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ORIGINAL ARTICLE

A likelihood-based assignment method for multiple criteria decision analysis with interval type-2 fuzzy information

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Abstract The purpose of this paper was to develop a likelihood-based assignment method based on interval type-2 fuzzy sets and apply it to decision-making problems involving multiple criteria evaluation and the ranking/selection of alternatives. The linear assignment method is a well-known outranking method in the field of multiple criteria decision analysis. The theory of interval type-2 fuzzy sets is useful for addressing the uncertainty and imprecision associated with a subjective environment. In this paper, the key feature of the proposed method is the incorporation of the extended concept of likelihoods of fuzzy preference relations between interval type-2 trapezoidal fuzzy numbers into the main structure of the linear assignment methodology. An effective ranking procedure using the optimal membership degree determination method is proposed to determine criterion-wise preference rankings of the alternatives. The proposed method establishes the novel concepts of an (adjusted) rank frequency matrix and an (adjusted) rank contribution matrix to combine the relative performances of the alternatives in terms of each criterion. Based on a signed distance comparison approach, this paper constructs a likelihood-based assignment model to obtain an aggregate ranking of the alternatives that is in the closest agreement with the criterion-wise preferences of the alternatives. The feasibility and applicability of the proposed method are illustrated with two practical multiple criteria decision-making applications concerning the selection of landfill sites and the selection of treatment options. Finally, a comparative analysis with other relevant methods is conducted to validate the effectiveness and advantages of the current methods in decision aiding.

Keywords Likelihood · Interval type-2 fuzzy set · Multiple criteria decision analysis · Rank frequency matrix · Rank contribution matrix

1 Introduction

The concept of type-2 fuzzy sets (T2 FSs), initially introduced by Zadeh [67], is an extension of a type-1 fuzzy set (T1 FS) in which the membership function falls into a fuzzy set in the interval [0, 1] [45, 70]. Type-2 fuzzy logic has a great ability to handle higher degrees of uncertainty [50]. However, the computational complexity of T2 FSs is very high, which makes it very difficult to employ them in practical applications [16, 29, 70, 73]. Therefore, considerable concern has arisen over interval type-2 fuzzy sets (IT2 FSs) as a special case of T2 FSs in practical fields [1, 34, 70]. The computations associated with IT2 FSs are manageable because their membership values take the form of crisp intervals [17]. Additionally, IT2 FSs can model the effects of uncertainties [53, 58] and handle imprecision and imperfect information in real-world applications [13, 14, 55, 72]; thus, they are gaining increasing popularity. At present, the interval type-2 fuzzy logic theory is in a mature state and has many areas of application [31].

Decision information is often determined according to decision-makers' opinions or assessments in multiple criteria decision-making situations [65]. Uncertainty is common in many real-life decision-making problems [46, 47]

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because decision-makers are not always certain about their given decision or preference information, and they often have some degree of uncertainty [48]. Accordingly, IT2 FSs have been successfully applied to tackle the issue of multiple criteria decision analysis (MCDA) in recent years [8, 16–18, 26, 30, 33, 39, 55, 59, 61, 69]. In particular, considerable studies have been conducted on fuzzy decision-making methods with interval type-2 trapezoidal fuzzy numbers (IT2 TrFNs) [3, 13–17, 20, 60, 62, 71, 72]. Furthermore, IT2 TrFNs can efficiently express linguistic evaluations or assessments by objectively transforming them into numerical variables [13, 72].

It is common for the evaluations of alternatives and the relative preference of criterion importance to be guided by decision-makers' subjective judgments in real-world situations. To address linguistic or numerical uncertainties associated with a subjective environment, the evaluative ratings of alternatives and the importance weights of criteria used in MCDA can be appropriately expressed as IT2 TrFNs. Most existing MCDA methodologies within the IT2 TrFN environment are characterized as scoring or compromising methods; there has thus far been relatively little research on outranking methods [16]. The present study is an attempt to supplement existing MCDA models and methods based on IT2 TrFNs. Because the extended versions of outranking methods have not been thoroughly investigated in the context of IT2 TrFNs [16], this paper is primarily concerned with the development of a new outranking model using IT2 TrFNs based on the main structure of the linear assignment methodology.

The linear assignment method, introduced by Bernardo and Blin [6], is a well-known outranking model for solving multiple criteria decision-making problems. Based on a set of criterion-wise rankings and a set of criterion weights, the linear assignment method combines the criterion-wise rankings into an overall ranking that reaches a best compromise among all criterion-wise rankings in a coherent and non-heuristic framework [11]. The linear assignment methodology has been enriched by several valuable developments, such as a fuzzy linear assignment approach [49]; a combined model using the analytic hierarchy process (AHP), the technique for order preference by similarity to an ideal solution (TOPSIS), and the linear assignment method [2]; a linear assignment method for ranking materials of engineering components [38]; a fuzzy linear assignment method for an initial order [4]; an interactive fuzzy linear assignment method [5]; a fuzzy group linear assignment method for ranking electronic business process management best practices based on the perspectives of a balanced scorecard [68]; a linear assignment model for group decision making with uncertain preference information [66]; a nonlinear assignment-based model with interval-valued intuitionistic fuzzy sets under incomplete and/or inconsistent preference information [11]; a wide range of sensitivity analyses for the linear assignment method [56]; a modified assignment method considering differences among criteria values or various stochastic elements [7]; and an interval type-2 fuzzy linear assignment method based on signed distances [13].

An increasing number of theoretical developments and applications have been conducted to improve the linear assignment methodology in the decision-making field. However, relatively few studies have investigated the linear assignment methodology within the interval type-2 fuzzy environment. At present, only Chen [13] used interval type-2 fuzzy numbers to capture imprecise or uncertain decision information in the fields that require the MCDA and developed the interval type-2 fuzzy linear assignment model based on the comparisons using signed distances. Moreover, Chen illustrated and discussed the proposed method by applying it to a case in which a landfill site is selected. Chen has demonstrated the usefulness and effectiveness of extending the linear assignment method to the interval type-2 fuzzy context. On the whole, within the literature on the linear assignment methodology, comparatively little research has focused on the development of the assignment models based on IT2 TrFNs in the interval type-2 fuzzy context. Therefore, this paper attempts to promote a novel and useful assignment method to handle MCDA problems based on IT2 TrFNs for the sake of capturing more imprecise or uncertain decision information for practical applications.

The purpose of this paper is to develop a likelihood-based assignment method for addressing MCDA problems within the decision environment of IT2 TrFNs. The proposed method features prominently in incorporating the extended concept of likelihoods of fuzzy preference relations between IT2 TrFNs into the core structure of the linear assignment methods. As a whole, the proposed method makes four main contributions to the linear assignment methodology. First, the extended concepts of lower likelihoods, upper likelihoods, and mean likelihoods regarding fuzzy preference relations between IT2 TrFNs are incorporated into the linear assignment structure. It follows that processing sophisticated IT2 TrFN data becomes simple and easy via the comparison approach using the likelihoods between IT2 TrFN evaluative ratings. Second, based on the optimal membership degree determination method, an effective ranking procedure is developed to determine criterion-wise preference rankings of the alternatives. Third, an innovative method using the concepts of an (adjusted) rank frequency matrix and an (adjusted) rank contribution matrix is proposed to establish a new structure of the likelihood-based assignment method. Fourth, a signed distances approach is utilized to construct a simple linear assignment model to acquire an overall preference ranking of the alternatives. The proposed method is new and unique compared with the linear assignment methodology that has previously been developed. Furthermore, to fully utilize information contained in the evaluative ratings and fuzzy preference relations between IT2 TrFNs, this paper determines the (adjusted) rank contribution matrix based on the IT2 TrFN importance weights and the optimal degrees of membership for making the information content complementary.

