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ORIGINAL ARTICLE

Oscillation of impulsive neutral delay generalized high-order Hopfield neural networks

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Abstract In this paper, the existence and the exponential stability of piecewise differentiable pseudo-almost periodic solutions for a class of impulsive neutral high-order Hopfield neural networks with mixed time-varying delays and leakage delays are established by employing the fixed point theorem, Lyapunov functional method and differential inequality. Numerical example with graphical illustration is given to illuminate our main results.

Keywords Neutral High-order Hopfield neural networks · Impulse · Piecewise differentiable pseudo-almost periodic function · Mixed time-varying delays · Leakage delays

1 Introduction

In this paper, we consider piecewise differentiable pseudo-almost periodic solutions of a class of impulsive neutral delay generalized high-order Hopfield neural networks with mixed delays. The mixed delays include leakage delay, time-varying delays and continuously distributed delays. To investigate the existence of solutions of the above-mentioned problem, we consider the following:

$$\begin{cases} x_i'(t) &= -a_i(t)x_i(t-\rho(t)) + \sum_{j=1}^{n} b_{ij}(t)f_j(x_j'(t-\tau_{ij}(t))) \\ & + \sum_{j=1}^{n} c_{ij}(t)\int_0^\infty d_{ij}(u)f_j(x_j'(t-u))\mathrm{d}u \\ & + \sum_{j=1}^{n}\sum_{l=1}^{n} \alpha_{ijl}(t)f_j(x_j(t-\sigma_{ij}(t)))f_l(x_l(t-v_{ij}(t))) \\ & + \sum_{j=1}^{n}\sum_{l=1}^{n} \beta_{ijl}(t)\int_0^\infty h_{ijl}(u) \\ & \quad f_j(x_j(t-u))\mathrm{d}u \int_0^\infty k_{ijl}(u)f_l(x_l(t-u))\mathrm{d}u \\ & + J_i(t), \quad t\in\mathbb{R},\ t\neq t_k,\ k\in\mathbb{Z} \\ \Delta x_i(t_k) &= x_i(t_k^+) - x_i(t_k^-) = I_k(x_i(t_k)) \end{cases} \tag{1}$$

in which n corresponds to the number of units in a neural network, $x_i(t)$ corresponds to the state vector of the ith unit at the time $t, a_i(t) > 0$ represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs at the time $t, b_{ij}(.), c_{ij}(.)$ and $\alpha_{ijl}(.), \beta_{ijl}(.)$ are, respectively, the first-order connection weights and the second-order connection weights of the neural network, $0 \leq \rho(.) \leq \rho^+, 0 \leq \tau_{ij}(.), \sigma_{ij}(.), v_{ij}(.) \leq \tau^+$ correspond to the transmission delays, $J_i(t)$ denote the external inputs at time t, and f_j is the activation function of signal transmission. The sequence $\{t_k\}$ has no finite accumulation point and $I_k : \mathbb{R}^n \longrightarrow \mathbb{R}$.

As an important research field of dynamic systems, it is well known that high-order neural networks received much attention and have been applied in a wide range of practical fields such as signal processing, pattern recognition, associative memories, optimization problems, image processing, associative memories, speed detection of moving objects, optimization problems and many other fields [1–9, 13, 15, 37, 38]. This is due to the fact that high-order

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neural networks have stronger approximation property, faster convergence rate, greater storage capacity, and higher fault tolerance than lower-order neural networks.

In addition, from the real-world application angle, time delay is inevitably encountered in the implementation of networks [1, 5, 10–14, 19–25]. According to the way it occurs, time delay can be classified as two types: discrete and distributed. Time delays in the neural networks are often one of the main sources to cause poor performance, make the dynamic behaviors become more complex, may destabilize the stable equilibria and admit oscillations, bifurcation and chaos. Therefore, it is of prime importance to consider the delay effects on the stability of neural networks. In particular, the time delay in the negative feedback terms which is known as leakage has a tendency to destabilize the system [26–29] and has great impact on the dynamical behavior of neural networks. This is to say, it is necessary to consider the effect of leakage delays when studying the stability of state estimation of neural networks.

Recently, another type of time delays, namely neutral-type time delays, has drawn much research attention [18, 27, 41]. Many practical delay systems can be modeled as differential systems of neutral type, whose differential expression includes the derivative term of the past state, such as partial element equivalent circuits and transmission lines in electrical engineering, population dynamics and controlled constrained manipulators in mechanical engineering [27]. Moreover, it has been shown that the existing neural network models in many cases cannot characterize the properties of a neural reaction process precisely due to the complicated dynamic properties of the neural cells in the real world, and it is natural and necessary that systems will contain some information about the derivative of the past state to further describe and model the dynamics for such complex neural reactions [41].

However, it is well known that the dynamics of evolving processes is usually subjected to suddenly changes such as shocks, harvesting and natural disasters [16–18, 42]. Often these short-term perturbations are treated as having acted instantaneously or in the form of impulses. The theory of impulsive differential equations represents a more natural framework for mathematical modeling of many real-world phenomena, such as population dynamic system and the neural networks. High-order recurrent neural networks are often subject to impulsive perturbations that in turn affect dynamical behaviors of the systems [6].

Also, it is well known that studies on neuron dynamic systems not only involve a discussion of stability properties, but also involve many dynamic behaviors such as periodic oscillatory behavior, almost periodic oscillatory properties, pseudo-almost periodic oscillatory properties, chaos and bifurcation [30–36, 44]. In applications, the assumption of pseudo-almost periodicity, which was introduced by Zhang [30, 31], is more realistic and more important than that of periodicity and is a natural and good generalization of the classical almost periodic functions in the sense of Bohr. Liu and Zhang [35] introduced the concept of piecewise pseudo-almost periodic functions and gave some properties including the composition theorem.

To the best of our knowledge, there are no published papers considering the piecewise differentiable pseudo-almost periodic solutions for impulsive neutral high-order Hopfield neural networks with time-varying delays in the leakage terms. In other words, we have never studied the existence and the exponential stability of piecewise differentiable pseudo-almost periodic solutions for impulsive neutral high-order Hopfield neural networks with time-varying delays in the leakage terms.

The main aim of this article is to establish some sufficient conditions for the existence, the uniqueness and the exponential stability of piecewise differentiable pseudo-almost periodic solutions of Eq. (1).

Throughout this paper, for $i, j, l = 1, 2, \ldots, n$, it will be assumed that $\rho(.), \tau_{ij}(.), \sigma_{ij}(.), v_{ij}(.)$ are almost periodic functions, such that $1-\dot{\rho}(t)>0,\ 1-\dot{\tau}_{ij}(t)>0,\ 1-\dot{\sigma}_{ij}(t)\ >0,\ 1-\dot{v}_{ij}(t)>0$ for all $t\in\mathbb{R}, b_{ij}, c_{ij}, \alpha_{ijl}, \beta_{ijl}, J_i : \mathbb{R}\longrightarrow\mathbb{R}$ are pseudo-almost periodic functions, and let the positive constant $a_{i*}, a_i^+, \overline{b}_{ij}, \overline{c}_{ij}, \overline{\alpha}_{ijl}, \overline{\beta}_{ijl}$ and $\overline{J}_i$ such that

$$a_{i*} = \inf_{t\in\mathbb{R}} a_i(t),\ \ a_i^+ = \sup_{t\in\mathbb{R}} a_i(t),$$

$$\overline{b}_{ij} = \sup_{t\in\mathbb{R}} \mid b_{ij}(t) \mid, \overline{c}_{ij} = \sup_{t\in\mathbb{R}} \mid c_{ij}(t) \mid$$

$$\overline{\alpha}_{ijl} = \sup_{t\in\mathbb{R}} \mid \alpha_{ijl}(t) \mid, \overline{\beta}_{ijl} = \sup_{t\in\mathbb{R}} \mid \beta_{ijl}(t) \mid,\ \overline{J}_i = \sup_{t\in\mathbb{R}} \mid J_i(t) \mid.$$

We also assume that the following conditions (H1)–(H5) hold.

(H1) For each $j=\{1,2,\ldots,n\}$, there exist nonnegative constants L_j^f and M_j^f such that

$$f_j(0)=0,\ \mid f_j(u)-f_j(v) \mid\ \leq L_j^f \mid u-v \mid,$$
$$\text{and } \mid f_j(u) \mid\ \leq M_j^f,\ \ \text{for all } u, v\in\mathbb{R}.$$

(H2) For $i, j, l \in \{1, 2, \ldots, n\}$, the delay kernels, $d_{ij}, h_{ijl}, k_{ijl} : [0,\infty)\longrightarrow\mathbb{R}$ are continuous, and there exist nonnegative constants $d_{ij}^+, h_{ijl}^+, k_{ijl}^+, \eta_d, \eta_h, \eta_k$ such that

$$\mid d_{ij}(u) \mid\ \leq d_{ij}^+ e^{-\eta_d u},\ \mid h_{ijl}(u) \mid\ \leq h_{ijl}^+ e^{-\eta_h u},$$
$$\mid k_{ijl}(u) \mid\ \leq k_{ijl}^+ e^{-\eta_k u}.$$

(H3) For all $1\leq i\leq n$ the functions $t\mapsto a_i(t)$ are almost periodic with $0<a_{i*}=\inf_{t\in\mathbb{R}}(a_i(t))$

(H4) $I_k\in PAP(\mathbb{Z},\mathbb{R}^n)$ and there exists a constant L_1 such that

$$\| I_k(u) - I_k(v) \| \le L_1 \| u - v \|, \; u, v \in \mathbb{R}^n, \; k \in \mathbb{Z}$$

(H5) Assume that there exist nonnegative constants $L, \widehat{p}$ and $\widehat{q}$ such that

$$\max_{1 \le i \le n} \max\left\{ \frac{\overline{J}_i}{a_{i*}}, (1 + \frac{a_i^+}{a_{i*}})\overline{J}_i \right\} = L$$

$$\widehat{p} = \max_{1 \le i \le n} \max\Bigg\{ \Bigg\{ a_{i*}^{-1}\Bigg[a_i^+ \rho^+ + \sum_{j=1}^n \overline{b}_{ij} L_j^g + \sum_{j=1}^n \overline{c}_{ij} \frac{d_{ij}^+}{\eta_d} L_j^f + \sum_{j=1}^n \sum_{l=1}^n \overline{\alpha}_{ijl} L_j^f M_l^f. + \sum_{j=1}^n \sum_{l=1}^n \overline{\beta}_{ijl} \frac{h_{ijl}^+}{\eta_h} \frac{k_{ijl}^+}{\eta_k} L_j^f M_l^f \Bigg] + \frac{L_1}{1 - e^{-a_{i*}}} \Bigg\},$$
$$\Bigg\{ (1 + \frac{a_i^+}{a_{i*}}) \Bigg[a_i^+ \rho^+ + \sum_{j=1}^n \overline{b}_{ij} L_j^f + \sum_{j=1}^n \overline{c}_{ij} \frac{d_{ij}^+}{\eta_d} L_j^f + \sum_{j=1}^n \sum_{l=1}^n \overline{\alpha}_{ijl} L_j^f M_l^f + \sum_{j=1}^n \sum_{l=1}^n \overline{\beta}_{ijl} \frac{h_{ijl}^+}{\eta_h} \frac{k_{ijl}^+}{\eta_k} L_j^f M_l^f \Bigg]. + \frac{a_i^+ L_1}{1 - e^{-a_{i*}}} \Bigg\} \Bigg\} < 1,$$

$$\widehat{q} = \max_{1 \le i \le n} max\Bigg\{ \Bigg\{ a_{i*}^{-1}[a_i^+ \rho^+ + \sum_{j=1}^n \overline{b}_{ij} L_j^f + \sum_{j=1}^n \overline{c}_{ij} \frac{d_{ij}^+}{\eta_d} L_j^f + \sum_{j=1}^n \sum_{l=1}^n \overline{\alpha}_{ijl} (L_j^f M_l^f + M_j^f L_l^f).. + \sum_{j=1}^n \sum_{l=1}^n \overline{\beta}_{ijl} \frac{h_{ijl}^+}{\eta_h} \frac{k_{ijl}^+}{\eta_k} (L_j^f M_l^f + M_j^f L_l^f)] + \frac{L_1}{1 - e^{-a_{i*}}} \Bigg\},$$
$$\Bigg\{ \left(1 + \frac{a_i^+}{a_{i*}}\right) [a_i^+ \rho^+ + \sum_{j=1}^n \overline{b}_{ij} L_j^f + \sum_{j=1}^n \overline{c}_{ij} \frac{d_{ij}^+}{\eta_d} L_j^f + \sum_{j=1}^n \sum_{l=1}^n \overline{\alpha}_{ijl} (L_j^f M_l^f + M_j^f L_l^f) + \sum_{j=1}^n \sum_{l=1}^n \overline{\beta}_{ijl} \frac{h_{ijl}^+}{\eta_h} \frac{k_{ijl}^+}{\eta_k} (L_j^f M_l^f + M_j^f L_l^f)] + \frac{a_i^+ L_1}{1 - e^{-a_{i*}}} \Bigg\} \Bigg\} < 1.$$

Throughout this paper, we will first recall some basic definitions and lemmas which are used in what follows.

- $\mathbb{N}, \mathbb{Z}$ and $\mathbb{R}$ stand for the set of natural numbers, integer numbers and real numbers, respectively.
- $C(\mathbb{R}, \mathbb{R}^n)$: the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}^n$.
- $BC(\mathbb{R}, \mathbb{R}^n)$: the set of bounded continued functions from $\mathbb{R}$ to $\mathbb{R}^n$. Note that $(BC(\mathbb{R}, \mathbb{R}^n), \| . \|_\infty)$ is a Banach space where $\| . \|_\infty$ denotes the sup norm

  $$\| f \|_\infty := \max_{1 \le i \le n} \sup_{t \in \mathbb{R}} | f_i(t) | .$$

- Let T be the set consisting of all real sequences $\{t_i\}_{i \in \mathbb{Z}}$ such that $\alpha = \inf_{i \in \mathbb{Z}}(t_{i+1} - t_i) > 0$. It is immediate that this condition implies that $\lim_{i \to +\infty} t_i = +\infty$ and $\lim_{i \to -\infty} t_i = -\infty$.
- $PC(\mathbb{R}, \mathbb{R}^n)$: the space formed by all piecewise continuous functions $f : \mathbb{R} \to \mathbb{R}^n$ such that $f(.)$ is continuous at t for any $t \notin \{t_i\}_{i \in \mathbb{Z}}$, $f(t_i^+)$, $f(t_i^-)$ exists and $f(t_i^-) = f(t_i)$ for all $i \in \mathbb{Z}$.
- $PC([-\tau, 0], \mathbb{R}^n) = \{f : [-\tau, 0] \to \mathbb{R}^n / f(t^-) = f(t)$, for $t \in [-\tau, 0], f(t^+)$ exists on $\mathbb{R}$ and $f(t^+) = f(t)$ for all but at most a finite number of points on $[-\tau, 0].\}$,
- $PC^1([-\tau, 0], \mathbb{R}^n) = \{f : [-\tau, 0] \to \mathbb{R}^n / f'(t^+)$ and $f'(t^-))$ exist ,$f'(t) = f'(t^-)$ for $t \in [-\tau, 0], f'(t^+) = f'(t)$ for all but at most a finite number of points on $[-\tau, 0].\}$,
- $l^\infty(\mathbb{Z}, \mathbb{R}^n) = \{x : \mathbb{Z} \to \mathbb{R}^n : \| x \| = \sup_{n \in \mathbb{Z}} \| x(n) \| < \infty\}$.

Definition 1 [36]. A function $f \in C(\mathbb{R}, \mathbb{R}^n)$ is called (Bohr) almost periodic if for each $\varepsilon > 0$ there exists $L(\varepsilon) > 0$ such that every interval of length $L(\varepsilon) > 0$ contains a number τ with the property that $\| f(t + \tau) - f(t) \|_\infty < \varepsilon$, for each $t \in \mathbb{R}$.

