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ORIGINAL ARTICLE

An extension of the ELECTRE approach with multi-valued neutrosophic information

Juan-juan Peng[1,2] · Jian-qiang Wang[2] · Xiao-hui Wu[1,2]

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Abstract In this paper, an extension Elimination and Choice Translating Reality (ELECTRE) method is introduced to handle multi-valued neutrosophic multi-criteria decision-making (MCDM) problems. First of all, some outranking relations for multi-valued neutrosophic numbers (MVNNs), which are based on traditional ELECTRE methods, are defined, and several properties are analyzed. In the next place, an outranking method to deal with MCDM problems similar to ELECTRE III, where weights and data are in the form of MVNNs, is developed. At last, an example is provided to demonstrate the proposed approach and testify its validity and feasibility. This study is supported by the comparison analysis with other existing methods.

Keywords Multi-criteria decision-making · Multi-valued neutrosophic sets · ELECTRE

1 Introduction

In recent years, multi-criteria decision-making (MCDM) method has always been wildly applied in the fields of psychology, artificial intelligence, sociology, data processing and other areas, etc. However, with the increasing uncertainty of problems and the complexity of human's cognitive information, decision-makers experience difficulty expressing a preference when attempting to solve MCDM problems. To solve this issue, Smarandache [1–3] developed the concept of neutrosophic sets (NSs) for the first time, which are the generalization of intuitionistic fuzzy sets (IFSs) initially introduced by Atanassov's [4]. NSs are characterized by a truth-membership, indeterminacy-membership and falsity-membership that express by crisp numbers in $]0^{-}, 1^{+}[$, the nonstandard unit interval. Later, the extensions of NSs', single-valued neutrosophic sets (SNSs) and interval neutrosophic sets (INSs), which are characterized by three numerical values and intervals, respectively, with the range [0, 1], were proposed [5, 6]. Recently, many of researchers have been done on MCDM problems where the evaluation values are in the form of NSs, SNSs and INSs [7–19], including aggregation operators, similarity measures and outranking methods. For example, Ye [7, 8, 12, 17] proposed some aggregation operators of SNSs and the similarity measures between SVNSs and INSs and applied them to solve MCDM problems. Deli and Subas [18] defined a novel ranking method with single-valued neutrosophic numbers (SNNs), Broumi and Deli [19] investigated the correlation measure for the neutrosophic refined sets, and Wu et al. [20] constructed some cross-entropy measures with SNSs and Peng et al. [21] developed some improved operations of SNSs and applied them to handle MCDM problems. Moreover, Liu and Wang [22] investigated the single-valued neutrosophic normalized weighted Bonferroni mean operator, Liu and Liu [23] defined the generalized neutrosophic number generalized weighted power averaging operator, and Liu et al. [24] developed some neutrosophic Hamacher aggregation operators and applied them to solve MCDM problems. Zhang et al. [25] presented a neutrosophic normal cloud and applied them to solve MCDM problems. Moreover, Zhang et al. [26] and Tian et al. [27] put forward an

✉ Jian-qiang Wang
jqwang@csu.edu.cn

1 School of Economics and Management, Hubei University of Automotive Technology, Shiyan 442002, China

2 School of Business, Central South University, Lushan South Road, Changsha 410083, China

MCDM method based on the weighted correlation coefficient and cross-entropy under an interval neutrosophic environment.

Considering some real-life decision situations where the decision-makers may hesitant among several values to evaluate an alternative, Wang and Li [28] and Ye [29] further extended the SNSs to develop the definition of multi-valued neutrosophic sets (MVNSs) and single-valued neutrosophic hesitant fuzzy sets (SVNHFSs) in 2015. Actually, both of MVNSs and SVNHFSs are extensions of SNSs and hesitant fuzzy sets (HFSs) first proposed by Torra [30] and Torra and Narukawa [31] and characterized by a truth-membership, indeterminacy-membership and falsity-membership that represented by a set of numerical numbers with range [0, 1]. Consequently, there exists no difference between MVNSs and SVNHFSs. SVNHFSs are actually MVNSs as well. Moreover, based on the definition of MVNSs, Peng et al. [32] further proposed some operations of MVNSs and multi-valued neutrosophic power aggregation operators and applied them to solve MCGDM problems.

However, those methods with SNSs, INSs and MVNSs always involve operations and measures which impact on the optimal decision could be momentous. The relation model, such as the Elimination and Choice Translating Reality (ELECTRE) methods, utilizes outranking relations or priority functions for ranking the alternatives in terms of priority among the criteria and could avoid these drawbacks. ELECTRE methods, including ELECTRE I, II, III, IV, IS, and TRI [33–36], were defined by Benayoun and Roy [33, 34]. They are always called non-compensatory MCDM methods where the values of performance indices cannot compensate for each other directly. That is, a very poor performance with respect to a criterion should not be justified by its good values in some other criteria. At present, ELECTRE methods can successfully be applied in various domains [37–44]. For instance, Vahdani et al. [41] suggested an extended ELECTRE method to handle MCDM problems where the evaluation values are expressed by interval values. Vahdani and Hadipour [42] presented an extended ELECTRE method to deal with MCDM problems with interval-valued fuzzy information. Peng et al. [10] and Zhang et al. [45] developed an outranking approach for MCDM problems with SNSs and INSs, respectively.

Apparently, the previous studies on ELECTRE methods cannot handle MCDM problems with multi-valued neutrosophic information. Moreover, those methods aforementioned earlier fail to deal with some decision-making problems where the data and criteria are expressed by MVNNs. In particular, if the number of values in three memberships increases, then the use of those methods based on aggregation operators makes the decision-making process very complex and may fail to obtain the distinct ranking results of the alternatives. Thus, the purpose of this paper is to develop a novel outranking method based on ELECTRE III with multi-valued neutrosophic information.

The rest of the article is organized as follows. Section 2 presents the preliminaries of NSs, SNSs and MVNSs. Then some outranking relations on MVNNs are defined, and some valuable properties are also analyzed in Sect. 3. Section 4 contains the extended ELECTRE method to solve the MCDM problems where the data and the weights of criteria are expressed by MVNNs. Section 5 provides an illustrative example and a comparison analysis to demonstrate the proposed approach. Section 6 presents the conclusions of the paper and the further research.

2 Preliminaries

In this section, a brief overview of the concepts of NSs, SNSs and MVNSs is provided required in subsequent sections.

2.1 Neutrosophic sets and simplified neutrosophic sets

Definition 1 [1] Let X be a non-empty set, with a generic element in X denoted by x. An NS A in X is characterized by the truth-membership function $T_A(x)$, the indeterminacy-membership function $I_A(x)$ and the falsity-membership function $F_A(x)$, respectively, as follows:

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in X\}. \quad (1)$$

Here $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of $]0^-, 1^+[$, and satisfy $0^- \le \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \le 3^+$.

Considering the applicability of NSs, Ye [7] developed the definition of SNSs, which is a special case of NSs.

Definition 2 [7] Let X be a non-empty set, with a generic element in X denoted by x. An SNS A in X is characterized by the truth-membership function $T_A(x)$, the indeterminacy-membership function $I_A(x)$ and the falsity-membership function $F_A(x)$, respectively, as follows:

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in X\}. \quad (2)$$

Here $T_A(x)$, $I_A(x)$ and $F_A(x)$ belong to the unit interval [0, 1]. In particular, if X has only one element, then A is called a simplified neutrosophic number (SNN), which can be denoted by $A = \langle T_A, I_A, F_A\rangle$ for convenience.

Definition 3 [11] The complement of a SNN A is defined as $A^C = \langle F_A, 1 - I_A, T_A\rangle$.

2.2 Multi-valued neutrosophic sets

Definition 4 [28, 29] Let X be a non-empty set, with a generic element in X denoted by x. An MVNSs $\tilde{A}$ in X is characterized by the truth-membership function $\tilde{T}_{\tilde{A}}(x)$, indeterminacy-membership function $\tilde{I}_{\tilde{A}}(x)$ and falsity-membership function $\tilde{F}_{\tilde{A}}(x)$, respectively, as follows:

$$\tilde{A} = \left\{ \left\langle x, \tilde{T}_{\tilde{A}}(x), \tilde{I}_{\tilde{A}}(x), \tilde{F}_{\tilde{A}}(x) \right\rangle | x \in X \right\}. \tag{3}$$

Here, $\tilde{T}_{\tilde{A}}(x), \tilde{I}_{\tilde{A}}(x)$ and $\tilde{F}_{\tilde{A}}(x)$ are three sets of numerical numbers with range [0, 1] and satisfy $0 \le \rho, \varsigma, \tau \le 1$, $0 \le \rho^+ + \varsigma^+ + \tau^+ \le 3$ where $\rho \in \tilde{T}_{\tilde{A}}(x)$, $\varsigma \in \tilde{I}_{\tilde{A}}(x)$, $\tau \in \tilde{F}_{\tilde{A}}(x), \rho^+ = \sup \tilde{T}_{\tilde{A}}(x), \varsigma^+ = \sup \tilde{I}_{\tilde{A}}(x)$ and $\tau^+ = \sup \tilde{F}_{\tilde{A}}(x)$. If X has only one element, then $\tilde{A}$ is called a MVNN, denoted by $\tilde{A} = \left\langle \tilde{T}_{\tilde{A}}(x), \tilde{I}_{\tilde{A}}(x), \tilde{F}_{\tilde{A}}(x) \right\rangle$. For convenience, a MVNN can be denoted by $\tilde{A} = \left\langle \tilde{T}_{\tilde{A}}, \tilde{I}_{\tilde{A}}, \tilde{F}_{\tilde{A}} \right\rangle$. The set of MVNNs are MVNNS.

Moreover, MVNSs are invariably called SVNHFSs in Ref. [29]. Obviously, MVNSs is an extension of NSs. If each of $\tilde{T}_{\tilde{A}}(x), \tilde{I}_{\tilde{A}}(x)$ and $\tilde{F}_{\tilde{A}}(x)$ for any x has one element, i.e., ρ, ς and τ, and $0 \le \rho + \varsigma + \tau \le 3$, then MVNSs are reduced to SNSs; if $\tilde{I}_{\tilde{A}}(x) = \emptyset$ for any x, then MVNSs are reduced to DHFSs; if $\tilde{I}_{\tilde{A}}(x) = \tilde{F}_{\tilde{A}}(x) = \emptyset$ for any x, then MVNSs are reduced to HFSs. Therefore, MVNSs are the extensions of SNSs, DHFSs and HFSs.

Definition 5 [32] The complement of a MVNN $\tilde{A}$ can be defined as follows:

$$\tilde{A}^C = \left\langle \cup_{\tau \in \tilde{F}_{\tilde{A}}} \{\tau\}, \cup_{\varsigma \in \tilde{I}_{\tilde{A}}} \{1 - \varsigma\}, \cup_{\rho \in \tilde{T}_{\tilde{A}}} \{\rho\} \right\rangle. \tag{4}$$

3 The outranking relations on MVNNs

Assume n alternatives denoted by $\beta = \{\beta_1, \ldots, \beta_i, \ldots, \beta_n\}$ and m criteria denoted by $\vartheta = \{\vartheta_1, \ldots, \vartheta_j, \ldots, \vartheta_m\}$. In ELECTRE methods, considering the j-th criterion for the alternative β_i, the preference $\tilde{p}_j$ is utilized to justify the preference in favor of one of the two alternatives; the indifference $\tilde{q}_j$ stands for the compatibility regarding the indifference between two alternatives; the veto $\tilde{v}_j$ is assigned to introduce discordance into the outranking relations. Three thresholds can be utilized to construct the concordance index and discordance index. In the following, a simple case where the thresholds $\tilde{p}_j, \tilde{q}_j$ and $\tilde{v}_j$ are constants under each criterion is considered. Actually, they can be generalized to the functions that vary with the value of the criterion $\vartheta_j(\beta_i)$; more details can be found in Refs. [33, 34].

Definition 6 [33, 34] Assume β_1 and β_2 are two alternatives and then the concordance index for a single criterion is defined on the basis of representing the degree of the majority criteria in favor of "β_1 is at least as good as β_2" as follows:

$$c_j(\beta_1, \beta_2) = \begin{cases} 1, & \vartheta_j(\beta_1) + \tilde{q}_j \ge \vartheta_j(\beta_2) \\ \dfrac{\vartheta_j(\beta_1) - \vartheta_j(\beta_2) + \tilde{p}_j}{\tilde{p}_j - \tilde{q}_j}, & \vartheta_j(\beta_1) + \tilde{q}_j < \vartheta_j(\beta_2) < \vartheta_j(\beta_1) + \tilde{p}_j \\ 0. & \vartheta_j(\beta_1) + \tilde{p}_j \le \vartheta_j(\beta_2) \end{cases} \tag{5}$$

Here $0 \le \tilde{q}_j < \tilde{p}_j$.

Definition 7 [33, 34] The discordance index $d(\beta_1, \beta_2)$ is constructed on the basis of representing the degree of the minority criteria against "β_1 is at least as good as β_2" as follows:

$$d_j(\beta_1, \beta_2) = \begin{cases} 0, & \vartheta_j(\beta_2) - \vartheta_j(\beta_1) \le \tilde{p}_j \\ \dfrac{\vartheta_j(\beta_2) - \vartheta_j(\beta_1) - \tilde{p}_j}{\tilde{v}_j - \tilde{p}_j}, & \tilde{p}_j < \vartheta_j(\beta_2) - \vartheta_j(\beta_1) < \tilde{v}_j. \\ 1 & \vartheta_j(\beta_2) - \vartheta_j(\beta_1) \ge \tilde{v}_j \end{cases} \tag{6}$$

Here $0 \le \tilde{q}_j < \tilde{p}_j < \tilde{v}_j$.