As mentioned previously, the interval type-2 fuzzy linear assignment model developed by Chen [13] is currently the most relevant linear assignment methodology for addressing MCDA problems within the IT2 TrFN environment. Chen [13] examined the feasibility and applicability of her proposed methods with a practical MCDA problem of landfill site selection. To maintain methodological relevance with Chen [13], this paper chooses the same problem of landfill site selection to not only illustrate the proposed method but also to allow for comparative discussion. Alternately, the linear assignment methodology belongs to outranking models in the decision-making field. Thus, this paper also conducts comparative analyses with other well-known outranking methods, including the method of elimination and choice expressing reality (ELECTRE) and the qualitative flexible multiple criteria method (QUALIFLEX), to examine the applicability and advantages of the proposed method.

Specifically, to demonstrate the feasibility and effectiveness of the proposed likelihood-based assignment method, we present an illustrative application to the selection problem of landfill sites and compare with the interval type-2 fuzzy linear assignment method based on signed distances [13]. Furthermore, the proposed method is applied to a medical decision-making problem that addresses the selection of treatment options. Comparative analyses are also provided to examine the validity and advantages of the proposed method relative to the existing outranking methods within the interval type-2 fuzzy environment, including the extended QUALIFLEX method using a signed distance-based approach [20] and the extended ELECTRE method [20]. Finally, this paper conducts a comparative discussion with the widely used simple additive weighting (SAW) method to further examine the distinct advantage of the proposed method.

This paper is organized as follows: Sect. 2 reviews the concept of IT2 FSs and IT2 TrFNs; Sect. 3 describes an MCDA problem within an environment of IT2 TrFNs and presents the concept of likelihoods of fuzzy preference relations between IT2 TrFNs; Sect. 4 develops a likelihood-based assignment method for MCDA involving the multiple criteria evaluation/selection of alternatives; Sect. 5 demonstrates the feasibility and the applicability of the proposed methodology using two practical MCDA applications concerning the selection of landfill sites and the selection of treatment options and conducts comparisons with other relevant methods, and Sect. 6 presents the conclusions.

2 Preliminaries

This section reviews selected relevant definitions of IT2 FSs [51, 52, 54, 57] and IT2 TrFNs [9, 13–17, 62] that are used throughout the paper.

Definition 1 [51, 54, 57] Let X be a crisp set. A mapping A: $X \to [0, 1]^{[0, 1]}$ is called a T2 FS defined on the universe of discourse X, and it is denoted by:

$$A = \{(x, A(x))|x \in X, A(x) = \{(u, Y_x(u))|u \in J_x \subseteq [0, 1], Y_x(u) \in [0, 1]\}\}, \tag{1}$$

where x indicates a primary variable, and $A(x)$ denotes the fuzzy membership value of x in A. Note that $A(x)$ is also called a secondary membership function or a secondary set. Additionally, $Y_x(u)$ denotes the secondary membership (grade), where u indicates the primary membership (grade) of x. $J_x \subseteq [0, 1]$ denotes the domain of $Y_x(u)$ and represents the primary membership values of $x \in X$.

Definition 2 [51, 54] Let A be a T2 FS on X. When $Y_x(u) = 1$ for all $u \in J_x$, A is known as an IT2 FS on X and can be represented by:

$$A = \{(x, A(x))|x \in X, A(x) = \{(u, 1)|A^-(x) \le u \le A^+(x), [A^-(x), A^+(x)] \subseteq [0, 1]\}\}, \tag{2}$$

where $A(x)$ is referred to as an interval membership value.

Definition 3 [51, 52, 54] Let A be an IT2 FS on X. The IT2 FS A can be fully characterized by its footprint of uncertainty (FOU), which is defined as the union of all primary memberships as follows:

$$FOU(A) = \bigcup_{x \in X} [A^-(x), A^+(x)]. \tag{3}$$

Moreover, $FOU(A)$ is a bounded region that represents the uncertainty associated with the membership grades of A.

Definition 4 [36, 51, 73, 74] Let a lower membership function (LMF) and an upper membership function (UMF) denote two type-1 membership functions that are bounds for the $FOU(A)$ of an IT2 FS A on X. Let two T1 FSs, A^-: $X \to [0, 1]$ and A^+: $X \to [0, 1]$, be the lower and upper fuzzy sets, respectively, with respect to A. The LMF $A^-(x)$ and the UMF $A^+(x)$ are associated with the lower bound $FOU^-(A)$ and the upper bound $FOU^+(A)$, respectively, of $FOU(A)$, and they are defined as follows:

$$A^- = FOU^-(A) = \{(x, A^-(x))|x \in X\}, \tag{4}$$

$$A^{+} = FOU^{+}(A) = \{(x, A^{+}(x)) | x \in X\}, \tag{5}$$

where $0 \le A^{-}(x) \le A^{+}(x) \le 1$ for all $x \in X$.

Definition 5 [9, 13–17, 62] Let $A^{-}(= (a_1^-, a_2^-, a_3^-, a_4^-; h_A^-))$ and $A^{+}(= (a_1^+, a_2^+, a_3^+, a_4^+; h_A^+))$ be the lower and upper trapezoidal fuzzy numbers, respectively, with respect to an IT2 FS A on X, where $a_1^- \le a_2^- \le a_3^- \le a_4^-$, $a_1^+ \le a_2^+ \le a_3^+ \le a_4^+$, $0 \le h_A^- \le h_A^+ \le 1$, $a_1^+ \le a_1^-$, $a_4^- \le a_4^+$, and $A^- \subseteq A^+$ (if and only if $\forall x \in X$, $A^-(x) \le A^+(x)$). The LMF $A^-(x)$ and the UMF $A^+(x)$ are lower and upper bounds, respectively, for the $FOU(A)$ of A; they are defined as follows:

$$A^{-}(x) = \begin{cases} \dfrac{h_A^-\left(x - a_1^-\right)}{a_2^- - a_1^-} & \text{if } a_1^- \le x \le a_2^-, \\ h_A^- & \text{if } a_2^- \le x \le a_3^-, \\ \dfrac{h_A^-\left(a_4^- - x\right)}{a_4^- - a_3^-} & \text{if } a_3^- \le x \le a_4^-, \\ 0 & \text{otherwise}; \end{cases} \tag{6}$$

$$A^{+}(x) = \begin{cases} \dfrac{h_A^+\left(x - a_1^+\right)}{a_2^+ - a_1^+} & \text{if } a_1^+ \le x \le a_2^+, \\ h_A^+ & \text{if } a_2^+ \le x \le a_3^+, \\ \dfrac{h_A^+\left(a_4^+ - x\right)}{a_4^+ - a_3^+} & \text{if } a_3^+ \le x \le a_4^+, \\ 0 & \text{otherwise}. \end{cases} \tag{7}$$

Then, A is an IT2 TrFN on X (see Fig. 1) and can be represented by:

$$\begin{aligned} A &= [A^-, A^+] \\ &= \left[(a_1^-, a_2^-, a_3^-, a_4^-; h_A^-), (a_1^+, a_2^+, a_3^+, a_4^+; h_A^+)\right]. \end{aligned} \tag{8}$$

3 Decision context based on IT2 TrFNs

This section first formulates an MCDA problem within a decision environment based on IT2 TrFNs and then presents the concept of likelihoods of fuzzy preference relations between IT2 TrFNs.

