Neural Comput & Applic (2018) 30:799–823
https://doi.org/10.1007/s00521-016-2660-6

ORIGINAL ARTICLE

Frank prioritized Bonferroni mean operator with single-valued neutrosophic sets and its application in selecting third-party logistics providers

Pu Ji[1] · Jian-qiang Wang[1] · Hong-yu Zhang[1]

Received: 26 January 2016 / Accepted: 24 October 2016 / Published online: 22 November 2016

Abstract The selection of third-party logistics (TPL) providers can greatly influence the development of a company, and it has therefore become a hot topic for research. The process of selecting TPL providers is complicated because it includes fuzzy information that can be depicted by single-valued neutrosophic numbers (SVNNs). This paper establishes a new method for selecting TPL providers under single-valued neutrosophic environments. This method simultaneously takes into account the interrelationships among criteria and the different priority levels of criteria. To construct the method, we first introduce Frank operations of SVNNs, considering that extant operations for SVNNs lack flexibility and robustness. Then, we develop the single-valued neutrosophic Frank normalized prioritized Bonferroni mean (SVNFNPBM) operator utilizing the proposed Frank operations. The SVNFNPBM operator considers the interrelationships among criteria and their distinct priority levels by combining the Bonferroni mean and the prioritized average operators. Subsequently, we construct a method for selecting TPL providers based on the proposed operator. Finally, we describe a numerical example of selecting a TPL provider and conduct a comparative analysis to verify the applicability and feasibility of the proposed method.

Keywords Single-valued neutrosophic set · Frank operation · Prioritized average operator · Bonferroni mean · Selecting third-party logistics providers

✉ Hong-yu Zhang
hyzhang@csu.edu.cn

[1] School of Business, Central South University, Changsha 410083, China

1 Introduction

Many companies consider logistics seriously because of the great influence it exerts on economic development. Sometimes, companies choose to outsource logistics to third-party logistics (TPL) providers, mainly because TPL providers can render professional services at lower costs compared with internal logistics. The process of selecting TPL providers has attracted the attention of researchers [1, 2]. In essence, the problem of selecting a TPL provider involves multi-criteria decision-making (MCDM). Several criteria must be considered when selecting a TPL provider, including the cost of service, operational experience in the industry, customer satisfaction, and market reputation. In this respect, methods for selecting TPL providers can be described through MCDM models.

The process of selecting TPL providers involves fuzzy information that cannot be addressed by crisp values. Zadeh [3] developed fuzzy logic and fuzzy sets (FSs) to handle fuzzy information. However, FSs have some limits; for example, they cannot depict uncertain information. Numerous extensions of FSs have been proposed [4–6] to tackle the deficiencies of FSs. To address the issue that FSs cannot depict information uncertainty, Turksen [7] developed interval-valued FSs. Basing the work on FSs, Atanassov [8] introduced the notion of nonmembership and proposed intuitionistic fuzzy sets (IFSs). Subsequently, Atanassov and Gargov [9] presented interval-valued IFSs based on IFSs. Furthermore, Torra and Narukawa [10, 11] defined hesitant fuzzy sets (HFSs) to depict decision makers' hesitancy, which cannot be expressed by FSs. Chen et al. [12] then developed interval-valued HFSs (IVHFSs). However, HFSs and IVHFSs are unable to deal with indeterminacy in decision-making processes. To overcome this problem, Smarandache [13, 14] introduced neutrosophic sets (NSs), which depict fuzzy information using the functions of truth, indeterminacy, and falsity. Notably, the function of indeterminacy is independent of the functions of truth and falsity [15]. Nevertheless, NSs are hard to apply in practical problems because the values of the truth, indeterminacy and falsity functions lie within $]0^{-}, 1^{+}[$ [16]. Therefore, Wang et al. [17] presented single-valued neutrosophic sets (SVNSs), in which truth, indeterminacy, and falsity functions lie between zero and one. Since then, SVNSs have been studied [18, 19] and extended by combining them with other theories [20–24]. In addition, SVNSs and various other extensions of FSs have been applied in various fields, including decision-making [25–33], medical diagnosis [34–37], green product development [38], tourists' restaurant selection [39], and cloud service selection [40].

The aggregation operator represents an important tool for constructing MCDM methods [41]. Numerous researchers [42, 43] have studied aggregation operators under all sorts of fuzzy environments, including single-valued neutrosophic environments. Most aggregation operators suppose that the elements integrated are mutually independent. However, criteria may be correlative in some practical problems, like the selection of TPL providers. In these situations, operational experience in the industry and market reputation may influence the cost of service, and customer satisfaction may both depend on and affect operational experience in the industry. In solving these kinds of practical problems, it is important to consider interrelationships among criteria, which can be accomplished by the strategy of the Bonferroni mean (BM) operator [44]. Researchers have studied the BM and weighted BM (WBM) operators in several fuzzy environments [45], including intuitionistic fuzzy environments [46–48], hesitant fuzzy environments [49], and neutrosophic environments [50]. However, Zhou and He [51] pointed out that WBM operators do not satisfy reducibility; to conquer this shortcoming, they developed the normalized WBM (NWBM) operator [51] for intuitionistic fuzzy environments. Subsequently, the NWBM operator has been applied in a few other fuzzy environments, like hesitant fuzzy environments [52] and single-valued neutrosophic environments [53].

In some practical MCDM problems, criteria have different priority levels. For instance, the four criteria involved in selecting TPL providers can have distinct degrees of priority. A company may attach more importance to customer satisfaction than the other three criteria. That company may set the cost of service as the second most important criterion, followed by operational experience in the industry. In this situation, market reputation represents the least significant criterion. When tackling this kind of problem, the distinct priorities of the criteria should be taken into consideration. This necessitates the introduction of the prioritized average (PA) operator [54], which considers criteria priority levels by assigning each criterion a weight calculated according to its priority. The higher the priority of a criterion is, the greater its weight will be. The PA operator has been applied in a variety of fuzzy environments, including intuitionistic fuzzy environments [55–57], hesitant fuzzy environments [58, 59], 2-tuple fuzzy environments [60, 61], and interval-valued neutrosophic environments [62].

All the aggregation operators described above are defined on the basis of algebraic product and sum. However, the algebraic operational laws lack flexibility and robustness. Frank operational laws [63] can be utilized to address this defect. Frank operational laws are generalizations of algebraic operational laws, but they are more flexible for involving a parameter. Researchers have studied Frank operational laws under fuzzy environments. For example, Zhang et al. [64] explored Frank operational laws in the context of IFSs and presented several intuitionistic fuzzy power aggregation operators along with the proposed laws. Zhang [65] further proposed Frank operational laws for interval-valued IFSs and defined two Frank aggregation operators for interval-valued IFSs. Similarly, Qin et al. [66] defined Frank operational laws for HFSs and developed several aggregation operators based on the proposed laws. However, Frank operational laws have barely been studied in the context of NSs.

SVNSs can depict fuzzy information in the process of selecting TPL providers as fully as possible. Criteria for selecting TPL providers are correlative and have different priorities. BM operators can consider the interrelationships among criteria, and PA operators can take into account the criteria's distinct priorities. Furthermore, as mentioned above, Frank operational laws are flexible and robust. However, Frank operational laws for SVNSs have not been studied, and BM and PA operators based on Frank operational laws under single-valued neutrosophic environments have not been developed. In addition, extant methods for the selection of TPL providers cannot simultaneously take into consideration the interrelationships among criteria and their different priorities. To overcome these shortcomings, this paper puts forth a comprehensive method for selecting TPL providers. To achieve this goal, we first define Frank operational laws for SVNSs. Then, we develop a single-valued neutrosophic Frank normalized prioritized BM (SVNFNPBM) operator by combining PA and BM operators with the proposed Frank operational laws. Finally, we establish a method for selecting TPL providers based on the proposed SVNFNPBM operator.

The remainder of this paper is organized as follows. Section 2 reviews some basic concepts concerning SVNSs, Frank operations, and PA and BM operators. Section 3 defines Frank operations for single-valued neutrosophic numbers (SVNNs) and explores some properties of the proposed operations. Section 4 develops the SVNFNPBM operator and discusses its properties. Section 5 constructs a novel method for selecting TPL providers under single-valued neutrosophic environments. Section 6 describes a numerical example of selecting TPL providers and conducts a comparative analysis to demonstrate the applicability and feasibility of the proposed method. Section 6 also discusses the influence of parameters on the final ranking order. Finally, Sect. 7 outlines conclusions and suggests several directions for future research.

2 Preliminaries

This section reviews some basic concepts about SVNSs, Frank operations, and PA and BM operators, which will be utilized throughout the rest of the paper.

2.1 SVNSs

Definition 1 [13, 14] Let X be a finite set of points (objects), and let x denote a generic element in X. An NS A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. For any arbitrary x in X, $T_A(x) \in]0^-, 1^+[$, $I_A(x) \in]0^-, 1^+[$, and $F_A(x) \in]0^-, 1^+[$. There is no constraint on the sum of $T_A(x)$, $I_A(x)$, and $F_A(x)$, that is, $0^- \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$.

However, NSs are difficult to apply in practical problems. To overcome this challenge, Wang et al. [17] defined SVNSs.

Definition 2 [17] Let X be a finite set of points (objects), and let x denote a generic element in X. An SVNS A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. For any arbitrary x in X, $T_A(x) \in [0, 1]$, $I_A(x) \in [0, 1]$, and $F_A(x) \in [0, 1]$. An SVNS A can be defined as:

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in X\}.$$

An SVNS is a subclass of NS. The sum of $T_A(x)$, $I_A(x)$, and $F_A(x)$ lies between zero and three, that is, $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

For convenience, we use $x = (T_x, I_x, F_x)$ to represent an SVNN, which is an element in an SVNS.

Definition 3 [17] Let $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle |\ x \in X\}$ be an SVNS. The complement A^c of A is defined as $A^c = \{\langle x, F_A(x),\ 1 - I_A(x), T_A(x)\rangle | x \in X\}$.

In order to compare two SVNNs, Şahin [42] defined a comparison method with the proposed score and accuracy functions.

Definition 4 [42] Let $x = (T_x, I_x, F_x)$ be an SVNN. The score function sc of SVNN x can be defined as follows:

$$sc(x) = \frac{1 + T_x - 2I_x - F_x}{2}, \tag{1}$$

where $sc(x) \in [-1, 1]$.

Definition 5 [42] Let $x = (T_x, I_x, F_x)$ be an SVNN. The accuracy function l of SVNN x can be defined as follows:

$$l(x) = T_x - I_x(1 - T_x) - F_x(1 - I_x), \tag{2}$$

where $l(x) \in [-1, 1]$.

Definition 6 [42] Let $x = (T_x, I_x, F_x)$ and $y = (T_y, I_y, F_y)$ be two SVNNs. The comparison method between x and y can be defined as follows:

(1) when $sc(x) < sc(y)$, $x \prec y$;
(2) when $sc(x) = sc(y)$ and $l(x) < l(y)$, $x \prec y$; and
(3) when $sc(x) = sc(y)$ and $l(x) = l(y)$, $x \sim y$.

Furthermore, the algebraic operations of SVNNs have been defined as follows.

Definition 7 [53] Let $x = (T_x, I_x, F_x)$ and $y = (T_y, I_y, F_y)$ be two SVNNs, and $\lambda > 0$. The algebraic operations can be defined as follows:

(1) $x \oplus y = (T_x + T_y - T_xT_y, I_xI_y, F_xF_y)$;
(2) $x \otimes y = (T_xT_y, I_x + I_y - I_xI_y, F_x + F_y - F_xF_y)$;
(3) $\lambda x = (1 - (1 - T_x)^\lambda, I_x^\lambda, F_x^\lambda)$; and
(4) $x^\lambda = (T_x^\lambda, 1 - (1 - I_x)^\lambda, 1 - (1 - F_x)^\lambda)$.

2.2 Frank operations

Frank operations include the Frank product and Frank sum. The Frank product $\otimes_F$ is a triangular norm, and the Frank sum $\oplus_F$ is a triangular conorm.

Definition 8 [63] Let u and z be two real numbers. The Frank product $\otimes_F$ and Frank sum $\oplus_F$ between u and z are defined as follows:

$$u \oplus_F z = 1 - \log_\gamma\left(1 + \frac{(\gamma^{1-u} - 1)(\gamma^{1-z} - 1)}{\gamma - 1}\right) \quad \forall (u, z) \in [0, 1] \times [0, 1], \tag{3}$$

$$u \otimes_F z = \log_\gamma\left(1 + \frac{(\gamma^{u} - 1)(\gamma^{z} - 1)}{\gamma - 1}\right) \quad \forall (u, z) \in [0, 1] \times [0, 1], \tag{4}$$

where $\gamma > 1$.

Wang and He [67] discussed some properties of the Frank product and Frank sum:

$$(u \oplus_F z) + (u \otimes_F z) = u + z, \tag{5}$$

$$\frac{\partial(u \oplus_F z)}{\partial u} + \frac{\partial(u \otimes_F z)}{\partial u} = 1. \tag{6}$$

Based on the limit theory, it can be proved that when $\gamma \to 1$, $u \oplus_F z \to u + z - uz$ and $u \otimes_F z \to uz$ [67]. The Frank product and Frank sum reduce to the probabilistic product and probabilistic sum.

2.3 PA and BM operators

Yager [54] proposed the PA operator, which considers the priority levels of criteria. The PA operator is defined as follows.

Definition 9 [54] Let $C = \{C_1, C_2, \ldots, C_n\}$ be a set of criteria, and a prioritization $C_1 \succ C_2 \succ \cdots \succ C_n$ exists among the criteria. The performance value of object y under criterion C_i is denoted by $C_i(y)$ $(i = 1, 2, \ldots, n)$, and $C_i(y) \in [0, 1]$. The PA operator is defined as follows:

$$\mathrm{PA}(C_1(y), C_2(y), \ldots, C_n(y)) = \sum_{i=1}^{n} w_i C_i(y), \tag{7}$$

where $w_i = \frac{H_i}{\sum_{t=1}^{n} H_t}$, $H_i = \prod_{k=1}^{i-1} C_k(y)$ $(i \geq 2)$, and $H_1 = 1$.

In some practical situations, criteria are interdependent. In order to consider the interrelationships among criteria, Bonferroni [44] developed the BM operator.

Definition 10 [44] Let $p, q > 0$ and b_i $(i = 1, 2, \ldots, n)$ be a collection of nonnegative real numbers. The BM operator is defined as follows:

$$\mathrm{BM}^{p,q}(b_1, b_2, \ldots, b_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1,\\ i \neq j}}^{n} b_i^p b_j^q \right)^{1/p+q}. \tag{8}$$

Considering the different relative importance of each element, Xu and Yager [68] further defined the WBM operator on the basis of the BM operator.

Definition 11 [68] Let $p, q > 0$ and b_i $(i = 1, 2, \ldots, n)$ be a collection of nonnegative real numbers. $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ is the weight vector of b_i $(i = 1, 2, \ldots, n)$, $\sum_{i=1}^{n} w_i = 1$, and $w_i \in [0, 1]$. The WBM operator is defined as follows:

$$\mathrm{WBM}^{p,q}(b_1, b_2, \ldots, b_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1,\\ i \neq j}}^{n} (w_i b_i)^p (w_j b_j)^q \right)^{1/p+q}. \tag{9}$$

Nevertheless, the WBM operator does not have the reducibility. To address this defect, Zhou and He [51] proposed the NWBM operator as follows.

Definition 12 [51] Let $p, q > 0$ and b_i $(i = 1, 2, \ldots, n)$ be a collection of nonnegative real numbers. $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ is the weight vector of b_i $(i = 1, 2, \ldots, n)$, $\sum_{i=1}^{n} w_i = 1$, and $w_i \in [0, 1]$. The NWBM operator is defined as

$$\mathrm{NWBM}^{p,q}(b_1, b_2, \ldots, b_n) = \left(\sum_{\substack{i,j=1,\\ i \neq j}}^{n} \frac{w_i w_j}{1 - w_i} b_i^p b_j^q \right)^{1/p+q}. \tag{10}$$

3 Frank operations of SVNNs

This section introduces Frank operations of SVNNs based on the algebraic operations of SVNNs described in Definition 7 and the Frank operations described in Definition 8. Furthermore, this section discusses some properties of the proposed Frank operations of SVNNs.