Following the rules of the ELECTRE methods, a concordance index and a discordance index and the outranking relations for MVNNs are presented in the following.

Definition 8 Assume $\tilde{A}_1 = \left\langle \tilde{T}_{\tilde{A}_1}, \tilde{I}_{\tilde{A}_1}, \tilde{F}_{\tilde{A}_1} \right\rangle$ and $\tilde{A}_2 = \left\langle \tilde{T}_{\tilde{A}_2}, \tilde{I}_{\tilde{A}_2}, \tilde{F}_{\tilde{A}_2} \right\rangle$ are two MVNNs and $\tilde{p}$ and $\tilde{q}(0 \le \tilde{q} < \tilde{p})$ are two thresholds. The truth-membership, indeterminacy-membership and falsity-membership concordance indices, respectively, of two MVNNs are defined by

$$\begin{aligned} \mu_{\tilde{p},\tilde{q}}\left(\tilde{T}_{\tilde{A}_1}, \tilde{T}_{\tilde{A}_2}\right) &= \frac{1}{l\left(\tilde{T}_{\tilde{A}_1}\right)} \sum_{\rho_{\tilde{A}_1} \in \tilde{T}_{\tilde{A}_1}} \min_{\rho_B \in \tilde{T}_{\tilde{A}_2}} \left\{ c_{\tilde{p},\tilde{q}}\left(\rho_{\tilde{A}_1}, \rho_{\tilde{A}_2}\right) \right\}, \\ \mu_{\tilde{p},\tilde{q}}\left(\tilde{I}_{\tilde{A}_1}, \tilde{I}_{\tilde{A}_2}\right) &= \frac{1}{l\left(\tilde{I}_{\tilde{A}_1}\right)} \sum_{\varsigma_{\tilde{A}_1} \in \tilde{I}_{\tilde{A}_1}} \min_{\varsigma_{\tilde{A}_2} \in \tilde{I}_{\tilde{A}_2}} \left\{ c_{\tilde{p},\tilde{q}}\left(\varsigma_{\tilde{A}_1}, \varsigma_{\tilde{A}_2}\right) \right\}, \\ \mu_{\tilde{p},\tilde{q}}\left(\tilde{F}_{\tilde{A}_1}, \tilde{F}_{\tilde{A}_2}\right) &= \frac{1}{l\left(\tilde{F}_{\tilde{A}_1}\right)} \sum_{\tau_{\tilde{A}_1} \in \tilde{F}_{\tilde{A}_1}} \min_{\tau_{\tilde{A}_2} \in \tilde{F}_{\tilde{A}_2}} \left\{ c_{\tilde{p},\tilde{q}}\left(\tau_{\tilde{A}_1}, \tau_{\tilde{A}_2}\right) \right\}. \end{aligned} \tag{7}$$

Then the concordance index of two MVNNs $\mu_{\tilde{p},\tilde{q}}(\tilde{A}_1, \tilde{A}_2)$ is defined by

$$\mu_{\tilde{p},\tilde{q}}(\tilde{A}_1, \tilde{A}_2) = \frac{1}{3}\left(\mu_{\tilde{p},\tilde{q}}\left(\tilde{T}_{\tilde{A}_1}, \tilde{T}_{\tilde{A}_2}\right) + \mu_{\tilde{p},\tilde{q}}\left(\tilde{I}_{\tilde{A}_1}, \tilde{I}_{\tilde{A}_2}\right) + \mu_{\tilde{p},\tilde{q}}\left(\tilde{F}_{\tilde{A}_1}, \tilde{F}_{\tilde{A}_2}\right) \right). \tag{8}$$

Here $l(\cdot)$ represents the number of elements in a set, and $c_{\tilde{p},\tilde{q}}(\rho_{\tilde{A}_1}, \rho_{\tilde{A}_2}), c_{\tilde{p},\tilde{q}}(\varsigma_{\tilde{A}_1}, \varsigma_{\tilde{A}_2})$ and $c_{\tilde{p},\tilde{q}}(\tau_{\tilde{A}_1}, \tau_{\tilde{A}_2})$ are the concordance index for the values $\rho_{\tilde{A}_1}$ and $\rho_{\tilde{A}_2}, \varsigma_{\tilde{A}_1}$ and $\varsigma_{\tilde{A}_2}$, and $\tau_{\tilde{A}_1}$ and $\tau_{\tilde{A}_2}$ under the indifferent threshold $\tilde{q}$ and the preference threshold $\tilde{p}$, respectively.

In particular, if $\tilde{T}_{\tilde{A}_1}, \tilde{I}_{\tilde{A}_1}$ and $\tilde{F}_{\tilde{A}_1}, \tilde{T}_{\tilde{A}_2}, \tilde{I}_{\tilde{A}_2}$ and $\tilde{F}_{\tilde{A}_2}$ have only one value, then $\mu_{\tilde{p},\tilde{q}}(\tilde{T}_{\tilde{A}_1}, \tilde{T}_{\tilde{A}_2}), \mu_{\tilde{p},\tilde{q}}(\tilde{I}_{\tilde{A}_1}, \tilde{I}_{\tilde{A}_2})$, and $\mu_{\tilde{p},\tilde{q}}(\tilde{F}_{\tilde{A}_1}, \tilde{F}_{\tilde{A}_2})$ are reduced to a concordance index introduced in Definition 6.

Based on Definition 8, the following properties could easily be obtained.

Property 1 *Assume* $\tilde{A}_1 = \left\langle \tilde{T}_{\tilde{A}_1}, \tilde{I}_{\tilde{A}_1}, \tilde{F}_{\tilde{A}_1} \right\rangle$ *and* $\tilde{A}_2 = \left\langle \tilde{T}_{\tilde{A}_2}, \tilde{I}_{\tilde{A}_2}, \tilde{F}_{\tilde{A}_2} \right\rangle$ *are two MVNNs and* $\tilde{q}$ *and* $\tilde{p} (0 \le \tilde{q} < \tilde{p})$ *are two thresholds, and then the followings can be true.*

(1) $0 \le \mu_{\tilde{p},\tilde{q}}\left(\tilde{T}_{\tilde{A}_1}, \tilde{T}_{\tilde{A}_2}\right) \le 1$;

(2) $0 \le \mu_{\tilde{p},\tilde{q}}\left(\tilde{I}_{\tilde{A}_1}, \tilde{I}_{\tilde{A}_2}\right) \le 1$;

(3) $0 \le \mu_{\tilde{p},\tilde{q}}\left(\tilde{F}_{\tilde{A}_1}, \tilde{F}_{\tilde{A}_2}\right) \le 1$;

(4) $0 \le \mu_{\tilde{p},\tilde{q}}\left(\tilde{A}_1, \tilde{A}_2\right) \le 1$.

Definition 9 The strong dominance relation, weak dominance relation and indifferent relation of MVNNs are defined as follows.

(1) If $\mu_{\tilde{p},\tilde{q}}(\tilde{A}_1, \tilde{A}_2) - \mu_{\tilde{p},\tilde{q}}(\tilde{A}_2, \tilde{A}_1) = 1$, then $\tilde{A}_1$ strongly dominates $\tilde{A}_2$ ($\tilde{A}_2$ is strongly dominated by $\tilde{A}_1$), denoted by $\tilde{A}_1 >_S \tilde{A}_2$;

(2) If $\mu_{\tilde{p},\tilde{q}}(\tilde{A}_1, \tilde{A}_2) - \mu_{\tilde{p},\tilde{q}}(\tilde{A}_2, \tilde{A}_1) = 0$, then $\tilde{A}_1$ is indifferent to $\tilde{A}_2$, denoted by $\tilde{A}_1 \sim \tilde{A}_2$;

(3) If $0 < \mu_{\tilde{p},\tilde{q}}(\tilde{A}_1, \tilde{A}_2) - \mu_{\tilde{p},\tilde{q}}(\tilde{A}_2, \tilde{A}_1) < 1$, then $\tilde{A}_1$ weakly dominates $\tilde{A}_2$ ($\tilde{A}_2$ is weakly dominated by $\tilde{A}_1$), denoted by $\tilde{A}_1 >_W \tilde{A}_2$;

(4) If $0 < \mu_{\tilde{p},\tilde{q}}(\tilde{A}_2, \tilde{A}_1) - \mu_{\tilde{p},\tilde{q}}(\tilde{A}_1, \tilde{A}_2) < 1$, then $\tilde{A}_2$ weakly dominates $\tilde{A}_1$ ($\tilde{A}_1$ is weakly dominated by $\tilde{A}_2$), denoted by $\tilde{A}_2 >_W \tilde{A}_1$.

Example 1 Let $\tilde{p} = 0.06$ and $\tilde{q} = 0.05$.

(1) If $\tilde{A}_1 = \langle \{0.5, 0.6\}, \{0.2\}, \{0.3\} \rangle$ and $\tilde{A}_2 = \langle \{0.2\}, \{0.1\}, \{0.2\} \rangle$ are two MVNNs, then $\mu_{\tilde{p},\tilde{q}}(\tilde{A}_1, \tilde{A}_2) - \mu_{\tilde{p},\tilde{q}}(\tilde{A}_2, \tilde{A}_1) = 1$. Thus, $\tilde{A}_1 >_S \tilde{A}_2$;

(2) If $\tilde{A}_1 = \langle \{0.3\}, \{0.1\}, \{0.2\} \rangle$ and $\tilde{A}_2 = \langle \{0.3, 0.31\}, \{0.15\}, \{0.15\} \rangle$ are two MVNNs, then $\mu_{\tilde{p},\tilde{q}}(\tilde{A}_1, \tilde{A}_2) - \mu_{\tilde{p},\tilde{q}}(\tilde{A}_2, \tilde{A}_1) = 0$. Thus, $\tilde{A}_1 \sim \tilde{A}_2$;

(3) If $\tilde{A}_1 = \langle \{0.2, 0.3\}, \{0.2\}, \{0.3\} \rangle$ and $\tilde{A}_2 = \langle \{0.2\}, \{0.3\}, \{0.2\} \rangle$ are two MVNNs, then $\mu_{\tilde{p},\tilde{q}}(\tilde{A}_1, \tilde{A}_2) - \mu_{\tilde{p},\tilde{q}}(\tilde{A}_2, \tilde{A}_1) = \frac{1}{3}$. Thus, $\tilde{A}_1 >_W \tilde{A}_2$.

Property 2 *Let* $\tilde{A}_1, \tilde{A}_2 \in$ *MVNNS, and* $\tilde{p}$ *and* $\tilde{q}$ $(0 \le \tilde{q} < \tilde{p})$ *be two thresholds.* $\tilde{A}_1 >_S \tilde{A}_2$ *if and only if* $\min\left\{\rho_{\tilde{A}_1} | \rho_{\tilde{A}_1} \in \tilde{T}_{\tilde{A}_1}\right\} - \max\left\{\rho_{\tilde{A}_2} | \rho_{\tilde{A}_2} \in \tilde{T}_{\tilde{A}_2}\right\} \ge \tilde{p}$, $\min\left\{\varsigma_{\tilde{A}_1} | \varsigma_{\tilde{A}_1} \in \tilde{I}_{\tilde{A}_1}\right\} - \max\left\{\varsigma_{\tilde{A}_2} | \varsigma_{\tilde{A}_2} \in \tilde{I}_{\tilde{A}_2}\right\} \ge \tilde{p}$ *and* $\min\left\{\tau_{\tilde{A}_1} | \tau_{\tilde{A}_1} \in \tilde{F}_{\tilde{A}_1}\right\} - \max\left\{\tau_{\tilde{A}_2} | \tau_{\tilde{A}_2} \in \tilde{F}_{\tilde{A}_2}\right\} \ge \tilde{p}$.