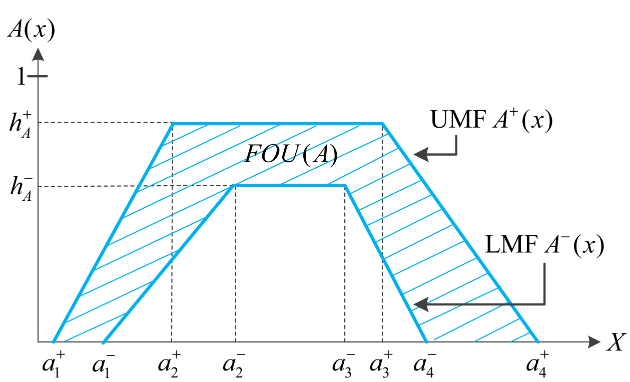

Fig. 1 Geometrical interpretation of an IT2 TrFN A

3.1 Decision environment of IT2 TrFNs

This subsection formulates an MCDA problem using IT2 TrFNs. Define $Z = \{z_1, z_2, \ldots, z_m\}$ as a set of decision alternatives, where m is the number of alternatives. Define $C = \{c_1, c_2, \ldots, c_n\}$ as a criterion set that contains the criteria by which the alternative performances are measured, where n is the number of criteria. Let x_j denote the value corresponding to the criterion c_j, where $x_j \in X$ (i.e., the universe of discourse) for $j = 1, 2, \ldots, n$. The set C can be generally divided into two sets, C_{I} and C_{II}, where C_{I} denotes a collection of benefit criteria (i.e., larger values of x_j indicate a greater preference) and C_{II} denotes a collection of cost criteria (i.e., smaller values of x_j indicate a greater preference). Note that $C_{\mathrm{I}} \cap C_{\mathrm{II}} = \emptyset$ and $C_{\mathrm{I}} \cup C_{\mathrm{II}} = C$.

The IT2 TrFN data required in an MCDA problem can be appropriately established by employing the linguistic scales with the corresponding IT2 TrFNs. Decision-makers often express their judgments using linguistic variables in many practical situations. The decision-maker's linguistic responses can be appropriately represented by IT2 TrFNs using a specific linguistic rating system, such as three-point linguistic scales [22, 35, 37, 69], four-point linguistic scales [22], five-point linguistic scales [22, 37, 55], seven-point linguistic scales [23, 27, 33, 37, 61, 72], and nine-point linguistic scales [9, 10, 12, 14, 15, 20, 21, 63]. Most of the IT2 TrFNs corresponding to linguistic terms are nonnegative [9, 10, 12, 14, 15, 20, 21, 23, 55, 61, 63, 72]. Therefore, this paper also constructs an MCDA problem using nonnegative IT2 TrFNs.

Consider an MCDA problem wherein the ratings of alternative evaluations and the importance weights of criteria are expressed as nonnegative IT2 TrFNs. Let a nonnegative IT2 TrFN A_{ij} denote the evaluative rating of the alternative $z_i \in Z$ in terms of the criterion $c_j \in C$. Let the two nonnegative T1 FSs A_{ij}^-: $X \to [0, 1]$ and A_{ij}^+: $X \to [0, 1]$ denote the lower and upper trapezoidal fuzzy numbers, respectively, with respect to A_{ij}, where $A_{ij}^- = (a_{1ij}^-, a_{2ij}^-, a_{3ij}^-, a_{4ij}^-; h_{A_{ij}}^-)$ and $A_{ij}^+ = (a_{1ij}^+, a_{2ij}^+, a_{3ij}^+, a_{4ij}^+; h_{A_{ij}}^+)$. The IT2 TrFN evaluative rating A_{ij} is expressed as follows:

$$A_{ij} = [A_{ij}^-, A_{ij}^+] = \left[(a_{1ij}^-, a_{2ij}^-, a_{3ij}^-, a_{4ij}^-; h_{A_{ij}}^-), (a_{1ij}^+, a_{2ij}^+, a_{3ij}^+, a_{4ij}^+; h_{A_{ij}}^+)\right], \tag{9}$$

where $0 \le a_{1ij}^- \le a_{2ij}^- \le a_{3ij}^- \le a_{4ij}^-$, $0 \le a_{1ij}^+ \le a_{2ij}^+ \le a_{3ij}^+ \le a_{4ij}^+$, $a_{1ij}^+ \le a_{1ij}^-$, $a_{4ij}^- \le a_{4ij}^+$, $0 \le h_{A_{ij}}^- \le h_{A_{ij}}^+ \le 1$, and $A_{ij}^- \subseteq A_{ij}^+$.

In a similar manner, let a nonnegative IT2 TrFN W_j denote the importance weight of the criterion $c_j \in C$. Let $W_j^- = (w_{1j}^-, w_{2j}^-, w_{3j}^-, w_{4j}^-; h_{W_j}^-)$ and $W_j^+ = (w_{1j}^+, w_{2j}^+, w_{3j}^+, w_{4j}^+; h_{W_j}^+)$ be the lower and upper trapezoidal fuzzy numbers, respectively, with respect to W_j. Then, W_j is expressed as follows:

$$W_j = [W_j^-, W_j^+] \\ = \left[(w_{1j}^-, w_{2j}^-, w_{3j}^-, w_{4j}^-; h_{W_j}^-), (w_{1j}^+, w_{2j}^+, w_{3j}^+, w_{4j}^+; h_{W_j}^+)\right], \tag{10}$$

where $0 \le w_{1j}^- \le w_{2j}^- \le w_{3j}^- \le w_{4j}^-$, $0 \le w_{1j}^+ \le w_{2j}^+ \le w_{3j}^+ \le w_{4j}^+$, $w_{1j}^+ \le w_{1j}^-$, $w_{4j}^- \le w_{4j}^+$, and $0 \le h_{W_j}^- \le h_{W_j}^+ \le 1$, and $W_j^- \subseteq W_j^+$.

3.2 Likelihood of fuzzy preference relations between IT2 TrFNs

By extending the fuzzy preference relations [22] and the likelihoods between trapezoidal fuzzy numbers [23, 24, 40], Chen [19] and Wang et al. [62] proposed the extended concept of likelihoods of fuzzy preference relations between IT2 TrFNs. This paper adopts the likelihoods presented by Chen [19] and Wang et al. [62] to develop an approach to generating criterion-wise rankings of the alternatives.

Lee and Chen [40] developed a fuzzy decision-making method based on the likelihood-based comparison relations on intervals introduced by Xu and Da [64]. Lee and Chen [40] and Chen and Lee [24] presented the likelihood-based comparison relations using T1 FSs and IT2 FSs. Chen and Lee [24] further proposed the concepts of likelihood-based comparison relations of fuzzy sets based on the α-cut representation. Chen and Lee [22] provided a method for handling fuzzy multiple criteria hierarchical group decision-making problems based on arithmetic operations and fuzzy preference relations of IT2 FSs. Chen and Lee [23] presented a ranking method of trapezoidal IT2 FSs using the concept of likelihoods. They employed the fuzzy preference relations based on IT2 FSs to establish the upper and lower fuzzy preference matrices. Chen [19] and Wang et al. [62] extended the likelihoods proposed by Chen and Lee [22, 23] to present the modified concepts of lower and upper likelihoods for determining the likelihood of a fuzzy preference relation in the context of IT2 TrFNs.