Definition 13 Let $x = (T_x, I_x, F_x)$ and $y = (T_y, I_y, F_y)$ be two SVNNs, $\gamma > 1$, and $\lambda > 0$. Frank operations of SVNNs are defined as follows:

(1) $x \oplus_F y = \left(1 - \log_\gamma \left(1 + \frac{(\gamma^{1-T_x} - 1)(\gamma^{1-T_y} - 1)}{\gamma - 1} \right), \log_\gamma \left(1 + \frac{(\gamma^{I_x} - 1)(\gamma^{I_y} - 1)}{\gamma - 1} \right), \log_\gamma \left(1 + \frac{(\gamma^{F_x} - 1)(\gamma^{F_y} - 1)}{\gamma - 1} \right) \right);$

(2) $x \otimes_F y = \left(\log_\gamma \left(1 + \frac{(\gamma^{T_x} - 1)(\gamma^{T_y} - 1)}{\gamma - 1} \right), 1 - \log_\gamma \left(1 + \frac{(\gamma^{1-I_x} - 1)(\gamma^{1-I_y} - 1)}{\gamma - 1} \right), 1 - \log_\gamma \left(1 + \frac{(\gamma^{1-F_x} - 1)(\gamma^{1-F_y} - 1)}{\gamma - 1} \right) \right);$

(3) $\lambda \cdot_F x = \left(1 - \log_\gamma \left(1 + \frac{(\gamma^{1-T_x} - 1)^\lambda}{(\gamma - 1)^{\lambda - 1}} \right), \log_\gamma \left(1 + \frac{(\gamma^{I_x} - 1)^\lambda}{(\gamma - 1)^{\lambda - 1}} \right), \log_\gamma \left(1 + \frac{(\gamma^{F_x} - 1)^\lambda}{(\gamma - 1)^{\lambda - 1}} \right) \right);$

and

(4) $x^{\wedge_F \lambda} = \left(\log_\gamma \left(1 + \frac{(\gamma^{T_x} - 1)^\lambda}{(\gamma - 1)^{\lambda - 1}} \right), 1 - \log_\gamma \left(1 + \frac{(\gamma^{1-I_x} - 1)^\lambda}{(\gamma - 1)^{\lambda - 1}} \right), 1 - \log_\gamma \left(1 + \frac{(\gamma^{1-F_x} - 1)^\lambda}{(\gamma - 1)^{\lambda - 1}} \right) \right).$

Example 1 Let $x = (0.4, 0.2, 0.3)$ and $y = (0.6, 0.1, 0.2)$ be two SVNNs. Suppose $\gamma = 2$, then by the Frank operations listed in Definition 13, we have $x \oplus_F y = (1 - \log_2 (1 + (2^{1-0.4} - 1)(2^{1-0.6} - 1)), \log_2 (1 + (2^{0.2} - 1)(2^{0.1} - 1)), \log_2 (1 + (2^{0.3} - 1)(2^{0.2} - 1))) =$

(0.7799, 0.0153, 0.0488), and $x \otimes_F y = (\log_2(1 + (2^{0.4} - 1)(2^{0.6} - 1)), \log_2(1 + (2^{1-0.2} - 1)(2^{1-0.1} - 1)), \log_2(1 + (2^{1-0.3} - 1)(2^{1-0.2} - 1))) = (0.2201, 0.7153, 0.0.5488)$.

Moreover, the Frank operations of SVNNs in Definition 13 have the following properties.

Theorem 1 *Let x, y, and z be three SVNNs, and $\lambda_1, \lambda_2 > 0$. The Frank operations of SVNNs satisfy the following properties*:

($FP1$) $x \oplus_F y = y \oplus_F x$;
($FP2$) $x \otimes_F y = y \otimes_F x$;
($FP3$) $\lambda_1 \cdot_F (x \oplus_F y) = \lambda_1 \cdot_F x \oplus_F \lambda_1 \cdot_F y$;
($FP4$) $(x \otimes_F y)^{\wedge_F \lambda_1} = x^{\wedge_F \lambda_1} \otimes_F y^{\wedge_F \lambda_1}$;
($FP5$) $\lambda_1 \cdot_F x \oplus_F \lambda_2 \cdot_F x = (\lambda_1 + \lambda_2) \cdot_F x$;
($FP6$) $x^{\wedge_F \lambda_1} \otimes_F x^{\wedge_F \lambda_2} = x^{\wedge_F (\lambda_1 + \lambda_2)}$;
($FP7$) $\lambda_1 \cdot_F (\lambda_2 \cdot_F x) = \lambda_2 \cdot_F (\lambda_1 \cdot_F x) = (\lambda_1 \lambda_2) \cdot_F x$; and
($FP8$) $(x \oplus_F y) \oplus_F z = x \oplus_F (y \oplus_F z)$.

Proof Assume that $x = (T_x, I_x, F_x)$, $y = (T_y, I_y, F_y)$, and $z = (T_z, I_z, F_z)$.

($FP1$) Based on operation (1) in Definition 13, it is true that

$$\begin{aligned} x \oplus_F y = &\left(1 - \log_\gamma\left(1 + \frac{(\gamma^{1-T_x} - 1)(\gamma^{1-T_y} - 1)}{\gamma - 1}\right),\right. \\ &\log_\gamma\left(1 + \frac{(\gamma^{I_x} - 1)(\gamma^{I_y} - 1)}{\gamma - 1}\right), \\ &\left.\log_\gamma\left(1 + \frac{(\gamma^{F_x} - 1)(\gamma^{F_y} - 1)}{\gamma - 1}\right)\right) \\ = &\left(1 - \log_\gamma\left(1 + \frac{(\gamma^{1-T_y} - 1)(\gamma^{1-T_x} - 1)}{\gamma - 1}\right),\right. \\ &\log_\gamma\left(1 + \frac{(\gamma^{I_y} - 1)(\gamma^{I_x} - 1)}{\gamma - 1}\right), \\ &\left.\log_\gamma\left(1 + \frac{(\gamma^{F_y} - 1)(\gamma^{F_x} - 1)}{\gamma - 1}\right)\right) \\ = &\, y \oplus_F x. \end{aligned}$$

Therefore, $x \oplus_F y = y \oplus_F x$ holds.

($FP2$) Based on operation (2) in Definition 13, it follows that

$$\begin{aligned} x \otimes_F y = &\left(\log_\gamma\left(1 + \frac{(\gamma^{T_x} - 1)(\gamma^{T_y} - 1)}{\gamma - 1}\right),\right. \\ &1 - \log_\gamma\left(1 + \frac{(\gamma^{1-I_x} - 1)(\gamma^{1-I_y} - 1)}{\gamma - 1}\right) \\ &\left.1 - \log_\gamma\left(1 + \frac{(\gamma^{1-F_x} - 1)(\gamma^{1-F_y} - 1)}{\gamma - 1}\right)\right) \\ = &\left(\log_\gamma\left(1 + \frac{(\gamma^{T_y} - 1)(\gamma^{T_x} - 1)}{\gamma - 1}\right),\right. \\ &1 - \log_\gamma\left(1 + \frac{(\gamma^{1-I_y} - 1)(\gamma^{1-I_x} - 1)}{\gamma - 1}\right), \\ &\left.1 - \log_\gamma\left(1 + \frac{(\gamma^{1-F_y} - 1)(\gamma^{1-F_x} - 1)}{\gamma - 1}\right)\right) \\ = &\, y \otimes_F x. \end{aligned}$$

Hence, $x \otimes_F y = y \otimes_F x$ is proven to be right.

($FP3$) Based on operations (1) and (3) in Definition 13,

$$\begin{aligned} x \oplus_F y = &\left(1 - \log_\gamma\left(1 + \frac{(\gamma^{1-T_x} - 1)(\gamma^{1-T_y} - 1)}{\gamma - 1}\right),\right. \\ &\log_\gamma\left(1 + \frac{(\gamma^{I_x} - 1)(\gamma^{I_y} - 1)}{\gamma - 1}\right), \\ &\left.\log_\gamma\left(1 + \frac{(\gamma^{F_x} - 1)(\gamma^{F_y} - 1)}{\gamma - 1}\right)\right), \end{aligned}$$

$$\begin{aligned} \lambda_1 \cdot_F x = &\left(1 - \log_\gamma\left(1 + \frac{(\gamma^{1-T_x} - 1)^{\lambda_1}}{(\gamma - 1)^{\lambda_1 - 1}}\right),\right. \\ &\log_\gamma\left(1 + \frac{(\gamma^{I_x} - 1)^{\lambda_1}}{(\gamma - 1)^{\lambda_1 - 1}}\right), \\ &\left.\log_\gamma\left(1 + \frac{(\gamma^{F_x} - 1)^{\lambda_1}}{(\gamma - 1)^{\lambda_1 - 1}}\right)\right), \end{aligned}$$

and

$$\begin{aligned} \lambda_1 \cdot_F y = &\left(1 - \log_\gamma\left(1 + \frac{(\gamma^{1-T_y} - 1)^{\lambda_1}}{(\gamma - 1)^{\lambda_1 - 1}}\right),\right. \\ &\log_\gamma\left(1 + \frac{(\gamma^{I_y} - 1)^{\lambda_1}}{(\gamma - 1)^{\lambda_1 - 1}}\right) \\ &\left.\log_\gamma\left(1 + \frac{(\gamma^{F_y} - 1)^{\lambda_1}}{(\gamma - 1)^{\lambda_1 - 1}}\right)\right). \end{aligned}$$

Therefore, we have

$$\lambda_1 \cdot_F (x \oplus_F y) = \left(1 - \log_\gamma\left(1 + \frac{(\gamma^{1-T_x}-1)^{\lambda_1}(\gamma^{1-T_y}-1)^{\lambda_1}}{(\gamma-1)^{2\lambda_1-1}}\right),\right.$$
$$\log_\gamma\left(1 + \frac{(\gamma^{I_x}-1)^{\lambda_1}(\gamma^{I_y}-1)^{\lambda_1}}{(\gamma-1)^{2\lambda_1-1}}\right),$$
$$\left.\log_\gamma\left(1 + \frac{(\gamma^{F_x}-1)^{\lambda_1}(\gamma^{F_y}-1)^{\lambda_1}}{(\gamma-1)^{2\lambda_1-1}}\right)\right),$$

and

$$\lambda_1 \cdot_F x \oplus_F \lambda_1 \cdot_F y = \left(1 - \log_\gamma\left(1 + \frac{(\gamma^{1-T_x}-1)^{\lambda_1}(\gamma^{1-T_y}-1)^{\lambda_1}}{(\gamma-1)^{2\lambda_1-1}}\right),\right.$$
$$\log_\gamma\left(1 + \frac{(\gamma^{I_x}-1)^{\lambda_1}(\gamma^{I_y}-1)^{\lambda_1}}{(\gamma-1)^{2\lambda_1-1}}\right),$$
$$\left.\log_\gamma\left(1 + \frac{(\gamma^{F_x}-1)^{\lambda_1}(\gamma^{F_y}-1)^{\lambda_1}}{(\gamma-1)^{2\lambda_1-1}}\right)\right).$$

Hence, $\lambda_1 \cdot_F (x \oplus_F y) = \lambda_1 \cdot_F x \oplus_F \lambda_1 \cdot_F y$.

($FP4$) Based on operations (2) and (4) in Definition 13,

$$x \otimes_F y = \left(\log_\gamma\left(1 + \frac{(\gamma^{T_x}-1)(\gamma^{T_y}-1)}{\gamma-1}\right),\right.$$
$$1 - \log_\gamma\left(1 + \frac{(\gamma^{1-I_x}-1)(\gamma^{1-I_y}-1)}{\gamma-1}\right),$$
$$\left.1 - \log_\gamma\left(1 + \frac{(\gamma^{1-F_x}-1)(\gamma^{1-F_y}-1)}{\gamma-1}\right)\right),$$

$$x^{\wedge_F \lambda_1} = \left(\log_\gamma\left(1 + \frac{(\gamma^{T_x}-1)^{\lambda_1}}{(\gamma-1)^{\lambda_1-1}}\right),\right.$$
$$1 - \log_\gamma\left(1 + \frac{(\gamma^{1-I_x}-1)^{\lambda_1}}{(\gamma-1)^{\lambda_1-1}}\right),$$
$$\left.1 - \log_\gamma\left(1 + \frac{(\gamma^{1-F_x}-1)^{\lambda_1}}{(\gamma-1)^{\lambda_1-1}}\right)\right),$$

and

$$y^{\wedge_F \lambda_1} = \left(\log_\gamma\left(1 + \frac{(\gamma^{T_y}-1)^{\lambda_1}}{(\gamma-1)^{\lambda_1-1}}\right),\right.$$
$$1 - \log_\gamma\left(1 + \frac{(\gamma^{1-I_y}-1)^{\lambda_1}}{(\gamma-1)^{\lambda_1-1}}\right),$$
$$\left.1 - \log_\gamma\left(1 + \frac{(\gamma^{1-F_y}-1)^{\lambda_1}}{(\gamma-1)^{\lambda_1-1}}\right)\right).$$

Therefore,

$$(x \otimes_F y)^{\wedge_F \lambda_1} = \left(\log_\gamma\left(1 + \frac{(\gamma^{T_x}-1)^{\lambda_1}(\gamma^{T_y}-1)^{\lambda_1}}{(\gamma-1)^{2\lambda_1-1}}\right),\right.$$
$$1 - \log_\gamma\left(1 + \frac{(\gamma^{1-I_x}-1)^{\lambda_1}(\gamma^{1-I_y}-1)^{\lambda_1}}{(\gamma-1)^{2\lambda_1-1}}\right),$$
$$\left.1 - \log_\gamma\left(1 + \frac{(\gamma^{1-F_x}-1)^{\lambda_1}(\gamma^{1-F_y}-1)^{\lambda_1}}{(\gamma-1)^{2\lambda_1-1}}\right)\right),$$

and

$$x^{\wedge_F \lambda_1} \otimes_F y^{\wedge_F \lambda_1} = \left(\log_\gamma\left(1 + \frac{(\gamma^{T_x}-1)^{\lambda_1}(\gamma^{T_y}-1)^{\lambda_1}}{(\gamma-1)^{2\lambda_1-1}}\right),\right.$$
$$1 - \log_\gamma\left(1 + \frac{(\gamma^{1-I_x}-1)^{\lambda_1}(\gamma^{1-I_y}-1)^{\lambda_1}}{(\gamma-1)^{2\lambda_1-1}}\right),$$
$$\left.1 - \log_\gamma\left(1 + \frac{(\gamma^{1-F_x}-1)^{\lambda_1}(\gamma^{1-F_y}-1)^{\lambda_1}}{(\gamma-1)^{2\lambda_1-1}}\right)\right).$$

This makes it clear that $(x \otimes_F y)^{\wedge_F \lambda_1} = x^{\wedge_F \lambda_1} \otimes_F y^{\wedge_F \lambda_1}$.

($FP5$) Based on operations (1) and (3) in Definition 13, we have

$$\lambda_1 \cdot_F x = \left(1 - \log_\gamma\left(1 + \frac{(\gamma^{1-T_x}-1)^{\lambda_1}}{(\gamma-1)^{\lambda_1-1}}\right),\right.$$
$$\log_\gamma\left(1 + \frac{(\gamma^{I_x}-1)^{\lambda_1}}{(\gamma-1)^{\lambda_1-1}}\right),$$
$$\left.\log_\gamma\left(1 + \frac{(\gamma^{F_x}-1)^{\lambda_1}}{(\gamma-1)^{\lambda_1-1}}\right)\right),$$

$$\lambda_2 \cdot_F x = \left(1 - \log_\gamma\left(1 + \frac{(\gamma^{1-T_x}-1)^{\lambda_2}}{(\gamma-1)^{\lambda_2-1}}\right),\right.$$
$$\log_\gamma\left(1 + \frac{(\gamma^{I_x}-1)^{\lambda_2}}{(\gamma-1)^{\lambda_2-1}}\right),$$
$$\left.\log_\gamma\left(1 + \frac{(\gamma^{F_x}-1)^{\lambda_2}}{(\gamma-1)^{\lambda_2-1}}\right)\right),$$

and

$$(\lambda_1 + \lambda_2) \cdot_F x = \left(1 - \log_\gamma\left(1 + \frac{(\gamma^{1-T_x}-1)^{\lambda_1+\lambda_2}}{(\gamma-1)^{\lambda_1+\lambda_2-1}}\right),\right.$$
$$\log_\gamma\left(1 + \frac{(\gamma^{I_x}-1)^{\lambda_1+\lambda_2}}{(\gamma-1)^{\lambda_1+\lambda_2-1}}\right),$$
$$\left.\log_\gamma\left(1 + \frac{(\gamma^{F_x}-1)^{\lambda_1+\lambda_2}}{(\gamma-1)^{\lambda_1+\lambda_2-1}}\right)\right).$$

Therefore,

$$\lambda_1 \cdot_F x \oplus_F \lambda_2 \cdot_F x = \left(1-\log_\gamma\left(1+\frac{(\gamma^{1-T_x}-1)^{\lambda_1+\lambda_2}}{(\gamma-1)^{\lambda_1+\lambda_2-1}}\right), \log_\gamma\left(1+\frac{(\gamma^{I_x}-1)^{\lambda_1+\lambda_2}}{(\gamma-1)^{\lambda_1+\lambda_2-1}}\right), \log_\gamma\left(1+\frac{(\gamma^{F_x}-1)^{\lambda_1+\lambda_2}}{(\gamma-1)^{\lambda_1+\lambda_2-1}}\right)\right) = (\lambda_1+\lambda_2) \cdot_F x.$$

Hence, $\lambda_1 \cdot_F x \oplus_F \lambda_2 \cdot_F x = (\lambda_1+\lambda_2) \cdot_F x$ holds.

($FP6$) Based on operations (2) and (4) in Definition 13,

$$x^{\wedge_F \lambda_1} = \left(\log_\gamma\left(1+\frac{(\gamma^{T_x}-1)^{\lambda_1}}{(\gamma-1)^{\lambda_1-1}}\right), 1-\log_\gamma\left(1+\frac{(\gamma^{1-I_x}-1)^{\lambda_1}}{(\gamma-1)^{\lambda_1-1}}\right), 1-\log_\gamma\left(1+\frac{(\gamma^{1-F_x}-1)^{\lambda_1}}{(\gamma-1)^{\lambda_1-1}}\right)\right),$$

$$x^{\wedge_F \lambda_2} = \left(\log_\gamma\left(1+\frac{(\gamma^{T_x}-1)^{\lambda_2}}{(\gamma-1)^{\lambda_2-1}}\right), 1-\log_\gamma\left(1+\frac{(\gamma^{1-I_x}-1)^{\lambda_2}}{(\gamma-1)^{\lambda_2-1}}\right), 1-\log_\gamma\left(1+\frac{(\gamma^{1-F_x}-1)^{\lambda_2}}{(\gamma-1)^{\lambda_2-1}}\right)\right),$$

and

$$x^{\wedge_F (\lambda_1+\lambda_2)} = \left(\log_\gamma\left(1+\frac{(\gamma^{T_x}-1)^{\lambda_1+\lambda_2}}{(\gamma-1)^{\lambda_1+\lambda_2-1}}\right), 1-\log_\gamma\left(1+\frac{(\gamma^{1-I_x}-1)^{\lambda_1+\lambda_2}}{(\gamma-1)^{\lambda_1+\lambda_2-1}}\right), 1-\log_\gamma\left(1+\frac{(\gamma^{1-F_x}-1)^{\lambda_1+\lambda_2}}{(\gamma-1)^{\lambda_1+\lambda_2-1}}\right)\right).$$

Hence,

$$x^{\wedge_F \lambda_1} \otimes_F x^{\wedge_F \lambda_2} = \left(\log_\gamma\left(1+\frac{(\gamma^{T_x}-1)^{\lambda_1+\lambda_2}}{(\gamma-1)^{\lambda_1+\lambda_2-1}}\right), 1-\log_\gamma\left(1+\frac{(\gamma^{1-I_x}-1)^{\lambda_1+\lambda_2}}{(\gamma-1)^{\lambda_1+\lambda_2-1}}\right), 1-\log_\gamma\left(1+\frac{(\gamma^{1-F_x}-1)^{\lambda_1+\lambda_2}}{(\gamma-1)^{\lambda_1+\lambda_2-1}}\right)\right) = x^{\hat{F}(\lambda_1+\lambda_2)}.$$

Therefore, $x^{\wedge_F \lambda_1} \otimes_F x^{\wedge_F \lambda_2} = x^{\wedge_F (\lambda_1+\lambda_2)}$ is true.