Proof

(1) Necessity:

$$\tilde{A}_1 \not>_S \tilde{A}_2 \Rightarrow \begin{cases} \min\left\{\rho_{\tilde{A}_1} | \rho_{\tilde{A}_1} \in \tilde{T}_{\tilde{A}_1}\right\} - \max\left\{\rho_{\tilde{A}_2} | \rho_{\tilde{A}_2} \in \tilde{T}_{\tilde{A}_2}\right\} \ge \tilde{p} \\ \min\left\{\varsigma_{\tilde{A}_1} | \varsigma_{\tilde{A}_1} \in \tilde{I}_{\tilde{A}_1}\right\} - \max\left\{\varsigma_{\tilde{A}_2} | \varsigma_{\tilde{A}_2} \in \tilde{I}_{\tilde{A}_2}\right\} \ge \tilde{p} \\ \min\left\{\tau_{\tilde{A}_1} | \tau_{\tilde{A}_1} \in \tilde{F}_{\tilde{A}_1}\right\} - \max\left\{\tau_{\tilde{A}_2} | \tau_{\tilde{A}_2} \in \tilde{F}_{\tilde{A}_2}\right\} \ge \tilde{p} \end{cases}.$$

According to Definition 9, if $\tilde{A}_1 >_S \tilde{A}_2$, then $\mu_{\tilde{p},\tilde{q}}(\tilde{A}_1, \tilde{A}_2) - \mu_{\tilde{p},\tilde{q}}(\tilde{A}_2, \tilde{A}_1) = 1$. Since $0 < \mu_{\tilde{p},\tilde{q}} (\tilde{A}_1, \tilde{A}_2) < 1$ and $0 < \mu_{\tilde{p},\tilde{q}}(\tilde{A}_2, \tilde{A}_1) < 1$, then $\mu_{\tilde{p},\tilde{q}}(\tilde{A}_2, \tilde{A}_1) = 0$ can be obtained. Thus, $\mu_{\tilde{p},\tilde{q}}(\tilde{T}_{\tilde{A}_1}, \tilde{T}_{\tilde{A}_2}) = \mu_{\tilde{p},\tilde{q}}(\tilde{I}_{\tilde{A}_1}, \tilde{I}_{\tilde{A}_2}) = \mu_{\tilde{p},\tilde{q}}(\tilde{F}_{\tilde{A}_1}, \tilde{F}_{\tilde{A}_2}) = 0$, i.e., $\frac{1}{l(\tilde{T}_{\tilde{A}_2})} \sum_{\rho_{\tilde{A}_2} \in \tilde{T}_{\tilde{A}_2}} \min_{\rho_{\tilde{A}_1} \in \tilde{T}_{\tilde{A}_1}} c_{\tilde{p},\tilde{q}}(\rho_{\tilde{A}_2}, \rho_{\tilde{A}_1}) = 0, \frac{1}{l(\tilde{I}_{\tilde{A}_2})} \sum_{\varsigma_{\tilde{A}_2} \in \tilde{I}_{\tilde{A}_2}} \min_{\varsigma_{\tilde{A}_1} \in \tilde{I}_{\tilde{A}_1}} c_{\tilde{p},\tilde{q}}(\varsigma_{\tilde{A}_2}, \varsigma_{\tilde{A}_1}) = 0$ and $\frac{1}{l(\tilde{F}_{\tilde{A}_2})} \sum_{\tau_{\tilde{A}_2} \in \tilde{F}_{\tilde{A}_2}} \min_{\tau_{\tilde{A}_1} \in \tilde{F}_{\tilde{A}_1}} c_{\tilde{p},\tilde{q}}(\tau_{\tilde{A}_2}, \tau_{\tilde{A}_1}) = 0$ are obtained. Derived from Definition 6, $c_{\tilde{p},\tilde{q}}(\rho_{\tilde{A}_2}, \rho_{\tilde{A}_1})$, $c_{\tilde{p},\tilde{q}}(\varsigma_{\tilde{A}_2}, \varsigma_{\tilde{A}_1})$, $c_{\tilde{p},\tilde{q}}(\tau_{\tilde{A}_2}, \tau_{\tilde{A}_1}) \in [0, 1]$, so $c_{\tilde{p},\tilde{q}}(\rho_{\tilde{A}_2}, \rho_{\tilde{A}_1}) = 0, c_{\tilde{p},\tilde{q}}(\varsigma_{\tilde{A}_2}, \varsigma_{\tilde{A}_1}) = 0$ and $c_{\tilde{p},\tilde{q}}(\tau_{\tilde{A}_2}, \tau_{\tilde{A}_1}) = 0$. Hence, $\rho_{\tilde{A}_1} - \rho_{\tilde{A}_2} \ge \tilde{p}$ for any $\rho_{\tilde{A}_1} \in \tilde{T}_{\tilde{A}_1}, \rho_{\tilde{A}_2} \in \tilde{T}_{\tilde{A}_2}, \varsigma_{\tilde{A}_1} - \varsigma_{\tilde{A}_2} \ge \tilde{p}$ for any $\varsigma_{\tilde{A}_1} \in \tilde{I}_{\tilde{A}_1}, \varsigma_{\tilde{A}_2} \in \tilde{I}_{\tilde{A}_2}, \tau_{\tilde{A}_1} - \tau_{\tilde{A}_2} \ge \tilde{p}$ for any $\tau_{\tilde{A}_1} \in \tilde{F}_{\tilde{A}_1}, \tau_{\tilde{A}_2} \in \tilde{F}_{\tilde{A}_2}$. Therefore, $\min\{\rho_{\tilde{A}_1} | \rho_{\tilde{A}_1} \in \tilde{T}_{\tilde{A}_1}\} - \max\{\rho_{\tilde{A}_2} | \rho_{\tilde{A}_2} \in \tilde{T}_{\tilde{A}_2}\} \ge \tilde{p}$, $\min\{\varsigma_{\tilde{A}_1} | \varsigma_{\tilde{A}_1} \in \tilde{I}_{\tilde{A}_1}\} - \max\{\varsigma_{\tilde{A}_2} | \varsigma_{\tilde{A}_2} \in \tilde{I}_{\tilde{A}_2}\} \ge \tilde{p}$ and $\min\{\tau_{\tilde{A}_1} | \tau_{\tilde{A}_1} \in \tilde{F}_{\tilde{A}_1}\} - \max\{\tau_{\tilde{A}_2} | \tau_{\tilde{A}_2} \in \tilde{F}_{\tilde{A}_2}\} \ge \tilde{p}$ are certainly valid

(2) Sufficiency:

$$\left.\begin{array}{l} \min\left\{\rho_{\tilde{A}_1} | \rho_{\tilde{A}_1} \in \tilde{T}_{\tilde{A}_1}\right\} - \max\left\{\rho_{\tilde{A}_2} | \rho_{\tilde{A}_2} \in \tilde{T}_{\tilde{A}_2}\right\} \ge \tilde{p} \\ \min\left\{\varsigma_{\tilde{A}_1} | \varsigma_{\tilde{A}_1} \in \tilde{I}_{\tilde{A}_1}\right\} - \max\left\{\varsigma_{\tilde{A}_2} | \varsigma_{\tilde{A}_2} \in \tilde{I}_{\tilde{A}_2}\right\} \ge \tilde{p} \\ \min\left\{\tau_{\tilde{A}_1} | \tau_{\tilde{A}_1} \in \tilde{F}_{\tilde{A}_1}\right\} - \max\left\{\tau_{\tilde{A}_2} | \tau_{\tilde{A}_2} \in \tilde{F}_{\tilde{A}_2}\right\} \ge \tilde{p} \end{array}\right\} \Rightarrow \tilde{A}_1 >_S \tilde{A}_2.$$

Since $\min\{\rho_{\tilde{A}_1} | \rho_{\tilde{A}_1} \in \tilde{T}_{\tilde{A}_1}\} - \max\{\rho_{\tilde{A}_2} | \rho_{\tilde{A}_2} \in \tilde{T}_{\tilde{A}_2}\} \ge \tilde{p}$, then $\rho_{\tilde{A}_1} - \rho_{\tilde{A}_2} \ge \tilde{p}$ for any $\rho_{\tilde{A}_1} \in \tilde{T}_{\tilde{A}_1}, \rho_{\tilde{A}_2} \in \tilde{T}_{\tilde{A}_2}$. Derived from Definition 6, $c_{\tilde{p},\tilde{q}}(\rho_{\tilde{A}_2}, \rho_{\tilde{A}_1}) = 0$ and $c_{\tilde{p},\tilde{q}}(\rho_{\tilde{A}_1}, \rho_{\tilde{A}_2}) = 1$ can be obtained. Therefore, $\frac{1}{l(\tilde{T}_{\tilde{A}_1})} \sum_{\rho_{\tilde{A}_1} \in \tilde{T}_{\tilde{A}_1}} \min_{\rho_{\tilde{A}_2} \in \tilde{T}_{\tilde{A}_2}} c_{\tilde{p},\tilde{q}} (\rho_{\tilde{A}_1}, \rho_{\tilde{A}_2}) = 1$ and $\frac{1}{l(\tilde{T}_{\tilde{A}_2})} \sum_{\rho_{\tilde{A}_2} \in \tilde{T}_{\tilde{A}_2}} \min_{\rho_{\tilde{A}_1} \in \tilde{T}_{\tilde{A}_1}} c_{\tilde{p},\tilde{q}} (\rho_{\tilde{A}_2}, \rho_{\tilde{A}_1}) = 0$, which indicates $\mu_{\tilde{p},\tilde{q}}(\tilde{T}_{\tilde{A}_1}, \tilde{T}_{\tilde{A}_2}) = 1$ and $\mu_{\tilde{p},\tilde{q}}(\tilde{T}_{\tilde{A}_2}, \tilde{T}_{\tilde{A}_1}) = 0$ based on Definition 8. Similarly, $\mu_{\tilde{p},\tilde{q}}(\tilde{I}_{\tilde{A}_1}, \tilde{I}_{\tilde{A}_2}) = 1$

and $\mu_{\tilde{p},\tilde{q}}(\tilde{I}_{\tilde{A}_2},\tilde{I}_{\tilde{A}_1})=0$, $\mu_{\tilde{p},\tilde{q}}(\tilde{F}_{\tilde{A}_1},\tilde{F}_{\tilde{A}_2})=1$ and $\mu_{\tilde{p},\tilde{q}}(\tilde{F}_{\tilde{A}_2},\tilde{F}_{\tilde{A}_1})=0$ can be achieved. Therefore, we have $\mu_{\tilde{p},\tilde{q}}(\tilde{A}_1,\tilde{A}_2)-\mu_{\tilde{p},\tilde{q}}(\tilde{A}_2,\tilde{A}_1)=1$. Based on Definition 9, $\tilde{A}_1 >_S \tilde{A}_2$ is obtained.

Property 3 *Let $\tilde{A}_1,\tilde{A}_2,\tilde{A}_3 \in MVNNS$, and $\tilde{p}$ and $\tilde{q}(0\le\tilde{q}<\tilde{p})$ be two thresholds. If $\tilde{A}_1 >_S \tilde{A}_2$ and $\tilde{A}_2 >_S \tilde{A}_3$, then $\tilde{A}_1 >_S \tilde{A}_3$.*

Proof According to Property 2, if $\tilde{A}_1 >_S \tilde{A}_2$, then $\min\{\rho_{\tilde{A}_1}|\rho_{\tilde{A}_1}\in\tilde{T}_{\tilde{A}_1}\}-\max\{\rho_{\tilde{A}_2}|\rho_{\tilde{A}_2}\in\tilde{T}_{\tilde{A}_2}\}\ge\tilde{p}$, $\min\{\varsigma_{\tilde{A}_1}|\varsigma_{\tilde{A}_1}\in\tilde{I}_{\tilde{A}_1}\}-\max\{\varsigma_{\tilde{A}_2}|\varsigma_{\tilde{A}_2}\in\tilde{I}_{\tilde{A}_2}\}\ge\tilde{p}$ and $\min\{\tau_{\tilde{A}_1}|\tau_{\tilde{A}_1}\in\tilde{F}_{\tilde{A}_1}\}-\max\{\tau_{\tilde{A}_2}|\tau_{\tilde{A}_2}\in\tilde{F}_{\tilde{A}_2}\}\ge\tilde{p}$. If $\tilde{A}_2 >_S \tilde{A}_3$, then $\min\{\rho_{\tilde{A}_2}|\rho_{\tilde{A}_2}\in\tilde{T}_{\tilde{A}_2}\}-\max\{\rho_{\tilde{A}_3}|\rho_{\tilde{A}_3}\in\tilde{T}_{\tilde{A}_3}\}\ge\tilde{p}$, $\min\{\varsigma_{\tilde{A}_2}|\varsigma_{\tilde{A}_2}\in\tilde{I}_{\tilde{A}_2}\}-\max\{\varsigma_{\tilde{A}_3}|\varsigma_{\tilde{A}_3}\in\tilde{I}_{\tilde{A}_3}\}\ge\tilde{p}$ and $\min\{\tau_{\tilde{A}_2}|\ \tau_{\tilde{A}_2}\in\tilde{F}_{\tilde{A}_2}\}-\max\{\tau_{\tilde{A}_3}|\tau_{\tilde{A}_3}\in\tilde{F}_{\tilde{A}_3}\}\ge\tilde{p}$ can be achieved. So $\max\{\rho_{\tilde{A}_2}|\rho_{\tilde{A}_2}\in\tilde{T}_{\tilde{A}_2}\}-\max\{\rho_{\tilde{A}_3}|\rho_{\tilde{A}_3}\in\tilde{T}_{\tilde{A}_3}\}\ge\tilde{p}$, $\max\{\varsigma_{\tilde{A}_2}|\varsigma_{\tilde{A}_2}\in\tilde{I}_{\tilde{A}_2}\}-\max\{\varsigma_{\tilde{A}_3}|\varsigma_{\tilde{A}_3}\in\tilde{I}_{\tilde{A}_3}\}\ge\tilde{p}$ and $\max\{\tau_{\tilde{A}_2}|\ \tau_{\tilde{A}_2}\in\tilde{F}_{\tilde{A}_2}\}-\max\{\tau_{\tilde{A}_3}|\tau_{\tilde{A}_3}\in\tilde{F}_{\tilde{A}_3}\}\ge\tilde{p}$ can be obtained.