In the decision context of IT2 TrFNs, let A_{ij} $(= [A_{ij}^-, A_{ij}^+] = [(a_{1ij}^-, a_{2ij}^-, a_{3ij}^-, a_{4ij}^-; h_{A_{ij}}^-), (a_{1ij}^+, a_{2ij}^+, a_{3ij}^+, a_{4ij}^+; h_{A_{ij}}^+)])$ and $A_{i'j}$ $(= [A_{i'j}^-, A_{i'j}^+] = [(a_{1i'j}^-, a_{2i'j}^-, a_{3i'j}^-, a_{4i'j}^-; h_{A_{i'j}}^-), (a_{1i'j}^+, a_{2i'j}^+, a_{3i'j}^+, a_{4i'j}^+; h_{A_{i'j}}^+)])$ denote the evaluative ratings of the alternatives z_i and $z_{i'}$, respectively, with respect to the criterion $c_j \in C$. As mentioned before, A_{ij} and $A_{i'j}$ are expressed as two nonnegative IT2 TrFNs defined on X. Let a fuzzy preference relation $A_{ij} \ge A_{i'j}$ denote the evaluative rating of the alternative z_i not being smaller than that of the alternative $z_{i'}$ in regard to a specific criterion c_j. Let $L(A_{ij} \ge A_{i'j})$ denote the likelihood of the fuzzy preference relation $A_{ij} \ge A_{i'j}$ for each pair of alternatives $(z_i, z_{i'})$. Lee and Chen [40] and Chen and Lee [22–24] determined the likelihood of a fuzzy preference relation between two lower fuzzy sets. Moreover, they calculated the likelihood for two upper fuzzy sets. The corresponding fuzzy preference matrices were separately constructed for lower and upper fuzzy sets. More specifically, they determined the lower and upper likelihoods via the relations $A_{ij}^- \ge A_{i'j}^-$ and $A_{ij}^+ \ge A_{i'j}^+$, respectively. Next, they obtained the likelihood $L(A_{ij} \ge A_{i'j})$ using the mean of the computed lower and upper likelihoods. However, the obtained likelihoods between two lower fuzzy sets (A_{ij}^- and $A_{i'j}^-$) and between two upper fuzzy sets (A_{ij}^+ and $A_{i'j}^+$) are not true values of the lower and upper likelihoods, respectively, for a fuzzy preference relation between IT2 TrFNs.

In general, the minimal possibility of the event $A_{ij} \ge A_{i'j}$ occurs in the comparison of the lower fuzzy set A_{ij}^- and the upper fuzzy set $A_{i'j}^+$, and the maximal possibility of the event $A_{ij} \ge A_{i'j}$ occurs in the comparison of the upper fuzzy set A_{ij}^+ and the lower fuzzy set $A_{i'j}^-$. Accordingly, the lower and upper likelihoods of $L(A_{ij} \ge A_{i'j})$ should be determined via the relations $A_{ij}^- \ge A_{i'j}^+$ and $A_{ij}^+ \ge A_{i'j}^-$, respectively. Therefore, based on the concept of likelihoods proposed by Lee and Chen [40] and Chen and Lee [22–24], Chen [19] and Wang et al. [62] developed a modified concept of likelihoods of fuzzy preference relations between IT2 TrFNs. Assume that at least one of $h_{A_{ij}}^- \ne h_{A_{i'j}}^+$, $a_{4ij}^- \ne a_{1ij}^-$, $a_{4i'j}^+ \ne a_{1i'j}^+$, and $a_{\eta ij}^- \ne a_{\eta i'j}^+$ holds and that at least one of $h_{A_{ij}}^+ \ne h_{A_{i'j}}^-$, $a_{4ij}^+ \ne a_{1ij}^+$, $a_{4i'j}^- \ne a_{1i'j}^-$, and $a_{\eta ij}^+ \ne a_{\eta i'j}^-$ holds, where $\eta \in \{1, 2, 3, 4\}$. This paper employs their modified likelihoods to present an effective ranking procedure for comparing IT2 TrFN evaluative ratings.

Definition 6 [19, 62] Let A_{ij} and $A_{i'j}$ be two IT2 TrFN evaluative ratings of the alternatives z_i and $z_{i'}$, respectively, with respect to the criterion $c_j \in C$. The lower likelihood $L^-(A_{ij} \ge A_{i'j})$ and the upper likelihood $L^+(A_{ij} \ge A_{i'j})$ of a fuzzy preference relation $A_{ij} \ge A_{i'j}$ are defined as follows:

$$L^-(A_{ij} \ge A_{i'j}) = \max\left\{1 - \max\left[\frac{\sum_{\eta=1}^4 \max\left(a_{\eta i'j}^+ - a_{\eta ij}^-, 0\right) + \left(a_{4i'j}^+ - a_{1ij}^-\right) + 2\max\left(h_{A_{i'j}}^+ - h_{A_{ij}}^-, 0\right)}{\sum_{\eta=1}^4 \left|a_{\eta i'j}^+ - a_{\eta ij}^-\right| + \left(a_{4ij}^- - a_{1ij}^-\right) + \left(a_{4i'j}^+ - a_{1i'j}^+\right) + 2\left|h_{A_{i'j}}^+ - h_{A_{ij}}^-\right|}, 0\right], 0\right\}, \tag{11}$$

$$L^{+}(A_{ij} \geq A_{i'j}) = \max\left\{1 - \max\left[\frac{\sum_{\eta=1}^{4}\max\left(a_{\eta i'j}^{-} - a_{\eta ij}^{+}, 0\right) + \left(a_{4i'j}^{-} - a_{1ij}^{+}\right) + 2\max\left(h_{A_{i'j}}^{-} - h_{A_{ij}}^{+}, 0\right)}{\sum_{\eta=1}^{4}\left|a_{\eta i'j}^{-} - a_{\eta ij}^{+}\right| + \left(a_{4ij}^{+} - a_{1ij}^{+}\right) + \left(a_{4i'j}^{-} - a_{1i'j}^{-}\right) + 2\left|h_{A_{i'j}}^{-} - h_{A_{ij}}^{+}\right|}, 0\right], 0\right\}. \quad (12)$$

The likelihood $L(A_{ij} \geq A_{i'j})$ of $A_{ij} \geq A_{i'j}$ is defined as follows:

$$L(A_{ij} \geq A_{i'j}) = \frac{L^{-}(A_{ij} \geq A_{i'j}) + L^{+}(A_{ij} \geq A_{i'j})}{2}. \quad (13)$$

Theorem 1 [19, 62] *Let A_{ij} and $A_{i'j}$ be two IT2 TrFN evaluative ratings. The lower likelihood $L^{-}(A_{ij} \geq A_{i'j})$ and the upper likelihood $L^{+}(A_{ij} \geq A_{i'j})$ satisfy the following properties:*

(T1.1) $0 \leq L^{-}(A_{ij} \geq A_{i'j}) \leq 1$;

(T1.2) $0 \leq L^{+}(A_{ij} \geq A_{i'j}) \leq 1$;

(T1.3) $L^{-}(A_{ij} \geq A_{i'j}) + L^{+}(A_{i'j} \geq A_{ij}) = 1$;

(T1.4) $L^{-}(A_{ij} \geq A_{i'j}) = 0$ *and* $L^{+}(A_{i'j} \geq A_{ij}) = 1$ *if* $a_{4ij}^{+} \leq a_{1i'j}^{+}$ *and* $h_{A_{ij}}^{-} \leq h_{A_{i'j}}^{+}$;

(T1.5) $L^{+}(A_{ij} \geq A_{i'j}) = 0$ *and* $L^{-}(A_{i'j} \geq A_{ij}) = 1$ *if* $a_{1i'j}^{-} - a_{4ij}^{+} \geq 2\max\left(h_{A_{ij}}^{+} - h_{A_{i'j}}^{-}, 0\right)$.