The number τ above is called an ε-translation number of f, and the collection of all such functions will be denoted as $AP(\mathbb{R}, \mathbb{R}^n)$.

Definition 2 [36]. A sequence $\{x_n\}$ is called almost periodic if for any $\varepsilon > 0$, there exists a relatively dense set of its ε-periods, i.e., there exists a natural number $l = l(\varepsilon)$, such that for $k \in \mathbb{Z}$, there is at least one number p in $[k, k + l]$, for which inequality $\| x_{n+p} - x_n \| < \varepsilon$ holds for all $n \in \mathbb{N}$. Denote by $AP(\mathbb{Z}, \mathbb{R}^n)$, the set of such sequences.

Define

$$PAP_0(\mathbb{Z}, \mathbb{R}^n) := \left\{ x \in l^\infty(\mathbb{Z}, \mathbb{R}^n) : \lim_{n \longrightarrow \infty} \frac{1}{2n} \sum_{k=-n}^n \| x(k) \| = 0 \right\}.$$

Remark 1 Notice that

1. A sequence vanishing at infinity is a $PAP_0(\mathbb{Z}, \mathbb{R})$ sequence.
2. The sequence $(x(n))_{n \in \mathbb{Z}}$ defined by

   $$x(n) = \begin{cases} 1, & n = 2^k, \\ 0, & n \neq 2^k, \end{cases}$$

   is an example of a $PAP_0(\mathbb{Z}, \mathbb{R})$ sequence which not vanishing at infinity.
3. For $k \in \mathbb{N}$ the sequence $(x(n))_{n \in \mathbb{Z}}$ defined by

   $$x(n) = \begin{cases} k, & n = 2^{k^2}, \\ 0, & n \neq 2^{k^2}, \end{cases}$$

   is an example of an unbounded $PAP_0(\mathbb{Z}, \mathbb{R})$ sequence.

Definition 3 [36]. A sequence $\{x_n\}_{n \in \mathbb{Z}} \in l^\infty(\mathbb{Z}, \mathbb{R}^n)$ is called pseudo-almost periodic if $x_n = x_n^1 + x_n^2$, where

$x_n^1 \in AP(\mathbb{Z}, \mathbb{R}^n), x_n^2 \in PAP_0(\mathbb{Z}, \mathbb{R}^n)$. Denote by $PAP(\mathbb{Z}, \mathbb{R}^n)$ the set of such sequences.

For $\{t_i\}_{n\in\mathbb{Z}} \in T, \{t_i^j\}$ is defined by

$$\{t_i^j = t_{i+j} - t_i\}, \quad i,j \in \mathbb{Z}.$$

It is easy to verify that the numbers t_i^j satisfy

$$t_{i+k}^j - t_i^j = t_{i+j}^k - t_i^k, t_i^j - t_i^k = t_{i+k}^{j-k}, \quad \text{for } i,j,k \in \mathbb{Z}.$$

Definition 4 [34]. A function $f \in PC(\mathbb{R}, \mathbb{R}^n)$ is said to be piecewise almost periodic if the following conditions are fulfilled:

1. $\{t_i^j = t_{i+j} - t_i\}, i,j \in \mathbb{Z}$ are equipotentially almost periodic, that is, for any $\varepsilon > 0$, there exists a relatively dense set in $\mathbb{R}$ of ε-almost periods common for all of the sequences $\{t_i^j\}$.
2. For any $\varepsilon > 0$, there exists a positive number $\delta = \delta(\varepsilon)$ such that if the points t' and t'' belong to the same interval of continuity of f and $\mid t' - t'' \mid < \delta$, then $\| f(t') - f(t'') \| < \varepsilon$.
3. For any $\varepsilon > 0$, there exists a relatively dense set Ω_ε in $\mathbb{R}$ such that if $\tau \in \Omega_\varepsilon$, then

   $$\| f(t+\tau) - f(t) \| < \varepsilon$$

   for all $t \in \mathbb{R}$ which satisfy condition $\mid t - t_i \mid > \varepsilon, i \in \mathbb{Z}$.

We denote by $AP_T(\mathbb{R}, \mathbb{R}^n)$ the space of all piecewise almost periodic functions. Obviously, $AP_T(\mathbb{R}, \mathbb{R}^n)$ endowed with the supremum norm is a Banach space. Throughout the rest of this paper, we always assume that $\{t_i^j\}$ are equipotentially almost periodic. Let $UPC(\mathbb{R}, \mathbb{R}^n)$ be the space of all functions $f \in PC(\mathbb{R}, \mathbb{R}^n)$ such that f satisfies the condition (2) in Definition 4.

Define

$$PAP_T^0(\mathbb{R}, \mathbb{R}^n) = \left\{ f \in PC(\mathbb{R}, \mathbb{R}^n), \lim_{r\longrightarrow\infty} \frac{1}{2r} \int_{-r}^{r} \| f(t) \| = 0 \right\}.$$

Definition 5 [36]. A function $f \in PC(\mathbb{R}, \mathbb{R}^n)$ is said to be piecewise pseudo-almost periodic if it can be decomposed $f = g + h$, where $g \in AP_T(\mathbb{R}, \mathbb{R}^n)$ and $h \in PAP_T^0(\mathbb{R}, \mathbb{R}^n)$. Denote by $PAP_T(\mathbb{R}, \mathbb{R}^n)$ the set of all such functions. $PAP_T(\mathbb{R}, \mathbb{R}^n)$ is a Banach space when endowed with the supremum norm.

Remark 2 The functions g and h in Definition 5 are, respectively, called the almost periodic component and the ergodic perturbation of the pseudo-almost periodic function f. The decomposition given in Definition 5 is unique. Further, $(PAP_T(\mathbb{R}, \mathbb{R}^n), \| . \|_\infty)$ is a Banach space which contains strictly the set of almost periodic functions. For instance, the function

$$f(t) = \begin{cases} \sin^2(\sqrt{3}t) + \cos^2(\pi t) + \dfrac{1}{1+t^2}, & t \neq t_k, k \in \mathbb{Z}, \\ \dfrac{1}{4}\sin(\sqrt{3}t_k) + \dfrac{1}{1+t_k^2}, & t = t_k, k \in \mathbb{Z}, \end{cases}$$

is a piecewise pseudo-almost periodic function, where

$$t_k = k + \frac{1}{6} \mid \sin k - \sin \sqrt{2}k \mid .$$

Hence, it is easy to see that $f(t)$ is more general than our traditional piecewise almost periodic functions since the ergodic perturbations are introduced.

Definition 6 [40]. Suppose that both functions f and its derivative f' are in $PAP(\mathbb{R}, \mathbb{R})$. That is, $f = g + h$ and $f' = g' + h'$, where $g, g' \in AP(\mathbb{R}, \mathbb{R})$ and $h, h' \in PAP_0(\mathbb{R}, \mathbb{R})$. Then the functions g and h are continuous differentiable.

Remark 3 Let $E = \{f \mid f, f' \in PAP(\mathbb{R}, \mathbb{R}^n)\}$ equipped with the induced norm defined by

$$\| f \|_E = \max\{\| f \|_\infty, \| f' \|_\infty\}\}.$$

Follows from [40] that $(PAP(\mathbb{R}, \mathbb{R}^n), \| . \|_E)$ is a Banach space.

The initial conditions associated with (1) are of the form

$$x_i(s) = \varphi_i(s), \ s \in (-\infty, 0], \ i = 1, 2, \ldots, n,$$

where $\varphi(.)$ are real-valued piecewise continuous functions defined on $(-\infty, 0]$.

Lemma 1 [32]. *Let $c_i(t)$ be an almost periodic function on $\mathbb{R}$ and*

$$M[c_i] = \lim_{T\to+\infty} \int_t^{t+T} c_i(s)\mathrm{d}s > 0, \ \ i = 1, \ldots, n.$$

Then the linear system

$$x'(t) = diag(-c_1(t), -c_2(t), \ldots, -c_n(t))x(t) \tag{2}$$

admits an exponential dichotomy on $\mathbb{R}$.

Lemma 2 [39]. *The inhomogeneous linear system*

$$x'(t) = -c(t)x(t) + f(t)$$

has a unique bounded solution for a vector $f \in C(\mathbb{R}, \mathbb{R})$ if and only if the inhomogeneous linear system (2) has exponential dichotomy.

The rest of this paper is organized as follows. The existence and the uniqueness of piecewise differentiable pseudo-almost periodic solutions of Eq. (1) in the suitable convex set are discussed in Sect. 2. Some sufficient conditions on the global exponential stability of piecewise differentiable pseudo-almost periodic solutions of Eq. (1) are established in Sect. 3. A numerical example is given in

Sect. 4 to illustrate the effectiveness of our results. Finally, we draw conclusion in Sect. 5.

2 Existence of piecewise differentiable pseudo-almost periodic solution

In this section, we establish some results for the existence of the piecewise differentiable pseudo-almost periodic solution of (1). To obtain the existence of piecewise differentiable pseudo-almost periodic solution of system (1), we shall introduce the following lemmas:

Lemma 3 [34]. *If $\phi(.) \in PAP_T(\mathbb{R}, \mathbb{R}^n)$ and for any $h \in \mathbb{R}$, then $\phi(. - h) \in PAP_T(\mathbb{R}, \mathbb{R}^n)$.*

Lemma 4 [34]. *If $\phi, \psi \in PAP_T(\mathbb{R}, \mathbb{R})$, then $\phi \times \psi \in PAP_T(\mathbb{R}, \mathbb{R})$.*

Lemma 5 *If $f_j(.) \in C(\mathbb{R}, \mathbb{R})$ satisfies the Lipschitz condition, $\phi(.) \in PAP_T(\mathbb{R}, \mathbb{R}), \phi'(.) \in PAP_T(\mathbb{R}, \mathbb{R})$ and $\beta(.) \in AP_T(\mathbb{R}, \mathbb{R})$* such that $1 - \beta'(t) > 0$ for all $t \in \mathbb{R}$ then $f_j(\phi(. - \beta(.))) \in PAP_T(\mathbb{R}, \mathbb{R})$

Proof We have $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1 \in AP_T(\mathbb{R}, \mathbb{R})$ and $\varphi_2 \in PAP_T^0(\mathbb{R}, \mathbb{R})$. Let

$$\begin{aligned} M(t) &= f_j(\phi(t - \beta(t))) = f_j(\phi_1(t - \beta(t))) \\ &\quad + [f_j(\phi_1(t - \beta(t)) + \phi_2(t - \beta(t))) \\ &\quad - f_j(\phi_1(t - \beta(t)))] \\ &= M_1(t) + M_2(t). \end{aligned}$$

Firstly, it follows from (Theorem 2.11, [33]) that $M_1(.) \in AP_T(\mathbb{R}, \mathbb{R})$. Then, we show that $M_2(.) \in PAP_T^0(\mathbb{R}, \mathbb{R})$ because

$$\begin{aligned} &\lim_{T \longrightarrow \infty} \frac{1}{2T} \int_{-T}^{T} | M_2(t) | \mathrm{d}t \\ &= \lim_{T \longrightarrow \infty} \frac{1}{2T} \int_{-T}^{T} | f_j(\phi_1(t - \beta(t)) + \phi_2(t - \beta(t))) \\ &\quad - f_j(\phi_1(t - \beta(t))) | \mathrm{d}t \\ &\leq \lim_{T \longrightarrow \infty} \frac{L_j^f}{2T} \int_{-T}^{T} | \phi_2(t - \beta(t)) | \mathrm{d}t = 0. \end{aligned}$$

Thus $M_2(.) \in PAP_T^0(\mathbb{R}, \mathbb{R})$. So, $f_j(\phi(. - \beta(.))) \in PAP_T(\mathbb{R}, \mathbb{R})$ and $f_j(\phi'(. - \beta(.))) \in PAP_T(\mathbb{R}, \mathbb{R})$. The proof is complete. □

Theorem 1 *Under the conditions (H1)–(H2), and for all $1 \leq j \leq n$, $x_j(.) \in PAP_T(\mathbb{R}, \mathbb{R}), x_j'(.) \in PAP_T(\mathbb{R}, \mathbb{R})$, then for all $1 \leq i \leq n$, the function $\phi_i : t \longmapsto \int_{-\infty}^{t} d_{ij}(t - s) f_j(x_j'(s)) \mathrm{d}s$ belongs to $PAP_T(\mathbb{R}, \mathbb{R})$.*

Proof For $x_j(.) \in PAP_T(\mathbb{R}, \mathbb{R})$,it is not difficult to see that $f_j(x_j'(.)) \in PAP_T(\mathbb{R}, \mathbb{R})$ by Lemma 5. Let $f_j(x_j'(.)) = u_j(.) + v_j(.)$, where $u_j \in AP_T(\mathbb{R}, \mathbb{R})$ and $v_j \in PAP_T^0(\mathbb{R}, \mathbb{R})$, then

$$\begin{aligned} \phi_i(t) &= \int_{-\infty}^{t} d_{ij}(t - s) f_j(x_j(s)) \mathrm{d}s = \int_{-\infty}^{t} d_{ij}(t - s) u_j(s) \mathrm{d}s \\ &\quad + \int_{-\infty}^{t} d_{ij}(t - s) v_j(s) \mathrm{d}s \\ &:= \phi_i^1(t) + \phi_i^2(t). \end{aligned}$$

First, it is not difficult to see that $\phi_i^1 \in UPC(\mathbb{R}, \mathbb{R})$. Let $t_k < t \leq t_{k+1}$.

For $\varepsilon > 0$, let Ω_ε be a relatively dense set of $\mathbb{R}$ formed by ε-periods of u_j. For $\tau \in \Omega_\varepsilon$ and $0 < h < \min\{\varepsilon, \frac{\alpha}{2}\}$,

$$\begin{aligned} &| \phi_i^1(t + \tau) - \phi_i^1(t) | \\ &\leq \int_{-\infty}^{t} | d_{ij}(t - s) || u_j(s + \tau) - u_j(s) | \mathrm{d}s \\ &\leq \sum_{w=-\infty}^{k-1} \int_{t_w + h}^{t_{w+1} - h} | d_{ij}(t - s) || u_j(s + \tau) - u_j(s) | \mathrm{d}s \\ &\quad + \sum_{w=-\infty}^{k-1} \int_{t_j}^{t_j + h} | d_{ij}(t - s) || u_j(s + \tau) - u_j(s) | \mathrm{d}s \\ &\quad + \sum_{w=-\infty}^{k-1} \int_{t_{w+1} - h}^{t_{w+1}} | d_{ij}(t - s) || u_j(s + \tau) - u_j(s) | \mathrm{d}s \\ &\quad + \int_{t_k}^{t} | d_{ij}(t - s) || u_j(s + \tau) - u_j(s) | \mathrm{d}s. \end{aligned}$$

Since $u_j \in AP_T(\mathbb{R}, \mathbb{R})$, one has

$| u_j(t + \tau) - u_j(t) | \leq \varepsilon$, for all $t \in [t_w + h, t_{w+1} - h]$ and $w \in \mathbb{Z}, w \leq k$,

then

$$\begin{aligned} &\sum_{w=-\infty}^{k-1} \int_{t_w + h}^{t_{w+1} - h} | d_{ij}(t - s) || u_j(s + \tau) - u_j(s) | \mathrm{d}s \\ &\leq \varepsilon \sum_{w=-\infty}^{k-1} \int_{t_w + h}^{t_{w+1} - h} | d_{ij}(t - s) | \mathrm{d}s \\ &\leq \varepsilon d_{ij}^+ \sum_{w=-\infty}^{k-1} \int_{t_w + h}^{t_{w+1} - h} e^{-\mu_d (t - s)} \mathrm{d}s \\ &\leq \frac{\varepsilon d_{ij}^+}{\mu_d} \sum_{w=-\infty}^{k-1} e^{-\mu_d (t - t_{w+1} + h)} \\ &\leq \frac{\varepsilon d_{ij}^+}{\mu_d} \sum_{w=-\infty}^{k-1} e^{-\mu_d \alpha (k - w - 1)} \\ &\leq \frac{\varepsilon d_{ij}^+}{\mu_d (1 - e^{-\mu_d \alpha})}. \end{aligned}$$

On the other hand,

$$\sum_{w=-\infty}^{k-1}\int_{t_w}^{t_w+h}\mid d_{ij}(t-s)\mid\mid u_j(s+\tau)-u_j(s)\mid \mathrm{d}s$$
$$\le 2d_{ij}\mid u_j\mid_\infty \sum_{w=-\infty}^{w-1}\int_{t_w}^{t_w+h} e^{-\mu_d(t-s)}\mathrm{d}s$$
$$\le 2d_{ij}^{+}\mid u_j\mid_\infty \varepsilon e^{\mu_d h}\sum_{w=-\infty}^{k-1} e^{-\mu_d(t-t_w)}\mathrm{d}s$$
$$\le 2d_{ij}^{+}\mid u_j\mid_\infty \varepsilon e^{\mu_d h}e^{-\mu\alpha(t-t_k)}\sum_{w=-\infty}^{k-1} e^{-\mu_d\alpha(i-j)}\mathrm{d}s$$
$$\le \frac{2d_{ij}^{+}\mid u_j\mid_\infty \varepsilon e^{\frac{(\mu_d\alpha)}{2}}}{1-e^{-\mu_d\alpha}}$$

Similarly, one has

$$\sum_{w=-\infty}^{k-1}\int_{t_{w+1}-h}^{t_{w+1}}\mid d_{ij}(t-s)\mid\mid u_j(s+\tau)-u_j(s)\mid \mathrm{d}s\le E_1\varepsilon,$$
$$\int_{t_k}^{t}\mid d_{ij}(t-s)\mid\mid u_j(s+\tau)-u_j(s)\mid \mathrm{d}s\le E_2\varepsilon,$$

where E_1, E_2 are some positive constants. Hence, $\phi_i^1(t)\in AP_T(\mathbb{R},\mathbb{R})$.