Therefore, the further derivations are:

$$\left.\begin{array}{l}\min\left\{\rho_{\tilde{A}_1}|\rho_{\tilde{A}_1}\in\tilde{T}_{\tilde{A}_1}\right\}-\max\left\{\rho_{\tilde{A}_2}|\rho_{\tilde{A}_2}\in\tilde{T}_{\tilde{A}_2}\right\}\ge\tilde{p}\\ \max\left\{\rho_{\tilde{A}_2}|\rho_{\tilde{A}_2}\in\tilde{T}_{\tilde{A}_2}\right\}-\max\left\{\rho_{\tilde{A}_3}|\rho_{\tilde{A}_3}\in\tilde{T}_{\tilde{A}_3}\right\}\ge\tilde{p}\end{array}\right\}$$
$$\Rightarrow \min\left\{\rho_{\tilde{A}_1}|\rho_{\tilde{A}_1}\in\tilde{T}_{\tilde{A}_1}\right\} - \max\left\{\rho_{\tilde{A}_3}|\rho_{\tilde{A}_3}\in\tilde{T}_{\tilde{A}_3}\right\}\ge 2\tilde{p}>\tilde{p},$$

$$\left.\begin{array}{l}\min\left\{\varsigma_{\tilde{A}_1}|\varsigma_{\tilde{A}_1}\in\tilde{T}_{\tilde{A}_1}\right\}-\max\left\{\varsigma_{\tilde{A}_2}|\varsigma_{\tilde{A}_2}\in\tilde{T}_{\tilde{A}_2}\right\}\ge\tilde{p}\\ \max\left\{\varsigma_{\tilde{A}_2}|\varsigma_{\tilde{A}_2}\in\tilde{T}_{\tilde{A}_2}\right\}-\max\left\{\varsigma_{\tilde{A}_3}|\varsigma_{\tilde{A}_3}\in\tilde{T}_{\tilde{A}_3}\right\}\ge\tilde{p}\end{array}\right\}$$
$$\Rightarrow \min\left\{\varsigma_{\tilde{A}_1}|\varsigma_{\tilde{A}_1}\in\tilde{T}_{\tilde{A}_1}\right\} - \max\left\{\varsigma_{\tilde{A}_3}|\varsigma_{\tilde{A}_3}\in\tilde{T}_{\tilde{A}_3}\right\}\ge 2\tilde{p}>\tilde{p},$$

and

$$\left.\begin{array}{l}\min\left\{\tau_{\tilde{A}_1}|\tau_{\tilde{A}_1}\in\tilde{T}_{\tilde{A}_1}\right\}-\max\left\{\tau_{\tilde{A}_2}|\tau_{\tilde{A}_2}\in\tilde{T}_{\tilde{A}_2}\right\}\ge\tilde{p}\\ \max\left\{\tau_{\tilde{A}_2}|\tau_{\tilde{A}_2}\in\tilde{T}_{\tilde{A}_2}\right\}-\max\left\{\tau_{\tilde{A}_3}|\tau_{\tilde{A}_3}\in\tilde{T}_{\tilde{A}_3}\right\}\ge\tilde{p}\end{array}\right\}$$
$$\Rightarrow \min\left\{\tau_{\tilde{A}_1}|\tau_{\tilde{A}_1}\in\tilde{T}_{\tilde{A}_1}\right\} - \max\left\{\tau_{\tilde{A}_3}|\tau_{\tilde{A}_3}\in\tilde{T}_{\tilde{A}_3}\right\}\ge 2\tilde{p}>\tilde{p}.$$

Therefore, $\tilde{A}_1 >_S \tilde{A}_3$.

Property 4 *Let $\tilde{A}_1,\tilde{A}_2,\tilde{A}_3 \in MVNNS,\tilde{p}$ and $\tilde{q}$ ($0\le\tilde{q}<\tilde{p}$) be two thresholds.*

(1) The strongly dominant relations have the following properties.
① irreflexivity: $\forall\tilde{A}_1\in MVNNS,\ \tilde{A}_1 >_S \tilde{A}_1$;
② asymmetry: $\forall\tilde{A}_1,\tilde{A}_2\in MVNNS,\ \tilde{A}_1 >_S \tilde{A}_2\Rightarrow\neg(\tilde{A}_2 >_S \tilde{A}_1)$;
③ transitivity: $\forall\,\tilde{A}_1,\ \tilde{A}_2,\ \tilde{A}_3\in MVNNS,\ \tilde{A}_1 >_S \tilde{A}_2,\ \tilde{A}_2 >_S \tilde{A}_3\Rightarrow\tilde{A}_1 >_S \tilde{A}_3$.

(2) The weakly dominant relations have the following properties.
④ irreflexivity: $\forall\tilde{A}_1\in MVNNS,\ \tilde{A}_1 >_W \tilde{A}_1$;
⑤ asymmetry: $\forall\tilde{A}_1,\tilde{A}_2\in MVNNS,\ \tilde{A}_1 >_W \tilde{A}_2\Rightarrow\neg(\tilde{A}_2 >_W \tilde{A}_1)$;
⑥ non-transitivity: $\exists\,\tilde{A}_1,\ \tilde{A}_2,\ \tilde{A}_3\in MVNNS,\ \tilde{A}_1 >_W \tilde{A}_2,\ \tilde{A}_2 >_W \tilde{A}_3\not\Rightarrow\tilde{A}_1 >_W \tilde{A}_3$.

(3) The indifferent relations have the following properties.
⑦ reflexivity: $\forall\tilde{A}_1\in MVNNS,\ \tilde{A}_1\sim\tilde{A}_1$;
⑧ symmetry: $\forall\tilde{A}_1,\tilde{A}_2\in MVNNS,\ \tilde{A}_1\sim\tilde{A}_2\Rightarrow\tilde{A}_2\sim\tilde{A}_1$;
⑨ non-transitivity: $\exists\,\tilde{A}_1,\ \tilde{A}_2,\ \tilde{A}_3\in MVNNS,\ \tilde{A}_1\sim\tilde{A}_2,\ \tilde{A}_2\sim\tilde{A}_3\not\Rightarrow\tilde{A}_1\sim\tilde{A}_3$.

According to Definitions 8, 9, it can be seen that ①–⑤, ⑦ and ⑧ are true, and ⑥ and ⑨ can be exemplified.

Example 2 Let $\tilde{p}=0.06$ and $\tilde{q}=0.05$. ⑥ and ⑨ can be exemplified as follows.

(1) If $\tilde{A}_1=\langle\{0.2,0.16\},\{0.2\},\{0.15\}\rangle$, $\tilde{A}_2=\langle\{0.15,0.16\},\{0.14\},\{0.15\}\rangle$ and $\tilde{A}_3=\langle\{0.1\},\{0.1\},\{0.09\}\rangle$ are three MVNNs, then $\mu_{\tilde{p},}\ \tilde{q}(\tilde{A}_1,\tilde{A}_2)-\mu_{\tilde{p},\tilde{q}}(\tilde{A}_2,\tilde{A}_1)=0.3333$, $\mu\quad\tilde{p},\tilde{q}(\tilde{A}_2,\tilde{A}_3)-\mu_{\tilde{p},\tilde{q}}(\tilde{A}_3,\tilde{A}_2)=0.3333$ and $\mu_{\tilde{p},\tilde{q}}(\tilde{A}_1,\tilde{A}_3)-\mu_{\tilde{p},\tilde{q}}(\tilde{A}_3,\tilde{A}_1)=1$. We have $\tilde{A}_1 >_W \tilde{A}_2,\tilde{A}_2 >_W \tilde{A}_3$ but $\tilde{A}_1 >_S \tilde{A}_3$. Thus, the weak dominance relations are non-transitive.

(2) If $\tilde{A}_1=\langle\{0.12,0.18\},\{0.1\},\{0.2\}\rangle,\tilde{A}_2=\langle\{0.14,0.16\},\{0.1\},\{0.2\}\rangle$ and $\tilde{A}_3=\langle\{0.12,0.14\},\{0.1\},\{0.2\}\rangle$ are three MVNNs, then $\mu_{\tilde{p},\tilde{q}}(\tilde{A}_1,\tilde{A}_2)-\mu_{\tilde{p},\tilde{q}}(\tilde{A}_2,\tilde{A}_1)=0,\mu_{\tilde{p},\tilde{q}}(\tilde{A}_2,\tilde{A}_3)-\mu_{\tilde{p},\tilde{q}}(\tilde{A}_3,\tilde{A}_2)=0$ and $\mu_{\tilde{p},\tilde{q}}(\tilde{A}_1,\tilde{A}_3)-\mu_{\tilde{p},\tilde{q}}(\tilde{A}_3,\tilde{A}_1)=0.1667$. We have $\tilde{A}_1\sim\tilde{A}_2,\tilde{A}_2\sim\tilde{A}_3$ but $\tilde{A}_1 >_W \tilde{A}_3$. Thus, the indifferent relations are non-transitive.

Similar to dominance relations, the strong opposition relation, weak opposition relation and indifferent opposition relation can be defined.

Definition 10 Let $\tilde{A}_1=\langle\tilde{T}_{\tilde{A}_1},\tilde{I}_{\tilde{A}_1},\tilde{F}_{\tilde{A}_1}\rangle$ and $\tilde{A}_2=\langle\tilde{T}_{\tilde{A}_2},\tilde{I}_{\tilde{A}_2},\tilde{F}_{\tilde{A}_2}\rangle$ be two MVNNs, $\tilde{p}$ and $\tilde{v}(\tilde{p}<\tilde{v})$ be two thresholds. The truth-membership, indeterminacy-membership and falsity-membership discordance index, respectively, of two MVNNs are defined as follows:

$$\varpi_{\tilde{p},\tilde{v}}\left(\tilde{T}_{\tilde{A}_1},\tilde{T}_{\tilde{A}_2}\right)=\frac{1}{l\left(\tilde{T}_{\tilde{A}_1}\right)}\sum_{\rho_{\tilde{A}_1}\in\tilde{T}_{\tilde{A}_1}}\min_{\rho_{\tilde{A}_2}\in\tilde{T}_{\tilde{A}_2}}\left\{d_{\tilde{p},\tilde{v}}\left(\rho_{\tilde{A}_1},\rho_{\tilde{A}_2}\right)\right\},$$
$$\varpi_{\tilde{p},\tilde{v}}\left(\tilde{I}_{\tilde{A}_1},\tilde{I}_{\tilde{A}_2}\right)=\frac{1}{l\left(\tilde{I}_{\tilde{A}_1}\right)}\sum_{\varsigma_{\tilde{A}_1}\in\tilde{I}_{\tilde{A}_1}}\min_{\varsigma_{\tilde{A}_2}\in\tilde{I}_{\tilde{A}_2}}\left\{d_{\tilde{p},\tilde{v}}\left(\varsigma_{\tilde{A}_1},\varsigma_{\tilde{A}_2}\right)\right\},$$
$$\varpi_{\tilde{p},\tilde{v}}\left(\tilde{F}_{\tilde{A}_1},\tilde{F}_{\tilde{A}_2}\right)=\frac{1}{l\left(\tilde{F}_{\tilde{A}_1}\right)}\sum_{\tau_{\tilde{A}_1}\in\tilde{F}_{\tilde{A}_1}}\min_{\tau_{\tilde{A}_2}\in\tilde{F}_{\tilde{A}_2}}\left\{d_{\tilde{p},\tilde{v}}\left(\tau_{\tilde{A}_1},\tau_{\tilde{A}_2}\right)\right\}. \tag{9}$$

Then the discordance index of two MVNNs $\varpi_{\tilde{p},\tilde{v}}(\tilde{A}_1,\ \tilde{A}_2)$ is defined by

$$\varpi_{\tilde{p},\tilde{v}}(\tilde{A}_1,\ \tilde{A}_2)=\frac{1}{3}\left(\varpi_{\tilde{p},\tilde{v}}\left(\tilde{T}_{\tilde{A}_1},\tilde{T}_{\tilde{A}_2}\right)+\varpi_{\tilde{p},\tilde{v}}\left(\tilde{I}_{\tilde{A}_1},\tilde{I}_{\tilde{A}_2}\right)+\varpi_{\tilde{p},\tilde{v}}\left(\tilde{F}_{\tilde{A}_1},\tilde{F}_{\tilde{A}_2}\right)\right). \tag{10}$$

Here $l(\cdot)$ represents the number of elements in a set, and $d_{\tilde{p},\tilde{v}}(\rho_{\tilde{A}_1},\rho_{\tilde{A}_2}), d_{\tilde{p},\tilde{v}}(\varsigma_{\tilde{A}_1},\varsigma_{\tilde{A}_2})$ and $d_{\tilde{p},\tilde{v}}(\tau_{\tilde{A}_1},\tau_{\tilde{A}_2})$ are the discordance index for the values $\rho_{\tilde{A}_1}$ and $\rho_{\tilde{A}_2}, \varsigma_{\tilde{A}_1}$ and $\varsigma_{\tilde{A}_2}$, and $\tau_{\tilde{A}_1}$ and $\tau_{\tilde{A}_2}$ under the preference threshold $\tilde{p}$ and the veto threshold $\tilde{v}$, respectively.