See Chen [19] *for detailed proofs.*

Theorem 2 [19] *Let A_{ij} and $A_{i'j}$ be two IT2 TrFN evaluative ratings. The likelihood $L(A_{ij} \geq A_{i'j})$ satisfies the following properties:*

(T2.1) $0 \leq L(A_{ij} \geq A_{i'j}) \leq 1$;

(T2.2) $L(A_{ij} \geq A_{i'j}) + L(A_{i'j} \geq A_{ij}) = 1$;

(T2.3) $L(A_{ij} \geq A_{i'j}) = L(A_{i'j} \geq A_{ij}) = \frac{1}{2}$ *if* $L(A_{ij} \geq A_{i'j}) = L(A_{i'j} \geq A_{ij})$;

(T2.4) $L(A_{ij} \geq A_{ij}) = \frac{1}{2}$;

(T2.5) $\sum_{i=1}^{m}\sum_{i'=1}^{m} L(A_{ij} \geq A_{i'j}) = \frac{m^2}{2}$.

Proof See Chen [19] for detailed proofs of (T2.1)–(T2.4). (T2.5) can be easily verified using (T2.2) and (T2.4). □

The likelihoods of pair-wise IT2 TrFN evaluative ratings with respect to each criterion $c_j \in C$ can be concisely expressed in the following matrix format:

$$\mathrm{L}_j = \begin{bmatrix} L(A_{1j} \geq A_{1j}) & L(A_{1j} \geq A_{2j}) & \cdots & L(A_{1j} \geq A_{mj}) \\ L(A_{2j} \geq A_{1j}) & L(A_{2j} \geq A_{2j}) & \cdots & L(A_{2j} \geq A_{mj}) \\ \vdots & \vdots & \ddots & \vdots \\ L(A_{mj} \geq A_{1j}) & L(A_{mj} \geq A_{2j}) & \cdots & L(A_{mj} \geq A_{mj}) \end{bmatrix} \quad (14)$$

for $j = 1, 2, \cdots, n$.

Theorem 3 *Assume that L_j is the likelihood matrix whose entries $L(A_{ij} \geq A_{i'j})$ are given by (13). Then, L_j is a fuzzy complementary judgment matrix for $j = 1, 2, \cdots, n$.*

Proof According to (T2.1) and (T2.2), with respect to the criterion $c_j \in C$, it is known that $0 \leq L(A_{ij} \geq A_{i'j}) \leq 1$ and $L(A_{ij} \geq A_{i'j}) + L(A_{i'j} \geq A_{ij}) = 1$ for $i, i' = 1, 2, \ldots, m$. They are the conditions that a fuzzy complementary judgment matrix should satisfy [28, 32, 43]. Therefore, L_j is proven to be a fuzzy complementary judgment matrix for $j = 1, 2, \ldots, n$. □

Example 1 Consider three alternatives z_1, z_2, and z_3 with respect to a criterion x_1. Suppose that the IT2 TrFN evaluative ratings $A_{11} = [(0.0075, 0.0075, 0.015, 0.0525; 0.8)\ (0.0, 0.0, 0.02, 0.07; 1.0)]$, $A_{21} = [(0.2325, 0.255, 0.325, 0.3575; 0.8), (0.17, 0.22, 0.36, 0.42; 1.0)]$, and $A_{31} = [(0.4025, 0.4525, 0.5375, 0.5675; 0.8), (0.32, 0.41, 0.58, 0.65; 1.0)]$. The likelihoods of pair-wise evaluative ratings can be computed according to (11)–(13) of Definition 6. For example, we obtain $L^{-}(A_{11} \geq A_{21}) = L^{-}(A_{11} \geq A_{31}) = 0$ and $L^{-}(A_{21} \geq A_{31}) = 0.0228$ using (11), $L^{+}(A_{11} \geq A_{21}) = 0.1418$, $L^{+}(A_{11} \geq A_{31}) = 0.0269$, and $L^{+}(A_{21} \geq A_{31}) = 0.2601$ using (12), and $L(A_{11} \geq A_{21}) = 0.0709$, $L(A_{11} \geq A_{31}) = 0.0135$, and $L(A_{21} \geq A_{31}) = 0.1415$ using (13). Next, according to (14), we construct the likelihood matrix as follows:

$$L_1 = \begin{bmatrix} 0.5 & 0.0709 & 0.0135 \\ 0.9291 & 0.5 & 0.1415 \\ 0.9865 & 0.8585 & 0.5 \end{bmatrix}.$$

Obviously, L_1 is a fuzzy complementary judgment matrix.

4 The likelihood-based assignment method

This section employs the extended concept of likelihoods of fuzzy preference relations between IT2 TrFNs to determine criterion-wise rankings of the alternatives. To address MCDA problems containing imprecise and uncertain type-2 fuzzy properties, this section develops a novel likelihood-based assignment method to determine an optimal preference ranking of the alternatives in accordance with the obtained criterion-wise rankings and the importance weights of criteria within the decision environment of IT2 TrFNs.

4.1 Proposed method

Based on the likelihoods of fuzzy preference relations between IT2 TrFNs, this subsection presents a ranking procedure using the optimal membership degree determination method to determine a criterion-wise preference of the alternatives. Then, a likelihood-based assignment model is constructed using the concept of an adjusted rank contribution matrix to determine the priority order of various alternatives.

Li et al. [44] and Li [41] used the likelihood of an alternative not being inferior to another alternative to construct a likelihood matrix for determining the optimal degrees of membership. Li [42, 43] employed an inclusion comparison probability to make comparisons between alternatives and proposed the optimal membership degree determination method. In a manner similar to that of Li et al. [44] and Li [41–43], we employ the optimal membership degree determination method to compare any two IT2 TrFN evaluative ratings. The m alternatives can then be ranked in terms of each criterion $c_j \in C$ according to the comparison results of the optimal degrees of membership.

