table 9. Next we compute the collective decision matrix in Table 10 using IVHFOWA operator and the weight vector $w = \{w_1, w_2, w_3, w_4\} = \{0.24659, 0.25457, 0.25276, 0.24608\}$. Those weights are derived from the normalized correlation efficiency of individual IVHFSS. Each of the alternatives, given in Table 10, is aggregated using IVHFWA operator with the attribute weight $\omega = \{\omega_1, \omega_2, \omega_3, \omega_4\} = \{0.4, 0.3, 0.2, 0.1\}$ to obtain the aggregated IVHFEs for each of the alternatives $A = \{A_1, A_2, A_3, A_4\}$, which are

$$\widehat{h}_{A_1} = \{(0.5, 0.6), (0.3, 0.5), (0.2, 0.3), (0.2, 0.3)\},$$

$$\widehat{h}_{A_2} = \{(0.7, 1.0), (0.4, 0.6), (0.3, 0.5), (0.2, 0.4)\},$$

$$\widehat{h}_{A_3} = \{(0.5, 0.8), (0.4, 0.6), (0.3, 0.4), (0.2, 0.4)\},$$

and $\widehat{h}_{A_4} = \{(0.6, 1), (0.4, 0.6), (0.2, 0.4), (0.2, 0.4)\}$ as shown in Table 11. Then we calculate the score values $s\left(\widehat{h}_{A_i}\right) (i = 1, 2, 3, 4)$ of those IVHFEs, which are

$$S(\widehat{h}_{A_1}) = (0.3, 0.43), S(\widehat{h}_{A_2}) = (0.4, 0.63), S(\widehat{h}_{A_3}) = (0.35, 0.55), S(\widehat{h}_{A_4}) = (0.35, 0.6).$$

Since $S(\widehat{h}_{A_2}) > S(\widehat{h}_{A_4}) > S(\widehat{h}_{A_3}) > S(\widehat{h}_{A_1})$, so $A_2 > A_4 > A_3 > A_1$, thus the most desirable alternative for investment is A_2, i.e., McDonald's as per opinion of the group of experts, which is similar to case study I.