It can be easily concluded that if $\tilde{T}_{\tilde{A}_1},\tilde{I}_{\tilde{A}_1}$ and $\tilde{F}_{\tilde{A}_1},\tilde{T}_{\tilde{A}_2},\tilde{I}_{\tilde{A}_2}$ and $\tilde{F}_{\tilde{A}_2}$ have only one value, then $\varpi_{\tilde{p},\tilde{q}}(\tilde{T}_{\tilde{A}_1},\tilde{T}_{\tilde{A}_2})$, $\varpi_{\tilde{p},\tilde{q}}(\tilde{I}_{\tilde{A}_1},\tilde{I}_{\tilde{A}_2})$ and $\varpi_{\tilde{p},\tilde{q}}(\tilde{F}_{\tilde{A}_1},\tilde{F}_{\tilde{A}_2})$ will reduce to a discordance index introduced in Definition 7.

Based on Definition 10, it can get the following properties.

Property 5 *Let* $\tilde{A}_1=\langle\tilde{T}_{\tilde{A}_1},\tilde{I}_{\tilde{A}_1},\tilde{F}_{\tilde{A}_1}\rangle$ *and* $\tilde{A}_2=\langle\tilde{T}_{\tilde{A}_2},\tilde{I}_{\tilde{A}_2},\tilde{F}_{\tilde{A}_2}\rangle$ *be two MVNNs and* $\tilde{p}$ *and* $\tilde{v}(\ \tilde{p}<\tilde{v})$ *be two thresholds. Then the followings can be true.*

(1) $0\le\varpi_{\tilde{p},\tilde{v}}(\tilde{T}_{\tilde{A}_1},\tilde{T}_{\tilde{A}_2})\le 1$;
(2) $0\le\varpi_{\tilde{p},\tilde{v}}(\tilde{I}_{\tilde{A}_1},\tilde{I}_{\tilde{A}_2})\le 1$;
(3) $0\le\varpi_{\tilde{p},\tilde{v}}(\tilde{F}_{\tilde{A}_1},\tilde{F}_{\tilde{A}_2})\le 1$;
(4) $0\le\varpi_{\tilde{p},\tilde{v}}(\tilde{A}_1,\tilde{A}_2)\le 1$;

Definition 11 The strong opposition relation, weak opposition relation and indifferent opposition relation for MVNNs are defined as follows.

(1) If $\varpi_{\tilde{p},\tilde{v}}(\tilde{A}_1,\tilde{A}_2)-\varpi_{\tilde{p},\tilde{v}}(\tilde{A}_2,\tilde{A}_1)=1$, then $\tilde{A}_1$ strongly opposes $\tilde{A}_2$ ($\tilde{A}_2$ is strongly opposed by $\tilde{A}_1$), denoted by $\tilde{A}_1>_{SO}\tilde{A}_2$;
(2) If $\varpi_{\tilde{p},\tilde{v}}(\tilde{A}_1,\tilde{A}_2)-\varpi_{\tilde{p},\tilde{v}}(\tilde{A}_2,\tilde{A}_1)=0$, then $\tilde{A}_1$ is indifferently opposed to $\tilde{A}_2$, denoted by $\tilde{A}_1\sim_O\tilde{A}_2$;
(3) If $0<\varpi_{\tilde{p},\tilde{v}}(\tilde{A}_1,\tilde{A}_2)-\varpi_{\tilde{p},\tilde{v}}(\tilde{A}_2,\tilde{A}_1)<1$, then $\tilde{A}_1$ weakly opposes $\tilde{A}_2$ ($\tilde{A}_2$ is weakly opposed by $\tilde{A}_1$), denoted by $\tilde{A}_1>_{WO}\tilde{A}_2$;
(4) If $0<\varpi_{\tilde{p},\tilde{v}}(\tilde{A}_2,\tilde{A}_1)-\varpi_{\tilde{p},\tilde{v}}(\tilde{A}_1,\tilde{A}_2)<1$, then $\tilde{A}_2$ weakly opposes $\tilde{A}_1$ ($\tilde{A}_1$ is weakly opposed by $\tilde{A}_2$), denoted by $\tilde{A}_2>_{WO}\tilde{A}_1$.

Example 3 Let $\tilde{p}=0.1$ and $\tilde{v}=0.2$.

(1) If $\tilde{A}_1=\langle\{0.2,0.3\},\{0.1\},\{0.2\}\rangle$ and $\tilde{A}_2=\langle\{0.5,0.7\},\{0.3\},\{0.4\}\rangle$ are two MVNNs, then $\varpi_{\tilde{p},\tilde{v}}(\tilde{A}_1,\tilde{A}_2)-\varpi_{\tilde{p},\tilde{v}}(\tilde{A}_2,\tilde{A}_1)=1$. Thus, $\tilde{A}_1>_{SO}\tilde{A}_2$;
(2) If $\tilde{A}_1=\langle\{0.2,0.5\},\{0.1\},\{0.15\}\rangle$ and $\tilde{A}_2=\langle\{0.2,0.6\},\{0.1\},\{0.2\}\rangle$ are two MVNNs, then $\varpi_{\tilde{p},\tilde{v}}(\tilde{A}_1,\tilde{A}_2)-\varpi_{\tilde{p},\tilde{v}}(\tilde{A}_2,\tilde{A}_1)=0$. Thus, $\tilde{A}_1\sim_O\tilde{A}_2$;
(3) If $\tilde{A}_1=\langle\{0.2,0.5\},\{0.1\},\{0.2\}\rangle$ and $\tilde{A}_2=\langle\{0.25,0.65\},\{0.3\},\{0.2\}\rangle$ are two MVNNs, then $\varpi_{\tilde{p},\tilde{v}}(\tilde{A}_1,\tilde{A}_2)-\varpi_{\tilde{p},\tilde{v}}(\tilde{A}_2,\tilde{A}_1)=0.4167$. Thus, $\tilde{A}_1>_{WO}\tilde{A}_2$.

Based on Definitions 7 and 10, the following properties can be true.

Property 6 *Let* $\tilde{A}_1,\tilde{A}_2\in MVNNS$*, and* $\tilde{p}$ *and* $\tilde{v}(0<\tilde{p}<\tilde{v})$ *be two thresholds. Then* $\tilde{A}_1>_{SO}\tilde{A}_2$ *if and only if* $\min\{\rho_{\tilde{A}_2}|\rho_{\tilde{A}_2}\in\tilde{T}_{\tilde{A}_2}\}-\max\{\rho_{\tilde{A}_1}|\rho_{\tilde{A}_1}\in\tilde{T}_{\tilde{A}_1}\}\ \ge\tilde{v}$, $\min\ \{\varsigma_{\tilde{A}_2}|\varsigma_{\tilde{A}_2}\in\tilde{I}_{\tilde{A}_2}\}-\max\{\varsigma_{\tilde{A}_1}|\varsigma_{\tilde{A}_1}\in\tilde{I}_{\tilde{A}_1}\}\ge\tilde{v}$ *and* $\min\{\tau_{\tilde{A}_2}|\tau_{\tilde{A}_2}\in\tilde{F}_{\tilde{A}_2}\}-\max\{\tau_{\tilde{A}_1}|\tau_{\tilde{A}_1}\in\tilde{F}_{\tilde{A}_1}\}\ge\tilde{v}$.

Proof

(1) Necessity:

$$\tilde{A}_1>_{SO}\tilde{A}_2\Rightarrow\begin{cases}\min\left\{\rho_{\tilde{A}_2}|\rho_{\tilde{A}_2}\in\tilde{T}_{\tilde{A}_2}\right\}-\max\left\{\rho_{\tilde{A}_1}|\rho_{\tilde{A}_1}\in\tilde{T}_{\tilde{A}_1}\right\}\ge\tilde{v}\\ \min\left\{\varsigma_{\tilde{A}_2}|\varsigma_{\tilde{A}_2}\in\tilde{I}_{\tilde{A}_2}\right\}-\max\left\{\varsigma_{\tilde{A}_1}|\varsigma_{\tilde{A}_1}\in\tilde{I}_{\tilde{A}_1}\right\}\ge\tilde{v}\\ \min\left\{\tau_{\tilde{A}_2}|\tau_{\tilde{A}_2}\in\tilde{F}_{\tilde{A}_2}\right\}-\max\left\{\tau_{\tilde{A}_1}|\tau_{\tilde{A}_1}\in\tilde{F}_{\tilde{A}_1}\right\}\ge\tilde{v}\end{cases}.$$

According to Definition 11, if $\tilde{A}_1>_{SO}\tilde{A}_2$, then $\varpi_{\tilde{p},\tilde{v}}(\tilde{A}_1,\tilde{A}_2)-\varpi_{\tilde{p},\tilde{v}}(\tilde{A}_2,\tilde{A}_1)=1$. Since $0<\varpi_{\tilde{p},\tilde{v}}(\tilde{A}_1,\tilde{A}_2)<1$ and $0<\varpi_{\tilde{p},\tilde{v}}(\tilde{A}_2,\tilde{A}_1)<1, \varpi_{\tilde{p},\tilde{v}}(\tilde{A}_1,\tilde{A}_2)=1$ and $\varpi_{\tilde{p},\tilde{v}}(\tilde{A}_2,\tilde{A}_1)=0$ can be obtained. Thus, $\varpi_{\tilde{p},\tilde{v}}(\tilde{T}_{\tilde{A}_1},\tilde{T}_{\tilde{A}_2})=\varpi_{\tilde{p},\tilde{v}}(\tilde{I}_{\tilde{A}_1},\tilde{I}_{\tilde{A}_2})=\varpi_{\tilde{p},\tilde{v}}(\tilde{F}_{\tilde{A}_1},\tilde{F}_{\tilde{A}_2})=1$, i.e., $\frac{1}{l(\tilde{T}_{\tilde{A}_1})}\sum_{\rho_{\tilde{A}_1}\in\tilde{T}_{\tilde{A}_1}}\ \min_{\rho_{\tilde{A}_2}\in\tilde{T}_{\tilde{A}_2}} d_{\tilde{p},}\ \tilde{v}(\rho\ \tilde{A}_1,\rho\ \tilde{A}_2)=\frac{1}{l(\tilde{I}_{\tilde{A}_1})}\sum\ \varsigma_{\tilde{A}_1}\in\tilde{I}_{\tilde{A}}\ \ 1\min\ \ \varsigma_{\tilde{A}_2}\in\tilde{I}_{\tilde{A}_2}d_{\tilde{p},\tilde{v}}(\varsigma_{\tilde{A}_1},\varsigma\ \ \tilde{A}_2)=\frac{1}{l(\tilde{F}_{\tilde{A}_1})}\sum_{\tau_{\tilde{A}_1}\in\tilde{F}_{\tilde{A}_1}}\min_{\tau_{\tilde{A}_2}\in\tilde{F}_{\tilde{A}_2}}d_{\tilde{p},\tilde{v}}(\tau_{\tilde{A}_1},\tau_{\tilde{A}_2})=1$. Derived from Definition 7, $d_{\tilde{p},\tilde{v}}(\rho_{\tilde{A}_1},\rho_{\tilde{A}_2})$, $d_{\tilde{p},\tilde{v}}(\varsigma_{\tilde{A}_1},\varsigma_{\tilde{A}_2}), d_{\tilde{p},\tilde{v}}(\tau_{\tilde{A}_1},\tau_{\tilde{A}_2})\in[0,1]$ so $d_{\tilde{p},\tilde{v}}(\rho_{\tilde{A}_1},\rho_{\tilde{A}_2})=d_{\tilde{p},\tilde{v}}(\varsigma_{\tilde{A}_1},\varsigma_{\tilde{A}_2})=d_{\tilde{p},\tilde{v}}(\tau_{\tilde{A}_1},\tau_{\tilde{A}_2})=1$. Hence, $\rho_{\tilde{A}_2}-\rho_{\tilde{A}_1}\ge\tilde{v}$ for any $\rho_{\tilde{A}_1}\in\tilde{T}_{\tilde{A}_1},\rho_{\tilde{A}_2}\in\tilde{T}_{\tilde{A}_2},\varsigma_{\tilde{A}_2}-\varsigma_{\tilde{A}_1}\ge\tilde{v}$ for any $\varsigma_{\tilde{A}_1}\in\tilde{I}_{\tilde{A}_1},\varsigma_{\tilde{A}_2}\in\tilde{I}_{\tilde{A}_2},\tau_{\tilde{A}_2}-\tau_{\tilde{A}_1}\ge\tilde{v}$, for any $\tau_{\tilde{A}_1}\in\tilde{F}_{\tilde{A}_1},\tau_{\tilde{A}_2}\in\tilde{F}_{\tilde{A}_2}$. Therefore, $\min\{\rho_{\tilde{A}_2}|\rho_{\tilde{A}_2}\in\tilde{T}_{\tilde{A}_2}\}-\max\{\rho_{\tilde{A}_1}|\rho_{\tilde{A}_1}\in\tilde{T}_{\tilde{A}_1}\}\ge\tilde{v}$,

$\min\{\varsigma_{\tilde{A}_2}|\varsigma_{\tilde{A}_2} \in \tilde{I}_{\tilde{A}_2}\} - \max\{\varsigma_{\tilde{A}_1}|\varsigma_{\tilde{A}_1} \in \tilde{I}_{\tilde{A}_1}\} \geq \tilde{v}$ and $\min\{\tau_{\tilde{A}_2}|\tau_{\tilde{A}_2} \in \tilde{F}_{\tilde{A}_2}\} - \max\{\tau_{\tilde{A}_1}|\tau_{\tilde{A}_1} \in \tilde{F}_{\tilde{A}_1}\} \geq \tilde{v}$ are certainly valid.