($FP7$) Based on operation (3) in Definition 13,

$$\lambda_2 \cdot_F x = \left(1-\log_\gamma\left(1+\frac{(\gamma^{T_x}-1)^{\lambda_2}}{(\gamma-1)^{\lambda_2-1}}\right), \log_\gamma\left(1+\frac{(\gamma^{1-I_x}-1)^{\lambda_2}}{(\gamma-1)^{\lambda_2-1}}\right), \log_\gamma\left(1+\frac{(\gamma^{1-F_x}-1)^{\lambda_2}}{(\gamma-1)^{\lambda_2-1}}\right)\right),$$

$$\lambda_1 \cdot_F x = \left(1-\log_\gamma\left(1+\frac{(\gamma^{T_x}-1)^{\lambda_1}}{(\gamma-1)^{\lambda_1-1}}\right), \log_\gamma\left(1+\frac{(\gamma^{1-I_x}-1)^{\lambda_1}}{(\gamma-1)^{\lambda_1-1}}\right), \log_\gamma\left(1+\frac{(\gamma^{1-F_x}-1)^{\lambda_1}}{(\gamma-1)^{\lambda_1-1}}\right)\right),$$

and

$$(\lambda_1\lambda_2) \cdot_F x = \left(1-\log_\gamma\left(1+\frac{(\gamma^{T_x}-1)^{\lambda_1\lambda_2}}{(\gamma-1)^{\lambda_1\lambda_2-1}}\right), \log_\gamma\left(1+\frac{(\gamma^{1-I_x}-1)^{\lambda_1\lambda_2}}{(\gamma-1)^{\lambda_1\lambda_2-1}}\right), \log_\gamma\left(1+\frac{(\gamma^{1-F_x}-1)^{\lambda_1\lambda_2}}{(\gamma-1)^{\lambda_1\lambda_2-1}}\right)\right).$$

Therefore,

$$\lambda_1 \cdot_F (\lambda_2 \cdot_F x) = \left(1-\log_\gamma\left(1+\frac{(\gamma^{T_x}-1)^{\lambda_1\lambda_2}}{(\gamma-1)^{\lambda_1\lambda_2-1}}\right), \log_\gamma\left(1+\frac{(\gamma^{1-I_x}-1)^{\lambda_1\lambda_2}}{(\gamma-1)^{\lambda_1\lambda_2-1}}\right), \log_\gamma\left(1+\frac{(\gamma^{1-F_x}-1)^{\lambda_1\lambda_2}}{(\gamma-1)^{\lambda_1\lambda_2-1}}\right)\right) = (\lambda_1\lambda_2) \cdot_F x,$$

and

$$\lambda_2 \cdot_F (\lambda_1 \cdot_F x) = \left(1-\log_\gamma\left(1+\frac{(\gamma^{T_x}-1)^{\lambda_1\lambda_2}}{(\gamma-1)^{\lambda_1\lambda_2-1}}\right), \log_\gamma\left(1+\frac{(\gamma^{1-I_x}-1)^{\lambda_1\lambda_2}}{(\gamma-1)^{\lambda_1\lambda_2-1}}\right), \log_\gamma\left(1+\frac{(\gamma^{1-F_x}-1)^{\lambda_1\lambda_2}}{(\gamma-1)^{\lambda_1\lambda_2-1}}\right)\right) = (\lambda_1\lambda_2) \cdot_F x.$$

Hence, $\lambda_1 \cdot_F (\lambda_2 \cdot_F x) = \lambda_2 \cdot_F (\lambda_1 \cdot_F x) = (\lambda_1\lambda_2) \cdot_F x$.

($FP8$) Based on operation (1) in Definition 13,

$$x \oplus_F y = \left(1 - \log_\gamma\left(1 + \frac{(\gamma^{1-T_x} - 1)(\gamma^{1-T_y} - 1)}{\gamma - 1}\right), \log_\gamma\left(1 + \frac{(\gamma^{I_x} - 1)(\gamma^{I_y} - 1)}{\gamma - 1}\right), \log_\gamma\left(1 + \frac{(\gamma^{F_x} - 1)(\gamma^{F_y} - 1)}{\gamma - 1}\right)\right),$$

and

$$y \oplus_F z = \left(1 - \log_\gamma\left(1 + \frac{(\gamma^{1-T_y} - 1)(\gamma^{1-T_z} - 1)}{\gamma - 1}\right), \log_\gamma\left(1 + \frac{(\gamma^{I_y} - 1)(\gamma^{I_z} - 1)}{\gamma - 1}\right), \log_\gamma\left(1 + \frac{(\gamma^{F_y} - 1)(\gamma^{F_z} - 1)}{\gamma - 1}\right)\right).$$

Therefore, we have

$$(x \oplus_F y) \oplus_F z = \left(1 - \log_\gamma\left(1 + \frac{(\gamma^{1-T_x} - 1)(\gamma^{1-T_y} - 1)(\gamma^{1-T_z} - 1)}{(\gamma - 1)}\right), \log_\gamma\left(1 + \frac{(\gamma^{I_x} - 1)(\gamma^{I_y} - 1)(\gamma^{I_z} - 1)}{(\gamma - 1)}\right), \log_\gamma\left(1 + \frac{(\gamma^{F_x} - 1)(\gamma^{F_y} - 1)(\gamma^{F_z} - 1)}{(\gamma - 1)}\right)\right),$$

and

$$x \oplus_F (y \oplus_F z) = \left(1 - \log_\gamma\left(1 + \frac{(\gamma^{1-T_x} - 1)(\gamma^{1-T_y} - 1)(\gamma^{1-T_z} - 1)}{(\gamma - 1)}\right), \log_\gamma\left(1 + \frac{(\gamma^{I_x} - 1)(\gamma^{I_y} - 1)(\gamma^{I_z} - 1)}{(\gamma - 1)}\right), \log_\gamma\left(1 + \frac{(\gamma^{F_x} - 1)(\gamma^{F_y} - 1)(\gamma^{F_z} - 1)}{(\gamma - 1)}\right)\right).$$

Hence, $(x \oplus_F y) \oplus_F z = x \oplus_F (y \oplus_F z)$ is true.

Thus, Theorem 1 holds.

4 SVNFNPBM operator

This section presents the definition of the single-valued neutrosophic Frank BM (SVNFBM) operator. Subsequently, we define the SVNFNPBM operator on the basis of the PA and NWBM operators. The SVNFNPBM operator simultaneously considers the priority levels of aggregated elements and the interrelationships among these elements by integrating the PA and WBM operators. Then, we discuss several properties of the SVNFNPBM operator.

Definition 14 Let $p, q > 0$, and $x_i = (T_i, I_i, F_i)$ ($i = 1, 2, \ldots, n$) be a set of SVNNs. The SVNFBM operator is defined as follows:

$$\mathrm{SVNFBM}^{p,q}(x_1, x_2, \ldots, x_n) = \left(\frac{1}{n(n-1)} \cdot_F \underset{\substack{i,j=1,\\ i\neq j}}{\overset{n}{\oplus_F}} \left((x_i)^{\wedge_F p} \otimes_F (x_j)^{\wedge_F q}\right)\right)^{\wedge_F \frac{1}{p+q}}. \tag{11}$$

This paper develops the SVNFNPBM operator based on the PA and NWBM operators as defined in Eqs. (7) and (10). The SVNFNPBM operator is defined as follows.

Definition 15 Let $p, q > 0$, and $C = \{C_1, C_2, \ldots, C_n\}$ be a set of criteria, such that a prioritization $C_1 \succ C_2 \succ \cdots \succ C_n$ exists among the criteria. The performance value of object x under criterion C_i is denoted by SVNN $x_i = (T_i, I_i, F_i)$ ($i = 1, 2, \ldots, n$). The SVNFNPBM operator is defined as follows:

$$\mathrm{SVNFNPBM}^{p,q}(x_1, x_2, \ldots, x_n) = \left(\underset{\substack{i,j=1,\\ i\neq j}}{\overset{n}{\oplus_F}} \left(\frac{w_i w_j}{1 - w_i} \cdot_F \left((x_i)^{\wedge_F p} \otimes_F (x_j)^{\wedge_F q}\right)\right)\right)^{\wedge_F \frac{1}{p+q}}, \tag{12}$$

where $w_i = \frac{H_i}{\sum_{t=1}^n H_t}$, $H_t = \prod_{k=1}^{t-1} sc(x_k)$ $(t \geq 2)$, $H_1 = 1$, and $sc(x_k)$ is the score value of SVNN x_k obtained by Eq. (1).

Theorem 2 *Let $p, q > 0$, and let $C = \{C_1, C_2, \ldots, C_n\}$ be a set of criteria, such that a prioritization $C_1 \succ C_2 \succ \cdots \succ C_n$ exists among the criteria. The performance value of object x under criterion C_i is denoted by* SVNN *$x_i = (T_i, I_i, F_i)$ ($i = 1, 2, \ldots, n$). The aggregated value of x_i ($i = 1, 2, \ldots, n$) by* SVNFNPBM *operator in* Eq. (12) *is still an* SVNN, *and*

$$\begin{aligned}
&\mathrm{SVNFPBM}^{p,q}(x_1,x_2,\ldots,x_n)\\
&=\left(\log_\gamma\left(1+(\gamma-1)\left(\frac{1-\prod\limits_{\substack{i,j=1,\\ i\neq j}}^{n}\left(\frac{(\gamma-1)^{p+q}-\left(\gamma^{T_{x_i}}-1\right)^p\left(\gamma^{T_{x_j}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)\left(\gamma^{T_{x_i}}-1\right)^p\left(\gamma^{T_{x_j}}-1\right)^q}\right)^{\frac{w_iw_j}{1-w_i}}}{1+(\gamma-1)\prod\limits_{\substack{i,j=1,\\ i\neq j}}^{n}\left(\frac{(\gamma-1)^{p+q}-\left(\gamma^{T_{x_i}}-1\right)^p\left(\gamma^{T_{x_j}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)\left(\gamma^{T_{x_i}}-1\right)^p\left(\gamma^{T_{x_j}}-1\right)^q}\right)^{\frac{w_iw_j}{1-w_i}}}\right)^{\frac{1}{p+q}}\right)\right.,\\
&\quad 1-\log_\gamma\left(1+(\gamma-1)\left(\frac{1-\prod\limits_{\substack{i,j=1,\\ i\neq j}}^{n}\left(\frac{(\gamma-1)^{p+q}-\left(\gamma^{1-I_{x_i}}-1\right)^p\left(\gamma^{1-I_{x_j}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)\left(\gamma^{1-I_{x_i}}-1\right)^p\left(\gamma^{1-I_{x_j}}-1\right)^q}\right)^{\frac{w_iw_j}{1-w_i}}}{1+(\gamma-1)\prod\limits_{\substack{i,j=1,\\ i\neq j}}^{n}\left(\frac{(\gamma-1)^{p+q}-\left(\gamma^{1-I_{x_i}}-1\right)^p\left(\gamma^{1-I_{x_j}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)\left(\gamma^{1-I_{x_i}}-1\right)^p\left(\gamma^{1-I_{x_j}}-1\right)^q}\right)^{\frac{w_iw_j}{1-w_i}}}\right)^{\frac{1}{p+q}}\right),\\
&\quad \left.1-\log_\gamma\left(1+(\gamma-1)\left(\frac{1-\prod\limits_{\substack{i,j=1,\\ i\neq j}}^{n}\left(\frac{(\gamma-1)^{p+q}-\left(\gamma^{1-F_{x_i}}-1\right)^p\left(\gamma^{1-F_{x_j}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)\left(\gamma^{1-F_{x_i}}-1\right)^p\left(\gamma^{1-F_{x_j}}-1\right)^q}\right)^{\frac{w_iw_j}{1-w_i}}}{1+(\gamma-1)\prod\limits_{\substack{i,j=1,\\ i\neq j}}^{n}\left(\frac{(\gamma-1)^{p+q}-\left(\gamma^{1-F_{x_i}}-1\right)^p\left(\gamma^{1-F_{x_j}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)\left(\gamma^{1-F_{x_i}}-1\right)^p\left(\gamma^{1-F_{x_j}}-1\right)^q}\right)^{\frac{w_iw_j}{1-w_i}}}\right)^{\frac{1}{p+q}}\right)\right),
\end{aligned}\tag{13}$$

where $w_i=\frac{H_i}{\sum_{t=1}^{n}H_t}$, $H_t=\prod_{k=1}^{t-1}sc(x_k)\ (t\geq 2)$, $H_1=1$, *and* $sc(x_k)$ *is the score value of* SVNN x_k *obtained by* Eq. (1).

Theorem 2 can be proved by mathematical induction as follows.

Proof (1) Firstly, the following equation can be proved.

$$\begin{aligned}
&\bigoplus_{\substack{i,j=1,\\ i\neq j}}^{n}{}_F\left(\frac{w_iw_j}{1-w_i}\cdot_F\left((x_i)^{\wedge_F p}\otimes_F\left(x_j\right)^{\wedge_F q}\right)\right)\\
&=\left(1-\log_\gamma\left(1+(\gamma-1)\prod_{\substack{i,j=1,\\ i\neq j}}^{n}\left(\frac{(\gamma-1)^{p+q}-(\gamma^{T_{x_i}}-1)^p\left(\gamma^{T_{x_j}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)(\gamma^{T_{x_i}}-1)^p\left(\gamma^{T_{x_j}}-1\right)^q}\right)^{\frac{w_iw_j}{1-w_i}}\right)\right.,\\
&\quad\log_\gamma\left(1+(\gamma-1)\prod_{\substack{i,j=1,\\ i\neq j}}^{n}\left(\frac{(\gamma-1)^{p+q}-(\gamma^{1-I_{x_i}}-1)^p\left(\gamma^{1-I_{x_j}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)(\gamma^{1-I_{x_i}}-1)^p\left(\gamma^{1-I_{x_j}}-1\right)^q}\right)^{\frac{w_iw_j}{1-w_i}}\right),\\
&\quad\left.\log_\gamma\left(1+(\gamma-1)\prod_{\substack{i,j=1,\\ i\neq j}}^{n}\left(\frac{(\gamma-1)^{p+q}-(\gamma^{1-F_{x_i}}-1)^p\left(\gamma^{1-F_{x_j}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)(\gamma^{1-F_{x_i}}-1)^p\left(\gamma^{1-F_{x_j}}-1\right)^q}\right)^{\frac{w_iw_j}{1-w_i}}\right)\right).
\end{aligned}\tag{14}$$

(a) When $n = 2$, based on operations (2) and (4) in Definition 13, we can determine that

$$(x_1)^{\wedge_F p} = \left(\log_\gamma\left(1 + \frac{(\gamma^{T_{x_1}} - 1)^p}{(\gamma - 1)^{p-1}}\right),\right.$$
$$1 - \log_\gamma\left(1 + \frac{(\gamma^{1-I_{x_1}} - 1)^p}{(\gamma - 1)^{p-1}}\right),$$
$$\left.1 - \log_\gamma\left(1 + \frac{(\gamma^{1-F_{x_1}} - 1)^p}{(\gamma - 1)^{p-1}}\right)\right),$$

$$(x_1)^{\wedge_F q} = \left(\log_\gamma\left(1 + \frac{(\gamma^{T_{x_1}} - 1)^p}{(\gamma - 1)^{p-1}}\right),\right.$$
$$1 - \log_\gamma\left(1 + \frac{(\gamma^{1-I_{x_1}} - 1)^q}{(\gamma - 1)^{p-1}}\right),$$
$$\left.1 - \log_\gamma\left(1 + \frac{(\gamma^{1-F_{x_1}} - 1)^q}{(\gamma - 1)^{q-1}}\right)\right),$$

$$(x_2)^{\wedge_F p} = \left(\log_\gamma\left(1 + \frac{(\gamma^{T_{x_2}} - 1)^p}{(\gamma - 1)^{p-1}}\right),\right.$$
$$1 - \log_\gamma\left(1 + \frac{(\gamma^{1-I_{x_2}} - 1)^p}{(\gamma - 1)^{p-1}}\right),$$
$$\left.1 - \log_\gamma\left(1 + \frac{(\gamma^{1-F_{x_2}} - 1)^p}{(\gamma - 1)^{p-1}}\right)\right),$$

$$(x_2)^{\wedge_F q} = \left(\log_\gamma\left(1 + \frac{(\gamma^{T_{x_2}} - 1)^q}{(\gamma - 1)^{q-1}}\right),\right.$$
$$1 - \log_\gamma\left(1 + \frac{(\gamma^{1-I_{x_2}} - 1)^q}{(\gamma - 1)^{q-1}}\right),$$
$$\left.1 - \log_\gamma\left(1 + \frac{(\gamma^{1-F_{x_2}} - 1)^q}{(\gamma - 1)^{q-1}}\right)\right),$$

$$(x_1)^{\wedge_F p} \otimes_F (x_2)^{\wedge_F q} = \left(\log_\gamma\left(1 + \frac{(\gamma^{T_{x_1}} - 1)^p(\gamma^{T_{x_2}} - 1)^q}{(\gamma - 1)^{p+q-1}}\right),\right.$$
$$1 - \log_\gamma\left(1 + \frac{(\gamma^{1-I_{x_1}} - 1)^p(\gamma^{1-I_{x_2}} - 1)^q}{(\gamma - 1)^{p+q-1}}\right),$$
$$\left.1 - \log_\gamma\left(1 + \frac{(\gamma^{1-F_{x_1}} - 1)^p(\gamma^{1-F_{x_2}} - 1)^q}{(\gamma - 1)^{p+q-1}}\right)\right),$$

and

$$(x_2)^{\wedge_F p} \otimes_F (x_1)^{\wedge_F q} = \left(\log_\gamma\left(1 + \frac{(\gamma^{T_{x_2}} - 1)^p(\gamma^{T_{x_1}} - 1)^q}{(\gamma - 1)^{p+q-1}}\right),\right.$$
$$1 - \log_\gamma\left(1 + \frac{(\gamma^{1-I_{x_2}} - 1)^p(\gamma^{1-I_{x_1}} - 1)^q}{(\gamma - 1)^{p+q-1}}\right),$$
$$\left.1 - \log_\gamma\left(1 + \frac{(\gamma^{1-F_{x_2}} - 1)^p(\gamma^{1-F_{x_1}} - 1)^q}{(\gamma - 1)^{p+q-1}}\right)\right).$$