The concepts of relative difference indices and the relative difference matrix are proposed to determine the optimal degree of membership. Consider any two IT2 TrFN evaluative ratings A_{ij} and $A_{i'j}$ for $i,\ i' = 1, 2, \ldots, m$ and $i \neq i'$. To construct the likelihood matrix L_j, we first compute the lower likelihood $L^-(A_{ij} \geq A_{i'j})$, the upper likelihood $L^+(A_{ij} \geq A_{i'j})$, and the likelihood $L(A_{ij} \geq A_{i'j})$ of a fuzzy preference relation $A_{ij} \geq A_{i'j}$ for each criterion $c_j \in C$. Additionally, we know that $L(A_{ij} \geq A_{ij}) = 0.5$ for $i = 1, 2, \ldots, m$ according to (T2.4). Let $\gamma^j_{ii'}$ denote the relative difference index representing a linear transformation of the difference in respective sums of the likelihoods $L(A_{ij} \geq A_{i''j})$ ($i'' = 1, 2, \ldots, m$) in the i-th row of L_j and $L(A_{i'j} \geq A_{i''j})$ ($i'' = 1, 2, \ldots, m$) in the i'-th row of L_j. It is defined as follows:

$$\gamma^j_{ii'} = \frac{\sum_{i''=1}^{m} L(A_{ij} \geq A_{i''j}) - \sum_{i''=1}^{m} L(A_{i'j} \geq A_{i''j})}{2(m-1)} + \frac{1}{2}. \qquad (15)$$

Example 2 Let us look at Example 1 again. There are three alternatives in Example 1; thus, $m = 3$. According to the definition in (15), the relative difference index γ^1_{12}, for example, is computed as follows:

$$\begin{aligned}\gamma^1_{12} &= \frac{\sum_{i''=1}^{3} L(A_{11} \geq A_{i''1}) - \sum_{i''=1}^{3} L(A_{21} \geq A_{i''1})}{2(3-1)} + \frac{1}{2} \\ &= \frac{L(A_{11} \geq A_{11}) + L(A_{11} \geq A_{21}) + L(A_{11} \geq A_{31}) - (L(A_{21} \geq A_{11}) + L(A_{21} \geq A_{21}) + L(A_{21} \geq A_{31}))}{4} + \frac{1}{2} \\ &= \frac{0.5 + 0.0709 + 0.0135 - (0.9291 + 0.5 + 0.1415)}{4} + \frac{1}{2} = 0.2535.\end{aligned}$$

The relative difference index $\gamma^j_{ii'}$ of the pair-wise IT2 TrFN evaluative ratings A_{ij} and $A_{i'j}$ with respect to each criterion $c_j \in C$ can be concisely expressed in the following matrix format:

$$\Upsilon_j = \begin{bmatrix} \gamma^j_{11} & \gamma^j_{12} & \cdots & \gamma^j_{1m} \\ \gamma^j_{21} & \gamma^j_{22} & \cdots & \gamma^j_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma^j_{m1} & \gamma^j_{m2} & \cdots & \gamma^j_{mm} \end{bmatrix} \qquad (16)$$

for $j = 1, 2, \cdots, n$.

Theorem 4 *Assume that Υ_j is the relative difference matrix whose entries $\gamma^j_{ii'}$ are given by (15). Then, Υ_j is a fuzzy complementary and consistent judgment matrix for $j = 1, 2, \ldots, n$.*

Proof According to (T2.1), (T2.2), and (T2.4), it is easily observed that

$$\sum_{i''=1}^{m} L(A_{ij} \geq A_{i''j}) = \sum_{i''=1, i'' \neq i}^{m} L(A_{ij} \geq A_{i''j}) + \frac{1}{2} \leq (m-1) + \frac{1}{2} \quad \textit{and}$$
$$\sum_{i''=1}^{m} L(A_{i'j} \geq A_{i''j}) = \sum_{i''=1, i'' \neq i'}^{m} L(A_{i'j} \geq A_{i''j}) + \frac{1}{2} \leq (m-1) + \frac{1}{2}$$

for $i,\ i' = 1, 2, \ldots, m$. Next, it follows that

$$\frac{1}{2} \leq \sum_{i''=1}^{m} L(A_{ij} \geq A_{i''j}) \leq (m-1) + \frac{1}{2}, \qquad (17)$$

$$\frac{1}{2} \leq \sum_{i''=1}^{m} L(A_{i'j} \geq A_{i''j}) \leq (m-1) + \frac{1}{2}. \qquad (18)$$

Combining (17) and (18), we obtain

$$-(m-1) \leq \sum_{i''=1}^{m} L(A_{ij} \geq A_{i''j}) - \sum_{i''=1}^{m} L(A_{i'j} \geq A_{i''j}) \leq m-1$$

for $i,\ i' = 1, 2, \cdots, m$. It implies that $-1 \leq \frac{\sum_{i''=1}^{m} L(A_{ij} \geq A_{ij}) - \sum_{i''=1}^{m} L(A_{i'j} \geq A_{i''j})}{m-1} \leq 1$ because $m \geq 1$; furthermore,

$$0 \leq \frac{\sum_{i''=1}^{m} L(A_{ij} \geq A_{i''j}) - \sum_{i''=1}^{m} L(A_{i'j} \geq A_{i''j})}{2(m-1)} + \frac{1}{2} \leq 1.$$

Therefore, it is proven that Υ_j is fuzzy because

$$0 \le \gamma^j_{ii'} \le 1 \tag{19}$$

for i, $i' = 1, 2, \ldots, m$. Next, it is easy to verify that Υ_j is complementary because

$$\begin{aligned} &\gamma^j_{ii'} + \gamma^j_{i'i} \\ &= \frac{\sum_{i''=1}^m L(A_{ij} \ge A_{i''j}) - \sum_{i''=1}^m L(A_{i'j} \ge A_{i''j})}{2(m-1)} + \frac{1}{2} \\ &\quad + \frac{\sum_{i''=1}^m L(A_{i'j} \ge A_{i''j}) - \sum_{i''=1}^m L(A_{ij} \ge A_{i''j})}{2(m-1)} + \frac{1}{2} = 1 \end{aligned} \tag{20}$$

for i, $i' = 1, 2, \ldots, m$. Furthermore, one can easily obtain

$$\begin{aligned} &\left(\gamma^j_{ii''} - \frac{1}{2}\right) + \left(\gamma^j_{i''i'} - \frac{1}{2}\right) \\ &= \frac{\sum_{i^-=1}^m L(A_{ij} \ge A_{i'''j}) - \sum_{i^-=1}^m L(A_{i''j} \ge A_{i^-j})}{2(m-1)} \\ &\quad + \frac{\sum_{i^-=1}^m L(A_{i''j} \ge A_{i''j}) - \sum_{i'''=1}^m L(A_{i'j} \ge A_{i^-j})}{2(m-1)} \\ &= \frac{\sum_{i^-=1}^m L(A_{ij} \ge A_{i^-j}) - \sum_{i'''=1}^m L(A_{i'j} \ge A_{i^-j})}{2(m-1)} \\ &= \gamma^j_{ii'} - \frac{1}{2} \end{aligned} \tag{21}$$

for i, $i' = 1, 2, \ldots, m$. Thus, Υ_j is additive transitive [28, 32, 43] because $(\gamma^j_{ii''} - 0.5) + (\gamma^j_{i''i'} - 0.5) = \gamma^j_{ii'} - 0.5$. According to (19)–(21), Υ_j is proven to be a fuzzy complementary and consistent judgment matrix for $j = 1, 2, \ldots, n$. □

Example 3 Continue Example 2. According to the definition in (16), the relative difference matrix Υ_1 is constructed as follows:

$$\Upsilon_1 = \begin{bmatrix} 0.5 & 0.2535 & 0.0598 \\ 0.7465 & 0.5 & 0.3064 \\ 0.9402 & 0.6936 & 0.5 \end{bmatrix}.$$

Obviously, Υ_1 is a fuzzy complementary and consistent judgment matrix.