(2) Sufficiency:

$$\left.\begin{array}{r}\min\left\{\rho_{\tilde{A}_2}|\rho_{\tilde{A}_2} \in \tilde{T}_{\tilde{A}_2}\right\} - \max\left\{\rho_{\tilde{A}_1}|\rho_{\tilde{A}_1} \in \tilde{T}_{\tilde{A}_1}\right\} \geq \tilde{v}\\ \min\left\{\varsigma_{\tilde{A}_2}|\varsigma_{\tilde{A}_2} \in \tilde{I}_{\tilde{A}_2}\right\} - \max\left\{\varsigma_{\tilde{A}_1}|\varsigma_{\tilde{A}_1} \in \tilde{I}_{\tilde{A}_1}\right\} \geq \tilde{v}\\ \min\left\{\tau_{\tilde{A}_2}|\tau_{\tilde{A}_2} \in \tilde{F}_{\tilde{A}_2}\right\} - \max\left\{\tau_{\tilde{A}_1}|\tau_{\tilde{A}_1} \in \tilde{F}_{\tilde{A}_1}\right\} \geq \tilde{v}\end{array}\right\}$$
$$\Rightarrow \tilde{A}_1 >_{SO} \tilde{A}_2.$$

Since $\min\{\rho_{\tilde{A}_2}|\rho_{\tilde{A}_2} \in \tilde{T}_{\tilde{A}_2}\} - \max\{\rho_{\tilde{A}_1}|\rho_{\tilde{A}_1} \in \tilde{T}_{\tilde{A}_1}\} \geq \tilde{v}$, $\rho_{\tilde{A}_2} - \rho_{\tilde{A}_1} \geq \tilde{v}$ for any $\rho_{\tilde{A}_1} \in \tilde{T}_{\tilde{A}_1}, \rho_{\tilde{A}_2} \in \tilde{T}_{\tilde{A}_2}$. Derived from Definition 7, $d_{\tilde{p},\tilde{v}}(\rho_{\tilde{A}_1}, \rho_{\tilde{A}_2}) = 1$ and $d_{\tilde{p},\tilde{v}}(\rho_{\tilde{A}_2}, \rho_{\tilde{A}_1}) = 0$ can be obtained. Therefore $\frac{1}{l(\tilde{T}_{\tilde{A}_1})}\sum_{\rho_{\tilde{A}_1}\in\tilde{T}_{\tilde{A}_1}} \min_{\rho_{\tilde{A}_2}\in\tilde{T}_{\tilde{A}_2}} d_{\tilde{p},\tilde{v}}(\rho_{\tilde{A}_1}, \rho_{\tilde{A}_2}) = 1$ and $\frac{1}{l(\tilde{T}_{\tilde{A}_2})}\sum_{\rho_{\tilde{A}_2}\in\tilde{T}_{\tilde{A}_2}} \min_{\rho_{\tilde{A}_1}\in\tilde{T}_{\tilde{A}_1}} d_{\tilde{p},\tilde{v}}(\rho_{\tilde{A}_2}, \rho_{\tilde{A}_1}) = 1$, which indicates $\varpi_{\tilde{p},\tilde{v}}(\tilde{T}_{\tilde{A}_1}, \tilde{T}_{\tilde{A}_2}) = 1$ and $\varpi_{\tilde{p},\tilde{v}}(\tilde{T}_{\tilde{A}_2}, \tilde{T}_{\tilde{A}_1}) = 0$ based on Definition 10. Similarly, $\varpi_{\tilde{p},\tilde{v}}(\tilde{I}_{\tilde{A}_1}, \tilde{I}_{\tilde{A}_2}) = 1$ and $\varpi_{\tilde{p},\tilde{v}}(\tilde{I}_{\tilde{A}_2}, \tilde{I}_{\tilde{A}_1}) = 0, \varpi_{\tilde{p},\tilde{v}}(\tilde{F}_{\tilde{A}_1}, \tilde{F}_{\tilde{A}_2}) = 1$ and $\varpi_{\tilde{p},\tilde{v}}(\tilde{F}_{\tilde{A}_2}, \tilde{F}_{\tilde{A}_1}) = 0$ can be achieved. Therefore, based on Definition 11, $\varpi_{\tilde{p},\tilde{v}}(\tilde{A}_1,\tilde{A}_2) - \varpi_{\tilde{p},\tilde{v}}(\tilde{A}_2,\tilde{A}_1) = 1$ and $\tilde{A}_1 >_{SO} \tilde{A}_2$ are obtained.

Property 7 *Let $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3 \in MVNNS$, and $\tilde{p}$ and $\tilde{v}(\tilde{p} < \tilde{v})$ be two thresholds. If $\tilde{A}_1 >_{SO} \tilde{A}_2$ and $\tilde{A}_2 >_{SO} \tilde{A}_3$, then $\tilde{A}_1 >_{SO} \tilde{A}_3$.*

Proof According to Property 6, if $\tilde{A}_1 >_{SO} \tilde{A}_2$, then $\min\{\rho_{\tilde{A}_2}|\rho_{\tilde{A}_2} \in \tilde{T}_{\tilde{A}_2}\} - \max\{\rho_{\tilde{A}_1}|\rho_{\tilde{A}_1} \in \tilde{T}_{\tilde{A}_1}\} \geq \tilde{v}$, $\min\{\varsigma_{\tilde{A}_2}|\varsigma_{\tilde{A}_2} \in \tilde{I}_{\tilde{A}_2}\} - \max\{\varsigma_{\tilde{A}_1}|\varsigma_{\tilde{A}_1} \in \tilde{I}_{\tilde{A}_1}\} \geq \tilde{v}$ and $\min\{\tau_{\tilde{A}_2}|\tau_{\tilde{A}_2} \in \tilde{F}_{\tilde{A}_2}\} - \max\{\tau_{\tilde{A}_1}|\tau_{\tilde{A}_1} \in \tilde{F}_{\tilde{A}_1}\} \geq \tilde{v}$. So $\max\{\rho_{\tilde{A}_2}|\ \rho_{\tilde{A}_2} \in \tilde{T}_{\tilde{A}_2}\} - \max\{\rho_{\tilde{A}_1}|\rho_{\tilde{A}_1} \in \tilde{T}_{\tilde{A}_1}\} \geq \tilde{v}$, $\max\{\varsigma_{\tilde{A}_2}|\varsigma_{\tilde{A}_2} \in \tilde{I}_{\tilde{A}_2}\} - \max\{\varsigma_{\tilde{A}_1}|\varsigma_{\tilde{A}_1} \in \tilde{I}_{\tilde{A}_1}\} \geq \tilde{v}$ and $\max\{\tau_{\tilde{A}_2}|\tau_{\tilde{A}_2} \in \tilde{F}_{\tilde{A}_2}\} - \max\{\tau_{\tilde{A}_1}|\tau_{\tilde{A}_1} \in \tilde{F}_{\tilde{A}_1}\} \geq \tilde{v}$ can be obtained.

If $\tilde{A}_2 >_{SO} \tilde{A}_3$, then $\min\{\rho_{\tilde{A}_3}|\rho_{\tilde{A}_3} \in \tilde{T}_{\tilde{A}_3}\} - \max\{\rho\ \tilde{A}_2|\rho_{\tilde{A}_2} \in \tilde{T}\ \ \tilde{A}_2\} \geq \tilde{v}$, $\min\{\varsigma_{\tilde{A}_3}|\varsigma_{\tilde{A}_3} \in \tilde{I}_{\tilde{A}_3}\} - \max\{\varsigma_{\tilde{A}_2}|\varsigma_{\tilde{A}_2} \in \tilde{I}_{\tilde{A}_2}\} \geq \tilde{v}$ and $\min\{\tau_{\tilde{A}_3}|\tau_{\tilde{A}_3} \in \tilde{F}_{\tilde{A}_3}\} - \max\{\tau_{\tilde{A}_2}|\tau_{\tilde{A}_2} \in \tilde{F}_{\tilde{A}_2}\} \geq \tilde{v}$ can be achieved.

Therefore, the further derivations are:

$$\left.\begin{array}{r}\max\left\{\rho_{\tilde{A}_2}|\rho_{\tilde{A}_2} \in \tilde{T}_{\tilde{A}_2}\right\} - \max\left\{\rho_{\tilde{A}_1}|\rho_{\tilde{A}_1} \in \tilde{T}_{\tilde{A}_1}\right\} \geq \tilde{v}\\ \min\left\{\rho_{\tilde{A}_3}|\rho_{\tilde{A}_3} \in \tilde{T}_{\tilde{A}_3}\right\} - \max\left\{\rho_{\tilde{A}_2}|\rho_{\tilde{A}_2} \in \tilde{T}_{\tilde{A}_2}\right\} \geq \tilde{v}\end{array}\right\}$$
$$\Rightarrow \min\left\{\rho_{\tilde{A}_3}|\rho_{\tilde{A}_3} \in \tilde{T}_{\tilde{A}_3}\right\} - \max\left\{\rho_{\tilde{A}_1}|\rho_{\tilde{A}_1} \in \tilde{T}_{\tilde{A}_1}\right\} \geq 2\tilde{v} > \tilde{v},$$

$$\left.\begin{array}{r}\max\left\{\varsigma_{\tilde{A}_2}|\varsigma_{\tilde{A}_2} \in \tilde{I}_{\tilde{A}_2}\right\} - \max\left\{\varsigma_{\tilde{A}_1}|\varsigma_{\tilde{A}_1} \in \tilde{I}_{\tilde{A}_1}\right\} \geq \tilde{v}\\ \min\left\{\varsigma_{\tilde{A}_3}|\varsigma_{\tilde{A}_3} \in \tilde{I}_{\tilde{A}_3}\right\} - \max\left\{\varsigma_{\tilde{A}_2}|\varsigma_{\tilde{A}_2} \in \tilde{I}_{\tilde{A}_2}\right\} \geq \tilde{v}\end{array}\right\}$$
$$\Rightarrow \min\left\{\varsigma_{\tilde{A}_3}|\varsigma_{\tilde{A}_3} \in \tilde{I}_{\tilde{A}_3}\right\} - \max\left\{\varsigma_{\tilde{A}_1}|\varsigma_{\tilde{A}_1} \in \tilde{I}_{\tilde{A}_1}\right\} \geq 2\tilde{v} > \tilde{v},$$

and

$$\left.\begin{array}{r}\max\left\{\tau_{\tilde{A}_2}|\tau_{\tilde{A}_2} \in \tilde{F}_{\tilde{A}_2}\right\} - \max\left\{\tau_{\tilde{A}_1}|\tau_{\tilde{A}_1} \in \tilde{F}_{\tilde{A}_1}\right\} \geq \tilde{v}\\ \min\left\{\tau_{\tilde{A}_3}|\tau_{\tilde{A}_3} \in \tilde{F}_{\tilde{A}_3}\right\} - \max\left\{\tau_{\tilde{A}_2}|\tau_{\tilde{A}_2} \in \tilde{F}_{\tilde{A}_2}\right\} \geq \tilde{v}\end{array}\right\}$$
$$\Rightarrow \min\left\{\tau_{\tilde{A}_3}|\tau_{\tilde{A}_3} \in \tilde{F}_{\tilde{A}_3}\right\} - \max\left\{\tau_{\tilde{A}_1}|\tau_{\tilde{A}_1} \in \tilde{F}_{\tilde{A}_1}\right\} \geq 2\tilde{v} > \tilde{v}.$$

Therefore, based on Property 6, $\tilde{A}_1 >_{SO} \tilde{A}_3$ can be obtained.

Property 8 Let $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3 \in MVNNS$, and $\tilde{p}$ and $\tilde{v}(\tilde{p} < \tilde{v})$ be two thresholds.

(1) The strictly opposed relations have the following properties.
① irreflexivity: $\forall\tilde{A}_1 \in MVNNS,\ \tilde{A}_1 >_S O\tilde{A}_1$;
② asymmetry: $\forall\tilde{A}_1, \tilde{A}_2 \in MVNNS,\ \tilde{A}_1 >_{SO} \tilde{A}_2 \Rightarrow \neg(\tilde{A}_2 >_{SO} \tilde{A}_1)$;
③ transitivity: $\forall\ \tilde{A}_1,\ \tilde{A}_2,\ \tilde{A}_3 \in MVNNS,\ \tilde{A}_1 >_{SO} \tilde{A}_2,\ \tilde{A}_2 >_{SO} \tilde{A}_3 \Rightarrow \tilde{A}_1 >_{SO} \tilde{A}_3$.

(2) The weakly opposed relations have the following properties.
④ irreflexivity: $\forall\tilde{A}_1 \in MVNNS,\ \tilde{A}_1 >_{WO}\tilde{A}_1$;
⑤ asymmetry: $\forall\tilde{A}_1, \tilde{A}_2 \in MVNNS,\ \tilde{A}_1 >_{WO} \tilde{A}_2 \Rightarrow \neg(\tilde{A}_2 >_{WO} \tilde{A}_1)$;
⑥ non-transitivity: $\exists\ \tilde{A}_1,\ \tilde{A}_2,\ \tilde{A}_3 \in MVNNS,\ \tilde{A}_1 > WO\,\tilde{A}_2,\ \tilde{A}_2 >_{WO} \tilde{A}_3 \nRightarrow \tilde{A}_1 >_{WO} \tilde{A}_3$.