Therefore, based on operation (3) in Definition 13, we can determine that

$$\frac{w_1 w_2}{1 - w_1} \cdot_F \left((x_1)^{\wedge_F p} \otimes_F (x_2)^{\wedge_F q}\right)$$
$$= \left(1 - \log_\gamma\left(1 + (\gamma - 1)\left(\frac{(\gamma - 1)^{p+q} - (\gamma^{T_{x_1}} - 1)^p(\gamma^{T_{x_2}} - 1)^q}{(\gamma - 1)^{p+q} + (\gamma - 1)(\gamma^{T_{x_1}} - 1)^p(\gamma^{T_{x_2}} - 1)^q}\right)^{\frac{w_1 w_2}{1 - w_1}}\right),\right.$$
$$\log_\gamma\left(1 + (\gamma - 1)\left(\frac{(\gamma - 1)^{p+q} - (\gamma^{1-I_{x_1}} - 1)^p(\gamma^{1-I_{x_2}} - 1)^q}{(\gamma - 1)^{p+q} + (\gamma - 1)(\gamma^{1-I_{x_1}} - 1)^p(\gamma^{1-I_{x_2}} - 1)^q}\right)^{\frac{w_1 w_2}{1 - w_1}}\right),$$
$$\left.\log_\gamma\left(1 + (\gamma - 1)\left(\frac{(\gamma - 1)^{p+q} - (\gamma^{1-F_{x_1}} - 1)^p(\gamma^{1-F_{x_2}} - 1)^q}{(\gamma - 1)^{p+q} + (\gamma - 1)(\gamma^{1-F_{x_1}} - 1)^p(\gamma^{1-F_{x_2}} - 1)^q}\right)^{\frac{w_1 w_2}{1 - w_1}}\right)\right),$$

and

$$\frac{w_1 w_2}{1-w_2} \cdot_F \left((x_2)^{\wedge_F p} \otimes_F (x_1)^{\wedge_F q}\right)$$
$$= \left(1 - \log_\gamma\left(1 + (\gamma-1)\left(\frac{(\gamma-1)^{p+q} - (\gamma^{T_{x_2}}-1)^p(\gamma^{T_{x_1}}-1)^q}{(\gamma-1)^{p+q} + (\gamma-1)(\gamma^{T_{x_2}}-1)^p(\gamma^{T_{x_1}}-1)^q}\right)^{\frac{w_1 w_2}{1-w_2}}\right),\right.$$
$$\log_\gamma\left(1 + (\gamma-1)\left(\frac{(\gamma-1)^{p+q} - \left(\gamma^{1-I_{x_2}}-1\right)^p\left(\gamma^{1-I_{x_1}}-1\right)^q}{(\gamma-1)^{p+q} + (\gamma-1)\left(\gamma^{1-I_{x_2}}-1\right)^p\left(\gamma^{1-I_{x_1}}-1\right)^q}\right)^{\frac{w_1 w_2}{1-w_2}}\right)$$
$$\left.\log_\gamma\left(1 + (\gamma-1)\left(\frac{(\gamma-1)^{p+q} - \left(\gamma^{1-F_{x_2}}-1\right)^p\left(\gamma^{1-F_{x_1}}-1\right)^q}{(\gamma-1)^{p+q} + (\gamma-1)\left(\gamma^{1-F_{x_2}}-1\right)^p\left(\gamma^{1-F_{x_1}}-1\right)^q}\right)^{\frac{w_1 w_2}{1-w_2}}\right)\right).$$

Then,

$$\bigoplus_{\substack{i,j=1,\\ i\neq j}}^{2}{}_F \left(\frac{w_i w_j}{1-w_i} \cdot_F \left((x_i)^{\wedge_F p} \otimes_F (x_j)^{\wedge_F q}\right)\right)$$
$$= \left(\frac{w_1 w_2}{1-w_1} \cdot_F \left((x_1)^{\hat{F}p} \otimes_F (x_2)^{\hat{F}q}\right)\right) \oplus_F \left(\frac{w_1 w_2}{1-w_2} \cdot_F \left((x_2)^{\hat{F}p} \otimes_F (x_1)^{\hat{F}q}\right)\right)$$
$$= \left(1 - \log_\gamma\left(1 + (\gamma-1)\left(\frac{(\gamma-1)^{p+q} - (\gamma^{T_{x_1}}-1)^p(\gamma^{T_{x_2}}-1)^q}{(\gamma-1)^{p+q} + (\gamma-1)(\gamma^{T_{x_1}}-1)^p(\gamma^{T_{x_2}}-1)^q}\right)^{\frac{w_1 w_2}{1-w_1}}\right.\right.$$
$$\left.\left(\frac{(\gamma-1)^{p+q} - (\gamma^{T_{x_2}}-1)^p(\gamma^{T_{x_1}}-1)^q}{(\gamma-1)^{p+q} + (\gamma-1)(\gamma^{T_{x_2}}-1)^p(\gamma^{T_{x_1}}-1)^q}\right)^{\frac{w_1 w_2}{1-w_2}}\right),$$
$$\log_\gamma\left(1 + (\gamma-1)\left(\frac{(\gamma-1)^{p+q} - \left(\gamma^{1-I_{x_1}}-1\right)^p\left(\gamma^{1-I_{x_2}}-1\right)^q}{(\gamma-1)^{p+q} + (\gamma-1)\left(\gamma^{1-I_{x_1}}-1\right)^p\left(\gamma^{1-I_{x_2}}-1\right)^q}\right)^{\frac{w_1 w_2}{1-w_1}}\right.$$
$$\left.\left(\frac{(\gamma-1)^{p+q} - \left(\gamma^{1-I_{x_2}}-1\right)^p\left(\gamma^{1-I_{x_1}}-1\right)^q}{(\gamma-1)^{p+q} + (\gamma-1)\left(\gamma^{1-I_{x_2}}-1\right)^p\left(\gamma^{1-I_{x_1}}-1\right)^q}\right)^{\frac{w_1 w_2}{1-w_2}}\right),$$
$$\log_\gamma\left((\gamma-1)\left(\frac{(\gamma-1)^{p+q} - \left(\gamma^{1-F_{x_1}}-1\right)^p\left(\gamma^{1-F_{x_2}}-1\right)^q}{(\gamma-1)^{p+q} + (\gamma-1)\left(\gamma^{1-F_{x_1}}-1\right)^p\left(\gamma^{1-F_{x_2}}-1\right)^q}\right)^{\frac{w_1 w_2}{1-w_1}}\right.$$
$$\left.\left.\left(\frac{(\gamma-1)^{p+q} - \left(\gamma^{1-F_{x_2}}-1\right)^p\left(\gamma^{1-F_{x_1}}-1\right)^q}{(\gamma-1)^{p+q} + (\gamma-1)\left(\gamma^{1-F_{x_2}}-1\right)^p\left(\gamma^{1-F_{x_1}}-1\right)^q}\right)^{\frac{w_1 w_2}{1-w_2}}\right)\right).$$

In other words, when $n = 2$, Eq. (14) holds.

(b) Assuming Eq. (14) holds for $n = k$, we can obtain the following equation:

$$\underset{\substack{i,j=1,\\ i\neq j}}{\overset{k}{\oplus_F}}\left(\frac{w_iw_j}{1-w_i}\cdot_F\left((x_i)^{\wedge_F p}\otimes_F(x_j)^{\wedge_F q}\right)\right)$$
$$=\left(1-\log_\gamma\left(1+(\gamma-1)\prod_{\substack{i,j=1,\\ i\neq j}}^{k}\left(\frac{(\gamma-1)^{p+q}-(\gamma^{T_{x_i}}-1)^p\left(\gamma^{T_{x_j}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)(\gamma^{T_{x_i}}-1)^p\left(\gamma^{T_{x_j}}-1\right)^q}\right)^{\frac{w_iw_j}{1-w_i}}\right),\right.$$
$$\log_\gamma\left(1+(\gamma-1)\prod_{\substack{i,j=1,\\ i\neq j}}^{n}\left(\frac{(\gamma-1)^{p+q}-(\gamma^{1-I_{x_i}}-1)^p\left(\gamma^{1-I_{x_j}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)(\gamma^{1-I_{x_i}}-1)^p\left(\gamma^{1-I_{x_j}}-1\right)^q}\right)^{\frac{w_iw_j}{1-w_i}}\right),$$
$$\left.\log_\gamma\left(1+(\gamma-1)\prod_{\substack{i,j=1,\\ i\neq j}}^{n}\left(\frac{(\gamma-1)^{p+q}-(\gamma^{1-F_{x_i}}-1)^p\left(\gamma^{1-F_{x_j}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)(\gamma^{1-F_{x_i}}-1)^p\left(\gamma^{1-F_{x_j}}-1\right)^q}\right)^{\frac{w_iw_j}{1-w_i}}\right)\right). \tag{15}$$

The following equation can be proven to calculate the value of $\underset{\substack{i,j=1,\\ i\neq j}}{\overset{k+1}{\oplus_F}}\left(\frac{w_iw_j}{1-w_i}\cdot_F\left((x_i)^{\wedge_F p}\otimes_F(x_j)^{\wedge_F q}\right)\right)$:

$$\underset{i=1}{\overset{k}{\oplus_F}}\left(\frac{w_iw_{k+1}}{1-w_i}\cdot_F\left((x_i)^{\wedge_F p}\otimes_F(x_{k+1})^{\wedge_F q}\right)\right)$$
$$=\left(1-\log_\gamma\left(1+(\gamma-1)\prod_{i=1}^{k}\left(\frac{(\gamma-1)^{p+q}-(\gamma^{T_{x_i}}-1)^p\left(\gamma^{T_{x_{k+1}}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)(\gamma^{1-T_{x_i}}-1)^p\left(\gamma^{1-T_{x_{k+1}}}-1\right)^q}\right)^{\frac{w_iw_{k+1}}{1-w_i}}\right),\right.$$
$$\log_\gamma\left(1+(\gamma-1)\prod_{i=1}^{k}\left(\frac{(\gamma-1)^{p+q}-(\gamma^{1-I_{x_i}}-1)^p\left(\gamma^{1-I_{x_{k+1}}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)(\gamma^{1-I_{x_i}}-1)^p\left(\gamma^{1-I_{x_{k+1}}}-1\right)^q}\right)^{\frac{w_iw_{k+1}}{1-w_i}}\right),$$
$$\left.\log_\gamma\left(1+(\gamma-1)\prod_{i=1}^{k}\left(\frac{(\gamma-1)^{p+q}-(\gamma^{1-F_{x_i}}-1)^p\left(\gamma^{1-F_{x_{k+1}}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)(\gamma^{1-F_{x_i}}-1)^p\left(\gamma^{1-F_{x_{k+1}}}-1\right)^q}\right)^{\frac{w_iw_{k+1}}{1-w_i}}\right)\right). \tag{16}$$

When $n = k + 1$, based on property ($FP8$) in Theorem 1, we have

$$\underset{\substack{i,j=1,\\ i\neq j}}{\overset{k+1}{\oplus_F}}\left(\frac{w_iw_j}{1-w_i}\cdot_F\left((x_i)^{\wedge_F p}\otimes_F(x_j)^{\wedge_F q}\right)\right)$$
$$=\underset{\substack{i,j=1,\\ i\neq j}}{\overset{k}{\oplus_F}}\left(\frac{w_iw_j}{1-w_i}\cdot_F\left((x_i)^{\wedge_F p}\otimes_F(x_j)^{\wedge_F q}\right)\right)$$
$$+\underset{i=1}{\overset{k}{\oplus_F}}\left(\frac{w_iw_{k+1}}{1-w_i}\cdot_F\left((x_i)^{\wedge_F p}\otimes_F(x_{k+1})^{\wedge_F q}\right)\right)$$
$$+\underset{j=1}{\overset{k}{\oplus_F}}\left(\frac{w_{k+1}w_j}{1-w_{k+1}}\cdot_F\left((x_{k+1})^{\wedge_F p}\otimes_F(x_j)^{\wedge_F q}\right)\right).$$

Equation (16) can be proven using mathematical induction.

(i) When $k = 2$, based on operations (2) and (4) in Definition 13, we can determine that

$$(x_1)^{\wedge_F p}=\left(\log_\gamma\left(1+\frac{(\gamma^{T_{x_1}}-1)^p}{(\gamma-1)^{p-1}}\right),\right.$$
$$1-\log_\gamma\left(1+\frac{(\gamma^{1-I_{x_1}}-1)^p}{(\gamma-1)^{p-1}}\right),$$
$$\left.1-\log_\gamma\left(1+\frac{(\gamma^{1-F_{x_1}}-1)^p}{(\gamma-1)^{p-1}}\right)\right),$$

$$(x_2)^{\wedge_F p} = \left(\log_\gamma\left(1+\frac{(\gamma^{T_{x_2}}-1)^p}{(\gamma-1)^{p-1}}\right), \right.$$
$$1-\log_\gamma\left(1+\frac{(\gamma^{1-I_{x_2}}-1)^p}{(\gamma-1)^{p-1}}\right),$$
$$\left. 1-\log_\gamma\left(1+\frac{(\gamma^{1-F_{x_2}}-1)^p}{(\gamma-1)^{p-1}}\right)\right),$$

$$(x_3)^{\wedge_F q} = \left(\log_\gamma\left(1+\frac{(\gamma^{T_{x_3}}-1)^q}{(\gamma-1)^{q-1}}\right), \right.$$
$$1-\log_\gamma\left(1+\frac{(\gamma^{1-I_{x_3}}-1)^q}{(\gamma-1)^{q-1}}\right),$$
$$\left. 1-\log_\gamma\left(1+\frac{(\gamma^{1-F_{x_3}}-1)^q}{(\gamma-1)^{q-1}}\right)\right),$$

$$(x_1)^{\wedge_F p}\otimes_F(x_3)^{\wedge_F q} = \left(\log_\gamma\left(1+\frac{(\gamma^{T_{x_1}}-1)^p(\gamma^{T_{x_3}}-1)^q}{(\gamma-1)^{p+q-1}}\right), \right.$$
$$1-\log_\gamma\left(1+\frac{(\gamma^{1-I_{x_1}}-1)^p(\gamma^{1-I_{x_3}}-1)^q}{(\gamma-1)^{p+q-1}}\right),$$
$$\left. 1-\log_\gamma\left(1+\frac{(\gamma^{1-F_{x_1}}-1)^p(\gamma^{1-F_{x_3}}-1)^q}{(\gamma-1)^{p+q-1}}\right)\right),$$

and

$$(x_2)^{\wedge_F p}\otimes_F(x_3)^{\wedge_F q} = \left(\log_\gamma\left(1+\frac{(\gamma^{T_{x_2}}-1)^p(\gamma^{T_{x_3}}-1)^q}{(\gamma-1)^{p+q-1}}\right), \right.$$
$$1-\log_\gamma\left(1+\frac{(\gamma^{1-I_{x_2}}-1)^p(\gamma^{1-I_{x_3}}-1)^q}{(\gamma-1)^{p+q-1}}\right),$$
$$\left. 1-\log_\gamma\left(1+\frac{(\gamma^{1-F_{x_2}}-1)^p(\gamma^{1-F_{x_3}}-1)^q}{(\gamma-1)^{p+q-1}}\right)\right).$$