Table 7 IVHFSMs for $(F_i, E),\ i = 1, 2, 3, 4$

	E_1	E_2	E_3	E_4
(F_1, E)				
A_1	(0.2, 0.3), (0.5, 0.7), (0.3, 0.4)	(0.2, 0.4), (0.0, 0.2), (0.4, 0.6), (0.6, 0.8)	(0.6, 0.8), (0.2, 0.4), (0.6, 0.8)	(0.3, 0.5), (0.1, 0.3)
A_2	(0.5, 0.6), (0.3, 0.4), (0.7, 0.8), (0.9, 1)	(0.3, 0.5), (0.0, 0.2), (0.5, 0.7)	(0.3, 0.5), (0.1, 0.3), (0.3, 0.5)	(0.5, 0.7), (0.3, 0.4), (0.6, 0.7), (0.1, 0.2)
A_3	(0.4, 0.5), (0.2, 0.4), (0.1, 0.3), (0.6, 0.8)	(0.7, 0.9), (0.2, 0.4), (0.4, 0.6)	(0.4, 0.6), (0.6, 0.8), (0.3, 0.5), (0.6, 0.8)	(0.4, 0.5), (0.6, 0.8), (0.1, 0.3)
A_4	(0.5, 0.7), (0.4, 0.6)	(0.0, 0.2), (0.5, 0.7), (0.2, 0.4)(0.8, 1.0)	(0.4, 0.5), (0.6, 0.8)	(0.4, 0.5), (0.6, 0.8), (0.2, 0.4)
(F_2, E)				
A_1	(0.3, 0.5), (0.2, 0.4), (0.1, 0.2), (0.2, 0.3)	(0.4, 0.6), (0.6, 0.7), (0.1, 0.3)	(0.4, 0.5), (0.2, 0.3), (0.6, 0.8), (0.1, 0.3)	(0.4, 0.5), (0.2, 0.4), (0.0, 0.2)
A_2	(0.3, 0.6), (0.4, 0.8)	(0.2, 0.4), (0.4, 0.6), (0.5, 0.7), (0.0, 0.1)	(0.3, 0.5), (0.5, 0.7), (0.0, 0.2)	(0.2, 0.4), (0.5, 0.7)
A_3	(0.7, 0.8), (0.3, 0.5), (0.7, 0.9)	(0.3, 0.5), (0.2, 0.3), (0.3, 0.5)	(0.6, 0.8), (0.2, 0.3), (0.3, 0.5)	(0.2, 0.4), (0.4, 0.6), (0.1, 0.3), (0.6, 0.8)
A_4	(0.1, 0.3), (0.4, 0.6), (0.1, 0.2), (0.5, 0.7)	(0.4, 0.6), (0.6, 0.8), (0.1, 0.2), (0.1, 0.2)	(0.3, 0.5), (0.1, 0.3), (0.6, 0.8)	(0.2, 0.4), (0.1, 0.2), (0.4, 0.6)
(F_3, E)				
A_1	(0.2, 0.4), (0.3, 0.5)	(0.2, 0.4), (0.1, 0.3), (0.5, 0.6)	(0.2, 0.4), (0.4, 0.6), (0.1, 0.2), (0.1, 0.2)	(0.4, 0.6), (0.2, 0.3)
A_2	(0.3, 0.5), (0.2, 0.3), (0.4, 0.6), (0.8, 1.0)	(0.3, 0.4), (0.7, 0.8)	(0.4, 0.5), (0.2, 0.3), (0.4, 0.6)	(0.5, 0.7), (0.4, 0.6), (0.1, 0.3)
A_3	(0.4, 0.6), (0.6, 0.8), (0.7, 0.8)	(0.2, 0.4), (0.4, 0.6), (0.6, 0.7), (0.2, 0.3)	(0.3, 0.5), (0.1, 0.3)	(0.4, 0.6), (0.5, 0.7), (0.2, 0.3), (0.1, 0.2)
A_4	(0.5, 0.6), (0.3, 0.5), (0.1, 0.3)	(0.2, 0.4), (0.5, 0.7)	(0.4, 0.6), (0.1, 0.3), (0.4, 0.5), (0.0, 0.2)	(0.8, 1.0), (0.7, 0.9), (0.6, 0.8)
(F_4, E)				
A_1	(0.4, 0.6), (0.2, 0.3), (0.1, 0.2), (0.4, 0.6)	(0.2, 0.4), (0.2, 0.3), (0.1, 0.2)	(0.3, 0.5), (0.4, 0.6), (0.2, 0.4)	(0.7, 0.9), (0.8, 1.0)
A_2	(0.7, 0.9), (0.6, 0.8)	(0.2, 0.4), (0.4, 0.6), (0.5, 0.7), (0.1, 0.3)	(0.3, 0.5), (0.2, 0.4), (0.4, 0.6), (0.0, 0.2)	(0.3, 0.5), (0.5, 0.7), (0.8, 1.0), (0.2, 0.3)
A_3	(0.3, 0.4), (0.2, 0.3), (0.1, 0.2)	(0.5, 0.8), (0.4, 0.7)	(0.4, 0.5), (0.2, 0.3)	(0.1, 0.2), (0.3, 0.5), (0.1, 0.3)
A_4	(0.3, 0.5), (0.4, 0.6), (0.1, 0.3), (0.4, 0.6)	(0.3, 0.4), (0.2, 0.3), (0.1, 0.2)	(0.7, 0.9), (0.6, 0.8), (0.8, 1.0)	(0.5, 0.7), (0.2, 0.4)

Table 8 Correlation measures of IVHFSS pairs

	(F_1, E)	(F_2, E)	(F_3, E)	(F_4, E)
(F_1, E)	6.761	4.912	4.685	4.693
(F_2, E)	4.912	5.351	4.571	4.352
(F_3, E)	4.685	4.571	5.052	4.179
(F_4, E)	4.693	4.352	4.179	5.387

9 Comparative analysis

Case study I present the role of correlation coefficient on HFSS and its impact in GDM process, whereas case study II shows the role of correlation coefficient on IVHFSS and its impact in GDM process. In both cases, correlation coefficient plays an important role to deploy the importance of HFSS/IVHFSS for decision-making paradigm. HFSSs are useful mainly in situations where decision makers hesitate to express their opinions about an element using a single membership value and they prefer a multiple number of membership values. The only constraint of HFSS is that all membership values are exact. When decision information about the elements is uncertain or fuzzy, instead of exact fuzzy values, decision makers prefer interval of fuzzy values. This kind of situations is well represented by IVHFSS. Our firstly proposed algorithm (described in Sect. 4.2) works in hesitant fuzzy environment, whereas the revised second algorithm (described in Sect. 7.2) works in interval-valued hesitant fuzzy environment. First algorithm uses aggregation operator, correlation coefficient and correlation efficiency in the context of HFSS, where the second algorithm uses the same in the

Table 9 Correlation coefficients of IVHFSS pairs and correlation efficiency

	(F_1, E)	(F_2, E)	(F_3, E)	(F_4, E)	Correlation efficiency	Normalized correlation efficiency
(F_1, E)	1	0.817	0.802	0.778	0.849	0.24659
(F_2, E)	0.817	1	0.879	0.811	0.877	0.25457
(F_3, E)	0.802	0.879	1	0.801	0.871	0.25276
(F_4, E)	0.778	0.811	0.801	1	0.848	0.24608