(3) The indifferently opposed relations have the following properties.
⑦ reflexivity: $\forall\tilde{A}_1 \in MVNNS,\ \tilde{A}_1 \sim_O \tilde{A}_1$;
⑧ symmetry: $\forall\tilde{A}_1, \tilde{A}_2 \in MVNNS,\ \tilde{A}_1 \sim_O \tilde{A}_2 \Rightarrow \tilde{A}_2 \sim_O \tilde{A}_1$;
⑨ non-transitivity: $\exists\ \tilde{A}_1,\ \tilde{A}_2,\ \ \tilde{A}_3 \in MVNNS,\ \tilde{A}_1 \sim_O \tilde{A}_2,\ \tilde{A}_2 \sim_O \tilde{A}_3 \nRightarrow \tilde{A}_1 \sim_O \tilde{A}_3$.

According to Definitions 7, 10, 11, it can be seen that ①–⑤, ⑦ and ⑧ are true and ⑥ and ⑨ can be exemplified.

Example 4 Let $\tilde{p} = 0.15$ and $\tilde{v} = 0.2$. ⑥ and ⑨ can be exemplified as follows.

(1) If $\tilde{A}_1 = \langle\{0.1, 0.2\}, \{0.05\}, \{0.1\}\rangle$, $\tilde{A}_2 = \langle\{0.3, 0.4\}, \{0.15\}, \{0.2\}\rangle$ and $\tilde{A}_3 = \langle\{0.5, 0.6\},\ \{0.25\}, \{0.3\}\rangle$ are three MVNNs, then $\varpi_{\tilde{p},\tilde{v}}(\tilde{A}_1,\ \tilde{A}_2) - \varpi_{\tilde{p},\tilde{v}}(\tilde{A}_2,\ \ \tilde{A}_1) = 0.1667$, $\varpi_{\tilde{p},\tilde{v}}(\tilde{A}_2, \tilde{A}_3) - \varpi_{\tilde{p},\tilde{v}}(\tilde{A}_3, \tilde{A}_2) = 0.1667$ and $\varpi_{\tilde{p},\tilde{v}}(\tilde{A}_1, \tilde{A}_3) - \varpi_{\tilde{p},\tilde{v}}(\tilde{A}_3, \tilde{A}_1) = 1$. We

have $\tilde{A}_1 >_{WO} \tilde{A}_2$, $\tilde{A}_2 >_{WO} \tilde{A}_3$ but $\tilde{A}_1 >_{WO} \tilde{A}_3$. Thus, the weak opposition relations are non-transitive.

(2) If $\tilde{A}_1 = \langle\{0.1\},\{0.2\},\{0.3\}\rangle$, $\tilde{A}_2 = \langle\{0.25\},\{0.1\},\{0.3\}\rangle$ and $\tilde{A}_3 = \langle\{0.3,0.4\},\{0.15\},\{0.25\}\rangle$ are three MVNNs, then $\varpi_{\tilde{p},\tilde{v}}(\tilde{A}_1,\tilde{A}_2) - \varpi_{\tilde{p},\tilde{v}}(\tilde{A}_2,\tilde{A}_1) = 0$, $\varpi_{\tilde{p},\tilde{v}}(\tilde{A}_2,\tilde{A}_3) - \varpi_{\tilde{p},\tilde{v}}(\tilde{A}_3,\tilde{A}_2) = 0$ and $\varpi_{\tilde{p},\tilde{v}}(\tilde{A}_1,\tilde{A}_3) - \varpi_{\tilde{p},\tilde{v}}(\tilde{A}_3,\tilde{A}_1) = 0.1667$. We have $\tilde{A}_1 >_O \tilde{A}_2, \tilde{A}_2 >_O \tilde{A}_3$, but $\tilde{A}_1 >_{WO} \tilde{A}_3$. Thus, the indifferent opposition relations are non-transitive.

4 An ELECTRE approach for MCDM problems where weights and data are in the form of MVNNs

In this section, an extended ELECTRE approach is proposed to solve the MCDM problems where both the assessments of alternatives with respect to criteria and the weights of criteria are in the form of MVNNs.

Let $\psi = \{\psi_1, \psi_2, \ldots, \psi_n\}$ be a set of alternatives and $C = \{c_1, c_2, \ldots, c_m\}$ be a set of criteria, and the weight of criterion w_j be expressed by MVNNs, i.e., $w_j = \langle \tilde{T}_{w_j}, \tilde{I}_{w_j}, \tilde{F}_{w_j}\rangle$ $(j = 1,2,\ldots,m)$. Assume that the evaluation value of $\tilde{\psi}_i$ for criterion $c_j$is characterized as a MVNNs $\psi_{ij} = \langle \tilde{T}_{\psi_{ij}}, \tilde{I}_{\psi_{ij}}, \tilde{F}_{\psi_{ij}}\rangle$. Here $\tilde{T}_{\psi_{ij}}$ represents the truth degree that the alternative ψ_i satisfies the criterion c_j, and truth-membership function, $\tilde{I}_{\psi_{ij}}$ represents the indeterminacy degree that the alternative ψ_i satisfies the criterion c_j, and $\tilde{F}_{\psi_{ij}}$ represents the falsity degree that the alternative ψ_i satisfies the criterion c_j. Thus, the multi-valued neutrosophic decision matrix can be denoted by $R = (\psi_{ij})_{n\times m}$.

Generally speaking, the criteria can be divided into two types, cost criteria and benefit criteria. Therefore, in order to unify all criteria, the cost criteria should be transformed into benefit criteria as follows [46]:

$$\tilde{\psi}_{ij} = \begin{cases} \psi_{ij}, & \text{for benefit criteria } c_j \\ (\psi_{ij})^c, & \text{for cost criteria } c_j \end{cases}, \quad (i = 1,2,\ldots,n;\ j = 1,2,\ldots,m). \tag{11}$$

Here $(\psi_{ij})^c$is the complement of ψ_{ij}.

Step 1 Construct the normalized decision matrix $R = (\psi_{ij})_{n\times m}$.

Based on Eq. (11), the multi-valued neutrosophic decision matrix $R = (\psi_{ij})_{n\times m}$ can be transformed into a normalized MVNN decision matrix $\tilde{R} = (\tilde{\psi}_{ij})_{n\times m}$.

If the criteria are the cost type, then

$$\tilde{\psi}_{ij} = (\psi_{ij})^c = \left\langle \cup_{\tau\in\tilde{F}_{\psi_{ij}}}\{\tau\}, \cup_{\varsigma\in\tilde{I}_{\psi_{ij}}}\{1-\varsigma\}, \cup_{\rho\in\tilde{T}_{\psi_{ij}}}\{\rho\}\right\rangle; \tag{12}$$

if the criteria are the benefit type, then

$$\tilde{\psi}_{ij} = \psi_{ij} = \left\langle \cup_{\rho\in\tilde{T}_{\psi_{ij}}}\{\rho\}, \cup_{\varsigma\in\tilde{I}_{\psi_{ij}}}\{\varsigma\}, \cup_{\tau\in\tilde{F}_{\psi_{ij}}}\{\tau\}\right\rangle. \tag{13}$$

Step 2 Construct the weighted normalized matrix.

Based on the weights of criteria and the operations (more details can be founded in Peng et al. [25]), the weighted normalized decision matrix is formulated as follows:

$$\tilde{\psi}^*_{ij} = \tilde{\psi}_{ij} \otimes w_j \quad (i = 1,2,\ldots,n;\ j = 1,2,\ldots,m). \tag{14}$$

where $\tilde{w}_j$ is the weight of the j-th criterion.

Step 3 Construct the concordance set of subscripts.

The concordance set of subscripts, which should satisfy the constraint $\tilde{\psi}^*_{ij} >_S \tilde{\psi}^*_{kj}$ or $\tilde{\psi}^*_{ij} >_W \tilde{\psi}^*_{kj}$ or $\tilde{\psi}^*_{ij} \sim \tilde{\psi}^*_{kj}$, is represented as:

$$O_{ik} = \left\{ j \middle| \mu_{\tilde{p},\tilde{q}}\left(\tilde{\psi}^*_{ij}, \tilde{\psi}^*_{kj}\right) - \mu_{\tilde{p},\tilde{q}}\left(\tilde{\psi}^*_{kj}, \tilde{\psi}^*_{ij}\right) \geq 0 \right\} \quad (i,k = 1,2,\ldots,n). \tag{15}$$

Here $\mu_{\tilde{p},\tilde{q}}(\tilde{\psi}^*_{ij}, \tilde{\psi}^*_{kj})$ is the concordance index between $\tilde{\alpha}^*_{ij}$ and $\tilde{\alpha}^*_{kj}$ and can be obtained by using Eq. (7) in Definition 8.

Step 4 Construct the concordance matrix.According to the weight vector w associated to the criteria, the concordance index $C(\psi_i, \psi_k)$ is obtained as follows:

$$C(\psi_i, \psi_k) = s(c^*(\psi_i, \psi_k)). \tag{16}$$

where $c^*(\psi_i, \psi_k) = \oplus_{j\in O_{ik}} w_j$ and $s(\cdot) = \frac{1}{l_{\tilde{T}_{(\cdot)}} \cdot l_{\tilde{I}_{(\cdot)}} \cdot l_{\tilde{F}_{(\cdot)}}} \sum_{\rho\in\tilde{T}_{(\cdot)},\varsigma\in\tilde{I}_{(\cdot)},\tau\in\tilde{F}_{(\cdot)}} (\rho+\varsigma+\tau)/3$. Here $\rho \in \tilde{T}_{(\cdot)}$, $\varsigma \in \tilde{I}_{(\cdot)}$ and $\tau \in \tilde{F}_{(\cdot)}$; $l_{\tilde{T}_{(\cdot)}}$, $l_{\tilde{I}_{(\cdot)}}$ and $l_{\tilde{F}_{(\cdot)}}$ represent the number of element in $\tilde{T}_{(\cdot)}$, $\tilde{I}_{(\cdot)}$ and $\tilde{F}_{(\cdot)}$,, respectively.

Therefore, the concordance matrix C is

$$C = \begin{pmatrix} - & c_{12} & \cdots & c_{1n} \\ c_{21} & - & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n1} & c_{n2} & \cdots & - \end{pmatrix}. \tag{17}$$

Step 5 Determine the credibility index of outranking relations.

$$\sigma(\psi_i, \psi_k) = C(\psi_i, \psi_k) \cdot \prod_{j=1}^{m} \delta_j(\psi_i, \psi_k), \tag{18}$$

where

$$\delta_j(\psi_i, \psi_k) = \begin{cases} \dfrac{1 - \varpi_{\tilde{p},\tilde{q}}\left(\tilde{\psi}^*_{ij}, \tilde{\psi}^*_{kj}\right)}{1 - C(\psi_i, \psi_k)} & \text{if } \varpi_{\tilde{p},\tilde{q}}\left(\tilde{\psi}^*_{ij}, \tilde{\psi}^*_{kj}\right) > C(\psi_i, \psi_k) \\ 1 & \text{otherwise} \end{cases}.$$

Here $\varpi_{\tilde{p},\tilde{q}}(\tilde{\psi}^*_{ij}, \tilde{\psi}^*_{kj})$ is the discordance index between $\tilde{\psi}^*_{ij}$ and $\tilde{\psi}^*_{kj}$ and can be obtained by using Eq. (9) in Definition 10.

Step 6 Determine the ranking of the alternatives' indices.

Based on descending and ascending distillations, the ranking of the alternatives' indices can be defined in two preorders. If $\lambda = \max_{\psi_i,\psi_k \in \psi} \sigma(\psi_i, \psi_k), \lambda - \kappa(\lambda)$ is a credibility value such that $\kappa(\lambda)$ is sufficiently close to λ, then considering that the distance between λ and $\kappa(\lambda)$ (more details about the values of $\kappa(\lambda)$ can be found in [33]) is sufficiently small, define S as:

$$S(\psi_i, \psi_k) = \begin{cases} 1, & \text{if } \sigma(\psi_i, \psi_k) > \lambda - \kappa(\lambda) \\ 0, & \text{otherwise} \end{cases}. \tag{19}$$

Based on Eq. (19), it can be seen that the final score for each alternative is the number of alternatives that are outranked by ψ_i, minus the number of alternatives that outrank ψ_i.

The descending distillation is realized by firstly retaining the alternative with the highest score and then applying the same procedure to the remaining alternatives. The ascending distillation is similar to the descending distillation, except that the lowest score is chosen instead of the highest score.

Step 7 Rank all the alternatives.