In fact, for $r>0$, one has

$$\lim_{r\longrightarrow\infty}\frac{1}{2r}\int_{-r}^{r}\mid \phi_i^2(t)\mid \mathrm{d}t$$
$$=\lim_{r\longrightarrow\infty}\frac{1}{2r}\int_{-r}^{r}\mid \int_{-\infty}^{t} d_{ij}(t-s)v_j(s)\mathrm{d}s\mid \mathrm{d}t$$
$$=\lim_{r\longrightarrow\infty}\frac{1}{2r}\int_{-r}^{r}\mid \int_{0}^{\infty} d_{ij}(s)v_j(t-s)\mathrm{d}s\mid \mathrm{d}t$$
$$\le\lim_{r\longrightarrow\infty}\frac{1}{2r}\int_{-r}^{r}\left(\int_{0}^{\infty} d_{ij}^{+}e^{-\mu_d s}\mid v_j(t-s)\mid \mathrm{d}s\right)\mathrm{d}t$$
$$\le\int_{0}^{\infty} d_{ij}^{+}e^{-\mu_d s}\left(\lim_{r\longrightarrow\infty}\frac{1}{2r}\int_{-r}^{r}\mid v_j(t-s)\mid \mathrm{d}t\right)\mathrm{d}s.$$

Since $v_i(t)\in PAP_T^0(\mathbb{R},\mathbb{R})$, it follows that $v_i(.-s)\in PAP_T^0(\mathbb{R},\mathbb{R})$ for each $s\in\mathbb{R}$ by Lemma 3. Using the Lebesgue dominated convergence theorem, we have $\phi_i^2(t)\in PAP_T^0(\mathbb{R},\mathbb{R})$. This completes the proof. □

Similarly, we can obtain:

Corollary 1 *Under the conditions (H1)–(H2), and for all* $1\le j\le n$, $x_j(.)\in PAP_T(\mathbb{R},\mathbb{R})$, *then for all* $1\le i\le n$, *the function* $\phi_i : t\longmapsto \int_{-\infty}^{t} h_{ijl}(t-s)f_j(x_j(s))\mathrm{d}s$ *belongs to* $PAP_T(\mathbb{R},\mathbb{R})$.

Corollary 2 *Under the conditions (H1)–(H2), and for all* $1\le l\le n$, $x_j(.)\in PAP_T(\mathbb{R},\mathbb{R})$, *then for all* $1\le i\le n$, *the function* $\phi_i : t\longmapsto \int_{-\infty}^{t} k_{ijl}(t-s)f_l(x_l(s))\mathrm{d}s$ *belongs to* $PAP_T(\mathbb{R},\mathbb{R})$.

Lemma 6 *Suppose that assumptions (H1)–(H3) hold. Define the nonlinear operator* $X_\varphi(.)$ *as follows, for each* $\varphi=(\varphi_1,\ldots,\varphi_n)\in PAP_T(\mathbb{R},\mathbb{R}^n)$ *and* $\varphi'=(\varphi'_1,\ldots,\varphi'_n)\in PAP_T(\mathbb{R},\mathbb{R}^n)$:

$$X_\varphi(t)=\begin{pmatrix}\int_{-\infty}^{t} e^{-\int_s^t a_1(u)\mathrm{d}u}F_1(s)\mathrm{d}s\\ \vdots\\ \int_{-\infty}^{t} e^{-\int_s^t a_n(u)\mathrm{d}u}F_n(s)\mathrm{d}s\end{pmatrix}$$

and

$$\begin{aligned}F_i(s)=&a_i(s)\int_{s-\rho(s)}^{s}\varphi_i'(u)\mathrm{d}u+\sum_{j=1}^{n}b_{ij}(s)f_j(\varphi_j'(s-\tau_{ij}(s)))\\&+\sum_{j=1}^{n}c_{ij}(s)\int_0^\infty d_{ij}(u)f_j(\varphi_j'(s-u))\mathrm{d}u\\&+\sum_{j=1}^{n}\sum_{l=1}^{n}\alpha_{ijl}(s)f_j(\varphi_j(s-\sigma_{ij}(s)))f_l(\varphi_l(s-v_{ij}(s)))\\&+\sum_{j=1}^{n}\sum_{l=1}^{n}\beta_{ijl}(s)\int_0^\infty h_{ijl}(u)f_j(\varphi_j(s-u))\mathrm{d}u\\&\int_0^\infty k_{ijl}(u)f_l(\varphi_l(s-u))\mathrm{d}u+J_i(s),\end{aligned}$$

then X_φ *maps* $PAP_T(\mathbb{R},\mathbb{R}^n)$ *into itself.*

Proof First, note that, for all $1\le i\le n$, the function

$$\begin{aligned}s\mapsto F_i(s)=&a_i(s)\int_{s-\rho(s)}^{s}\varphi_i'(u)\mathrm{d}u+\sum_{j=1}^{n}b_{ij}(s)f_j(\varphi_j'(s-\tau_{ij}(s)))\\&+\sum_{j=1}^{n}b_{ij}(s)\int_0^\infty d_{ij}(u)f_j(\varphi_j'(s-u))\mathrm{d}u\\&+\sum_{j=1}^{n}\sum_{l=1}^{n}\alpha_{ijl}(s)f_j(\varphi_j(s-\sigma_{ij}(s)))f_l(\varphi_l(s-v_{ij}(s)))\\&+\sum_{j=1}^{n}\sum_{l=1}^{n}\beta_{ijl}(s)\int_0^\infty h_{ijl}(u)f_j(\varphi_j(s-u))\mathrm{d}u\\&\times\int_0^\infty k_{ijl}(u)f_l(\varphi_l(s-u))\mathrm{d}u+J_i(s),\end{aligned}$$

is in $PAP_T(\mathbb{R},\mathbb{R})$, by using Lemmas 3, 4, 5, Theorem 1, Corollaries 1, 2. Consequently, for all $1\le i\le n, F_i$ can be expressed as

$$F_i=F_i^1+F_i^2,$$

where $F_i^1\in AP_T(\mathbb{R},\mathbb{R})$ and $F_i^2\in PAP_T^0(\mathbb{R},\mathbb{R})$. So

$$\begin{aligned}(X_i\varphi)(t)=&\int_{-\infty}^{t} e^{-\int_s^t a_i(u)\mathrm{d}u}F_i^1(s)\mathrm{d}s+\int_{-\infty}^{t} e^{-\int_s^t a_i(u)\mathrm{d}u}F_i^2(s)\mathrm{d}s\\=&H_i^1(t)+H_i^2(t).\end{aligned}$$

(i) $H_i^1(.)\in UPC(\mathbb{R},\mathbb{R})$. Let $t', t''\in(t_k,t_{k+1}), k\in\mathbb{Z}, t''<t'$, then

$$
\begin{aligned}
&| H_i^1(t') - H_i^1(t'') | \\
&= | \int_{-\infty}^{t'} e^{-\int_s^{t'} a_i(u)\mathrm{d}u} F_i^1(s)\mathrm{d}s - \int_{-\infty}^{t''} e^{-\int_s^{t''} a_i(u)\mathrm{d}u} F_i^1(s)\mathrm{d}s | \\
&\quad \le | \int_{-\infty}^{t''} [e^{-\int_s^{t'} a_i(u)\mathrm{d}u} - e^{-\int_s^{t''} a_i(u)\mathrm{d}u}] F_i^1(s)\mathrm{d}s | \\
&\quad + | \int_{t''}^{t'} e^{-\int_s^{t'} a_i(u)\mathrm{d}u} F_i^1(s)\mathrm{d}s | \\
&\quad \le | e^{-\int_{t''}^{t'} a_i(u)\mathrm{d}u} - 1 | \int_{-\infty}^{t''} e^{-\int_s^{t''} a_i(u)\mathrm{d}u} | F_i^1(s) | \mathrm{d}s \\
&\quad + \int_{t''}^{t'} e^{-\int_s^{t'} a_i(u)\mathrm{d}u} | F_i^1(s) | \mathrm{d}s \\
&\quad \le ((t' - t'')a_i^+) \int_{-\infty}^{t''} e^{-(t''-s)a_{i*}} | F_i^1(s) | \mathrm{d}s \\
&\quad + \int_{t''}^{t'} e^{-(t'-s)a_{i*}} | F_i^1(s) | \mathrm{d}s,
\end{aligned}
$$

it is easy to see that for any $\varepsilon > 0$, there exists

$$0 < \delta < \min\left\{ \frac{a_{i*}\varepsilon}{2a_i^+ | F_i^1 |_\infty}, \frac{\varepsilon}{2 | F_i^1 |_\infty} \right\}$$

and for a suitable t', t'' satisfying $0 < t' - t'' < \delta$ one has

$$
\begin{aligned}
&| H_i^1(t') - H_i^1(t'') | \\
&\le \left[(t' - t'')a_i^+ \int_{-\infty}^{t''} e^{-(t''-s)a_{i*}}\mathrm{d}s + \int_{t''}^{t'} e^{-(t'-s)a_{i*}}\mathrm{d}s \right] | F_i^1 |_\infty \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}
$$

which implies that $H_i^1(.) \in UPC(\mathbb{R}, \mathbb{R})$.

(ii) $H_i^1(.) \in AP_T(\mathbb{R}, \mathbb{R})$. Since $F_i^1 \in AP_T(\mathbb{R}, \mathbb{R})$, for $\varepsilon > 0$, there exists a relatively dense set Ω_ε such that for $\tau \in \Omega_\varepsilon, t \in \mathbb{R}, | t - t_k | > \varepsilon, k \in \mathbb{Z}$, then

$$
\begin{aligned}
&H_i^1(t+\tau) - H_i^1(t) \\
&= \int_{-\infty}^{t+\tau} e^{-\int_s^{t+\tau} a_i(u)\mathrm{d}u} F_i^1(s)\mathrm{d}s - \int_{-\infty}^{t} e^{-\int_s^{t} a_i(u)\mathrm{d}u} F_i^1(s)\mathrm{d}s \\
&= \int_{-\infty}^{t+\tau} e^{-\int_{s-\tau}^{t} a_i(\rho+\tau)d\rho} F_i^1(s)\mathrm{d}s - \int_{-\infty}^{t} e^{-\int_s^{t} a_i(u)\mathrm{d}u} F_i^1(s)\mathrm{d}s \\
&= \int_{-\infty}^{t} e^{-\int_s^{t} a_i(m+\tau)\mathrm{d}m} F_i^1(s+\tau)\mathrm{d}s - \int_{-\infty}^{t} e^{-\int_s^{t} a_i(m+\tau)\mathrm{d}m} F_i^1(s)\mathrm{d}s \\
&\quad + \int_{-\infty}^{t} e^{-\int_s^{t} a_i(m+\tau)\mathrm{d}m} F_i^1(s)\mathrm{d}s - \int_{-\infty}^{t} e^{-\int_s^{t} a_i(u)\mathrm{d}u} F_i^1(s)\mathrm{d}s \\
&= \int_{-\infty}^{t} e^{-\int_s^{t} a_i(u+\tau)\mathrm{d}u} (F_i^1(s+\tau) - F_i^1(s))\mathrm{d}s \\
&\quad + \int_{-\infty}^{t} (e^{-\int_s^{t} a_i(u+\tau)\mathrm{d}u} - e^{-\int_s^{t} a_i(u)\mathrm{d}u}) F_i^1(s)\mathrm{d}s
\end{aligned}
$$

So there exists $\theta \in]0, 1[$ such that

$$
\begin{aligned}
&|H_i^1(t+\tau) - H_i^1(t)| \\
&\quad \le |F_i^1|_\infty \int_{-\infty}^{t} \left(e^{-\int_s^t a_i(u+\tau)\mathrm{d}u} - e^{-\int_s^t a_i(u)\mathrm{d}u} \right) \mathrm{d}s \\
&\quad + \int_{-\infty}^{t} e^{-\int_s^t a_i(u+\tau)\mathrm{d}u} \left|F_i^1(s+\tau) - F_i^1(s)\right| \mathrm{d}s \\
&\quad \le \int_{-\infty}^{t} \left(e^{-\left[\int_s^t a_{ij}(u+\tau)\mathrm{d}u + \theta(\int_s^t a_i(u)\mathrm{d}u - \int_s^t a_i(u+\tau)\mathrm{d}u)\right]} \int_s^t |a_i(u) - a_i(u+\tau)|\mathrm{d}u \right) \mathrm{d}s |F_i^1|_\infty \\
&\qquad + \int_{-\infty}^{t} e^{-\int_s^t a_i(u+\tau)\mathrm{d}u} \left|F_i^1(s+\tau) - F_i^1(s)\right| \mathrm{d}s \\
&\quad \le \int_{-\infty}^{t} \left(e^{-\left[\int_s^t a_i(u+\tau)\mathrm{d}u + \theta(\int_s^t a_i(u)\mathrm{d}u - \int_s^t a_i(u+\tau)\mathrm{d}u)\right]} \int_s^t |a_i(u) - a_i(u+\tau)|\mathrm{d}u \right) \mathrm{d}s |F_i^1|_\infty \\
&\qquad + \int_{-\infty}^{t} e^{-a_{i*}(t-s)} \left|F_i^1(s+\tau) - F_i^1(s)\right| \mathrm{d}s \\
&\quad \le \int_{-\infty}^{t} \left\{ e^{-a_{i*}(t-s)} e^{-\theta\left(\int_s^t \left|a_i(u) - a_i(u+\tau)|\mathrm{d}u\right)\right.} \int_s^t |a_i(u) - a_i(u+\tau)|\mathrm{d}u \right\} \mathrm{d}s |F_i^1|_\infty \\
&\qquad + \int_{-\infty}^{t} e^{-a_{i*}(t-s)} \left|F_i^1(s+\tau) - F_i^1(s)\right| \mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\le |F_i^1|_\infty \int_{-\infty}^{t} \left\{ e^{-a_{i*}(t-s)} \int_s^t |a_i(u) - a_i(u+\tau)|\mathrm{d}u \right\} \mathrm{d}s \\
&\quad + \int_{-\infty}^{t} e^{-a_{i*}(t-s)} \left|F_i^1(s+\tau) - F_i^1(s)\right| \mathrm{d}s \\
&= \int_{-\infty}^{t} \Phi_i(t,s)\mathrm{d}s + \int_{-\infty}^{t} \Psi_i(t,s)\mathrm{d}s
\end{aligned}
$$

where

$$\Phi_i(t,s) = e^{-a_{i*}(t-s)} |F_i^1|_\infty \int_s^t |a_i(u) - a_i(u+\tau)|\mathrm{d}u$$

and

$$\Psi_i(t,s) = e^{-a_{i*}(t-s)} \left|F_i^1(s+\tau) - F_i^1(s)\right|,$$

we obtain immediately that, $H_i^1(.) \in AP_T(\mathbb{R}, \mathbb{R})$.