Based on the relative difference index $\gamma^j_{ii'}$ in the relative difference matrix Υ_j, we can determine the optimal degree of membership for each IT2 TrFN evaluative rating. First, the sum of all relative difference indices of each row in the matrix Υ_j is computed as follows:

$$\gamma^j_i = \sum_{i'=1}^m \gamma^j_{ii'} \tag{22}$$

for $i = 1, 2, \ldots, m$.

Let $\Gamma(A_{ij})$ denote the optimal degree of membership for the IT2 TrFN evaluative rating A_{ij}. $\Gamma(A_{ij})$ is defined by the normalized value of γ^j_i as follows:

$$\Gamma(A_{ij}) = \frac{\gamma^j_i}{\sum_{i'''=1}^m \gamma^j_{i'''}} \tag{23}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Theorem 5 *Let A_{ij} be an IT2 TrFN evaluative rating of the alternative z_i with respect to the criterion $c_j \in C$. The optimal degree of membership $\Gamma(A_{ij})$ of A_{ij} is determined by:*

$$\Gamma(A_{ij}) = \frac{1}{m(m-1)}\left(\sum_{i'=1}^m L(A_{ij} \ge A_{i'j}) + \frac{m}{2} - 1\right). \tag{24}$$

Proof As defined in (23), it is obvious that

$$\Gamma(A_{ij}) = \frac{\sum_{i''=1}^m \gamma^j_{ii''}}{\sum_{i^-=1}^m \sum_{i''=1}^m \gamma^j_{i^-i''}}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ using (22) and (15). According to (T2.5), it is known that $\sum_{i=1}^m \sum_{i'=1}^m L(A_{ij} \ge A_{i'j}) = m^2/2$. It follows that

$$\begin{aligned} \Gamma(A_{ij}) &= \frac{\sum_{i''=1}^m \gamma^j_{ii''}}{\sum_{1 \le i'' < i^- \le m}^m \left(\gamma^j_{i^-i''} + \gamma^j_{i''i^-}\right) + \frac{m}{2}} \\ &= \frac{\sum_{i''=1}^m \left[\frac{\sum_{i'=1}^m L(A_{ij} \ge A_{i'j}) - \sum_{i'=1}^m L(A_{i''j} \ge A_{i'j})}{2(m-1)} + \frac{1}{2}\right]}{\frac{m(m-1)}{2} + \frac{m}{2}} \\ &= \frac{m\sum_{i'=1}^m L(A_{ij} \ge A_{i'j}) - \sum_{i''=1}^m \sum_{i'=1}^m L(A_{i''j} \ge A_{i'j}) + m(m-1)}{m^2(m-1)} \\ &= \frac{\sum_{i'=1}^m L(A_{ij} \ge A_{i'j}) + \frac{m}{2} - 1}{m(m-1)}. \end{aligned}$$

Therefore, Theorem 5 is proven. □

Example 4 Continue Example 3. According to the definition in (24), the optimal degree of membership $\Gamma(A_{21})$ of A_{21}, for example, is computed as follows:

$$\begin{aligned} \Gamma(A_{21}) &= \frac{1}{3(3-1)}\left(\sum_{i'=1}^3 L(A_{21} \ge A_{i'1}) + \frac{3}{2} - 1\right) \\ &= \frac{1}{6}(0.9291 + 0.5 + 0.1415 + 0.5) = 0.3451. \end{aligned}$$

Of course, $\Gamma(A_{21})$ can also be computed using (22) and (23) as follows:

$$\Gamma(A_{21}) = \frac{\gamma_2^1}{\sum_{i'''=1}^{3}\gamma_{i'''}^1} = \frac{\sum_{i'=1}^{3}\gamma_{2i'}^1}{\sum_{i'''=1}^{3}\sum_{i'=1}^{3}\gamma_{i'''i'}^1}$$
$$= \frac{0.7465+0.5+0.3064}{(0.5+0.2535+0.0598)+(0.7465+0.5+0.3064)+(0.9402+0.6936+0.5)} = 0.3451.$$

Theorem 6 *The optimal degree of membership $\Gamma(A_{ij})$ of the alternative $z_i \in Z$ with respect to the criterion $c_j \in C$ satisfies the following properties:*

(T6.1) $\frac{1}{2m} \le \Gamma(A_{ij}) \le \frac{3}{2m}$;
(T6.2) $\sum_{i=1}^{m} \Gamma(A_{ij}) = 1$ *for* $c_j \in C$.

Proof Because $0 \le L(A_{ij} \ge A_{i'j}) \le 1$, $L(A_{ij} \ge A_{ij}) = 0.5$, and $L(A_{ij} \ge A_{i'j}) + L(A_{i'j} \ge A_{ij}) = 1$ via Theorem 2, one can easily obtain that $0.5 \le \sum_{i'=1}^{m} L(A_{ij} \ge A_{i'j}) \le (m-1) + 0.5$. When $\sum_{i'=1}^{m} L(A_{ij} \ge A_{i'j})$ is equal to the lower bound 0.5, $\Gamma(A_{ij})$ becomes

$$\Gamma(A_{ij}) = \frac{1}{m(m-1)}\left(\frac{1}{2} + \frac{m}{2} - 1\right) = \frac{1}{2m}. \tag{25}$$

When $\sum_{i'=1}^{m} L(A_{ij} \ge A_{i'j})$ is equal to the upper bound $(m-1) + 0.5$, $\Gamma(A_{ij})$ becomes

$$\Gamma(A_{ij}) = \frac{1}{m(m-1)}\left((m-1) + \frac{1}{2} + \frac{m}{2} - 1\right) = \frac{3}{2m}. \tag{26}$$

According to (25) and (26), $1/2m \le \Gamma(A_{ij}) \le 3/2m$ holds. Therefore, (T6.1) is valid. Additionally, it is obvious that $\sum_{i=1}^{m} \Gamma(A_{ij}) = 1$ because of the definition in (23). Accordingly, (T6.2) is valid. □

Example 5 Continue Example 4. Applying (24), the respective optimal degrees of membership of z_1, z_2, and z_3 with respect to x_1 are obtained as follows: $\Gamma(A_{11})= 0.1807$, $\Gamma(A_{21})= 0.3451$, and $\Gamma(A_{31})= 0.4742$. Obviously, the condition (T6.1) is satisfied because $1/(2\times 3) \le \Gamma(A_{i1}) \le 3/(2\times 3)$ for $i = 1, 2, 3$. The condition (T6.2) is also satisfied because $\Gamma(A_{11}) + \Gamma(A_{21}) + \Gamma(A_{31}) = 1$.

The m alternatives can be ranked with respect to each criterion based on the optimal degrees of membership. Specifically, in regard to each criterion $c_j \in C_{\mathrm{I}}$ (i.e., c_j is a benefit criterion), the ranking order of all m alternatives can be subsequently generated according to the descending order of the $\Gamma(A_{ij})$ values. Conversely, for each criterion $c_j \in C_{\mathrm{II}}$ (i.e., c_j is a cost criterion), the ranking order of all m alternatives is determined according to the ascending order of the $\Gamma(A_{ij})$ values.