Therefore, based on operation (3) in Definition 13, we can determine that

$$\frac{w_1 w_3}{1-w_1}\cdot_F\left((x_1)^{\wedge_F p}\otimes_F(x_3)^{\wedge_F q}\right)$$
$$= \left(1-\log_\gamma\left(1+(\gamma-1)\left(\frac{(\gamma-1)^{p+q}-(\gamma^{T_{x_1}}-1)^p(\gamma^{T_{x_3}}-1)^q}{(\gamma-1)^{p+q}+(\gamma-1)(\gamma^{T_{x_1}}-1)^p(\gamma^{T_{x_3}}-1)^q}\right)^{\frac{w_1w_3}{1-w_1}}\right),\right.$$
$$\log_\gamma\left(1+(\gamma-1)\left(\frac{(\gamma-1)^{p+q}-(\gamma^{1-I_{x_1}}-1)^p(\gamma^{1-I_{x_3}}-1)^q}{(\gamma-1)^{p+q}+(\gamma-1)(\gamma^{1-I_{x_1}}-1)^p(\gamma^{1-I_{x_3}}-1)^q}\right)^{\frac{w_1w_3}{1-w_1}}\right),$$
$$\left.\log_\gamma\left(1+(\gamma-1)\left(\frac{(\gamma-1)^{p+q}-(\gamma^{1-F_{x_1}}-1)^p(\gamma^{1-F_{x_3}}-1)^q}{(\gamma-1)^{p+q}+(\gamma-1)(\gamma^{1-F_{x_1}}-1)^p(\gamma^{1-F_{x_3}}-1)^q}\right)^{\frac{w_1w_3}{1-w_1}}\right)\right),$$

and

$$\frac{w_2 w_3}{1-w_3}\cdot_F\left((x_2)^{\wedge_F p}\otimes_F(x_3)^{\wedge_F q}\right)$$
$$= \left(1-\log_\gamma\left(1+(\gamma-1)\left(\frac{(\gamma-1)^{p+q}-(\gamma^{T_{x_2}}-1)^p(\gamma^{T_{x_3}}-1)^q}{(\gamma-1)^{p+q}+(\gamma-1)(\gamma^{T_{x_2}}-1)^p(\gamma^{T_{x_3}}-1)^q}\right)^{\frac{w_2w_3}{1-w_2}}\right),\right.$$
$$\log_\gamma\left(1+(\gamma-1)\left(\frac{(\gamma-1)^{p+q}-(\gamma^{1-I_{x_2}}-1)^p(\gamma^{1-I_{x_3}}-1)^q}{(\gamma-1)^{p+q}+(\gamma-1)(\gamma^{1-I_{x_2}}-1)^p(\gamma^{1-I_{x_3}}-1)^q}\right)^{\frac{w_2w_3}{1-w_2}}\right)$$
$$\left.\log_\gamma\left(1+(\gamma-1)\left(\frac{(\gamma-1)^{p+q}-(\gamma^{1-F_{x_2}}-1)^p(\gamma^{1-F_{x_3}}-1)^q}{(\gamma-1)^{p+q}+(\gamma-1)(\gamma^{1-F_{x_2}}-1)^p(\gamma^{1-F_{x_3}}-1)^q}\right)^{\frac{w_2w_3}{1-w_2}}\right)\right).$$

Then,

$$
\begin{aligned}
&\mathop{\oplus_F}_{i=1}^{2}\left(\frac{w_i w_3}{1-w_i}\cdot_F\left((x_i)^{\wedge_F p}\otimes_F(x_3)^{\wedge_F q}\right)\right)\\
&=\left(\frac{w_1 w_3}{1-w_1}\cdot_F\left((x_1)^{\wedge_F p}\otimes_F(x_3)^{\wedge_F q}\right)\right)\oplus_F\left(\frac{w_2 w_3}{1-w_2}\cdot_F\left((x_2)^{\wedge_F p}\otimes_F(x_3)^{\wedge_F q}\right)\right)\\
&=\left(1-\log_\gamma\left(1+(\gamma-1)\left(\frac{(\gamma-1)^{p+q}-(\gamma^{T_{x_1}}-1)^p(\gamma^{T_{x_3}}-1)^q}{(\gamma-1)^{p+q}+(\gamma-1)(\gamma^{T_{x_1}}-1)^p(\gamma^{T_{x_3}}-1)^q}\right)^{\frac{w_1w_3}{1-w_1}}\right.\right.\\
&\left.\left(\frac{(\gamma-1)^{p+q}-(\gamma^{T_{x_2}}-1)^p(\gamma^{T_{x_3}}-1)^q}{(\gamma-1)^{p+q}+(\gamma-1)(\gamma^{T_{x_2}}-1)^p(\gamma^{T_{x_3}}-1)^q}\right)^{\frac{w_2w_3}{1-w_2}}\right),\\
&\quad\log_\gamma\left(1+(\gamma-1)\left(\frac{(\gamma-1)^{p+q}-\left(\gamma^{1-I_{x_1}}-1\right)^p\left(\gamma^{1-I_{x_3}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)\left(\gamma^{1-I_{x_1}}-1\right)^p\left(\gamma^{1-I_{x_3}}-1\right)^q}\right)^{\frac{w_1w_3}{1-w_1}}\right.\\
&\left.\left(\frac{(\gamma-1)^{p+q}-\left(\gamma^{1-I_{x_2}}-1\right)^p\left(\gamma^{1-I_{x_3}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)\left(\gamma^{1-I_{x_2}}-1\right)^p\left(\gamma^{1-I_{x_3}}-1\right)^q}\right)^{\frac{w_2w_3}{1-w_2}}\right),\\
&\quad\log_\gamma\left((\gamma-1)\left(\frac{(\gamma-1)^{p+q}-\left(\gamma^{1-F_{x_1}}-1\right)^p\left(\gamma^{1-F_{x_3}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)\left(\gamma^{1-F_{x_1}}-1\right)^p\left(\gamma^{1-F_{x_3}}-1\right)^q}\right)^{\frac{w_1w_3}{1-w_1}}\right.\\
&\left.\left.\left(\frac{(\gamma-1)^{p+q}-\left(\gamma^{1-F_{x_2}}-1\right)^p\left(\gamma^{1-F_{x_3}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)\left(\gamma^{1-F_{x_2}}-1\right)^p\left(\gamma^{1-F_{x_3}}-1\right)^q}\right)^{\frac{w_2w_3}{1-w_2}}\right)\right),
\end{aligned}
$$

in other words, when $k = 2$, Eq. (16) holds.

(ii) Assuming Eq. (16) holds for $k = h$, we can determine that

$$
\begin{aligned}
&\mathop{\oplus_F}_{i=1}^{h}\left(\frac{w_i w_{k+1}}{1-w_i}\cdot_F\left((x_i)^{\wedge_F p}\otimes_F(x_{h+1})^{\wedge_F q}\right)\right)\\
&=\left(1-\log_\gamma\left(1+(\gamma-1)\prod_{i=1}^{h}\left(\frac{(\gamma-1)^{p+q}-(\gamma^{T_{x_i}}-1)^p\left(\gamma^{T_{x_{h+1}}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)(\gamma^{T_{x_i}}-1)^p\left(\gamma^{T_{x_{h+1}}}-1\right)^q}\right)^{\frac{w_iw_{h+1}}{1-w_i}}\right),\right.\\
&\quad\log_\gamma\left(1+(\gamma-1)\prod_{i=1}^{h}\left(\frac{(\gamma-1)^{p+q}-(\gamma^{1-I_{x_i}}-1)^p\left(\gamma^{1-I_{x_{h+1}}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)(\gamma^{1-I_{x_i}}-1)^p\left(\gamma^{1-I_{x_{h+1}}}-1\right)^q}\right)^{\frac{w_iw_{h+1}}{1-w_i}}\right),\\
&\quad\left.\log_\gamma\left(1+(\gamma-1)\prod_{i=1}^{h}\left(\frac{(\gamma-1)^{p+q}-(\gamma^{1-F_{x_i}}-1)^p\left(\gamma^{1-F_{x_{h+1}}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)(\gamma^{1-F_{x_i}}-1)^p\left(\gamma^{1-F_{x_{h+1}}}-1\right)^q}\right)^{\frac{w_iw_{h+1}}{1-w_i}}\right)\right).
\end{aligned}
$$

Moreover, it is true that

$$\mathop{\oplus_F}_{i=1}^{h+1}\left(\frac{w_i w_{h+2}}{1-w_i}\cdot_F\left((x_i)^{\wedge_F p}\otimes_F (x_{h+2})^{\wedge_F q}\right)\right) = \mathop{\oplus_F}_{i=1}^{h}\left(\frac{w_i w_{h+2}}{1-w_i}\cdot_F\left((x_i)^{\wedge_F p}\otimes_F (x_{h+1})^{\wedge_F q}\right)\right)\oplus_F \left(\frac{w_{h+1} w_{h+2}}{1-w_i}\cdot_F\left((x_{h+1})^{\wedge_F p}\otimes_F (x_{h+2})^{\wedge_F q}\right)\right).$$

Based on operations (2) and (4) in Definition 13,

$$(x_{h+1})^{\wedge_F p} = \left(\log_\gamma\left(1+\frac{\left(\gamma^{T_{x_{h+1}}}-1\right)^p}{(\gamma-1)^{p-1}}\right), 1-\log_\gamma\left(1+\frac{\left(\gamma^{1-I_{x_{h+1}}}-1\right)^p}{(\gamma-1)^{p-1}}\right), 1-\log_\gamma\left(1+\frac{\left(\gamma^{1-F_{x_{h+1}}}-1\right)^p}{(\gamma-1)^{p-1}}\right)\right),$$

$$(x_{h+2})^{\wedge_F q} = \left(\log_\gamma\left(1+\frac{\left(\gamma^{T_{x_{h+2}}}-1\right)^q}{(\gamma-1)^{q-1}}\right), 1-\log_\gamma\left(1+\frac{\left(\gamma^{1-I_{x_{h+2}}}-1\right)^q}{(\gamma-1)^{q-1}}\right), 1-\log_\gamma\left(1+\frac{\left(\gamma^{1-F_{x_{h+2}}}-1\right)^q}{(\gamma-1)^{q-1}}\right)\right),$$

$$(x_{h+1})^{\wedge_F p}\otimes_F (x_{h+2})^{\wedge_F q} = \left(\log_\gamma\left(1+\frac{\left(\gamma^{T_{x_{h+1}}}-1\right)^p\left(\gamma^{T_{x_{h+2}}}-1\right)^q}{(\gamma-1)^{p+q-1}}\right), 1-\log_\gamma\left(1+\frac{\left(\gamma^{1-I_{x_{h+1}}}-1\right)^p\left(\gamma^{1-I_{x_{h+2}}}-1\right)^q}{(\gamma-1)^{p+q-1}}\right), 1-\log_\gamma\left(1+\frac{\left(\gamma^{1-F_{x_{h+1}}}-1\right)^p\left(\gamma^{1-F_{x_{h+2}}}-1\right)^q}{(\gamma-1)^{p+q-1}}\right)\right).$$

Hence, it is clear that

$$\frac{w_{h+1} w_{h+2}}{1-w_{h+1}}\cdot_F\left((x_{h+1})^{\wedge_F p}\otimes_F (x_{h+2})^{\wedge_F q}\right) = \left(1-\log_\gamma\left(1+(\gamma-1)\left(\frac{(\gamma-1)^{p+q}-\left(\gamma^{T_{x_{h+1}}}-1\right)^p\left(\gamma^{T_{x_{h+2}}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)\left(\gamma^{T_{x_{h+1}}}-1\right)^p\left(\gamma^{T_{x_{h+2}}}-1\right)^q}\right)^{\frac{w_{h+1}w_{h+2}}{1-w_{h+1}}}\right),\right.$$
$$\log_\gamma\left(1+(\gamma-1)\left(\frac{(\gamma-1)^{p+q}-\left(\gamma^{1-I_{x_{h+1}}}-1\right)^p\left(\gamma^{1-I_{x_{h+2}}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)\left(\gamma^{1-I_{x_{h+1}}}-1\right)^p\left(\gamma^{1-I_{x_{h+2}}}-1\right)^q}\right)^{\frac{w_{h+1}w_{h+2}}{1-w_{h+1}}}\right),$$
$$\left.\log_\gamma\left(1+(\gamma-1)\left(\frac{(\gamma-1)^{p+q}-\left(\gamma^{1-F_{x_{h+1}}}-1\right)^p\left(\gamma^{1-F_{x_{h+2}}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)\left(\gamma^{1-F_{x_{h+1}}}-1\right)^p\left(\gamma^{1-F_{x_{h+2}}}-1\right)^q}\right)^{\frac{w_{h+1}w_{h+2}}{1-w_{h+1}}}\right)\right).$$

Hence, it is clear that

$$\mathop{\oplus_F}_{i=1}^{h+1}\left(\frac{w_i w_{h+2}}{1-w_i}\cdot_F\left((x_i)^{\wedge_F p}\otimes_F (x_{h+2})^{\wedge_F q}\right)\right) = \left(1-\log_\gamma\left(1+(\gamma-1)\prod_{i=1}^{h+1}\left(\frac{(\gamma-1)^{p+q}-(\gamma^{T_{x_i}}-1)^p\left(\gamma^{T_{x_{h+2}}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)(\gamma^{T_{x_i}}-1)^p\left(\gamma^{T_{x_{h+2}}}-1\right)^q}\right)^{\frac{w_i w_{h+2}}{1-w_i}}\right),\right.$$
$$\log_\gamma\left(1+(\gamma-1)\prod_{i=1}^{h+1}\left(\frac{(\gamma-1)^{p+q}-(\gamma^{1-I_{x_i}}-1)^p\left(\gamma^{1-I_{x_{h+2}}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)(\gamma^{1-I_{x_i}}-1)^p\left(\gamma^{1-I_{x_{h+2}}}-1\right)^q}\right)^{\frac{w_i w_{h+2}}{1-w_i}}\right),$$
$$\left.\log_\gamma\left(1+(\gamma-1)\prod_{i=1}^{h+1}\left(\frac{(\gamma-1)^{p+q}-(\gamma^{1-F_{x_i}}-1)^p\left(\gamma^{1-F_{x_{h+2}}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)(\gamma^{1-F_{x_i}}-1)^p\left(\gamma^{1-F_{x_{h+2}}}-1\right)^q}\right)^{\frac{w_i w_{h+2}}{1-w_i}}\right)\right),$$

in other words, it is true that Eq. (16) holds for $k = h + 1$.

Therefore, Eq. (16) is true for all k.

The following equation can be proven in a similar fashion.

in other words, it is true that Eq. (14) holds for $k = h + 1$.

Therefore, Eq. (14) is also true for all k.

$$\begin{aligned}
&\mathop{\oplus_F}\limits_{i=1}^{k}\left(\frac{w_{k+1}w_i}{1-w_{k+1}}\cdot_F\left((x_{k+1})^{\wedge_F p}\otimes_F(x_i)^{\wedge_F q}\right)\right)\\
&=\left(1-\log_\gamma\left(1+(\gamma-1)\prod_{i=1}^{k}\left(\frac{(\gamma-1)^{p+q}-\left(\gamma^{T_{x_{k+1}}}-1\right)^p\left(\gamma^{T_{x_i}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)\left(\gamma^{T_{x_{k+1}}}-1\right)^p\left(\gamma^{T_{x_i}}-1\right)^q}\right)^{\frac{w_iw_{k+1}}{1-w_{k+1}}}\right),\right.\\
&\quad\log_\gamma\left(1+(\gamma-1)\prod_{i=1}^{k}\left(\frac{(\gamma-1)^{p+q}-\left(\gamma^{1-I_{x_{k+1}}}-1\right)^p\left(\gamma^{1-I_{x_i}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)\left(\gamma^{1-I_{x_{k+1}}}-1\right)^p\left(\gamma^{1-I_{x_i}}-1\right)^q}\right)^{\frac{w_iw_{k+1}}{1-w_{k+1}}}\right),\\
&\quad\left.\log_\gamma\left(1+(\gamma-1)\prod_{i=1}^{k}\left(\frac{(\gamma-1)^{p+q}-\left(\gamma^{1-F_{x_{k+1}}}-1\right)^p\left(\gamma^{1-F_{x_i}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)\left(\gamma^{1-F_{x_{k+1}}}-1\right)^p\left(\gamma^{1-F_{x_i}}-1\right)^q}\right)^{\frac{w_iw_{k+1}}{1-w_{k+1}}}\right)\right).
\end{aligned}\tag{17}$$

Therefore, based on Eqs. (15), (16), and (17), we can determine that

$$\begin{aligned}
&\mathop{\oplus_F}\limits_{\substack{i,j=1,\\ i\neq j}}^{k+1}\left(\frac{w_iw_j}{1-w_i}\cdot_F\left((x_i)^{\wedge_F p}\otimes_F(x_j)^{\wedge_F q}\right)\right)\\
&=\left(1-\log_\gamma\left(1+(\gamma-1)\prod_{\substack{i,j=1,\\ i\neq j}}^{k+1}\left(\frac{(\gamma-1)^{p+q}-\left(\gamma^{T_{x_i}}-1\right)^p\left(\gamma^{T_{x_j}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)\left(\gamma^{T_{x_i}}-1\right)^p\left(\gamma^{T_{x_j}}-1\right)^q}\right)^{\frac{w_iw_j}{1-w_i}}\right),\right.\\
&\quad\log_\gamma\left(1+(\gamma-1)\prod_{\substack{i,j=1,\\ i\neq j}}^{k+1}\left(\frac{(\gamma-1)^{p+q}-\left(\gamma^{1-I_{x_i}}-1\right)^p\left(\gamma^{1-I_{x_j}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)\left(\gamma^{1-I_{x_i}}-1\right)^p\left(\gamma^{1-I_{x_j}}-1\right)^q}\right)^{\frac{w_iw_j}{1-w_i}}\right),\\
&\quad\left.\log_\gamma\left(1+(\gamma-1)\prod_{\substack{i,j=1,\\ i\neq j}}^{k+1}\left(\frac{(\gamma-1)^{p+q}-\left(\gamma^{1-F_{x_i}}-1\right)^p\left(\gamma^{1-F_{x_j}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)\left(\gamma^{1-F_{x_i}}-1\right)^p\left(\gamma^{1-F_{x_j}}-1\right)^q}\right)^{\frac{w_iw_j}{1-w_i}}\right)\right),
\end{aligned}$$