Now, we turn our attention to $H_i^2(.)$, so

$$
\begin{aligned}
&\lim_{r \longrightarrow \infty} \frac{1}{2r} \int_{-r}^{r} | H_i^2(t) | \mathrm{d}t \\
&\le \lim_{r \longrightarrow \infty} \frac{1}{2r} \int_{-r}^{r} \int_{-\infty}^{t} e^{-(t-s)a_{i*}} | F_i^2(s) | \mathrm{d}s\mathrm{d}t \\
&\le I_1 + I_2,
\end{aligned}
$$

where

$$I_1 = \lim_{r \longrightarrow \infty} \frac{1}{2r} \int_{-r}^{r} \left(\int_{-r}^{t} e^{-(t-s)a_{i*}} | F_i^2(s) | \mathrm{d}s \right) \mathrm{d}t$$

and

$$I_2 = \lim_{r\longrightarrow\infty} \frac{1}{2r}\int_{-r}^{r}\left(\int_{-\infty}^{-r} e^{-(t-s)a_{i*}} \mid F_i^2(s) \mid \mathrm{d}s\right)\mathrm{d}t.$$

Pose $m = t - s$, then by Fubini's theorem one has

$$\begin{aligned} I_1 &= \lim_{r\longrightarrow\infty} \frac{1}{2r}\int_{-r}^{r}\left(\int_{0}^{t+r} e^{-ma_{i*}} \mid F_i^2(t-m) \mid \mathrm{d}m\right)\mathrm{d}t \\ &\leq \lim_{r\longrightarrow\infty} \frac{1}{2r}\int_{-r}^{r}\left(\int_{0}^{+\infty} e^{-ma_{i*}} \mid F_i^2(t-m) \mid \mathrm{d}m\right)\mathrm{d}t \\ &\leq \int_{0}^{+\infty} e^{-ma_{i*}}\left(\lim_{r\longrightarrow\infty} \frac{1}{2r}\int_{-r}^{r} \mid F_i^2(t-m) \mid \mathrm{d}t\right)\mathrm{d}m \end{aligned}$$

since the function $F_i^2(.) \in PAP_T^0(\mathbb{R},\mathbb{R})$, and by the Lebesgue dominated convergence theorem, we obtain

$I_1 = 0.$

On the other hand, notice that$\mid F_i^2 \mid_\infty = \sup_{t\in\mathbb{R}} \mid F_i^2(t) \mid < \infty$ then

$$\begin{aligned} I_2 &= \lim_{r\longrightarrow\infty} \frac{1}{2r}\int_{-r}^{r}\left(\int_{-\infty}^{-r} e^{-(t-s)a_{i*}} \mid F_i^2(s) \mid \mathrm{d}s\right)\mathrm{d}t \\ &\leq \lim_{r\longrightarrow\infty} \frac{1}{2r}\int_{-\infty}^{-r} e^{sa_{i*}} \mid F_i^2(s) \mid \mathrm{d}s \int_{-r}^{r} e^{-ta_{i*}}\mathrm{d}t \\ &= \lim_{r\longrightarrow\infty} \frac{\mid F_i^2 \mid_\infty}{2ra_{i*}} \int_{-r}^{r} e^{-(r+t)a_{i*}}\mathrm{d}t \\ &=0, \end{aligned}$$

then

$$\lim_{r\longrightarrow\infty} \frac{1}{2r}\int_{-r}^{r} \mid \int_{-\infty}^{t} e^{-\int_s^t a_i(u)\mathrm{d}u} F_i^2(s)\mathrm{d}s \mid \mathrm{d}t = 0.$$

Consequently, the function H_i^2 belongs to $PAP_T^0(\mathbb{R},\mathbb{R})$. So X_φ belongs to $PAP_T(\mathbb{R},\mathbb{R}^n)$. □

Lemma 7 *Suppose that assumptions (H4) hold, Define the nonlinear operator, for each* $\varphi = (\varphi_1,\ldots,\varphi_n) \in PAP_T(\mathbb{R},\mathbb{R}^n)$, *we have*

$$\sum_{t_k<t} e^{-\int_{t_k}^t a_i(u)\mathrm{d}u} I_k(\varphi_i(t_k)) \in PAP_T(\mathbb{R},\mathbb{R}).$$

Proof We will show that $\sum_{t_k<t} e^{-\int_{t_k}^t a_i(u)\mathrm{d}u} I_k(\varphi_i(t_k)) \in PAP_T(\mathbb{R},\mathbb{R})$. It is not difficult to see that $\sum_{t_k<t} e^{-\int_{t_k}^t a_i(u)\mathrm{d}u} I_k (\varphi_i(t_k)) \in UPC(\mathbb{R},\mathbb{R})$. After by Corollary 2.1 (see [34]), $I_k(x_i(t_k)) \in PAP(\mathbb{Z},\mathbb{R})$, then let $I_k(x_i(t_k)) = I_k^1 + I_k^2$ where $I_k^1 \in AP(\mathbb{Z},\mathbb{R})$ and $I_k^2 \in PAP_0(\mathbb{Z},\mathbb{R})$, so

$$\begin{aligned} &\sum_{t_k<t} e^{-\int_{t_k}^t a_i(u)\mathrm{d}u} I_k(x_i(t_k)) \\ &= \sum_{t_k<t} e^{-\int_{t_k}^t a_i(u)\mathrm{d}u} I_k^1 + \sum_{t_k<t} e^{-\int_{t_k}^t a_i(u)\mathrm{d}u} I_k^2 = \Phi_1(t) + \Phi_2(t). \end{aligned}$$

Since $\{t_j^k\}, k,j \in \mathbb{Z}$ are equipotentially almost periodic, then by Lemma 3.2 (see [34]), for any $\varepsilon > 0$, there exists relative dense sets of real numbers Ω_ε and integers Q_ε, such that for $t_k < t \leq t_{k+1}, \tau \in \Omega_\varepsilon$, $q \in Q_\varepsilon, \mid t - t_k \mid > \varepsilon, \mid t - t_{k+1} \mid > \varepsilon, k \in \mathbb{Z}$, one has

$$\begin{aligned} &t + \tau > t_k + \varepsilon + \tau > t_{k+q}, \\ &\quad t_{k+q+1} > t_{k+1} - \varepsilon + \tau > t + \tau, \end{aligned}$$

that is $t_{k+q} > t + \tau > t_{k+q+1}$; then

$$\begin{aligned} &\parallel \Phi_1(t+\tau) - \Phi_1(t) \parallel \\ &\quad = \parallel \sum_{t_k<t+\tau} e^{-\int_{t_k}^{t+\tau} a_i(u)\mathrm{d}u} I_k^1 - \sum_{t_k<t} e^{-\int_{t_k}^{t} a_i(u)\mathrm{d}u} I_k^1 \parallel \\ &\quad \leq \sum_{t_i<t} e^{-\int_{t_i}^{t} a_i(u)\mathrm{d}u} \parallel I_{i+q}^1 - I_i^1 \parallel \\ &\quad \leq \varepsilon \sum_{t_k<t} e^{-(t-t_k)a_{i*}} \\ &\quad \leq \varepsilon \frac{1}{1 - e^{-a_{i*}}}, \end{aligned}$$

so, $\Phi_1(t) \in AP_T(\mathbb{R},\mathbb{R})$.

Next, we show that $\Phi_2(t) \in PAP_T^0(\mathbb{R},\mathbb{R})$. For a given $k \in \mathbb{Z}$, define the function $\chi(t)$ by

$$\chi(t) = e^{-\int_{t_k}^t a_i(u)\mathrm{d}u} I_k^2, \quad t_k < t \leq t_{k+1},$$

then

$$\begin{aligned} &\lim_{t\to\infty} \parallel \chi(t) \parallel = \lim_{t\to\infty} \parallel e^{-\int_{t_k}^t a_i(u)\mathrm{d}u} I_k^2 \parallel \\ &\quad \leq \lim_{t\to\infty} e^{-(t-t_k)a_{i*}} \sup_{k\in\mathbb{Z}} \parallel I_k^2 \parallel = 0, \end{aligned}$$

then $\chi \in PAP_T^0(\mathbb{R},\mathbb{R})$. Define $\chi_n : \mathbb{R} \to \mathbb{R}$ by

$$\chi_n(t) = e^{-\int_{t_{k-n}}^t a_i(u)\mathrm{d}u} I_{k-n}^2, \quad t_k < t \leq t_{k+1}, \ n \in \mathbb{N}.$$

So $\chi_n \in PAP_T^0(\mathbb{R},\mathbb{R})$. Moreover,

$$\begin{aligned} \parallel \chi_n(t) \parallel &= \parallel e^{-\int_{t_{k-n}}^t a_i(u)\mathrm{d}u} I_{k-n}^2 \parallel \\ &\leq e^{-(t-t_{k-n})a_{i*}} \sup_{k\in\mathbb{Z}} \parallel I_k^2 \parallel \\ &\leq e^{-(t-t_k)a_{i*}} e^{-a_{i*}\alpha n} \sup_{k\in\mathbb{Z}} \parallel I_k^2 \parallel. \end{aligned}$$

therefore, the series $\sum_{n=1}^\infty \chi_n$ is uniformly convergent on $\mathbb{R}$. By Lemma 2.2 (see [34]), one has

$$\Phi_2(t) = \sum_{t_k<t} e^{-\int_{t_k}^t a_i(u)\mathrm{d}u} I_k^2 = \sum_{n=0}^{\infty} \chi_n \in PAP_T^0(\mathbb{R},\mathbb{R}).$$

So, $\sum_{t_k<t} e^{-\int_{t_k}^t a_i(u)\mathrm{d}u} I_k(x_i(t_k)) \in PAP_T(\mathbb{R},\mathbb{R})$. □

Theorem 2 Suppose that assumptions (H1)–(H4) hold. Define the nonlinear operator Γ *as follows, for each* $\varphi =$

$(\varphi_1,\ldots,\varphi_n)\in PAP_T(\mathbb{R},\mathbb{R}^n)$, *and* $\varphi'=(\varphi'_1,\ldots,\varphi'_n)\in PAP_T(\mathbb{R},\mathbb{R}^n)$,

$$(\Gamma_\varphi)_i(t):=(X_\varphi)_i(t)+\sum_{t_k<t}e^{-\int_{t_k}^t a_i(u)\mathrm{d}u}I_k(\varphi_i(t_k)),$$

then Γ *maps* $PAP_T(\mathbb{R},\mathbb{R}^n)$ *into itself and if*

$$(\Gamma_\varphi)'_i(t):=F_i(t)-a_i(t)\int_{-\infty}^t e^{-\int_s^t a_i(u)\mathrm{d}u}F_i(s)\mathrm{d}s - a_i(t)\sum_{t_k<t}e^{-\int_{t_k}^t a_i(u)\mathrm{d}u}I_k(\varphi_i(t_k)),$$

then Γ' maps $PAP_T(\mathbb{R},\mathbb{R}^n)$ *into itself.*

Theorem 3 *Let conditions (H1)–(H5) hold. Then, there exists a unique piecewise differentiable pseudo-almost periodic solution of system (1) in the region*

$$B=\{\varphi/\varphi,\varphi'\in PAP_T(\mathbb{R},\mathbb{R}^n),\|\varphi-\varphi_0\|_E\le\frac{\widehat{p}L}{1-\widehat{p}}\},$$

where

$$\varphi_0(t)=\left(\int_{-\infty}^t e^{-\int_s^t a_1(u)\mathrm{d}u}J_1(s)\mathrm{d}s,\ldots,\int_{-\infty}^t e^{-\int_s^t a_n(u)\mathrm{d}u}J_n(s)\mathrm{d}s\right)^T.$$

Proof It is easy to see that $B=\{\varphi/\varphi,\varphi'\in PAP_T(\mathbb{R},\mathbb{R}^n),\|\varphi-\varphi_0\|_E\le\frac{\widehat{p}L}{1-\widehat{p}}\}$ is a closed convex subset of $PAP_T(\mathbb{R},\mathbb{R}^n)$. According to the definition of the norm of Banach space $PAP_T(\mathbb{R},\mathbb{R}^n)$, we get

$$\begin{aligned}\|\varphi_0\|_E&=\max_{1\le i\le n}\max\sup_{t\in\mathbb{R}}\Big\{\Big|\int_{-\infty}^t e^{-\int_s^t a_i(u)\mathrm{d}u}J_i(s)\mathrm{d}s\Big|,\Big|J_i(t) - a_i(t)\int_{-\infty}^t e^{-\int_s^t a_i(u)\mathrm{d}u}J_i(s)\mathrm{d}s\Big|\Big\}\\&\le\max_{1\le i\le n}\max\{\frac{\overline{J}_i}{a_{i*}},(1+\frac{a_i^+}{a_{i*}})\overline{J}_i\}=L.\end{aligned}\tag{3}$$

Therefore, for $\forall\varphi\in B$, we have

$$\|\varphi\|_E\le\|\varphi-\varphi_0\|_E+\|\varphi_0\|_E\le\frac{\widehat{p}L}{1-\widehat{p}}+L=\frac{L}{1-\widehat{p}}.\tag{4}$$

In view of (H1), we have

$$|f_j(u)|\le L_j^f|u|,\text{forall}u\in\mathbb{R},j=1,2,\ldots,n.\tag{5}$$

Now, we prove that the mapping Γ is a self-mapping from B to B. In fact, for all $\varphi\in B$ by using the estimate just obtained together with (4), (5), Lemma 1, Lemma 2, Lemma 6 and Lemma 7 we obtain

$$\begin{aligned}&\|\Gamma_\varphi-\varphi_0\|_\infty\\&=\max_{1\le i\le n}\sup_{t\in\mathbb{R}}\Big\{\Big|\int_{-\infty}^t e^{-\int_s^t a_i(u)\mathrm{d}u}[a_i(s)\int_{s-\rho(s)}^s\varphi'_i(m)\mathrm{d}m\\&+\sum_{j=1}^n b_{ij}(s)f_j(\varphi'_j(s-\tau_{ij}(s)))+\sum_{j=1}^n c_{ij}(s)\int_0^\infty d_{ij}(u)f_j(\varphi'_j(s-u))\mathrm{d}u\\&+\sum_{j=1}^n\sum_{l=1}^n\alpha_{ijl}(s)f_j(\varphi_j(s-\sigma_{ij}(s)))f_l(\varphi_l(s-v_{ij}(s)))\\&+\sum_{j=1}^n\sum_{l=1}^n\beta_{ijl}(s)\int_0^\infty h_{ijl}(u)f_j(\varphi_j(s-u))\mathrm{d}u\\&\times\int_0^\infty k_{ijl}(u)f_l(\varphi_l(s-u))\mathrm{d}u]\mathrm{d}s+\sum_{t_k<t}e^{-\int_{t_k}^t a_i(u)\mathrm{d}u}I_k(\varphi_i(t_k))\Big|\Big\}\\&\le\max_{1\le i\le n}\sup_{t\in\mathbb{R}}\Big\{\int_{-\infty}^t e^{-a_{i*}(t-s)}[a_i^+\rho^+\|\varphi'\|_\infty\\&+\sum_{j=1}^n\overline{b}_{ij}L_j^f\|\varphi'\|_\infty+\sum_{j=1}^n\overline{c}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^f\|\varphi'\|_\infty\\&+\sum_{j=1}^n\sum_{l=1}^n\overline{\alpha}_{ijl}L_j^fM_l^f\|\varphi\|_\infty\\&+\sum_{j=1}^n\sum_{l=1}^n\overline{\beta}_{ijl}\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}L_j^fM_l^f\|\varphi\|_\infty]\mathrm{d}s+\sum_{t_k<t}e^{-(t-t_k)a_{i*}}L_1\|\varphi\|_\infty\Big\}\\&\le\max_{1\le i\le n}\Big\{a_{i*}^{-1}\Big[a_i^+\rho^++\sum_{j=1}^n\overline{b}_{ij}L_j^f+\sum_{j=1}^n\overline{c}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^f+\sum_{j=1}^n\sum_{l=1}^n\overline{\alpha}_{ijl}L_j^fM_l^f\\&+\sum_{j=1}^n\sum_{l=1}^n\overline{\beta}_{ijl}\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}L_j^fM_l^f\Big]+\frac{L_1}{1-e^{-a_{i*}}}\Big\}\|\varphi\|_E\end{aligned}$$