Table 10 Collective decision matrix

	E_1	E_2	E_3	E_4
A_1	(0.4, 0.6), (0.3, 0.5), (0.2, 0.3), (0.2, 0.3)	(0.5, 0.7), (0.3, 0.5), (0.1, 0.3), (0.1, 0.3)	(0.5, 0.7), (0.4, 0.6), (0.2, 0.3), (0.1, 0.2)	(0.5, 0.5), (0.4, 0.6), (0.3, 0.6), (0.3, 0.6)
A_2	(0.8, 1), (0.5, 0.7), (0.4, 0.6), (0.4, 0.6)	(0.6, 0.7), (0.4, 0.5), (0.2, 0.4), (0.1, 0.3)	(0.4, 0.6), (0.3, 0.6), (0.1, 0.3), (0.1, 0.3)	(0.6, 0.6), (0.4, 0.6), (0.2, 0.4), (0.2, 0.3)
A_3	(0.6, 0.8), (0.5, 0.7), (0.3, 0.4), (0.2, 0.4)	(0.5, 0.8), (0.4, 0.6), (0.3, 0.5), (0.3, 0.5)	(0.5, 0.7), (0.3, 0.5), (0.2, 0.4), (0.2, 0.4)	(0.5, 0.7), (0.3, 0.5), (0.2, 0.3), (0.1, 0.3)
A_4	(0.5, 0.7), (0.4, 0.6), (0.2, 0.4), (0.2, 0.4)	(0.6, 1), (0.3, 0.5), (0.2, 0.3), (0.1, 0.3)	(0.6, 1), (0.5, 0.7), (0.3, 0.5), (0.3, 0.5)	(0.6, 1), (0.4, 0.6), (0.3, 0.5), (0.3, 0.5)

Table 11 Aggregated alternatives and score values

Alternatives	Aggregated IVHFE	Score values
A_1	(0.5, 0.6), (0.3, 0.5), (0.2, 0.3), (0.2, 0.3)	(0.3, 0.43)
A_2	(0.7, 1), (0.4, 0.6), (0.3, 0.5), (0.2, 0.4)	(0.4, 0.63)
A_3	(0.5, 0.8), (0.4, 0.6), (0.3, 0.4), (0.2, 0.4)	(0.35, 0.55)
A_4	(0.6, 1), (0.4, 0.6), (0.2, 0.4), (0.2, 0.4)	(0.35, 0.6)

Table 12 Ordering of alternatives

Case study	Ordering of alternatives
Case study I	$A_2 > A_3 > A_4 > A_1,$
Case study II	$A_2 > A_4 > A_3 > A_1,$

context of IVHFSS. Our experimental result shows slightly different ordering of the alternatives for the two cases, although the best alternative is same, which is A_2, i.e., McDonald's as per opinion of the group of experts. Table 12 shows the ordering of the alternatives. For the comparison purpose, a similar kind of data sets has been used by both the cases, i.e., case I and case II. More specifically, the dataset, i.e., HFEs used by the experts in first case, have been generalized to interval-valued HFEs in the second case.

10 Conclusions

Correlation between two variables has wide applications in statistical analysis. In previous studies, researchers have shown the impact of correlation measure on various types of fuzzy sets including IFS, IVIFS, HFS, IVHFS, etc., to provide wider applications of correlation measures. A common deficiency of those studies is that the correlation measures are applied only in fuzzy sets, rather than the soft sets. In this paper, we have introduced correlation coefficients of HFSSs and some of their properties. We have investigated correlation efficiency for individual HFSS which reflects the significance of an HFSS in decision-making process. For the purpose of decision making, this paper has proposed correlation efficiency which is used to assign importance to the corresponding decision makers. This paper has also proposed a decision-making algorithm which presents the application of correlation coefficient in hesitant fuzzy environment. In order to generalize the HFSSs to a wide domain of hesitant fuzzy environments, we have presented IVHFSS and defined its correlation coefficient. In the framework of IVHFSSs, we have introduced correlation efficiency and revised our decision-making algorithm to show the usage of correlation coefficient in interval-valued hesitant fuzzy environment. Finally, two examples are given to validate our proposed approaches. In future, researchers might use correlation coefficient for uncertain decision-making problems where GDM is crucial due to lack of information, expertiseness of the experts, risk amendment, etc. Researchers may also introduce correlation coefficient for various types of soft sets and their hybridizations.

Compliance with ethical standards

Conflict of interest We declare that the authors have no conflict of interest.

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