(2) According to Eq. (14) and operation (4) in Definition 13,

$$\begin{aligned}
&\mathrm{SVNFNPBM}^{p,q}(x_1,x_2,\ldots,x_n)=\left(\mathop{\oplus_F}\limits_{\substack{i,j=1,\\ i\neq j}}^{n}\left(\frac{w_iw_j}{1-w_i}\cdot_F\left((x_i)^{\wedge_F p}\otimes_F(x_j)^{\wedge_F q}\right)\right)\right)^{\wedge_F\frac{1}{p+q}}\\
&=\left(\left(\log_\gamma\left(1+(\gamma-1)\left(\frac{1-\prod_{i=1}^{n}\left(\frac{(\gamma-1)^{p+q}-\left(\gamma^{T_{x_i}}-1\right)^p\left(\gamma^{T_{x_j}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)\left(\gamma^{T_{x_i}}-1\right)^p\left(\gamma^{T_{x_j}}-1\right)^q}\right)^{\frac{w_iw_j}{1-w_i}}}{1+(\gamma-1)\prod_{i=1}^{n}\left(\frac{(\gamma-1)^{p+q}-\left(\gamma^{T_{x_i}}-1\right)^p\left(\gamma^{T_{x_j}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)\left(\gamma^{T_{x_i}}-1\right)^p\left(\gamma^{T_{x_j}}-1\right)^q}\right)^{\frac{w_iw_j}{1-w_i}}}\right)^{\frac{1}{p+q}}\right)\right),\right.\\
&\quad 1-\log_\gamma\left(1+(\gamma-1)\left(\frac{1-\prod_{\substack{i,j=1,\\ i\neq j}}^{n}\left(\frac{(\gamma-1)^{p+q}-\left(\gamma^{1-I_{x_i}}-1\right)^p\left(\gamma^{1-I_{x_j}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)\left(\gamma^{1-I_{x_i}}-1\right)^p\left(\gamma^{1-I_{x_j}}-1\right)^q}\right)^{\frac{w_iw_j}{1-w_i}}}{1+(\gamma-1)\prod_{\substack{i,j=1,\\ i\neq j}}^{n}\left(\frac{(\gamma-1)^{p+q}-\left(\gamma^{1-I_{x_i}}-1\right)^p\left(\gamma^{1-I_{x_j}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)\left(\gamma^{1-I_{x_i}}-1\right)^p\left(\gamma^{1-I_{x_j}}-1\right)^q}\right)^{\frac{w_iw_j}{1-w_i}}}\right)^{\frac{1}{p+q}}\right),\\
&\quad\left.1-\log_\gamma\left(1+(\gamma-1)\left(\frac{1-\prod_{\substack{i,j=1,\\ i\neq j}}^{n}\left(\frac{(\gamma-1)^{p+q}-\left(\gamma^{1-F_{x_i}}-1\right)^p\left(\gamma^{1-F_{x_j}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)\left(\gamma^{1-F_{x_i}}-1\right)^p\left(\gamma^{1-F_{x_j}}-1\right)^q}\right)^{\frac{w_iw_j}{1-w_i}}}{1+(\gamma-1)\prod_{\substack{i,j=1,\\ i\neq j}}^{n}\left(\frac{(\gamma-1)^{p+q}-\left(\gamma^{1-F_{x_i}}-1\right)^p\left(\gamma^{1-F_{x_j}}-1\right)^q}{(\gamma-1)^{p+q}+(\gamma-1)\left(\gamma^{1-F_{x_i}}-1\right)^p\left(\gamma^{1-F_{x_j}}-1\right)^q}\right)^{\frac{w_iw_j}{1-w_i}}}\right)^{\frac{1}{p+q}}\right)\right).
\end{aligned}$$

In addition, the following inequalities are correct:

$$0 \le \log_\gamma \left(1 + (\gamma - 1) \left(\frac{1 - \prod_{i=1}^{n} \left(\frac{(\gamma-1)^{p+q} - \left(\gamma^{T_{x_i}} - 1\right)^p \left(\gamma^{T_{x_j}} - 1\right)^q}{(\gamma-1)^{p+q} + (\gamma-1)\left(\gamma^{T_{x_i}} - 1\right)^p \left(\gamma^{T_{x_j}} - 1\right)^q} \right)^{\frac{w_i w_j}{1-w_i}}}{1 + (\gamma - 1) \prod_{i=1}^{n} \left(\frac{(\gamma-1)^{p+q} - \left(\gamma^{T_{x_i}} - 1\right)^p \left(\gamma^{T_{x_j}} - 1\right)^q}{(\gamma-1)^{p+q} + (\gamma-1)\left(\gamma^{T_{x_i}} - 1\right)^p \left(\gamma^{T_{x_j}} - 1\right)^q} \right)^{\frac{w_i w_j}{1-w_i}}} \right)^{\frac{1}{p+q}} \right) \le 1,$$

$$0 \le 1 - \log_\gamma \left(1 + (\gamma - 1) \left(\frac{1 - \prod_{\substack{i,j=1,\\ i \ne j}}^{n} \left(\frac{(\gamma-1)^{p+q} - \left(\gamma^{1-I_{x_i}} - 1\right)^p \left(\gamma^{1-I_{x_j}} - 1\right)^q}{(\gamma-1)^{p+q} + (\gamma-1)\left(\gamma^{1-I_{x_i}} - 1\right)^p \left(\gamma^{1-I_{x_j}} - 1\right)^q} \right)^{\frac{w_i w_j}{1-w_i}}}{1 + (\gamma - 1) \prod_{\substack{i,j=1,\\ i \ne j}}^{n} \left(\frac{(\gamma-1)^{p+q} - \left(\gamma^{1-I_{x_i}} - 1\right)^p \left(\gamma^{1-I_{x_j}} - 1\right)^q}{(\gamma-1)^{p+q} + (\gamma-1)\left(\gamma^{1-I_{x_i}} - 1\right)^p \left(\gamma^{1-I_{x_j}} - 1\right)^q} \right)^{\frac{w_i w_j}{1-w_i}}} \right)^{\frac{1}{p+q}} \right) \le 1,$$

and

$$0 \le 1 - \log_\gamma \left(1 + (\gamma - 1) \left(\frac{1 - \prod_{\substack{i,j=1,\\ i \ne j}}^{n} \left(\frac{(\gamma-1)^{p+q} - \left(\gamma^{1-F_{x_i}} - 1\right)^p \left(\gamma^{1-F_{x_j}} - 1\right)^q}{(\gamma-1)^{p+q} + (\gamma-1)\left(\gamma^{1-F_{x_i}} - 1\right)^p \left(\gamma^{1-F_{x_j}} - 1\right)^q} \right)^{\frac{w_i w_j}{1-w_i}}}{1 + (\gamma - 1) \prod_{\substack{i,j=1,\\ i \ne j}}^{n} \left(\frac{(\gamma-1)^{p+q} - \left(\gamma^{1-F_{x_i}} - 1\right)^p \left(\gamma^{1-F_{x_j}} - 1\right)^q}{(\gamma-1)^{p+q} + (\gamma-1)\left(\gamma^{1-F_{x_i}} - 1\right)^p \left(\gamma^{1-F_{x_j}} - 1\right)^q} \right)^{\frac{w_i w_j}{1-w_i}}} \right)^{\frac{1}{p+q}} \right) \le 1,$$

which meet the requirements of an SVNN.

Therefore, Theorem 2 holds.

Example 2 Let $x_1 = (0.4, 0.2, 0.3)$ and $x_2 = (0.6, 0.1, 0.2)$ be two SVNNs, and let $p = q = 1$ and $C_1 \succ C_2$. Based on Eq. (1),

$$H_2 = s(x_1) = \frac{1 + 0.4 - 0.4 - 0.3}{2} = 0.35.$$

Therefore,

$$w_1 = \frac{1}{1 + 0.35} = 0.741$$

and

$$w_2 = \frac{0.35}{1 + 0.35} = 0.259.$$

Suppose that $\gamma = 2$, then, according to Eq. (13),

$$\begin{aligned}
&\mathrm{SVNFPBM}^{1,1}(x_1,x_2)\\
&= \left(\log_2\left(1+\left(\frac{1-\left(\frac{1-\left(2^{T_{x_1}}-1\right)\left(2^{T_{x_2}}-1\right)}{1+\left(2^{T_{x_1}}-1\right)\left(2^{T_{x_2}}-1\right)}\right)^{\frac{w_1w_2}{1-w_1}+\frac{w_1w_2}{1-w_2}}}{1+\left(\frac{1-\left(2^{T_{x_1}}-1\right)\left(2^{T_{x_2}}-1\right)}{1+\left(2^{T_{x_1}}-1\right)\left(2^{T_{x_2}}-1\right)}\right)^{\frac{w_1w_2}{1-w_1}+\frac{w_1w_2}{1-w_2}}}\right)^{\frac{1}{2}}\right),\right.\\
&\quad 1-\log_2\left(1+\left(\frac{1-\left(\frac{1-\left(2^{1-I_{x_1}}-1\right)\left(2^{1-I_{x_2}}-1\right)}{1+\left(2^{1-I_{x_1}}-1\right)\left(2^{1-I_{x_2}}-1\right)}\right)^{\frac{w_1w_2}{1-w_1}+\frac{w_1w_2}{1-w_2}}}{1+\left(\frac{1-\left(2^{1-I_{x_1}}-1\right)\left(2^{1-I_{x_2}}-1\right)}{1+\left(2^{1-I_{x_1}}-1\right)\left(2^{1-I_{x_2}}-1\right)}\right)^{\frac{w_1w_2}{1-w_1}+\frac{w_1w_2}{1-w_2}}}\right)^{\frac{1}{2}}\right),\\
&\quad \left. 1-\log_2\left(1+\left(\frac{1-\left(\frac{1-\left(2^{1-F_{x_1}}-1\right)\left(2^{1-F_{x_2}}-1\right)}{1+\left(2^{1-F_{x_1}}-1\right)\left(2^{1-F_{x_2}}-1\right)}\right)^{\frac{w_1w_2}{1-w_1}+\frac{w_1w_2}{1-w_2}}}{1+\left(\frac{1-\left(2^{1-F_{x_1}}-1\right)\left(2^{1-F_{x_2}}-1\right)}{1+\left(2^{1-F_{x_1}}-1\right)\left(2^{1-F_{x_2}}-1\right)}\right)^{\frac{w_1w_2}{1-w_1}+\frac{w_1w_2}{1-w_2}}}\right)^{\frac{1}{2}}\right)\right)\\
&= \left(\log_2\left(1+\left(\frac{1-\left(\frac{1-\left(2^{0.4}-1\right)\left(2^{0.6}-1\right)}{1+(2^{0.4}-1)(2^{0.6}-1)}\right)}{1+\left(\frac{1-\left(2^{T_{x_1}}-1\right)\left(2^{T_{x_2}}-1\right)}{1+\left(2^{T_{x_1}}-1\right)\left(2^{T_{x_2}}-1\right)}\right)}\right)^{\frac{1}{2}}\right),\right.\\
&\quad 1-\log_2\left(1+\left(\frac{1-\left(\frac{1-\left(2^{0.8}-1\right)\left(2^{0.9}-1\right)}{1+(2^{0.8}-1)(2^{0.9}-1)}\right)}{1+\left(\frac{1-(2^{0.8}-1)(2^{0.9}-1)}{1+(2^{0.8}-1)(2^{0.9}-1)}\right)}\right)^{\frac{1}{2}}\right),\\
&\quad \left. 1-\log_2\left(1+\left(\frac{1-\left(\frac{1-\left(2^{0.7}-1\right)\left(2^{0.8}-1\right)}{1+(2^{0.7}-1)(2^{0.8}-1)}\right)}{1+\left(\frac{1-(2^{0.7}-1)(2^{0.8}-1)}{1+(2^{0.7}-1)(2^{0.8}-1)}\right)}\right)^{\frac{1}{2}}\right)\right)\\
&= (0.492, 0.151, 0.251).
\end{aligned}$$

The following section investigates some additional properties of the SVNFNPBM operator.

Theorem 3 (Reducibility) *Let C_1, C_2, …, C_n have the same priority level; that is, $w_i = \frac{1}{n}$ $(i = 1, 2, \ldots, n)$. Then,* $\mathrm{SVNFNPBM}^{p,q}$ $(x_1, x_2, \ldots, x_n) = \mathrm{SVNFBM}^{p,q}$ $(x_1, x_2, \ldots, x_n)$.

Proof When $w_i = \frac{1}{n}$ $(i = 1, 2, \ldots, n)$, based on Eq. (12), we can determine that

$$\begin{aligned}
&\mathrm{SVNFNPBM}^{p,q}(x_1,x_2,\ldots,x_n)\\
&= \left(\bigoplus_{\substack{i,j=1,\\ i\neq j}}^{n}{}_F \left(\frac{w_iw_j}{1-w_i} \cdot_F \left((x_i)^{\wedge_F p} \otimes_F \left(x_j\right)^{\wedge_F q} \right) \right) \right)^{\wedge_F \frac{1}{p+q}}\\
&= \left(\bigoplus_{\substack{i,j=1,\\ i\neq j}}^{n}{}_F \left(\frac{1}{n(n-1)} \cdot_F \left((x_i)^{\wedge_F p} \otimes_F \left(x_j\right)^{\wedge_F q} \right) \right) \right)^{\wedge_F \frac{1}{p+q}}.
\end{aligned}$$

Based on the property ($FP3$) in Theorem 1,

$$\begin{aligned}
&\bigoplus_{\substack{i,j=1,\\ i\neq j}}^{n}{}_F \left(\frac{1}{n(n-1)} \cdot_F \left((x_i)^{\wedge_F p} \otimes_F \left(x_j\right)^{\wedge_F q} \right) \right)\\
&= \frac{1}{n(n-1)} \cdot_F \bigoplus_{\substack{i,j=1,\\ i\neq j}}^{n}{}_F \left((x_i)^{\wedge_F p} \otimes_F \left(x_j\right)^{\wedge_F q} \right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathrm{SVNFNPBM}^{p,q}(x_1,x_2,\ldots,x_n)\\
&= \left(\bigoplus_{\substack{i,j=1,\\ i\neq j}}^{n}{}_F \left(\frac{w_iw_j}{1-w_i} \cdot_F \left((x_i)^{\wedge_F} \otimes_F \left(x_j\right)^{\wedge_F q} \right) \right) \right)^{\wedge_F \frac{1}{p+q}}\\
&= \left(\frac{1}{n(n-1)} \cdot_F \bigoplus_{\substack{i,j=1,\\ i\neq j}}^{n}{}_F \left((x_i)^{\wedge_F p} \otimes_F \left(x_j\right)^{\wedge_F q} \right) \right)^{\wedge_F \frac{1}{p+q}}\\
&= \mathrm{SVNFBM}^{p,q}(x_1,x_2,\ldots,x_n).
\end{aligned}$$

Theorem 4 (Idempotency) *Let all* SVNNs x_i $(i = 1, 2, \ldots, n)$ *be equal, i.e.,* $x_i = x$. *Then,* $\mathrm{IVNBM}^{p,q}(x_1, x_2, \ldots, x_n) = x$.

Proof Since $x_i = x$ $(i = 1, 2, \ldots, n)$,

$$\begin{aligned}
&\bigoplus_{\substack{i,j=1,\\ i\neq j}}^{n}{}_F \left(\frac{w_iw_j}{1-w_i} \cdot_F \left((x_i)^{\wedge_F p} \otimes_F \left(x_j\right)^{\wedge_F q} \right) \right)\\
&= \bigoplus_{\substack{i,j=1,\\ i\neq j}}^{n}{}_F \left(\frac{w_iw_j}{1-w_i} \cdot_F \left((x)^{\wedge_F p} \otimes_F (x)^{\wedge_F q} \right) \right).
\end{aligned}$$

Based on the property ($FP6$) in Theorem 1,

$$\underset{\substack{i,j=1,\\ i\neq j}}{\overset{n}{\oplus_F}}\left(\frac{w_i w_j}{1-w_i}\cdot_F\left((x_i)^{\wedge_F p}\otimes_F(x_j)^{\wedge_F q}\right)\right)$$
$$=\underset{\substack{i,j=1,\\ i\neq j}}{\overset{n}{\oplus_F}}\left(\frac{w_i w_j}{1-w_i}\cdot_F (x)^{\wedge_F(p+q)}\right).$$

By applying Eq. (12) and property (*FP*5) in Theorem 1, we can determine that

$$\begin{aligned}&\mathrm{SVNFNPBM}^{p,q}(x_1,x_2,\ldots,x_n)\\ &=\left(\underset{\substack{i,j=1,\\ i\neq j}}{\overset{n}{\oplus_F}}\left(\frac{w_i w_j}{1-w_i}\cdot_F\left((x_i)^{\wedge_F p}\otimes_F(x_j)^{\wedge_F q}\right)\right)\right)^{\wedge_F\frac{1}{p+q}}\\ &=\left(\underset{\substack{i,j=1,\\ i\neq j}}{\overset{n}{\oplus_F}}\left(\frac{w_i w_j}{1-w_i}\cdot_F\left((x)^{\wedge_F(p+q)}\right)\right)\right)^{\wedge_F\frac{1}{p+q}}\\ &=\left(\left(\sum_{\substack{i,j=1,\\ i\neq j}}^{n}\left(\frac{w_i w_j}{1-w_i}\right)\right)\cdot_F\left((x)^{\wedge_F(p+q)}\right)\right)^{\wedge_F\frac{1}{p+q}}\\ &=\left((x)^{\wedge_F(p+q)}\right)^{\wedge_F\frac{1}{p+q}}=x.\end{aligned}$$

5 Method for selecting TPL providers

This section establishes a novel method for selecting TPL providers using SVNNs.