On the other hand

$$\begin{aligned}&\|(\Gamma_\varphi-\varphi_0)'\|_\infty\\&=\max_{1\le i\le n}\sup_{t\in\mathbb{R}}\Big\{\Big|a_i(t)\int_{t-\rho(t)}^t\varphi'_i(m)\mathrm{d}m+\sum_{j=1}^n b_{ij}(t)f_j(\varphi'_j(t-\tau_{ij}(t)))\\&+\sum_{j=1}^n c_{ij}(t)\int_0^\infty d_{ij}(u)f_j(\varphi'_j(t-u))\mathrm{d}u\\&+\sum_{j=1}^n\sum_{l=1}^n\alpha_{ijl}(t)f_j(\varphi_j(t-\sigma_{ij}(t)))f_l(\varphi_l(t-v_{ij}(t)))\\&+\sum_{j=1}^n\sum_{l=1}^n\beta_{ijl}(t)\int_0^\infty h_{ijl}(u)f_j(\varphi_j(t-u))\mathrm{d}u\\&\times\int_0^\infty k_{ijl}(u)f_l(\varphi_l(t-u))\mathrm{d}u\\&-a_i(t)\int_{-\infty}^t e^{-\int_s^t a_i(u)\mathrm{d}u}[a_i(s)\int_{s-\rho(s)}^s\varphi'_i(m)\mathrm{d}m\\&+\sum_{j=1}^n b_{ij}(s)f_j(\varphi'_j(s-\tau_{ij}(s)))+\sum_{j=1}^n c_{ij}(s)\int_0^\infty d_{ij}(u)f_j(\varphi'_j(s-u))\mathrm{d}u\\&+\sum_{j=1}^n\sum_{l=1}^n\alpha_{ijl}(s)f_j(\varphi_j(s-\sigma_{ij}(s)))f_l(\varphi_l(s-v_{ij}(s)))\\&+\sum_{j=1}^n\sum_{l=1}^n\beta_{ijl}(s)\int_0^\infty h_{ijl}(u)f_j(\varphi_j(s-u))\mathrm{d}u\\&\times\int_0^\infty k_{ijl}(u)f_l(\varphi_l(s-u))\mathrm{d}u]\mathrm{d}s-a_i(t)\sum_{t_k<t}e^{-\int_{t_k}^t a_i(u)\mathrm{d}u}I_k(\varphi_i(t_k))\Big|\Big\}\end{aligned}$$

$$\leq \max_{1\leq i\leq n}\sup_{t\in\mathbb{R}}\Bigg\{[a_i^+\rho^+ \parallel \varphi' \parallel_\infty + \sum_{j=1}^{n}\overline{b}_{ij}L_j^f \parallel \varphi' \parallel_\infty + \sum_{j=1}^{n}\overline{c}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^f \parallel \varphi' \parallel_\infty$$
$$+ \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\alpha}_{ijl}L_j^f M_l^f \parallel \varphi \parallel_\infty + \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\beta}_{ijl}\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}L_j^f M_l^f \parallel \varphi \parallel_\infty]$$
$$+ c_i^+\int_{-\infty}^{t} e^{-a_{i*}(t-s)}[a_i^+\rho^+ \parallel \varphi' \parallel_\infty + \sum_{j=1}^{n}\overline{b}_{ij}L_j^f \parallel \varphi \parallel_\infty$$
$$+ \sum_{j=1}^{n}\overline{c}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^f \parallel \varphi \parallel_\infty + \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\alpha}_{ijl}L_j^f M_l^f \parallel \varphi \parallel_\infty$$
$$+ \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\beta}_{ijl}\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}L_j^f M_l^f \parallel \varphi \parallel_\infty]\mathrm{d}s$$
$$+ a_i^+\sum_{t_k<t} e^{-(t-t_k)a_{i*}}L_1 \parallel \varphi \parallel_\infty \Bigg\}$$
$$\leq \max_{1\leq i\leq n}\Bigg\{(1+\frac{a_i^+}{a_{i*}})\Bigg[a_i^+\rho^+ + \sum_{j=1}^{n}\overline{b}_{ij}L_j^f + \sum_{j=1}^{n}\overline{c}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^f$$
$$+ \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\alpha}_{ijl}L_j^f M_l^f + \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\beta}_{ijl}\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}L_j^f M_l^f\Bigg] + \frac{a_i^+L_1}{1-e^{-a_{i*}}}\Bigg\} \parallel \varphi \parallel_E$$

where $i = 1, 2, \ldots, n$. So we can write

$$\parallel \Gamma_\varphi - \varphi_0) \parallel_E$$
$$= \max\{\parallel \Gamma_\varphi - \varphi_0 \parallel_\infty, \parallel (\Gamma_\varphi - \varphi_0)' \parallel_\infty\}$$
$$\leq \max_{1\leq i\leq n}\max\Bigg\{\{a_{i*}^{-1}\Bigg[a_i^+\rho^+ + \sum_{j=1}^{n}\overline{b}_{ij}L_j^f + \sum_{j=1}^{n}\overline{c}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^f$$
$$+ \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\alpha}_{ijl}L_j^f M_l^f + \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\beta}_{ijl}\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}L_j^f M_l^f\Bigg]$$
$$+ \frac{L_1}{1-e^{-a_{i*}}}\Bigg\}, \Bigg\{(1+\frac{a_i^+}{a_{i*}})\Bigg[a_i^+\rho^+ + \sum_{j=1}^{n}\overline{b}_{ij}L_j^f$$
$$+ \sum_{j=1}^{n}\overline{c}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^f + \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\alpha}_{ijl}L_j^f M_l^f$$
$$+ \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\beta}_{ijl}\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}L_j^f M_l^f\Bigg] + \frac{a_i^+L_1}{1-e^{-a_{i*}}}\}\Bigg\} \parallel \varphi \parallel_E$$
$$= \widehat{p} \parallel \varphi \parallel_E,$$

where $\widehat{p} < 1$, it implies that $\Gamma_\varphi(.) \in B$. So, the mapping Γ is a self-mapping from B to B. Next, we prove that the mapping Γ is a contraction mapping of the B. In fact, in view of (H1), $\forall \phi, \psi \in B$, we have

$$|\left(\Gamma_\phi(t) - \Gamma_\psi(t)\right)_i|$$
$$= |\int_{-\infty}^{t} e^{-\int_s^t a_i(u)\mathrm{d}u}[a_i(s)\int_{s-\rho(s)}^{s}(\phi_i'(m) - \psi'(m))\mathrm{d}m$$
$$+ \sum_{j=1}^{n} b_{ij}(s)(f_j(\phi_j'(s-\tau_{ij}(s))) - f_j(\psi_j'(s-\tau_{ij}(s))))$$
$$+ \sum_{j=1}^{n} c_{ij}(s)\int_0^\infty d_{ij}(u)(f_j(\phi_j'(s-u)) - f_j(\psi_j'(s-u)))\mathrm{d}u$$
$$+ \sum_{j=1}^{n}\sum_{l=1}^{n}\alpha_{ijl}(s)(f_j(\phi_j(s-\sigma_{ij}(s)))f_l(\phi_l(s-v_{ij}(s)))$$
$$- f_j(\psi_j(s-\sigma_{ij}(s)))f_l(\psi_l(s-v_{ij}(s))))$$
$$+ \sum_{j=1}^{n}\sum_{l=1}^{n}\beta_{ijl}(s)(\int_0^\infty h_{ijl}(u)f_j(\phi_j(s-u))\mathrm{d}u$$
$$\int_0^\infty k_{ijl}(u)f_l(\phi_l(s-u))\mathrm{d}u - \int_0^\infty h_{ijl}(u)f_j(\psi_j(s-u))\mathrm{d}u$$
$$\int_0^\infty k_{ijl}(u)f_l(\psi_l(s-u))\mathrm{d}u)]\mathrm{d}s$$
$$+ \sum_{t_k<t} e^{-\int_{t_k}^t a_i(u)\mathrm{d}u}(I_k(\varphi_i(t_k)) - I_k(\psi_i(t_k)))\ |$$
$$\leq \int_{-\infty}^{t} e^{-(t-s)a_{i*}}[a_i^+\rho^+ \parallel \phi' - \psi' \parallel_\infty + \sum_{j=1}^{n}\overline{b}_{ij}L_j^{f'} \parallel \phi - \psi' \parallel_\infty$$
$$+ \sum_{j=1}^{n}\overline{c}_{ij}d_{ij}^+L_j^f \parallel \phi' - \psi' \parallel_\infty$$
$$+ \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\alpha}_{ijl}\ |\ f_j(\phi_j(s-\sigma_{ij}(s)))f_l(\phi_l(s-v_{ij}(s)))$$
$$- f_j(\psi_j(s-\sigma_{ij}(s)))f_l(\phi_l(s-v_{ij}(s)))$$
$$+ f_j(\psi_j(s-\sigma_{ij}(s)))f_l(\phi_l(s-v_{ij}(s)))$$
$$- f_j(\psi_j(s-\sigma_{ij}(s)))f_l(\psi_l(s-v_{ij}(s)))\ |$$
$$+ \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\beta}_{ijl}\ |\int_0^\infty h_{ijl}(u)f_j(\phi_j(s-u))\mathrm{d}u$$
$$\int_0^\infty k_{ijl}(u)f_l(\phi_l(s-u))\mathrm{d}u$$
$$- \int_0^\infty h_{ijl}(u)f_j(\psi_j(s-u))\mathrm{d}u\int_0^\infty k_{ijl}(u)f_l(\phi_l(s-u))\mathrm{d}u$$
$$+ \int_0^\infty h_{ijl}(u)f_j(\psi_j(s-u))\mathrm{d}u\int_0^\infty k_{ijl}(u)f_l(\phi_l(s-u))\mathrm{d}u$$
$$- \int_0^\infty h_{ijl}(u)f_j(\psi_j(s-u))\mathrm{d}u\int_0^\infty k_{ijl}(u)f_l(\psi_l(s-u))\mathrm{d}u\ |]\mathrm{d}s$$
$$+ \sum_{t_k<t} e^{-(t-t_k)a_{i*}}\ |\ I_k(\varphi_i(t_k)) - I_k(\psi_i(t_k))\ |$$

$$\le \int_{-\infty}^{t} e^{-(t-s)a_{i*}}[a_i^+\rho^+ \parallel \phi' - \psi' \parallel_\infty + \sum_{j=1}^{n}\overline{b}_{ij}L_j^f \parallel \phi' - \psi' \parallel_\infty + \sum_{j=1}^{n}\overline{c}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^f \parallel \phi' - \psi' \parallel_\infty + \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\alpha}_{ijl}(L_j^f M_l^f + M_j^f L_l^f) \parallel \phi - \psi \parallel_\infty + \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\beta}_{ijl}\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}(L_j^f M_l^f + M_j^f L_l^f) \parallel \phi - \psi \parallel_\infty]\mathrm{d}s + \sum_{t_k<t} e^{-(t-t_k)a_{i*}}L_1 \parallel \phi - \psi \parallel_\infty$$

$$\le \{a_{i*}^{-1}[a_i^+\rho^+ + \sum_{j=1}^{n}\overline{b}_{ij}L_j^f + \sum_{j=1}^{n}\overline{c}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^f + \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\alpha}_{ijl}(L_j^f M_l^f + M_j^f L_l^f) + \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\beta}_{ijl}\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}(L_j^f M_l^f + M_j^f L_l^f)] + \frac{L_1}{1-e^{-a_{i*}}}\} \parallel \phi - \psi \parallel_E,$$

On the other hand

$$\begin{aligned}
&| \left(\Gamma_\phi(t) - \Gamma_\psi(t)\right)_i'| \\
&= | [a_i(t)\int_{t-\rho(t)}^{t}(\phi_i'(m) - \psi'(m))\mathrm{d}m \\
&+ \sum_{j=1}^{n} b_{ij}(t)(f_j(\phi_j'(t-\tau_{ij}(t))) - f_j(\psi_j'(t-\tau_{ij}(t)))) \\
&+ \sum_{j=1}^{n} c_{ij}(t)\int_0^\infty d_{ij}(u)(f_j(\phi_j'(t-u)) - f_j(\psi_j'(t-u)))\mathrm{d}u \\
&+ \sum_{j=1}^{n}\sum_{l=1}^{n}\alpha_{ijl}(t)(f_j(\phi_j(t-\sigma_{ij}(t)))f_l(\phi_l(t-v_{ij}(t))) \\
&- f_j(\psi_j(t-\sigma_{ij}(t)))f_l(\psi_l(t-v_{ij}(t)))) \\
&+ \sum_{j=1}^{n}\sum_{l=1}^{n}\beta_{ijl}(t)(\int_0^\infty h_{ijl}(u)f_j(\phi_j(t-u))\mathrm{d}u \\
&\times \int_0^\infty k_{ijl}(u)f_l(\phi_l(t-u))\mathrm{d}u - \int_0^\infty h_{ijl}(u)f_j(\psi_j(t-u))\mathrm{d}u \\
&\times \int_0^\infty k_{ijl}(u)f_l(\psi_l(t-u))\mathrm{d}u)] - a_i(t)\int_{-\infty}^{t} e^{-\int_s^t a_i(u)\mathrm{d}u}[a_i(s) \\
&\times \int_{s-\rho(s)}^{s}(\phi_i'(m) - \psi'(m))\mathrm{d}m \\
&+ \sum_{j=1}^{n} b_{ij}(s)(f_j(\phi_j'(s-\tau_{ij}(s))) - f_j(\psi_j'(s-\tau_{ij}(s)))) \\
&+ \sum_{j=1}^{n} c_{ij}(s)\int_0^\infty d_{ij}(u)(f_j(\phi_j'(s-u)) - f_j(\psi_j'(s-u)))\mathrm{d}u \\
&+ \sum_{j=1}^{n}\sum_{l=1}^{n}\alpha_{ijl}(s)(f_j(\phi_j(s-\sigma_{ij}(s)))f_l(\phi_l(s-v_{ij}(s))) \\
&- f_j(\psi_j(s-\sigma_{ij}(s)))f_l(\psi_l(s-v_{ij}(s)))) \\
&+ \sum_{j=1}^{n}\sum_{l=1}^{n}\beta_{ijl}(s)(\int_0^\infty h_{ijl}(u)f_j(\phi_j(s-u))\mathrm{d}u \\
&\times \int_0^\infty k_{ijl}(u)f_l(\phi_l(s-u))\mathrm{d}u - \int_0^\infty h_{ijl}(u)f_j(\psi_j(s-u))\mathrm{d}u \\
&\times \int_0^\infty k_{ijl}(u)f_l(\psi_l(s-u))\mathrm{d}u)]\mathrm{d}s \\
&- a_i(t)\sum_{t_k<t} e^{-\int_{t_k}^{t} a_i(u)\mathrm{d}u}(I_k(\varphi_i(t_k)) - I_k(\psi_i(t_k))) |
\end{aligned}$$

$$\begin{aligned}
&\le [a_i^+\rho^+ \parallel \phi' - \psi' \parallel_\infty + \sum_{j=1}^{n}\overline{b}_{ij}L_j^f \parallel \phi' - \psi' \parallel_\infty \\
&+ \sum_{j=1}^{n}\overline{c}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^f \parallel \phi' - \psi' \parallel_\infty \\
&+ \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\alpha}_{ijl} | f_j(\phi_j(t-\sigma_{ij}(t)))f_l(\phi_l(t-v_{ij}(t))) \\
&- f_j(\psi_j(s-\sigma_{ij}(t)))f_l(\phi_l(t-v_{ij}(t))) \\
&+ f_j(\psi_j(t-\sigma_{ij}(t)))f_l(\phi_l(t-v_{ij}(t))) \\
&- f_j(\psi_j(t-\sigma_{ij}(t)))f_l(\psi_l(t-v_{ij}(t))) | \\
&+ \sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\beta}_{ijl} | \int_0^\infty h_{ijl}(u)f_j(\phi_j(t-u))\mathrm{d}u \\
&\times \int_0^\infty k_{ijl}(u)f_l(\phi_l(t-u))\mathrm{d}u - \int_0^\infty h_{ijl}(u)f_j(\psi_j(t-u))\mathrm{d}u \\
&\times \int_0^\infty k_{ijl}(u)f_l(\phi_l(t-u))\mathrm{d}u + \int_0^\infty h_{ijl}(u)f_j(\psi_j(t-u))\mathrm{d}u \\
&\times \int_0^\infty k_{ijl}(u)f_l(\phi_l(t-u))\mathrm{d}u - \int_0^\infty h_{ijl}(u)f_j(\psi_j(t-u))\mathrm{d}u \\
&\times \int_0^\infty k_{ijl}(u)f_l(\psi_l(t-u))\mathrm{d}u |] \\
&+ a_i^+\int_{-\infty}^{t} e^{-(t-s)a_{i*}}[a_i^+\rho^+ \parallel \phi' - \psi' \parallel_\infty \\
&+ \sum_{j=1}^{n}\overline{b}_{ij}L_j^f \parallel \phi' - \psi' \parallel_\infty + \sum_{j=1}^{n}\overline{c}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^f \parallel \phi' - \psi' \parallel_\infty
\end{aligned}$$