5 Illustrative example

In this section, an example is provided (adapted from Wei [47]) for further illustration. The school of management in a Chinese university wants to recruit several excellent teachers from overseas to improve academic capability and strengthen the university's teaching quality. Then the dean of the management school and the human resource officer make up the panel of decision-makers and are mainly responsible for this recruitment. They make a strict evaluation for five alternatives, denoted by $\psi_1, \psi_2, \ldots, \psi_5$, according to the following four criteria: morality, research capability, teaching skills and educational background, denoted by c_1, c_2, c_3, c_4, with their corresponding weights being $w_1 = \langle\{0.3, 0.45\}, \{0.2\}, \{0.1\}\rangle$, $w_2 = \langle\{0.3\}, \{0.1\}, \{0.2\}\rangle$, $w_3 = \langle\{0.2\}, \{0.2\}, \{0.3\}\rangle$ and $w_4 = \langle\{0.3\}, \{0.3\}, \{0.2\}\rangle$. Moreover, the evaluation of five candidates $\psi_i (i = 1, 2, 3, 4, 5)$ is characterized by MVNNs by two decision-makers under the criterion $c_j (j = 1, 2, 3, 4)$. One decision-maker can provide several evaluation values for three membership degrees, respectively. In particular, if two decision-makers set the same value, then it is counted only once. Then, the multi-valued neutrosophic decision matrix $R = (\psi_{ij})_{5\times4}$ is obtained as follows:

$$R = \begin{pmatrix} \langle\{0.5,0.7\},\{0.3\},\{0.3\}\rangle & \langle\{0.4\},\{0.2\},\{0.1\}\rangle & \langle\{0.5\},\{0.3\},\{0.4\}\rangle & \langle\{0.5,0.6\},\{0.2\},\{0.3\}\rangle \\ \langle\{0.3\},\{0.2\},\{0.2\}\rangle & \langle\{0.6\},\{0.2\},\{0.3\}\rangle & \langle\{0.7\},\{0.2\},\{0.3\}\rangle & \langle\{0.4\},\{0.2\},\{0.1\}\rangle \\ \langle\{0.3\},\{0.2\},\{0.4\}\rangle & \langle\{0.4,0.5\},\{0.2\},\{0.2\}\rangle & \langle\{0.6\},\{0.1\},\{0.3\}\rangle & \langle\{0.5\},\{0.1\},\{0.2\}\rangle \\ \langle\{0.5\},\{0.3\},\{0.3\}\rangle & \langle\{0.6\},\{0.1\},\{0.2\}\rangle & \langle\{0.7\},\{0.2\},\{0.2,0.3\}\rangle & \langle\{0.6\},\{0.2\},\{0.2\}\rangle \\ \langle\{0.6\},\{0.2\},\{0.4\}\rangle & \langle\{0.7\},\{0.2\},\{0.4\}\rangle & \langle\{0.7\},\{0.3\},\{0.2\}\rangle & \langle\{0.5\},\{0.1\},\{0.3\}\rangle \end{pmatrix}.$$

5.1 Illustration of the proposed method

Step 1 Construct the normalized decision-making matrix.

Since c_1, c_2, c_3 and c_4 are of the benefit type, so the normalized multi-valued neutrosophic decision matrix $\tilde{R} = (\tilde{\psi}_{ij})_{4\times3} = (\psi_{ij})_{4\times3}$ can be obtained.

Step 2 Construct the weighted normalized matrix.

Based on the weights of criteria and the operations, the weighted normalized decision matrix is obtained as follows:

$$R = \begin{pmatrix} \langle\{0.15,0.225,0.21,0.315\},\{0.44\},\{0.37\}\rangle & \langle\{0.12\},\{0.28\},\{0.28\}\rangle & \langle\{0.1\},\{0.44\},\{0.58\}\rangle & \langle\{0.15,0.18\},\{0.44\},\{0.44\}\rangle \\ \langle\{0.09,0.135\},\{0.36\},\{0.28\}\rangle & \langle\{0.18\},\{0.28\},\{0.44\}\rangle & \langle\{0.14\},\{0.36\},\{0.51\}\rangle & \langle\{0.12\},\{0.44\},\{0.28\}\rangle \\ \langle\{0.09,0.135\},\{0.36\},\{0.46\}\rangle & \langle\{0.12,0.15\},\{0.28\},\{0.36\}\rangle & \langle\{0.12\},\{0.28\},\{0.51\}\rangle & \langle\{0.15\},\{0.37\},\{0.36\}\rangle \\ \langle\{0.15,0.225\},\{0.44\},\{0.37\}\rangle & \langle\{0.18\},\{0.19\},\{0.36\}\rangle & \langle\{0.14\},\{0.36\},\{0.44,0.51\}\rangle & \langle\{0.18\},\{0.44\},\{0.36\}\rangle \\ \langle\{0.18,0.27\},\{0.36\},\{0.46\}\rangle & \langle\{0.21\},\{0.28\},\{0.52\}\rangle & \langle\{0.14\},\{0.44\},\{0.44\}\rangle & \langle\{0.15\},\{0.37\},\{0.44\}\rangle \end{pmatrix}.$$

Step 3 Construct the concordance set of subscripts.

Let $\tilde{q}_j = 0.05$ and $\tilde{p}_j = 0.10$ and $\tilde{v}_j = 0.15$ be the thresholds for all criteria $c_j (j = 1, 2, 3, 4)$. By using

Eq. (15), since $\tilde{\psi}_{11}^{*} >_W \tilde{\psi}_{21}^{*}, \tilde{\psi}_{13}^{*} >_W \tilde{\psi}_{23}^{*}$ and $\tilde{\psi}_{14}^{*} >_W \tilde{\psi}_{24}^{*}$, we have $O_{12} = \{1, 3, 4\}$.

Thus, the concordance set of subscripts can be obtained:

$$O = (O_{ik}) = \begin{pmatrix} - & 1,3,4 & 1,3,4 & 1,2,3,4 & 1,3,4 \\ 2 & - & 2,3 & 2,3 & 3 \\ 2 & 1,4 & - & 2 & \emptyset \\ 2 & 1,4 & 1,3,4 & - & 3 \\ 2 & 1,2,4 & 1,2,3,4 & 1,2,4 & - \end{pmatrix}.$$

Step 4 Construct the concordance matrix.

By using Eq. (16), the concordance index c_{12} is obtained: $c^{*}(\psi_1, \psi_2) = w_1 \oplus w_3 \oplus w_4 = \langle\{0.608, 0.692\}, \{0.012\}, \{0.006\}\rangle$.

Thus, $c_{12} = s(c^{*}(\psi_1, \psi_2)) = 0.2227$.

Therefore, the concordance matrix is obtained and shown as below:

$$C = \begin{pmatrix} - & 0.2227 & 0.2227 & 0.2525 & 0.2227 \\ 0.2000 & - & 0.1733 & 0.1733 & 0.2333 \\ 0.2000 & 0.2142 & - & 0.2000 & 0 \\ 0.2000 & 0.2142 & 0.2227 & - & 0.2333 \\ 0.2000 & 0.2346 & 0.2525 & 0.2346 & - \end{pmatrix}.$$

Step 5 Determine the credibility index.

According to Step 4 and Eq. (18), the credibility index matrix is obtained:

$$\sigma = \begin{pmatrix} - & 0.1911 & 0.1640 & 0.2525 & 0.1911 \\ 0.1667 & - & 0.1398 & 0.1733 & 0.1764 \\ 0.1667 & 0.2142 & - & 0.2000 & 0 \\ 0.2000 & 0.2142 & 0.2227 & - & 0.2029 \\ 0.2000 & 0.2346 & 0.2525 & 0.2346 & - \end{pmatrix}.$$

Step 6 Determine the ranking of the alternatives' indices.

According to Step 5, $\lambda = \max_{\psi_i, \psi_j \in \psi} \sigma(\psi_i, \psi_j) = 0.2525$. If $\kappa(\lambda) = 0.05$, then

$$S(\psi_i, \psi_j) = \begin{pmatrix} - & 0 & 0 & 1 & 0 \\ 0 & - & 0 & 0 & 0 \\ 0 & 1 & - & 0 & 0 \\ 0 & 1 & 1 & - & 1 \\ 0 & 1 & 1 & 1 & - \end{pmatrix}.$$

Therefore the descending distillation is $\{\psi_5\} \to \{\psi_1, \psi_4\} \to \{\psi_3\} \to \{\psi_2\}$, the ascending distillation is $\{\psi_1, \psi_5\} \to \{\psi_4\} \to \{\psi_3\} \to \{\psi_2\}$, and the final ranking $\{\psi_5\} \to \{\psi_1\} \to \{\psi_4\} \to \{\psi_3\} \to \{\psi_2\}$ can be derived.

Step 7 Rank all the alternatives.

Thus the optimal alternative is ψ_5, while the worst alternative is ψ_2.

5.2 Comparison analysis

In order to verify the availability of the proposed approach based on outranking relations, a comparison analysis is conducted here. Since the methods outlined in Wang and Li [28], Ye [29] and Peng et al. [32] can deal with multi-valued neutrosophic MCDM or MCGDM problems, so these methods are selected to compare the developed approach. However, the methods in Wang and Li [28], Ye [29] and Peng et al. [32] fail to handle the MCDM or MCGDM problems where both of evaluation of alternative and the criteria are in the form of MVNNs. Therefore, the weight of criteria should be modified as $w = (0.15, 0.20, 0.35, 0.30)$ to facilitate the comparative analysis conducted on the same illustrative example.

With regard to the method in Wang and Li [28], it can be used to deal with the MCDM problem directly. Then the comprehensive value can be determined by using the TODIM method in Wang and Li [29]. For the method in Ye [29], the single-valued neutrosophic hesitant fuzzy weighted averaging (SVNHFWA) operator and single-valued neutrosophic hesitant fuzzy weighted geometric (SVNHFWG) operator are used, respectively, to aggregate the evaluation values, and then final ranking can be obtained based on the cosine measure. With regard to the method in Peng et al. [32], it is used to handle MCGDM problems. So the example in Sect. 5 can be seen as there are four decision-makers to make decisions and the weight of experts is determined as $\lambda = (0.25, 0.25, 0.25, 0.25)$. Therefore, two power aggregation operators were utilized to aggregate the multi-valued neutrosophic information first; and the score function and accuracy function were obtained and utilized to determine the final ranking of all the alternatives.

Therefore, if the methods in Wang and Li [28], Ye [29], Peng et al. [32] and the developed method are used to handle the modified example, then the final results are shown in Table 1.

Table 1 Compared results utilizing the different methods with MVNNs

Methods	The final ranking	The best alternative(s)	The worst alternative(s)
Wang and Li [28]	$\psi_5 \succ \psi_1 \succ \psi_3 \succ \psi_2 \succ \psi_4$	ψ_5	ψ_4
Ye [29]	$\psi_1 \succ \psi_5 \succ \psi_3 \succ \psi_4 \succ \psi_2$ or $\psi_5 \succ \psi_1 \succ \psi_3 \succ \psi_4 \succ \psi_2$	ψ_1 or ψ_4	ψ_2
Peng et al. [32]	$\psi_5 \succ \psi_1 \succ \psi_3 \succ \psi_2 \succ \psi_4$ or $\psi_1 \succ \psi_5 \succ \psi_3 \succ \psi_4 \succ \psi_2$	ψ_5 or ψ_1	ψ_2 or ψ_4
The proposed method	$\psi_5 \succ \psi_1 \succ \psi_3 \succ \psi_2 \succ \psi_4$	ψ_5	ψ_4

According to the results presented in Table 1, it can be seen that if the SVNHFWA and SVNHFWG operators presented in Ye [29] are used, respectively, then the final ranking is $\psi_1 \succ \psi_5 \succ \psi_3 \succ \psi_4 \succ \psi_2$ or $\psi_5 \succ \psi_1 \succ \psi_3$ - $\psi_4 \succ \psi_2$. If the TODIM method in Wang and Li [28] and the proposed approach are utilized, then the optimal alternative is ψ_5 while the worst alternative is ψ_4. If the power aggregation operators proposed by Peng et al. [32] are used, then the optimal alternative is ψ_5 or ψ_1, while the worst alternative is ψ_4 or ψ_2. Apparently, the final order obtained by the developed approach is different from the results by using the method in Ye [29]. However, it is same as those that use the method of Wang and Li [28] and the power weighted average operator of Peng et al. [32], and the final ranking is always $\psi_5 \succ \psi_1 \succ \psi_3 \succ \psi_2 \succ \psi_4$.

Based on the comparative analyses presented above, two mainly advantages of the proposed method can be summarized. In the first place, the methods in Ye [29] and Peng et al. [32] involve in different aggregation operators, which always lead to different final results. Moreover, the number of operations and the size of results will exponentially increase if more MVNNs are involved in the operations. However, the proposed approach could avoid these shortcomings and make the decision-making process simple. There is one more point, all the methods in Wang and Li [28], Ye [29] and Peng et al. [32] cannot handle the multi-valued neutrosophic problems in which the preference information of alternatives and the weights of criteria are expressed by MVNN, which may confine the application of the method. But the proposed method can describe the evaluation information more flexibly and hold the integrity of original decision-making data, which causes the final results to correspond to practical decision-making problems more closely.

6 Conclusions

MVNSs, as a combination of SNSs and HFSs, present the additional capability to deal with uncertainty, incomplete and imprecise information. Therefore, based on the traditional ELECTRE method and the related research achievements of SNSs and HFSs, some outranking relations of MVNNs were developed. Then their properties were investigated in detail. Moreover, those outranking relations are applied to MCDM problems, in which the evaluation values on criteria of alternatives and weights are characterized by MVNNs. At last, one illustrative example was presented to illustrate the effectiveness of the proposed approach. The advantage of this paper is that an outranking approach for MCDM problems with MVNNs can avoid the drawbacks of the existing methods as were discussed earlier and can handle MCDM problems where the data and weights of criteria are expressed by MVNNs. In future research, we will continue to working on the approach of obtaining the optimal values of $\tilde{p}$ and $\tilde{v}$ in ELECTRE by using a specific model.

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Compliance with ethical standards

Conflict of interest The authors declare that there is no conflict of interest regarding the publication of this paper.

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