Assume that there are m alternative TPL providers $P=\{P_1, P_2, \ldots, P_m\}$ and n criteria $C=\{C_1, C_2, \ldots, C_n\}$. The criteria are correlative, the priority levels of the n criteria are different, and the alternatives will be assessed by e decision makers. If $e=1$, selecting TPL provider is an MCDM problem. Let $U=(a_{ij})_{m\times n}$ be a single-valued neutrosophic decision matrix where $a_{ij}=\langle T_{a_{ij}}, I_{a_{ij}}, F_{a_{ij}}\rangle$ is an evaluation value denoted by an SVNN. Moreover, $T_{a_{ij}}$ in a_{ij} indicates the degree of truth-membership to which the provider P_i satisfies criterion C_j, $I_{a_{ij}}$ in a_{ij} indicates the degree of indeterminacy-membership to which the provider P_i satisfies criterion C_j, and $F_{a_{ij}}$ in a_{ij} indicates the degree of falsity-membership to which the provider P_i satisfies criterion C_j. If $e>1$, the selection of the TPL provider becomes a group decision-making problem. Let $U^k=(a^k_{ij})_{m\times n}$ be the single-valued neutrosophic decision matrix provided by the k-th ($k=1, 2, \ldots, e$) decision maker, and $a^k_{ij}=\left\langle T_{a^k_{ij}}, I_{a^k_{ij}}, F_{a^k_{ij}}\right\rangle$. Then, $\omega=(\omega_1,\omega_2,\ldots,\omega_e)^T$ will be the weight vector of the decision makers.

The following section proposes a method to rank TPL provider(s) and select the perfect one based upon the SVNFNPBM operator.

Step 1: Normalize the decision matrix.

Two kinds of different criteria can exist in problems of selecting TPL providers: benefit criteria and cost criteria. To unify all criteria, the decision matrix must be normalized. If $e=1$, the normalized decision matrix $N=(g_{ij})_{m\times n}$ can be obtained by

$$g_{ij}=\begin{cases} a_{ij}, & \text{if } C_j \text{ is a benefit criterion,}\\ a^{c}_{ij}, & \text{else,}\end{cases} \tag{18}$$

where a^c_{ij} is the complement of a_{ij}, and a^c_{ij} can be calculated according to Definition 3.

If $e>1$, the decision matrices U^k ($k=1, 2, \ldots, e$) can be normalized as $N^k=(g^k_{ij})_{m\times n}$ by

$$g^k_{ij}=\begin{cases} a^k_{ij}, & \text{if } C_j \text{ is a benefit criterion,}\\ a^{kc}_{ij}, & \text{else,}\end{cases} \tag{19}$$

where $g^k_{ij}=\langle T_{g^k_{ij}}, I_{g^k_{ij}}, F_{g^k_{ij}}\rangle$, a^{kc}_{ij} is the complement of a^k_{ij}, and a^{kc}_{ij} can be calculated according to Definition 3.

Step 2: Obtain the comprehensive decision matrix.

If $e=1$, the decision matrix does not need to be integrated, and the comprehensive decision matrix is $N^*=(b_{ij})_{m\times n}=N$. If $e>1$, based on the Frank weighted average operator of SVNSs, the comprehensive decision matrix $N^*=(b_{ij})_{m\times n}$ can be obtained by

$$\begin{aligned} b_{ij}&=\mathrm{SVNFWA}\left(g^1_{ij}, g^2_{ij}, \ldots, g^e_{ij}\right)\\ &=\left(1-\log_\gamma\left(1+\frac{\prod_{k=1}^{e}\left(\gamma^{1-T_{g^k_{ij}}}-1\right)^{\omega_k}}{(\gamma-1)\prod_{k=1}^{e}(\gamma-1)^{\omega_k-1}}\right),\right.\\ &\qquad \log_\gamma\left(1+\frac{\prod_{k=1}^{e}\left(\gamma^{I_{g^k_{ij}}}-1\right)^{\omega_k}}{(\gamma-1)\prod_{k=1}^{e}(\gamma-1)^{\omega_k-1}}\right),\\ &\qquad \left.\log_\gamma\left(1+\frac{\prod_{k=1}^{e}\left(\gamma^{F_{g^k_{ij}}}-1\right)^{\omega_k}}{(\gamma-1)\prod_{k=1}^{e}(\gamma-1)^{\omega_k-1}}\right)\right). \end{aligned} \tag{20}$$

Step 3: Obtain the score value sc_{ij} of each b_{ij} ($i=1, 2, \ldots, m$; $j=1, 2, \ldots, n$).

Using Eq. (1), we can obtain the score value sc_{ij} of each b_{ij}.

Step 4: Obtain the value of H_{ij} ($i=1, 2, \ldots, m$; $j=1, 2, \ldots, n$).

H_{ij} can be calculated based on the score value sc_{ij} of each b_{ij} using the following equation:

$$H_{ij}=\prod_{t=1}^{j-1} sc_{it}, \quad H_{i1}=1. \tag{21}$$

Step 5: Obtain the weight w_{ij} of each b_{ij} ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$).
w_{ij} can be calculated on the basis of H_{ij} using the following equation:

$$w_{ij} = \frac{H_{ij}}{\sum_{t=1}^{n} H_{it}}. \quad (22)$$

Step 6: Obtain the overall performance value r_i ($i = 1, 2, \ldots, m$) of TPL provider P_i.
The overall performance value r_i can be obtained using the SVNFNPBM operator in Eq. (13).
Step 7: Obtain the score value sc_i of each r_i ($i = 1, 2, \ldots, m$).
Using the score function of the SVNN in Eq. (1), we can obtain the score value sc_i of each r_i.
Step 8: Obtain the accuracy value l_i of each r_i ($i = 1, 2, \ldots, m$).
Based on the accuracy function of the SVNN in Eq. (2), we can obtain the accuracy value l_i of each r_i.
Step 9: Rank TPL providers according to the comparison method of SVNNs.
Using the comparison method presented in Definition 6, we can obtain the ranking order of alternative TPL providers.

6 Numerical example

6.1 The steps of the proposed method

This subsection uses a numerical example of selecting TPL providers with SVNNs to verify the applicability of the proposed approach.

The following example of selecting TPL providers is adapted from Ref. [69].

An electronic commerce retailer intends to find an appropriate TPL provider with whom to carry out long-term cooperation. Preliminary selection identifies four alternative providers $\{P_1, P_2, P_3, P_4\}$. These TPL providers are assessed by experts with respect to four criteria: (1) the cost of service (C_1); (2) operational experience in the industry (C_2); (3) customer satisfaction (C_3); and (4) market reputation (C_4). C_2, C_3, and C_4 are benefit criteria, while C_1 is a cost criterion. The electronic commerce retailer identifies the priority relationship of the four criteria as $C_3 \succ C_1 \succ C_4 \succ C_2$. Moreover, these four criteria are correlative. For the sake of reflecting reality in more detail and obtaining fuzzy and uncertain information, the evaluation values provided by one expert are transformed into SVNNs. Table 1 lists the transformed decision information.

Table 1 Transformed decision information

	C_1	C_2	C_3	C_4
P_1	⟨0.3, 0.9, 0.5⟩	⟨0.5, 0.1, 0.4⟩	⟨0.7, 0.1, 0.2⟩	⟨0.3, 0.2, 0.1⟩
P_2	⟨0.3, 0.8, 0.4⟩	⟨0.3, 0.2, 0.4⟩	⟨0.9, 0.0, 0.1⟩	⟨0.5, 0.3, 0.2⟩
P_3	⟨0.1, 0.7, 0.4⟩	⟨0.5, 0.1, 0.3⟩	⟨0.5, 0.0, 0.4⟩	⟨0.6, 0.2, 0.2⟩
P_4	⟨0.2, 0.9, 0.6⟩	⟨0.2, 0.2, 0.5⟩	⟨0.4, 0.3, 0.2⟩	⟨0.7, 0.2, 0.1⟩

Table 2 Normalized decision information

	C_1	C_2	C_3	C_4
P_1	⟨0.5, 0.1, 0.3⟩	⟨0.5, 0.1, 0.4⟩	⟨0.7, 0.1, 0.2⟩	⟨0.3, 0.2, 0.1⟩
P_2	⟨0.4, 0.2, 0.3⟩	⟨0.3, 0.2, 0.4⟩	⟨0.9, 0.0, 0.1⟩	⟨0.5, 0.3, 0.2⟩
P_3	⟨0.4, 0.3, 0.1⟩	⟨0.5, 0.1, 0.3⟩	⟨0.5, 0.0, 0.4⟩	⟨0.6, 0.2, 0.2⟩
P_4	⟨0.6, 0.1, 0.2⟩	⟨0.2, 0.2, 0.5⟩	⟨0.4, 0.3, 0.2⟩	⟨0.7, 0.2, .01⟩

Table 3 The score values

	C_1	C_2	C_3	C_4
P_1	0.5	0.45	0.65	0.4
P_2	0.35	0.25	0.9	0.35
P_3	0.35	0.55	0.55	0.5
P_4	0.6	0.15	0.3	0.6

Step 1: Normalize the decision matrix.
Since C_2, C_3, and C_4 are benefit criteria, while C_1 is a cost criterion, the decision matrix needs to be normalized using Eq. (18). Table 2 presents the normalized decision information.
Step 2: Obtain the comprehensive decision matrix.
Since $e = 1$, the decision matrix does not need to be integrated, and the comprehensive decision matrix is the same matrix presented in Table 2.
Step 3: Obtain the score value sc_{ij} of each b_{ij} ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$).
Using Eq. (1), the score value sc_{ij} of each b_{ij} can be obtained. Table 3 shows the corresponding score values sc_{ij}($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$).
Step 4: Obtain the value of H_{ij} ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$).
Using Eq. (21), the value of H_{ij} can be obtained. Table 4 gives the corresponding values.
Step 5: Obtain the weight w_{ij} of each b_{ij} ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$).
Using Eq. (22), the weight w_{ij} of each b_{ij} can be obtained. Table 5 depicts the corresponding weights w_{ij} ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$).
Step 6: Obtain the overall performance value r_i ($i = 1, 2, \ldots, m$) of provider P_i.

Table 4 The value of H_{ij}

	C_1	C_2	C_3	C_4
P_1	0.65	0.13	1	0.325
P_2	0.9	0.1102	1	0.315
P_3	0.55	0.0963	1	0.1925
P_4	0.3	0.108	1	0.18

Table 5 The corresponding weight matrix

	C_1	C_2	C_3	C_4
P_1	0.3088	0.0618	0.4751	0.1544
P_2	0.3871	0.0474	0.4301	0.1355
P_3	0.2991	0.0523	0.5438	0.1047
P_4	0.1889	0.068	0.6297	0.1134

According to the SVNFNPBM operator in Eq. (13), supposing that $\gamma = 2$ [66] and $p = q = 1$ [53], the overall performance value r_i of each provider P_i can be identified as follows: $r_1 = (0.5394, 0.1167, 0.2255)$, $r_2 = (0.589, 0.1331, 0.2096)$, $r_3 = (0.4789, 0.1388, 0.2549)$, and $r_4 = (0.4856, 0.2182, 0.2059)$.

Step 7: Obtain the score value sc_i of each r_i ($i = 1, 2, \ldots, m$).

According to Eq. (1), the score value sc_i of each r_i can be identified as $sc_1 = 0.5403$, $sc_2 = 0.5566$, $sc_3 = 0.5732$, and $sc_4 = 0.4216$.

Step 8: Obtain the accuracy value l_i of each r_i ($i = 1, 2, \ldots, m$).

Using Eq. (2), the accuracy value l_i of each r_i can be identified as $l_1 = 0.2865$, $l_2 = 0.3525$, $l_3 = 0.1871$, and $l_4 = 0.2124$.

Step 9: Rank the TPL providers using the comparison method for SVNNs.

Since $sc_3 > sc_2 > sc_1 > sc_4$, according to the comparison method in Definition 6, we can rank the TPL providers as $P_3 \succ P_2 \succ P_1 \succ P_4$. The most desirable TPL provider is P_3.

6.2 Influence of parameters

To investigate the influence of various parameters, this subsection calculates the ranking results with varying values for p, q, and γ. Two parts are included in this subsection. The first part discusses the impact of p and q on the ranking order, and the second one discusses the influence of γ.

To study the impact of p and q, in the first part, we determine and compare the ranking results using different pairs of p and q. The pairs of p and q can be distributed into three classes. The value of p is smaller than that of q in the first class, the value of p equals that of q in the second class, while the value of p is greater than that of q in the third class. In addition to changing the relationships between p and q, we also explore the effect of p and q on the ranking order of the four TPL providers by increasing the difference between p and q. Table 6 shows the significant pairs of p and q and their respective final ranking orders according to the proposed method.

As Table 6 shows, these three classes may produce different ranking orders. Furthermore, different ranking orders may exist within the same class in the presence of distinct differences between p and q. The first class yields two different rankings. When the value of p is smaller than 0.01, TPL provider P_3 performs better than TPL provider P_1. Meanwhile, the company prefers P_1 to P_3 when p is

Table 6 The ranking orders of the TPL providers with different p and q

p, q	Score value sc_i	Ranking order
$p = 0.001$, $q = 1$	$sc_1 = 0.5475$, $sc_2 = 0.7249$, $sc_3 = 0.6175$, $sc_4 = 0.4939$	$P_2 \succ P_3 \succ P_1 \succ P_4$
$p = 0.01$, $q = 1$	$sc_1 = 0.5469$, $sc_2 = 0.697$, $sc_3 = 0.587$, $sc_4 = 0.4919$	$P_2 \succ P_3 \succ P_1 \succ P_4$
$p = 0.1$, $q = 1$	$sc_1 = 0.5426$, $sc_2 = 0.6263$, $sc_3 = 0.525$, $sc_4 = 0.4751$	$P_2 \succ P_1 \succ P_3 \succ P_4$
$p = 1$, $q = 2$	$sc_1 = 0.5436$, $sc_2 = 0.5699$, $sc_3 = 0.4804$, $sc_4 = 0.4474$	$P_2 \succ P_1 \succ P_3 \succ P_4$
$p = 1$, $q = 5$	$sc_1 = 0.5701$, $sc_2 = 0.6524$, $sc_3 = 0.5313$, $sc_4 = 0.5021$	$P_2 \succ P_1 \succ P_3 \succ P_4$
$p = 1$, $q = 10$	$sc_1 = 0.6068$, $sc_2 = 0.7391$, $sc_3 = 0.5984$, $sc_4 = 0.5548$	$P_2 \succ P_1 \succ P_3 \succ P_4$
$p = 0.1$, $q = 0.1$	$sc_1 = 0.5365$, $sc_2 = 0.5524$, $sc_3 = 0.4712$, $sc_4 = 0.4148$	$P_2 \succ P_1 \succ P_3 \succ P_4$
$p = 1$, $q = 1$	$sc_1 = 0.5403$, $sc_2 = 0.5566$, $sc_3 = 0.4732$, $sc_4 = 0.4216$	$P_2 \succ P_1 \succ P_3 \succ P_4$
$p = 4$, $q = 4$	$sc_1 = 0.5594$, $sc_2 = 0.5777$, $sc_3 = 0.4917$, $sc_4 = 0.4505$	$P_2 \succ P_1 \succ P_3 \succ P_4$
$p = 10$, $q = 10$	$sc_1 = 0.5822$, $sc_2 = 0.6044$, $sc_3 = 0.5372$, $sc_4 = 0.4945$	$P_2 \succ P_1 \succ P_3 \succ P_4$
$p = 0.1$, $q = 0$	$sc_1 = 0.5704$, $sc_2 = 0.7629$, $sc_3 = 0.6199$, $sc_4 = 0.4054$	$P_2 \succ P_3 \succ P_1 \succ P_4$
$p = 0.5$, $q = 0$	$sc_1 = 0.5712$, $sc_2 = 0.7642$, $sc_3 = 0.6203$, $sc_4 = 0.4066$	$P_2 \succ P_3 \succ P_1 \succ P_4$
$p = 1$, $q = 0$	$sc_1 = 0.5736$, $sc_2 = 0.7682$, $sc_3 = 0.6216$, $sc_4 = 0.4099$	$P_2 \succ P_3 \succ P_1 \succ P_4$
$p = 10$, $q = 0$	$c_1 = 0.6293$, $sc_2 = 0.8488$, $sc_3 = 0.6744$, $sc_4 = 0.5204$	$P_2 \succ P_3 \succ P_1 \succ P_4$

Table 7 The ranking order of TPL providers with different γ

γ	Score value sc_i	Ranking order
$\gamma \to 1$	$sc_1 = 0.5393$, $sc_2 = 0.5513$, $sc_3 = 0.4706$, $sc_4 = 0.4198$	$P_2 \succ P_1 \succ P_3 \succ P_4$
$\gamma = 2$	$sc_1 = 0.5403$, $sc_2 = 0.5566$, $sc_3 = 0.4732$, $sc_4 = 0.4216$	$P_2 \succ P_1 \succ P_3 \succ P_4$
$\gamma = 5$	$sc_1 = 0.5415$, $sc_2 = 0.5621$, $sc_3 = 0.4757$, $sc_4 = 0.4237$	$P_2 \succ P_1 \succ P_3 \succ P_4$
$\gamma = 10$	$sc_1 = 0.5423$, $sc_2 = 0.5652$, $sc_3 = 0.477$, $sc_4 = 0.4251$	$P_2 \succ P_1 \succ P_3 \succ P_4$
$\gamma = 50$	$sc_1 = 0.5437$, $sc_2 = 0.5694$, $sc_3 = 0.4784$, $sc_4 = 0.4276$	$P_2 \succ P_1 \succ P_3 \succ P_4$

Table 8 The ranking orders of these five methods

Method	Ranking order	The best provider(s)	The worst provider(s)
Method using WAA [42]	$P_2 \succ P_3 \succ P_1 \succ P_4$	P_2	P_4
Method using WGA [42]	$P_3 \succ P_1 \succ P_2 \succ P_4$	P_3	P_4
Method using cross-entropy [70]	$P_3 \succ P_1 \succ P_2 \succ P_4$	P_3	P_4
Method using SVNNWBM [53]	$P_3 \succ P_1 \succ P_2 \succ P_4$	P_3	P_4
The proposed method	$P_3 \succ P_2 \succ P_1 \succ P_4$	P_3	P_4

greater than 0.1. In the second class, the ranking orders are the same, and P_1 is consistently preferred to P_3. The third class also produces the same ranking order, which is different from the order in the second class. In these three classes, the best TPL provider is always P_2 and the worst is always P_4.