$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\alpha}_{ijl}\mid f_j(\phi_j(s-\sigma_{ij}(s)))f_l(\phi_l(s-v_{ij}(s)))$$
$$-f_j(\psi_j(s-\sigma_{ij}(s)))f_l(\phi_l(s-v_{ij}(s)))$$
$$+f_j(\psi_j(s-\sigma_{ij}(s)))f_l(\phi_l(s-v_{ij}(s)))$$
$$-f_j(\psi_j(s-\sigma_{ij}(s)))f_l(\psi_l(s-v_{ij}(s)))\mid$$
$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\beta}_{ijl}\mid\int_0^\infty h_{ijl}(u)f_j(\phi_j(s-u))\mathrm{d}u$$
$$\int_0^\infty k_{ijl}(u)f_l(\phi_l(s-u))\mathrm{d}u-\int_0^\infty h_{ijl}(u)f_j(\psi_j(s-u))\mathrm{d}u$$
$$\int_0^\infty k_{ijl}(u)f_l(\phi_l(s-u))\mathrm{d}u+\int_0^\infty h_{ijl}(u)f_j(\psi_j(s-u))\mathrm{d}u$$
$$\int_0^\infty k_{ijl}(u)f_l(\phi_l(s-u))\mathrm{d}u-\int_0^\infty h_{ijl}(u)f_j(\psi_j(s-u))\mathrm{d}u$$
$$\int_0^\infty k_{ijl}(u)f_l(\psi_l(s-u))\mathrm{d}u\mid]\mathrm{d}s$$
$$+a_i^+\sum_{t_k<t}e^{-(t-t_k)a_{i*}}\mid I_k(\varphi_i(t_k))-I_k(\psi_i(t_k))\mid$$
$$\le[a_i^+\rho^+\parallel\phi'-\psi'\parallel_\infty+\sum_{j=1}^{n}\overline{b}_{ij}L_j^f\parallel\phi'-\psi'\parallel_\infty$$
$$+\sum_{j=1}^{n}\overline{c}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^f\parallel\phi'-\psi'\parallel_\infty$$
$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\alpha}_{ijl}(L_j^fM_l^f+M_j^fL_l^f)\parallel\phi-\psi\parallel_\infty$$
$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\beta}_{ijl}\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}(L_j^fM_l^f+M_j^fL_l^f)\parallel\phi-\psi\parallel_\infty]$$
$$+a_i^+\int_{-\infty}^{t}e^{-(t-s)a_{i*}}[a_i^+\rho^+\parallel\phi'-\psi'\parallel_\infty$$
$$+\sum_{j=1}^{n}\overline{b}_{ij}L_j^f\parallel\phi'-\psi'\parallel_\infty+\sum_{j=1}^{n}\overline{c}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^f\parallel\phi'-\psi'\parallel_\infty$$
$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\alpha}_{ijl}(L_j^fM_l^f+M_j^fL_l^f)\parallel\phi-\psi\parallel_\infty$$
$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\beta}_{ijl}\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}(L_j^fM_l^f+M_j^fL_l^f)\parallel\phi-\psi\parallel_\infty]\mathrm{d}s$$
$$+a_i^+\sum_{t_k<t}e^{-(t-t_k)a_{i*}}L_1\parallel\phi-\psi\parallel_\infty$$
$$\le\{(1+\frac{a_i^+}{a_{i*}})[a_i^+\rho^++\sum_{j=1}^{n}\overline{b}_{ij}L_j^f+\sum_{j=1}^{n}\overline{c}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^f$$
$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\alpha}_{ijl}(L_j^fM_l^f+M_j^fL_l^f)$$
$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\beta}_{ijl}\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}(L_j^fM_l^f+M_j^fL_l^f)]$$
$$+\frac{a_i^+L_1}{1-e^{-a_{i*}}}\}\parallel\phi-\psi\parallel_E,$$

where $i=1,2,\ldots,n$. It follows that

$$\parallel\Gamma_\phi-\Gamma_\psi\parallel_E\le\widehat{q}\parallel\phi-\psi\parallel_E$$

where

$$\widehat{q}=\max_{1\le i\le n}\max\Bigg\{\Bigg\{a_{i*}^{-1}[a_i^+\rho^++\sum_{j=1}^{n}\overline{b}_{ij}L_j^f+\sum_{j=1}^{n}\overline{c}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^f$$
$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\alpha}_{ijl}(L_j^fM_l^f+M_j^fL_l^f)$$
$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\beta}_{ijl}\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}(L_j^fM_l^f+M_j^fL_l^f)]+\frac{L_1}{1-e^{-a_{i*}}}\Bigg\},$$
$$\Bigg\{(1+\frac{a_i^+}{a_{i*}})[a_i^+\rho^++\sum_{j=1}^{n}\overline{b}_{ij}L_j^f$$
$$+\sum_{j=1}^{n}\overline{c}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^f+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\alpha}_{ijl}(L_j^fM_l^f+M_j^fL_l^f)$$
$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\overline{\beta}_{ijl}\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}(L_j^fM_l^f+M_j^fL_l^f)]+\frac{a_i^+L_1}{1-e^{-a_{i*}}}\Bigg\}\Bigg\}<1$$

It is clear that the mapping Γ is a contraction. Therefore the mapping Γ possesses a unique fixed point $z^*\in B$, $\Gamma(z^*)=z^*$. By (7), z^* satisfies (1). So z^* is a piecewise differentiable pseudo-almost periodic solution of system (1) in the region B. The proof is now complete. □

3 Exponential stability of piecewise differentiable pseudo-almost periodic solution

To study the exponential stability of (1), we need the following lemma and notations. So, for a continuous function $g(t)$, we denote $\overline{g}(t)=\sup_{t-\tau^+\le s\le t}\mid g(s)\mid$.

(H6) Assume that there exist positive constants p_i and q_i, such that

$$p_ia_i(t)-q_ia_i^+\rho^+-\sum_{j=1}^{n}q_j\mid b_{ij}(t)\mid L_j^f-\sum_{j=1}^{n}q_j\mid c_{ij}(t)\mid\frac{d_{ij}^+}{\eta_d}L_j^f$$
$$-\sum_{j=1}^{n}\sum_{l=1}^{n}\mid\alpha_{ijl}(t)\mid[p_jL_j^fM_l^f+p_lM_j^fL_l^f]$$
$$-\sum_{j=1}^{n}\sum_{l=1}^{n}\mid\beta_{ijl}(t)\mid\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}[p_jL_j^fM_l^f+p_lM_j^fL_l^f]>0,$$

$$q_i-p_ia_i(t)-q_ia_i^+\rho^+-\sum_{j=1}^{n}q_j\mid b_{ij}(t)\mid L_j^f-\sum_{j=1}^{n}q_j\mid c_{ij}(t)\mid\frac{d_{ij}^+}{\eta_d}L_j^f$$
$$-\sum_{j=1}^{n}\sum_{l=1}^{n}\mid\alpha_{ijl}(t)\mid[p_jL_j^fM_l^f+p_lM_j^fL_l^f]$$
$$-\sum_{j=1}^{n}\sum_{l=1}^{n}\mid\beta_{ijl}(t)\mid\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}[p_jL_j^fM_l^f+p_lM_j^fL_l^f]>0,$$

for $t\in[0,\infty), i=1,2,\ldots,n$

Lemma 8 *Let $\tau\ge0$ be a given real constant. Assume that $p(t)$ and $q_i(t)(i=1,2)$ be continuous functions on $[0,+\infty)$, $k(s)$ be nonnegative function on $[0,+\infty)$ and* *satisfies that* $\int_0^{+\infty} k(s)\mathrm{d}s \le k$ *and* $\int_0^{+\infty} k(s)e^{\mu s}\mathrm{d}s \le +\infty$ *for positive constant* μ.

Moreover, assume that there exist positive constants η *and* M *such that*

$$p(t) - q_1(t) - kq_2(t) \ge \eta > 0,\, 0 \le q_1(t) \le M,\, 0 \le q_2(t) \le M, \quad \forall t \ge 0,$$

then

$$\lambda^* = \inf_{t \ge 0}\{\lambda > 0, \lambda - p(t) + q_1(t)e^{\lambda\tau} + q_2(t)\int_0^{+\infty} k(s)e^{\lambda s}\mathrm{d}s = 0\} > 0.$$

Proof Consider the following equation:

$$G(\lambda) = \lambda - p(t) + q_1(t)e^{\lambda\tau} + q_2(t)\int_0^{+\infty} k(s)e^{\lambda s}\mathrm{d}s. \tag{6}$$

Because

$$G(0) = -p(t) + q_1(t) + kq_2(t) < 0,$$

$$\frac{dG}{d\lambda} = 1 + q_1(t)\tau e^{\lambda\tau} + q_2(t)\int_0^{+\infty} k(s)se^{\lambda s}\mathrm{d}s > 0$$

and $G(+\infty) > 0$, we follow that $G(\lambda)$ is a strictly monotone increasing function.

Therefore, for any $t \ge 0$, there is a unique positive $\lambda(t)$ such that

$$\lambda(t) - p(t) + q_1(t)e^{\lambda(t)\tau} + q_2(t)\int_0^{+\infty} k(s)e^{\lambda(t)s}\mathrm{d}s = 0.$$

Moreover, λ^* exists and $\lambda^* \ge 0$.

Now, we will prove $\lambda^* > 0$. Suppose this is not true. Pick $\varepsilon \in (0, \mu)$ such that $\varepsilon < \{\frac{\eta}{3}, \frac{1}{\tau}\ln(1 + \frac{\eta}{3M})\}$ and $\int_0^{+\infty} k(s)e^{\varepsilon s}\mathrm{d}s \le k + \frac{\eta}{3M}$. Then there exist $t^* > 0$ such that $\lambda^*(t^*) < \varepsilon$ and

$$\lambda^*(t^*) - p(t^*) + q_1(t^*)e^{\lambda^*(t^*)\tau} + q_2(t^*)\int_0^{+\infty} k(s)e^{\lambda^*(t^*)s}\mathrm{d}s = 0.$$

Now we have

$$\begin{aligned} 0 &= \lambda^*(t^*) - p(t^*) + q_1(t^*)e^{\lambda^*(t^*)\tau} + q_2(t^*)\int_0^{+\infty} k(s)e^{\lambda^*(t^*)s}\mathrm{d}s \\ &< \lambda^*(t^*) - p(t^*) + q_1(t^*)e^{\lambda^*(t^*)\tau} + q_2(t^*)\int_0^{+\infty} k(s)e^{\varepsilon s}\mathrm{d}s \\ &< \varepsilon - p(t^*) + q_1(t^*)\left(1 + \frac{\eta}{3M}\right) + q_2(t^*)\left(k + \frac{\eta}{3M}\right) \\ &< \frac{\eta}{3} - (p(t^*) - q_1(t^*) - kq_2(t^*)) + \left(q_1(t^*) + q_2(t^*)\right)\frac{\eta}{3M}\Big) \\ &\quad + q_2(t^*)\left(k + \frac{\eta}{3M}\right) \\ &< \frac{\eta}{3} - \eta + \frac{2\eta}{3} = 0, \end{aligned}$$

which is a contradiction. Hence, $\lambda^* > 0$. The proof of this lemma is completed. □

Then we have

Lemma 9 *Assume that (H1)–(H6) hold and there exist nonnegative vector functions* $(V_1(t), \ldots, V_n(t))^T$ *and* $(W_1(t), \ldots, W_n(t))^T \in PC([-\rho^+, 0], \mathbb{R}^n)$, *where* $V_i(t)$ *is continuous at* $t \ne t_k$ $(k \in \mathbb{N}^*)$, *such that*

$$\begin{aligned} D^- V_i(t^-) \le &- a_i(t)V_i(t^-) + a_i(t)\int_{t^- - \rho(t^-)}^{t^-} W_i(s)\mathrm{d}s \\ &+ \sum_{j=1}^n |b_{ij}(t)| L_j^f \overline{W}_j(t^-) \\ &+ \sum_{j=1}^n |c_{ij}(t)| \int_0^\infty |d_{ij}(u)| L_j^f W_j(t^- - u)\mathrm{d}u \\ &+ \sum_{j=1}^n \sum_{l=1}^n |\alpha_{ijl}(t)| [L_j^f \overline{V}_j(t^-)M_l^f + M_j^f L_l^f \overline{V}_l(t^-)] \\ &+ \sum_{j=1}^n \sum_{l=1}^n |\beta_{ijl}(t)| [\int_0^\infty |h_{ijl}(u)| \\ &L_j^f V_j(t^- - u)\mathrm{d}u \frac{k_{ijl}^+}{\eta_k} M_l^f \\ &+ \frac{h_{ijl}^+}{\eta_h} M_j^f \int_0^\infty |k_{ijl}(u)| L_l^f V_l(t^- - u)\mathrm{d}u], \end{aligned} \tag{7}$$

$$\begin{aligned} W_i(t^+) \le &\, a_i(t)V_i(t^+) + a_i(t)\int_{t^+ - \rho(t^+)}^{t^+} W_i(s)\mathrm{d}s \\ &+ \sum_{j=1}^n |b_{ij}(t)| L_j^f \overline{W}_j(t^+) \\ &+ \sum_{j=1}^n |c_{ij}(t)| \int_0^\infty |d_{ij}(u)| L_j^f W_j(t^+ - u)\mathrm{d}u \\ &+ \sum_{j=1}^n \sum_{l=1}^n |\alpha_{ijl}(t)| [L_j^f \overline{V}_j(t^+)M_l^f + M_j^f L_l^f \overline{V}_l(t^+)] \\ &+ \sum_{j=1}^n \sum_{l=1}^n |\beta_{ijl}(t)| [\int_0^\infty |h_{ijl}(u)| \\ &L_j^f V_j(t^+ - u)\mathrm{d}u \frac{k_{ijl}^+}{\eta_k} M_l^f \\ &+ \frac{h_{ijl}^+}{\eta_h} M_j^f \int_0^\infty |k_{ijl}(u)| L_l^f V_l(t^+ - u)\mathrm{d}u], \end{aligned} \tag{8}$$