These results suggest that the values of p and q can influence the ranking order of TPL providers. From this perspective, the proposed method for selecting TPL providers is a flexible one.

To investigate the influence of γ, in the second part, we determine and compare the ranking orders using different values of γ. Table 7 presents the significant values of γ and the respective ranking orders according to the proposed method.

As Table 7 shows, the score values on an individual criterion differ as the value of γ varies. The bigger the value of γ, the higher the score for an individual criterion. However, the ranking orders remain the same as the corresponding values of γ differ; in other words, the value of γ does not influence the ranking order of TPL providers. The worst provider is always P_4, and the best is always P_2. Decision makers can determine an appropriate value for γ based on their own preferences. In practice, the value of γ can be calculated using regression analysis and previous data provided by the decision makers.

6.3 Comparative analysis

This subsection compares the proposed method with four extant methods to verify its feasibility. Sahin [42] developed the first two methods, which make use of the weighted arithmetic average (WAA) operator and the weighted geometric average (WGA) operator, respectively, to aggregate values under every criterion. Furthermore, the two methods proposed by Sahin [42] rank alternatives using a score function that is identical to that in Eq. (1). Ye [70] detailed the third method, which was constructed with the proposed cross-entropy method for SVNSs. Liu and Wang [53] proposed the fourth method, which utilizes the SVNNWBM operator developed based on the algebraic operational laws. Table 8 lists the ranking orders according to these five methods.

Table 8 demonstrates that the proposed method produces a different ranking order from those obtained by the four extant methods. According to the method using WAA [42], the best TPL provider is P_2, while the other four methods identify P_3. The worst TPL provider is P_4 across the five methods. The method using WGA [42], the method using cross-entropy [70], and the method using SVNNWBM [53] produce the same ranking orders. Meanwhile, the method using WAA [42] and the proposed method produce two different ranking orders, which are also distinct from the ranking order obtained by the other three methods.

The reasons why these differences exist in Table 8 are as follows. The differences exist between the ranking order of the proposed method and those of the first three methods because of three possible reasons. The first one is that the first three methods do not take into account the interrelationships among criteria, while the proposed method considers these interrelationships using the BM operator. The second one lies in the criteria's priority levels. The proposed method considers the criteria's different priorities, while the first three methods assume that the criteria all have the same priority levels. The last one can be traced back to the fact that the proposed method is constructed based on Frank operations, while the first three methods use algebraic operational laws. Therefore, it is reasonable that the ranking orders of the first three methods differ from that of the proposed method. Meanwhile, we discuss two main reasons for the dissimilarities existing between the ranking orders respectively obtained by the method using the SVNNWBM [53] and the proposed method, even though both methods consider the interrelationships among criteria. One main reason is that different weight vectors of criteria are involved in two methods. The weights for the criteria in the method using the SVNNWBM [53] are subjectively provided by decision makers lacking of objectivity. Conversely, the proposed method takes the criteria priorities into account by objectively obtaining the weight vector of the criteria through the weight method using the PA operator. And the other main reason lies in the difference of operational laws. The proposed method utilizes Frank operational laws, while the SVNNWBM [53] method makes use of algebraic operational laws. As discussed in Sect. 1, Frank operational laws are more robust than algebraic operational laws.

Generally speaking, the proposed method can be successfully utilized to select TPL providers. By combining the BM operator and the PA operator, this method takes into consideration the interrelationships among criteria and their varying priority levels. In addition, the proposed method is more robust than extant methods based on algebraic operational laws. Compared with extant methods, the proposed method using the SVNFNPBM operator is more suitable for tackling practical problems in which criteria are independent and have different priority levels. The ranking order of the proposed method can identify a TPL provider that is more in line with decision makers' preferences than the providers selected by other methods.

7 Conclusion

SVNNs can comprehensively depict fuzzy information involved in the process of selecting TPL providers. This paper established a method based on the proposed Frank operations for selecting TPL providers under single-valued neutrosophic environments. The method took into account the interrelationships among criteria and the different priority levels of criteria. To do that, we developed an SVNFNPBM operator by combining the NWBM and PA operators. Moreover, we demonstrated the applicability and feasibility of the proposed method using a numerical example and a comparative analysis. In addition, this paper discussed the influence of the all involved parameters on ranking order.

The major contributions of this paper can be summarized as follows. First of all, this paper introduces Frank operations of SVNNs that are more flexible and robust than the extant algebraic operations of SVNNs. In addition, this paper presents and proves some properties of the proposed Frank operations. Secondly, this paper proposes a novel single-valued neutrosophic aggregation operator that combines the BM and PA operators to consider simultaneously the interrelationships among criteria and their distinct priority levels. The proposed aggregation operator is based on Frank operations, while most of the previous single-valued neutrosophic aggregation operators are based on algebraic operations. Last but not least, this paper constructs a decision-making method for the selection of TPL providers based on the proposed aggregation operator. The new method is then proven to be useful in selecting a TPL provider. The proposed MCDM method using the SVNFNPBM operator is more effective than extant methods in addressing situations with independent criteria that have different priority levels, which occur in most practical problems. In other words, the proposed method using the SVNFNPBM operator has a higher practical value than extant methods.

Several directions for future research may be promising. First, the aggregation operator proposed in this paper can be introduced into interval-valued neutrosophic environments. Second, applications of the proposed MCDM method can be explored to tackle practical problems in other areas, such as selecting hotels, cloud services, or renewable energy sources. The common feature of these practical problems is that multiple criteria involved are interdependent and have different priority levels. Third, the complexity of the proposed method can be improved with the help of computer technology. In the future, we will devote ourselves to reducing the complexity of the method as well as increasing accuracy.

Acknowledgements This work was supported by the National Natural Science Foundation of China (Nos. 71501192 and 71571193). The authors also would like to express appreciation to the anonymous reviewers and editors for their helpful comments that improved the paper.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interests regarding the publication of this paper.

References

1. Wan S-P, Wang F, Lin L-L, Dong J-Y (2015) An intuitionistic fuzzy linear programming method for logistics outsourcing provider selection. Knowl Based Syst 82:80–94
2. Yayla AY, Oztekin A, Gumus AT, Gunasekaran A (2015) A hybrid data analytic methodology for 3PL transportation provider evaluation using fuzzy multi-criteria decision making. Int J Prod Res 53(20):6097–6113
3. Zadeh LA (1965) Fuzzy sets. Inf Control 8(3):338–353
4. Broumi S, Smarandache F (2014) Single valued neutrosophic trapezoid linguistic aggregation operators based multi-attribute decision making. Bull Pure Appl Sci Math Stat 33(2):135–155

5. Broumi S, Smarandache F (2014) New distance and similarity measures of interval neutrosophic sets. In: Proceedings of the 17t[h] international conference on information fusion, Salamanca, Spain, pp 1–7
6. Broumi S, Smarandache F (2014) Cosine similarity measure of interval valued neutrosophic sets. Neutrosophic Sets Syst 5:15–20
7. Turksen IB (1986) Interval valued fuzzy sets based on normal forms. Fuzzy Set Syst 20(2):191–210
8. Atanassov KT (1986) Intuitionistic fuzzy sets. Fuzzy Set Syst 20(1):87–96
9. Atanassov K, Gargov G (1989) Interval valued intuitionistic fuzzy sets. Fuzzy Set Syst 31(3):343–349
10. Torra V (2010) Hesitant fuzzy sets. Int J Intell Syst 25(6):529–539
11. Torra V, Narukawa Y (2009) On hesitant fuzzy sets and decision. In: The 18t[h] IEEE international conference on fuzzy systems, Jeju Island, Korea, pp 1378–1382
12. Chen N, Xu ZS, Xia MM (2013) Interval-valued hesitant preference relations and their applications to group decision making. Knowl Based Syst 37:528–540
13. Smarandache F (1999) A unifying field in logics: neutrosophic logic. Neutrosophy, neutrosophic set, probability. American Research Press, Rehoboth
14. Smarandache F (1998) Neutrosophy: Neutrosophic probability, set, and logic. American Research Press, Rehoboth
15. Majumdar P, Samanta SK (2014) On similarity and entropy of neutrosophic sets. J Intell Fuzzy Syst 26(3):1245–1252
16. Rivieccio U (2008) Neutrosophic logics: prospects and problems. Fuzzy Set Syst 159(14):1860–1868
17. Wang H, Smarandache F, Zhang Y, Sunderraman R (2010) Single valued neutrosophic sets. Rev Air Force Acad 17(1):10–14
18. Şahin R, Liu PD (2016) Maximizing deviation method for neutrosophic multiple attribute decision making with incomplete weight information. Neural Comput Appl 27(7):2017–2029
19. Ye J (2013) Multicriteria decision-making method using the correlation coefficient under single-valued neutrosophic environment. Int J Gen Syst 42(4):386–394
20. Broumi S, Talea M, Bakali A, Smarandache F (2016) Single valued neutrosophic graphs. J New Theory 10:86–101
21. Broumi S, Talea M, Bakali A, Smarandache F (2016) On bipolar single valued neutrosophic graphs. J New Theory 11:84–102
22. Deli I, Broumi S (2015) Neutrosophic soft matrices and NSM-decision making. J Intell Fuzzy Syst 28(5):2233–2241
23. Deli I (2015) Npn-soft sets theory and their applications. Ann Fuzzy Math Inform 10(6):847–862
24. Zhang HY, Ji P, Wang JQ, Chen XH (2016) A neutrosophic normal cloud and its application in decision-making. Cogn Comput 8(4):649–669
25. Broumi S, Ye J, Smarandache F (2015) An extended TOPSIS method for multiple attribute decision making based on interval neutrosophic uncertain linguistic variables. Neutrosophic Sets Syst 8:23–32
26. Zhang H, Ji P, Wang J, Chen X (2015) An improved weighted correlation coefficient based on integrated weight for interval neutrosophic sets and its application in multi-criteria decision-making problems. Int J Comput Int Syst 8(6):1027–1043
27. Deli I (2014) Interval-valued neutrosophic soft sets and its decision making. Int J Mach Learn Cybern. doi:10.1007/s13042-015-0461-3
28. Deli I, Şubaş Y (2016) A ranking method of single valued neutrosophic numbers and its applications to multi-attribute decision making problems. Int J Mach Learn Cybern. doi:10.1007/s13042-016-0505-3
29. Tian ZP, Wang J, Wang JQ, Zhang HY (2016) An improved MULTIMOORA approach for multi-criteria decision-making based on interdependent inputs of simplified neutrosophic linguistic information. Neural Comput Appl. doi:10.1007/s00521-016-2378-5
30. Peng JJ, Wang JQ, Wu XH (2016) An extension of the ELECTRE approach with multi-valued neutrosophic information. Neural Comput Appl. doi:10.1007/s00521-016-2411-8
31. Wang JQ, Kuang JJ, Wang J, Zhang HY (2016) An extended outranking approach to rough stochastic multi-criteria decision-making problems. Cogn Comput. doi:10.1007/s12559-016-9417-5
32. Ji P, Zhang HY, Wang JQ (2016) A projection-based TODIM method under multi-valued neutrosophic environments and its application in personnel selection. Neural Comput Appl. doi:10.1007/s00521-016-2436-z
33. Zhou H, J-q Wang, Zhang HY (2016) Multi-criteria decision-making approaches based on distance measures for linguistic hesitant fuzzy sets. J Oper Res Soc. doi:10.1057/jors.2016.1041
34. Mak DK (2015) A fuzzy probabilistic method for medical diagnosis. J Med Syst 39(3):1–10
35. Ye J, Jing F (2015) Multi-period medical diagnosis method using a single valued neutrosophic similarity measure based on tangent function. Comput Methods Prog Biomed 123:142–149
36. Broumi S, Smarandache F (2014) Neutrosophic refined similarity measure based on cosine function. Neutrosophic Sets Syst 6:41–47
37. Ma YX, Wang JQ, Wang J, Wu XH (2016) An interval neutrosophic linguistic multi-criteria group decision-making method and its application in selecting medical treatment options. Neural Comput Appl. doi:10.1007/s00521-016-2203-1
38. Tian ZP, Wang J, Wang JQ, Zhang HY (2016) Simplified neutrosophic linguistic multi-criteria group decision-making approach to green product development. Group Decis Negot. doi:10.1007/s10726-016-9479-5
39. Zhang H, Ji P, Wang J, Chen X (2017) A novel decision support model for satisfactory restaurants utilizing social information: a case study of TripAdvisor.com. Tour Manag 59:281–297
40. Ma H, Hu Z, Li K, Zhang H (2016) Toward trustworthy cloud service selection: a time-aware approach using interval neutrosophic set. J Parallel Distrib Comput 96:75–94
41. Wei G, Lin R, Zhao X, Wang H (2014) An approach to multiple attribute decision making based on the induced Choquet integral with fuzzy number intuitionistic fuzzy information. J Bus Econ Manag 15(2):277–298
42. Şahin R (2014) Multi-criteria neutrosophic decision making method based on score and accuracy functions under neutrosophic environment. arXiv preprint arXiv:14125202
43. Wang J, Wang J-q, Zhang H-y (2016) A likelihood-based TODIM approach based on multi-hesitant fuzzy linguistic information for evaluation in logistics outsourcing. Comput Ind Eng 99:287–299
44. Bonferroni C (1950) Sulle medie multiple di potenze. Boll Mat Ital 5(3–4):267–270
45. Tian Z, Wang J, Wang J, Chen X (2015) Multi-criteria decision-making approach based on gray linguistic weighted Bonferroni mean operator. Int Trans Oper Res. doi:10.1111/itor.12220
46. Xu Z, Yager RR (2011) Intuitionistic fuzzy Bonferroni means. IEEE Trans Syst Man Cybern Part B Cybern A Publ IEEE Syst Man Cybern Soc 41(2):568–578
47. Xia M, Xu Z, Zhu B (2012) Generalized intuitionistic fuzzy Bonferroni means. Int J Intell Syst 27(1):23–47
48. Xia M, Xu Z, Zhu B (2013) Geometric Bonferroni means with their application in multi-criteria decision making. Knowl Based Syst 40(1):88–100
49. Zhu B, Xu ZS (2013) Hesitant fuzzy Bonferroni means for multi-criteria decision making. J Oper Res Soc 64(12):1831–1840

50. Liu P, Li H (2015) Multiple attribute decision-making method based on some normal neutrosophic Bonferroni mean operators. Neural Comput Appl. doi:10.1007/s00521-015-2048-z
51. Zhou W, He JM (2012) Intuitionistic fuzzy normalized weighted Bonferroni mean and its application in multicriteria decision making. J Appl Math. doi:10.1155/2012/136254
52. Zhou W (2014) On hesitant fuzzy reducible weighted Bonferroni mean and its generalized form for multicriteria aggregation. J Appl Math. doi:10.1155/2014/954520
53. Liu P, Wang Y (2014) Multiple attribute decision-making method based on single-valued neutrosophic normalized weighted Bonferroni mean. Neural Comput Appl 25(7–8):2001–2010
54. Yager RR (2008) Prioritized aggregation operators. Int J Approx Reason 48(1):263–274
55. Yu X, Xu Z (2007) Prioritized intuitionistic fuzzy aggregation operators. Inform Fusion 14(1):108–116
56. Yu D, Wu Y, Lu T (2012) Interval-valued intuitionistic fuzzy prioritized operators and their application in group decision making. Knowl Based Syst 30(6):57–66
57. Yu D (2012) Group decision making based on generalized intuitionistic fuzzy prioritized geometric operator. Int J Intell Syst 27(7):635–661
58. Wei G (2012) Hesitant fuzzy prioritized operators and their application to multiple attribute decision making. Knowl Based Syst 31:176–182
59. Yu D (2012) Hesitant fuzzy prioritized operators and their application in multi-criteria group decision making. Afr J Bus Manag 6(33):9370–9380
60. Zhang Z (2014) 2-tuple prioritized aggregation operators and their application to multiple attribute group decision making. Br J Math Comput Sci 4(16):2278–2297
61. Zhang Z (2014) Interval-valued 2-tuple prioritized aggregation operators and their application to multiple attribute group decision making. Br J Appl Sci Technol 4(22):3217–3236
62. Liu P, Wang Y (2015) Interval neutrosophic prioritized OWA operator and its application to multiple attribute decision making. J Syst Sci Complex 29(3):681–697
63. Frank MJ (1979) On the simultaneous associativity of F(x, y) and x + y − F(x, y). Aequous Math 19(1):194–226
64. Zhang X, Liu P, Wang Y (2015) Multiple attribute group decision making methods based on intuitionistic fuzzy frank power aggregation operators. J Intell Fuzzy Syst 29(5):2235–2246
65. Zhang Z (2016) Interval-valued intuitionistic fuzzy Frank aggregation operators and their applications to multiple attribute group decision making. Neural Comput Appl. doi:10.1007/s00521-015-2143-1
66. Qin J, Liu X, Pedrycz W (2016) Frank aggregation operators and their application to hesitant fuzzy multiple attribute decision making. Appl Soft Comput 41:428–452
67. W-s Wang, H-c He (2009) Research on flexible probability logic operator based on Frank T/S norms. Acta Electron Sin 37(5):1141–1145
68. Xu ZS, Yager RR (2011) Intuitionistic fuzzy Bonferroni means. IEEE Trans Syst Man Cybern Part B Cybern 41(2):568–578
69. Sun H-X, Yang H-X, Wu J-Z, Ouyang Y (2015) Interval neutrosophic numbers Choquet integral operator for multi-criteria decision making. J Intell Fuzzy Syst 28:2443–2455
70. Ye J (2014) Single valued neutrosophic cross-entropy for multicriteria decision making problems. Appl Math Model 38(3):1170–1175