$$V_i(t_k^+) \le L_1 V_i(t^+) \tag{9}$$

for $t > 0, i = 1, 2, \ldots, n$ *and* $k \in \mathbb{N}^*$. *Then for all* $t \ge 0$ *and* $i = 1, 2, \ldots, n$, *there exists a positive constant* $\widetilde{L}$ *such that*

$$V_i(t) \le \widetilde{L}\sum_{l=1}^n \max\{\overline{V}_l(0), \overline{W}_l(0)\}e^{-\lambda^* t}, \tag{10}$$

where λ^* is defined, respectively, as

$$\lambda^* = \min\{\lambda_i^*, \widehat{\lambda}_i^* \mid i = 1, 2, \ldots, n\}, \tag{11}$$

$$\begin{aligned}\lambda_i^* = \inf_{t \geq 0}\{&\lambda(t) > 0,\ \lambda(t) - a_i(t) + \frac{q_i}{p_i} a_i^+ \rho^+ e^{\lambda(t)\rho^+} \\ &+ \sum_{j=1}^n \frac{q_j}{p_i} \mid b_{ij}(t) \mid L_j^f e^{\lambda(t)\tau^+} \\ &+ \sum_{j=1}^n \frac{q_j}{p_i} \mid c_{ij}(t) \mid \int_0^\infty \mid d_{ij}(u) \mid L_j^f e^{\lambda(t)u} \mathrm{d}u \\ &+ \sum_{j=1}^n \sum_{l=1}^n \mid \alpha_{ijl}(t) \mid [\frac{p_j}{p_i} L_j^f M_l^f + \frac{p_l}{p_i} M_j^f L_l^f] e^{\lambda(t)\tau^+} \\ &+ \sum_{j=1}^n \sum_{l=1}^n \mid \beta_{ijl}(t) \mid [\frac{p_j}{p_i} \int_0^\infty \mid h_{ijl}(u) \mid L_j^f e^{\lambda(t)u} \mathrm{d}u \frac{k_{ijl}^+}{\eta_k} M_l^f \\ &+ \frac{p_l}{p_i} \frac{h_{ijl}^+}{\eta_h} M_j^f \int_0^\infty \mid k_{ijl}(u) \mid L_l^f e^{\lambda(t)u} \mathrm{d}u] = 0\} > 0,\end{aligned} \tag{12}$$

$$\begin{aligned}\widehat{\lambda}_i^* = \inf_{t \geq 0}\{&\lambda(t) > 0,\ -q_i + a_i(t) p_i + q_i a_i^+ \rho^+ e^{\lambda(t)\rho^+} \\ &+ \sum_{j=1}^n q_j \mid b_{ij}(t) \mid L_j^f e^{\lambda(t)\tan^+} \\ &+ \sum_{j=1}^n q_j \mid c_{ij}(t) \mid \int_0^\infty \mid d_{ij}(u) \mid L_j^f e^{\lambda(t)u} \mathrm{d}u \\ &+ \sum_{j=1}^n \sum_{l=1}^n \mid \alpha_{ijl}(t) \mid [p_j L_j^f M_l^f + p_l M_j^f L_l^f] e^{\lambda(t)\tau^+} \\ &+ \sum_{j=1}^n \sum_{l=1}^n \mid \beta_{ijl}(t) \mid [p_j \int_0^\infty \mid h_{ijl}(u) \mid L_j^f e^{\lambda(t)u} \mathrm{d}u \frac{k_{ijl}^+}{\eta_k} M_l^f \\ &+ p_l \frac{h_{ijl}^+}{\eta_h} M_j^f \int_0^\infty \mid k_{ijl}(u) \mid L_l^f e^{\lambda(t)u} \mathrm{d}u] = 0\} > 0,\end{aligned} \tag{13}$$

Proof By the similar analysis in Lemma 8, we can deduce that $\lambda_i^* > 0$ and $\widehat{\lambda}_i^* > 0$ exist uniquely.

Choose a positive constant θ such that

$$\min\{p_i, q_i | i = 1, 2, \ldots, n\}\theta > 1.$$

Let

$$\Phi_i(t) = \max\left\{\frac{1}{p_i} V_i(t), \frac{1}{q_i} W_i(t)\right\}, \quad i = 1, 2, \ldots, n,$$

$$\Psi(t) = \theta \sum_{l=1}^n \max\{\overline{V}_l(0), \overline{W}_l(0)\} e^{-\lambda^* t}. \tag{14}$$

Then for all $t \in (-\infty, 0]$ and $\gamma > 1$, we have

$$\gamma\Psi(t) = \gamma\theta \sum_{l=1}^n \max\{\overline{V}_l(0), \overline{W}_l(0)\} e^{-\lambda^* t} > \Phi_i(t). \tag{15}$$

Then

$$\Phi_i(t) < \gamma\Psi(t),\ t \in [0, \infty),\ i = 1, 2, \ldots, n. \tag{16}$$

For the sake of contradiction, assume that there exist $i \in \{1, 2, \ldots, n\}$ and $\bar{t} > 0$ such that

$$\Phi_i(\bar{t}^+) \geq \gamma\Psi(\bar{t}), \Phi_j(t) < \gamma\Psi(t), \mathit{for}\ \ t \in [0, \bar{t}), j \in \{1, 2, \ldots, n\}. \tag{17}$$

Then we have the following

(I) $(1/p_i)V_i(\bar{t}^+) \geq \gamma\Psi(\bar{t})$ then we have the following subcases.

(i) $\bar{t} \neq t_k, t_k \in\in \mathbb{N}^*$. So $V_i(t)$ is continuous at $\bar{t}$. By 17, we have

$$\frac{1}{p_i} V_i(\bar{t}) = \gamma\Psi(\bar{t}),\ \frac{1}{p_i} D^- V_i(\bar{t}) > \gamma\Psi'(\bar{t}) \tag{18}$$

From (H6), (17) and the definition of λ^*, we have

$$\begin{aligned}&\frac{1}{p_i} D^- V_i(\bar{t}) - \gamma\Psi'(\bar{t}) \\ &\leq -a_i(t)\gamma\Psi(\bar{t}) \\ &\quad + \frac{q_i}{p_i} a_i(\bar{t}) \int_{\bar{t}-\rho(\bar{t})}^{\bar{t}} \gamma\Psi(s)\mathrm{d}s + \sum_{j=1}^n \frac{q_j}{p_i} \mid b_{ij}(t) \mid L_j^f \gamma\Psi(\bar{t} - \tau^+) \\ &\quad + \sum_{j=1}^n \frac{q_j}{p_i} \mid c_{ij}(\bar{t}) \mid \int_0^\infty \mid d_{ij}(u) \mid L_j^f \gamma\Psi(\bar{t} - u)\mathrm{d}u \\ &\quad + \sum_{j=1}^n \sum_{l=1}^n \mid \alpha_{ijl}(\bar{t}) \mid [\frac{p_j}{p_i} L_j^f M_l^f + \frac{p_l}{p_i} M_j^f L_l^f] \gamma\Psi(\bar{t} - \tau^+) \\ &\quad + \sum_{j=1}^n \sum_{l=1}^n \mid \beta_{ijl}(\bar{t}) \mid [\frac{p_j}{p_i} \int_0^\infty h_{ijl}(u) \mid L_j^f \gamma\Psi(\bar{t} - u)\mathrm{d}u \frac{k_{ijl}^+}{\eta_k} M_l^f \\ &\quad + \frac{p_l}{p_i} \frac{h_{ijl}^+}{\eta_h} M_j^f \int_0^\infty k_{ijl}(u) \mid L_l^f \gamma\Psi(\bar{t} - u)\mathrm{d}u] + \lambda^*\gamma\Psi(\bar{t}) \\ &\leq \gamma\Psi(\bar{t})(\lambda^* - a_i(t) + \frac{q_i}{p_i} a_i^+ \rho^+ e^{\lambda^*\rho^+} + \sum_{j=1}^n \frac{q_j}{p_i} \mid b_{ij}(t) \mid L_j^f e^{\lambda^*\tau^+} \\ &\quad + \sum_{j=1}^n \frac{q_j}{p_i} \mid c_{ij}(\bar{t}) \mid \int_0^\infty \mid d_{ij}(u) \mid L_j^f e^{\lambda^* u}\mathrm{d}u \\ &\quad + \sum_{j=1}^n \sum_{l=1}^n \mid \alpha_{ijl}(\bar{t}) \mid [\frac{p_j}{p_i} L_j^f M_l^f + \frac{p_l}{p_i} M_j^f L_l^f] e^{\lambda^*\tau^+} \\ &\quad + \sum_{j=1}^n \sum_{l=1}^n \mid \beta_{ijl}(\bar{t}) \mid [\frac{p_j}{p_i} \int_0^\infty \mid h_{ijl}(u) \mid L_j^f e^{\lambda^* u}\mathrm{d}u \frac{k_{ijl}^+}{\eta_k} M_l^f \\ &\quad + \frac{p_l}{p_i} \frac{h_{ijl}^+}{\eta_h} M_j^f \int_0^\infty \mid k_{ijl}(u) \mid L_l^f e^{\lambda^* u}\mathrm{d}u]) < 0,\end{aligned} \tag{19}$$

which is a contradiction with (18).

(ii) There exists $k_0 \in \mathbb{N}^*$ such that $\bar{t} = t_k$. By (17), we have

$$\frac{1}{p_i} V_i(\bar{t}) \leq \gamma\Psi(\bar{t}) \leq \frac{1}{p_i} V_i(\bar{t}^+). \tag{20}$$

Noting $\frac{1}{p_i}V_i(\bar{t}^-) \neq \frac{1}{p_i}V_i(\bar{t}^+)$, we have $\frac{1}{p_i}V_i(\bar{t}^-) < \gamma\Psi(\bar{t})$ or $\gamma\Psi(\bar{t}) < \frac{1}{p_i}V_i(\bar{t}^+)$. Without loss of generality, we assume that $\gamma\Psi(\bar{t}) < \frac{1}{p_i}V_i(\bar{t}^+)$. from (9) and (20) we get that

$$\gamma\Psi(\bar{t}) < \frac{1}{p_i}V_i(\bar{t}^+) \leq \gamma L_1 \Psi(\bar{t}). \tag{21}$$

Simplifying (21), we obtain $L_1 > 1$, which contradict that $L_1 < 1$.

If (I) does not hold, then

(II)

$$\begin{aligned}&\frac{1}{q_i}W_i(\bar{t}^+) \geq \gamma\Psi(\bar{t}),\ \frac{1}{q_j}W_j(t) < \gamma\Psi(t),\\ &\frac{1}{p_j}W_j(t) \geq \gamma\Psi(t), fort \in [0,\bar{t}),\ j \in \mathbb{N}.\end{aligned} \tag{22}$$

Then from (8) and (H6) we have

$$\begin{aligned}0 \leq &-W_i(\bar{t}^+) + a_i(t)V_i(\bar{t}^+) + a_i(t^+)\int_{\bar{t}^+-\rho(t^+)}^{t^+} W_i(s)\mathrm{d}s\\ &+\sum_{j=1}^n |b_{ij}(t^+)| L_j^f \overline{W}_j(\bar{t}^+)\\ &+\sum_{j=1}^n |c_{ij}(t^+)| \int_0^\infty |d_{ij}(u)| L_j^f W_j(\bar{t}^+-u)\mathrm{d}u\\ &+\sum_{j=1}^n\sum_{l=1}^n |\alpha_{ijl}(t^+)| [L_j^f\overline{V}_j(\bar{t}^+)M_l^f + M_j^f L_l^f \overline{V}_l(\bar{t}^-)]\\ &+\sum_{j=1}^n\sum_{l=1}^n |\beta_{ijl}(t^+)| [\int_0^\infty |h_{ijl}(u)| L_j^f V_j(\bar{t}^+-u)\mathrm{d}u\frac{k_{ijl}^+}{\eta_k}M_l^f\\ &+\frac{h_{ijl}^+}{\eta_h}M_j^f\int_0^\infty |k_{ijl}(u)| L_l^f V_l(\bar{t}^+-u)\mathrm{d}u]\\ \leq &\gamma\Psi(\bar{t})(-q_i + a_i(t)p_i + q_i a_i^+\rho^+ e^{\lambda^*\rho^+} + \sum_{j=1}^n q_j |b_{ij}(t^+)| L_j^f e^{\lambda^*\tau^+}\\ &+\sum_{j=1}^n q_j |c_{ij}(t^+)| \int_0^\infty |d_{ij}(u)| L_j^f e^{\lambda^* u}\mathrm{d}u\\ &+\sum_{j=1}^n\sum_{l=1}^n |\alpha_{ijl}(t^+)| [p_j L_j^f M_l^f + p_l M_j^f L_l^f]e^{\lambda^*\tau^+}\\ &+\sum_{j=1}^n\sum_{l=1}^n |\beta_{ijl}(t^+)| [p_j\int_0^\infty |h_{ijl}(u)| L_j^f e^{\lambda^* u}\mathrm{d}u\frac{k_{ijl}^+}{\eta_k}M_l^f\\ &+p_l\frac{h_{ijl}^+}{\eta_h}M_j^f\int_0^\infty |k_{ijl}(u)| L_l^f e^{\lambda^* u}\mathrm{d}u]) < 0\end{aligned} \tag{23}$$

which is a contradiction. From (I) and (II), (16) holds. Letting $\gamma \to 1^+$ in (16), we have

$$\Phi_i(t) \leq \gamma\Psi(t),\ t \in [0,\infty),\ i = 1,2,\ldots,n. \tag{24}$$

So $\frac{1}{p_i}V_i(t) \leq \Psi(t)$ for all $t \in [0,\infty)$, $i = 1,2,\ldots,n$. Let $M = \max_{1\leq i\leq n}\{p_i\theta\}$ then for $t \geq 0$ and $i = 1,2,\ldots,n$, we have

$$V_i(t) \leq M\sum_{l=1}^n \max\{\overline{V}_l(0), \overline{W}_l(0)\}e^{-\lambda^* t}, \tag{25}$$

The proof is complete. □

Theorem 4 *Assume that (H1)–(H6) hold, then the unique piecewise differentiable pseudo-almost periodic solution of system (1) is globally exponentially stable.*

Proof It follows from Theorem 3 that system (1) has at least one piecewise differentiable pseudo-almost periodic solution $x^*(t) = (x_1^*(t),\ldots,x_n^*(t))^T \in \mathbb{B}$ with initial value $\phi^*(t)$. Let $x(t) = (x_1(t),\ldots,x_n(t))^T$ be an arbitrary solution of system (1) with initial value $\phi(t)$.

Let $V_i(t) = |x_i(t) - z_i(t)|, W_i(t) = |x_i'(t) - z_i'(t)|$ for $i = 1,\ldots,n$, Then,

$$\begin{aligned}D^-V_i(t^-) \leq &-a_i(t)V_i(t^-) + a_i(t)\int_{t^- -\rho(t^-)}^{t^-} W_i(s)\mathrm{d}s\\ &+\sum_{j=1}^n |b_{ij}(t)| L_j^f\overline{W}_j(t^-)\\ &+\sum_{j=1}^n |c_{ij}(t)| \int_0^\infty |d_{ij}(u)| L_j^f W_j(t^- - u)\mathrm{d}u\\ &+\sum_{j=1}^n\sum_{l=1}^n |\alpha_{ijl}(t)| [L_j^f\overline{V}_j(t^-)M_l^f + M_j^f L_l^f\overline{V}_l(t^-)]\\ &+\sum_{j=1}^n\sum_{l=1}^n |\beta_{ijl}(t)| [\int_0^\infty |h_{ijl}(u)|\\ &L_j^f V_j(t^- - u)\mathrm{d}u\frac{k_{ijl}^+}{\eta_k}M_l^f\\ &+\frac{h_{ijl}^+}{\eta_h}M_j^f\int_0^\infty |k_{ijl}(u)| L_l^f V_l(t^- - u)\mathrm{d}u],\end{aligned} \tag{26}$$

$$\begin{aligned}W_i(t^+) \leq &a_i(t)V_i(t^+) + a_i(t)\int_{t^+ -\rho(t^+)}^{t^+} W_i(s)\mathrm{d}s\\ &+\sum_{j=1}^n |b_{ij}(t^+)| L_j^f\overline{W}_j(t^+)\\ &+\sum_{j=1}^n |c_{ij}(t)| \int_0^\infty |d_{ij}(u)| L_j^f W_j(t^+ - u)\mathrm{d}u\\ &+\sum_{j=1}^n\sum_{l=1}^n |\alpha_{ijl}(t)| [L_j^f\overline{V}_j(t^+)M_l^f + M_j^f L_l^f\overline{V}_l(t^-)]\\ &+\sum_{j=1}^n\sum_{l=1}^n |\beta_{ijl}(t)| [\int_0^\infty |h_{ijl}(u)|\\ &L_j^f V_j(t^+ - u)\mathrm{d}u\frac{k_{ijl}^+}{\eta_k}M_l^f\\ &+\frac{h_{ijl}^+}{\eta_h}M_j^f\int_0^\infty |k_{ijl}(u)| L_l^f V_l(t^+ - u)\mathrm{d}u],\end{aligned} \tag{27}$$

By (9) and (H5) we have

$$V_i(t_k^+) \leq L_1 V_i(t^+), \ \ with \ L_1 < 1. \tag{28}$$

By (26)–(28), (H1)–(H6) and Lemma 9, there exists a positive constant M such that

$$V_i(t) \leq M \sum_{l=1}^{n} \max\{\overline{V}_l(0), \overline{W}_l(0)\} e^{-\lambda^* t}, \tag{29}$$

where λ^* is defined in (11). □

Remark 4 To the best of our knowledge, there have been no results of piecewise pseudo-almost periodic solutions for impulsive neutral high-order Hopfield neural networks with time-varying coefficients, mixed delays and leakage until now. Hence, the obtained results are essentially new and the investigation methods used in this paper can also be applied to study the piecewise pseudo-almost periodic solutions for some other types of neural networks.

Remark 5 If throughout this paper, for $i, j, l = 1, 2, \ldots, n$, it will be assumed that $a_i : \mathbb{R} \longrightarrow \mathbb{R}^+$ is almost periodic functions, $b_{ij}, c_{ij}, \alpha_{ijl}, \beta_{ijl}, J_i : \mathbb{R} \longrightarrow \mathbb{R}$ are almost periodic functions, then the investigation methods used here can also be applied to study the piecewise almost periodic solutions for some other types of impulsive neural networks.

4 Application

Consider the following impulsive neutral high-order Hopfield neural networks with time-varying coefficients, mixed delays and leakage:

$$\begin{cases} x_i'(t) &= -a_i(t)x_i(t-\rho(t)) + \sum_{j=1}^{2} b_{ij}(t) f_j(x_j'(t-\tau_{ij}(t))) \\ & + \sum_{j=1}^{2} c_{ij}(t) \int_0^\infty d_{ij}(u) f_j(x_j'(t-u))\mathrm{d}u \\ & + \sum_{j=1}^{2}\sum_{l=1}^{2} \alpha_{ijl}(t) f_j(x_j(t-\sigma_{ij}(t))) f_l(x_l(t-\nu_{ij}(t))) \\ & + \sum_{j=1}^{2}\sum_{l=1}^{2} \beta_{ijl}(t) \int_0^\infty h_{ijl}(u) f_j(x_j(t-u))\mathrm{d}u \\ & \int_0^\infty k_{ijl}(u) f_l(x_l(t-u))\mathrm{d}u \\ & + I_i(t), \ \ t \in \mathbb{R}, \ t \neq 2k, \ k \in \mathbb{Z} \\ \Delta x_i(t_k) &= x_i(t_k^+) - x_i(t_k^-) = I_k(x_i(t_k)) \end{cases} \tag{30}$$

where

$$a(t) = \begin{pmatrix} 4 + \cos^2(t) \\ 4 + \sin^2(t) \end{pmatrix} \Rightarrow a_{1*} = a_{2*} = 4,$$

for all $t \in \mathbb{R}$

$$f_1(t) = f_2(t) = \sin t \Rightarrow L_1^f = L_2^f = M_1^f = M_2^f = 1,$$

$$\tau_{ij}(t) = \sigma_{ij}(t) = \nu_{ij}(t) = \rho(t) = \frac{1}{80} \mid \sin t \mid, \text{for} i, j \in \{1, 2\}$$

$$d_{ij}(t) = h_{ijl}(t) = k_{ijl}(t) = e^{-t} \Rightarrow \frac{d_{ij}^+}{\eta_d} = \frac{h_{ijl}^+}{\eta_h} = \frac{k_{ijl}^+}{\eta_k} = 1,$$

for $i, j, l \in \{1, 2\}$

$$\begin{pmatrix} \Delta x_1(2k) \\ \Delta x_2(2k) \end{pmatrix} = \begin{pmatrix} -\frac{1}{80}x_1(2k) + \frac{1}{80}\sin(x_1(2k)) + \frac{1}{20} \\ -\frac{1}{80}x_2(2k) + \frac{1}{80}\cos(x_2(2k)) + \frac{1}{30} \end{pmatrix}$$
$$\Rightarrow L = \frac{1}{80}.$$

$$b(t) = \begin{pmatrix} 0.01\cos t + \frac{0.01}{1+t^2} & 0.03\sin t \\ 0.03\sin t + \frac{0.01}{1+t^2} & 0.01\sin\sqrt{2}t \end{pmatrix}$$
$$\Rightarrow \overline{b} = \begin{pmatrix} 0.02 & 0.03 \\ 0.04 & 0.01 \end{pmatrix},$$

$$c(t) = \begin{pmatrix} 0.01\sin t + \frac{0.01}{1+t^2} & 0.03\cos t \\ 0.02\sin t + \frac{0.01}{1+t^2} & 0.01\cos t + \frac{0.01}{1+t^2} \end{pmatrix}$$
$$\Rightarrow \overline{c} = \begin{pmatrix} 0.03 & 0.02 \\ 0.02 & 0.03 \end{pmatrix},$$

$$(\alpha_{1jl}(t))_{1\leq j,l\leq 2} = \begin{pmatrix} 0 & 0.04\sin t + \frac{0.01}{1+t^2} \\ 0 & 0 \end{pmatrix}$$
$$\Rightarrow (\overline{\alpha}_{1jl})_{1\leq j,l\leq 2} = \begin{pmatrix} 0 & 0.05 \\ 0 & 0 \end{pmatrix},$$

$$(\alpha_{2jl}(t))_{1\leq j,l\leq 2} = \begin{pmatrix} 0 & 0.06\cos t + \frac{0.01}{1+t^2} \\ 0 & 0 \end{pmatrix}$$
$$\Rightarrow (\overline{\alpha}_{2jl})_{1\leq j,l\leq 2} = \begin{pmatrix} 0 & 0.07 \\ 0 & 0 \end{pmatrix},$$

$$(\beta_{1jl}(t))_{1\leq j,l\leq 2} = \begin{pmatrix} 0 & 0.05\cos t + \frac{0.01}{1+t^2} \\ 0 & 0 \end{pmatrix}$$
$$\Rightarrow (\overline{\beta}_{1jl})_{1\leq j,l\leq 2} = \begin{pmatrix} 0 & 0.06 \\ 0 & 0 \end{pmatrix},$$

$$(\beta_{2jl}(t))_{1\leq j,l\leq 2} = \begin{pmatrix} 0 & 0.4\sin t + \frac{0.01}{1+t^2} \\ 0 & 0 \end{pmatrix}$$
$$\Rightarrow (\overline{\beta}_{2jl})_{1\leq j,l\leq 2} = \begin{pmatrix} 0 & 0.05 \\ 0 & 0 \end{pmatrix},$$

$$J(t)=\begin{pmatrix}0.8\cos t+\dfrac{0.1}{1+t^2}\\[2mm] 0.7\sin t+\dfrac{0.1}{1+t^2}\end{pmatrix}\Rightarrow \overline{J}=\begin{pmatrix}0.9\\0.8\end{pmatrix}.$$

Then

$$\max_{1\le i\le n}\max\left\{\frac{\overline{J}_i}{a_{i*}},(1+\frac{a_i^+}{a_{i*}})\overline{J}_i\right\}=2.0250=L$$

$$\begin{aligned}\widehat{p}=\max_{1\le i\le 2}\max\Bigg\{&\Bigg\{a_{i*}^{-1}\Bigg[a_i^+\rho^+ +\sum_{j=1}^{2}\overline{b}_{ij}L_j^f+\sum_{j=1}^{2}\overline{c}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^f\\
&+\sum_{j=1}^{2}\sum_{l=1}^{2}\overline{\alpha}_{ijl}L_j^fM_l^f+\sum_{j=1}^{2}\sum_{l=1}^{n}\overline{\beta}_{ijl}\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}L_j^fM_l^f\Bigg]+\frac{L_1}{1-e^{-a_{i*}}}\Bigg\},\\
&\Bigg\{(1+\frac{a_i^+}{a_{i*}})\Bigg[a_i^+\rho^+ +\sum_{j=1}^{2}\overline{b}_{ij}L_j^f\\
&+\sum_{j=1}^{2}\overline{c}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^f+\sum_{j=1}^{2}\sum_{l=1}^{2}\overline{\alpha}_{ijl}L_j^fM_l^f\\
&+\sum_{j=1}^{2}\sum_{l=1}^{2}\overline{\beta}_{ijl}\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}L_j^fM_l^f\Bigg]+\frac{a_i^+L_1}{1-e^{-a_{i*}}}\Bigg\}\Bigg\}\\
=&0.6712<1.\end{aligned}$$

$$\begin{aligned}\widehat{q}=\max_{1\le i\le 2}\max\Bigg\{&\Bigg\{a_{i*}^{-1}\Bigg[a_i^+\rho^+ +\sum_{j=1}^{2}\overline{b}_{ij}L_j^f+\sum_{j=1}^{2}\overline{c}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^f\\
&+\sum_{j=1}^{2}\sum_{l=1}^{2}\overline{\alpha}_{ijl}(L_j^fM_l^f+M_j^fL_l^f)\\
&+\sum_{j=1}^{2}\sum_{l=1}^{2}\overline{\beta}_{ijl}\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}(L_j^fM_l^f+M_j^fL_l^f)\Bigg]+\frac{L_1}{1-e^{-a_{i*}}}\Bigg\},\\
&\Bigg\{(1+\frac{a_i^+}{a_{i*}})\Bigg[a_i^+\rho^+ +\sum_{j=1}^{2}\overline{b}_{ij}L_j^f\\
&+\sum_{j=1}^{2}\overline{c}_{ij}\frac{d_{ij}^+}{\eta_d}L_j^f+\sum_{j=1}^{2}\sum_{l=1}^{2}\overline{\alpha}_{ijl}(L_j^fM_l^f+M_j^fL_l^f)\\
&+\sum_{j=1}^{2}\sum_{l=1}^{2}\overline{\beta}_{ijl}\frac{h_{ijl}^+}{\eta_h}\frac{k_{ijl}^+}{\eta_k}(L_j^fM_l^f+M_j^fL_l^f)\Bigg]+\frac{a_i^+L_1}{1-e^{-a_{i*}}}\Bigg\}\Bigg\}\\
=&0.9412<1.\end{aligned}$$

Let $p_1=p_2=1$ and $q_1=q_2=70$, and from the above assumption, the (H6) is satisfied. Therefore, all conditions from Theorems 3 and 4 are satisfied; then, the impulsive neutral high-order Hopfield neural networks with time-varying coefficients, mixed delays and leakage have a unique piecewise differentiable pseudo-almost periodic solution. Simulation results of Example 30 are depicted in Figs. 1, 2 and 3.

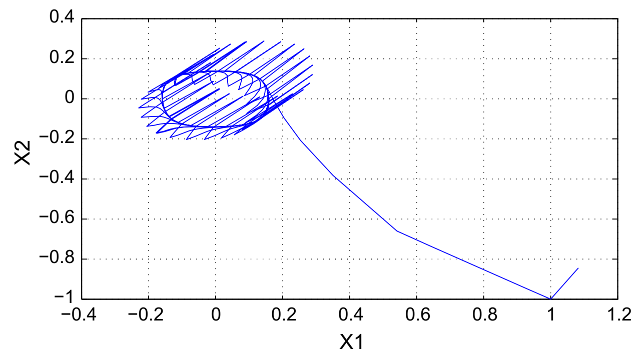

Fig. 1 The orbit of $X1$–$X2$ for the system

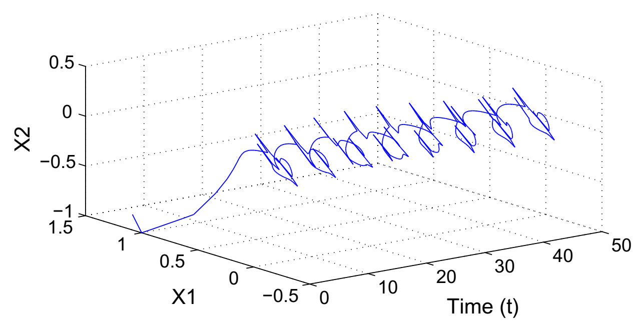

Fig. 2 The phase system for the system

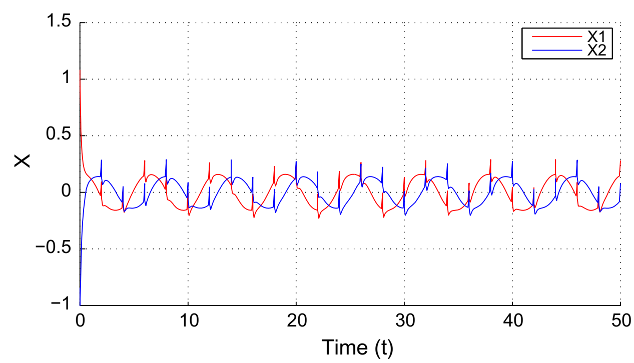

Fig. 3 Transient response of state variables $X1$ and $X2$ for the system

Figures 4 and 5 confirm that the proposed condition in Theorem 4 leads to globally exponentially stable piecewise differentiable pseudo-almost periodic solution for system 30.

5 Conclusion

In this paper we discuss the existence and the exponential stability of piecewise differentiable pseudo-almost periodic solutions for a class of impulsive neutral high-order

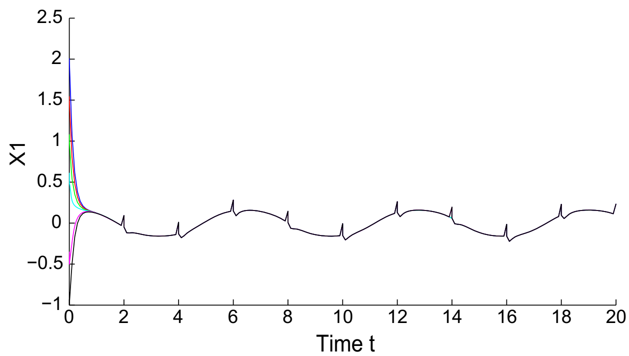

Fig. 4 Global exponential stability of state variables x_1 of system

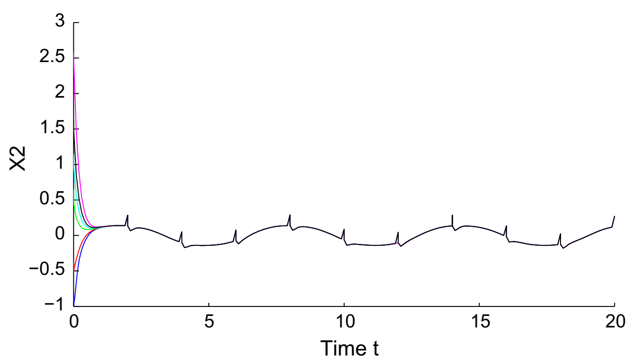

Fig. 5 Global exponential stability of state variables x_2 of system

Hopfield neural networks with mixed time-varying delays and leakage delays. We give several sufficient conditions for the existence and the exponential stability of the solution. The results of this paper are new, and they supplement previously known results. An example is given to illustrate the results.

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