To combine the obtained criterion-wise rankings into an overall preference ranking, we establish the main structure of the proposed likelihood-based assignment model using the concepts of an (adjusted) rank frequency matrix and an (adjusted) rank contribution matrix. In general, the MCDA problem involving multiple criteria evaluation and the ranking/selection of alternatives can be considered as that of assigning alternatives to a rank of order. Problems of this nature are assignment problems [6]. More specifically, we can give a rank frequency matrix whose entries measure the frequency that each alternative is assigned a particular rank among all criterion-wise preferences of the alternatives.

Employing the criterion-wise rankings of the m alternatives by comparing the values of $\Gamma(A_{ij})$, we define a rank frequency matrix F^0 that is similar to the concept of the product-attribute matrix introduced by Bernardo and Blin [6]. F^0 is an $m \times m$ square nonnegative matrix whose element f_{ik}^0 $(i, k = 1, 2, \ldots, m)$ represents the number of criterion-wise rankings where the alternative z_i is ranked k-th, as expressed in the following matrix:

$$\mathrm{F}^0 = \begin{array}{c} \\ z_1 \\ z_2 \\ \vdots \\ z_m \end{array}\!\!\begin{array}{c} \begin{array}{cccc} \text{1st} & \text{2nd} & \cdots & m\text{-th} \end{array} \\ \begin{bmatrix} f_{11}^0 & f_{12}^0 & \cdots & f_{1m}^0 \\ f_{21}^0 & f_{22}^0 & \cdots & f_{2m}^0 \\ \vdots & \vdots & \ddots & \vdots \\ f_{m1}^0 & f_{m2}^0 & \cdots & f_{mm}^0 \end{bmatrix} \end{array}. \tag{27}$$

Theorem 7 *The entry f_{ik}^0 ($i, k = 1, 2, \ldots, m$) in the rank frequency matrix F^0 satisfies the following properties:*

(T7.1) $\sum_{k=1}^{m} f_{ik}^0 = n$ *for* $i = 1, 2, \ldots, m$;
(T7.2) $\sum_{i=1}^{m} f_{ik}^0 = n$ *for* $k = 1, 2, \ldots, m$.

Proof (T7.1) and (T7.2) are obvious. □

Example 6 Let the set of alternatives $Z = \{z_1, z_2, z_3, z_4\}$ and the set of criteria $C = \{c_1, c_2, c_3\}$. Suppose that the criterion-wise rankings of all alternatives z_i $(i = 1, 2, 3, 4)$ based on each criterion c_j $(j = 1, 2, 3)$ are $z_3 \succ z_1 \succ$

$z_4 \succ z_2$, $z_2 \succ z_3 \succ z_4 \succ z_1$, and $z_4 \succ z_1 \succ z_2 \succ z_3$. According to (27), the rank frequency matrix F^0 is constructed as follows:

$$F^0 = \begin{array}{c} \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{array} \begin{array}{c} \begin{array}{cccc} \text{1st} & \text{2nd} & \text{3rd} & \text{4th} \end{array} \\ \begin{bmatrix} 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 \end{bmatrix} \end{array}.$$

Obviously, the condition (T7.1) is satisfied because $\sum_{k=1}^{4} f_{ik}^0 = 3$ for $i = 1, 2, 3, 4$. The condition (T7.2) is also satisfied because $\sum_{i=1}^{4} f_{ik}^0 = 3$ for $k = 1, 2, 3, 4$.

Note that equal importance is assigned to each criterion $c_j \in C$ in the rank frequency matrix F^0. Nevertheless, incorporating individual subjective preferences over the criteria into the MCDA process is crucial in generating a solution that is acceptable to the decision-makers. Therefore, we define a rank contribution matrix Π^0 that contains all of the information needed for an individual's decision (i.e., the subjective weights a decision-maker place on each criterion and the criterion-wise rankings of all alternatives according to the optimal degrees of membership). Let $c_{j_1}, c_{j_2}, \ldots$, and $c_{j_{f_{ik}^0}}$ represent the corresponding criteria for which the alternative z_i is ranked k-th. Let the symbol $\oplus$ represent the addition operation of IT2 TrFNs. The rank contribution matrix Π^0 is determined by aggregating the products of multiplying the IT2 TrFN importance weights $W_{j_1}, W_{j_2}, \ldots$, and $W_{j_{f_{ik}^0}}$ by the optimal degrees of membership $\Gamma(A_{ij_1}), \Gamma(A_{ij_2}), \ldots$, and $\Gamma(A_{ij_{f_{ik}^0}})$, respectively, corresponding to f_{ik}^0; it is defined as follows:

$$\Pi^0 = \begin{array}{c} \\ z_1 \\ z_2 \\ \vdots \\ z_m \end{array} \begin{array}{c} \begin{array}{cccc} \text{1st} & \text{2nd} & \cdots & m\text{-th} \end{array} \\ \begin{bmatrix} \Pi_{11}^0 & \Pi_{12}^0 & \cdots & \Pi_{1m}^0 \\ \Pi_{21}^0 & \Pi_{22}^0 & \cdots & \Pi_{2m}^0 \\ \vdots & \vdots & \ddots & \vdots \\ \Pi_{m1}^0 & \Pi_{m2}^0 & \cdots & \Pi_{mm}^0 \end{bmatrix} \end{array}, \tag{28}$$

where the meaning of each entry Π_{ik}^0 of Π^0 is a measure of the concordance among all of the criteria in assigning the alternative z_i a rank k, and

$$\Pi_{ik}^0 = \left(\Gamma(A_{ij_1}) \cdot W_{j_1}\right) \oplus \left(\Gamma(A_{ij_2}) \cdot W_{j_2}\right) \oplus \cdots \oplus \left(\Gamma(A_{ij_{f_{ik}^0}}) \cdot W_{j_{f_{ik}^0}}\right) \tag{29}$$

for $i, k = 1, 2, \ldots, m$. Applying Zadeh's extension principle [67] to the IT2 TrFN environment, we calculate Π_{ik}^0 in the following manner:

$$\Pi_{ik}^0 = \left[\left(\sum_{\vartheta=1}^{f_{ik}^0} \Gamma(A_{ij_\vartheta}) \cdot w_{1j_\vartheta}^-, \sum_{\vartheta=1}^{f_{ik}^0} \Gamma(A_{ij_\vartheta}) \cdot w_{2j_\vartheta}^-, \sum_{\vartheta=1}^{f_{ik}^0} \Gamma(A_{ij_\vartheta}) \cdot w_{3j_\vartheta}^-, \sum_{\vartheta=1}^{f_{ik}^0} \Gamma(A_{ij_\vartheta}) \cdot w_{4j_\vartheta}^-; \min_{\vartheta=1}^{f_{ik}^0} h_{W_{j_\vartheta}}^-\right), \left(\sum_{\vartheta=1}^{f_{ik}^0} \Gamma(A_{ij_\vartheta}) \cdot w_{1j_\vartheta}^+, \sum_{\vartheta=1}^{f_{ik}^0} \Gamma(A_{ij_\vartheta}) \cdot w_{2j_\vartheta}^+, \sum_{\vartheta=1}^{f_{ik}^0} \Gamma(A_{ij_\vartheta}) \cdot w_{3j_\vartheta}^+, \sum_{\vartheta=1}^{f_{ik}^0} \Gamma(A_{ij_\vartheta}) \cdot w_{4j_\vartheta}^+; \min_{\vartheta=1}^{f_{ik}^0} h_{W_{j_\vartheta}}^+\right)\right]. \tag{30}$$

Note that the operations of addition and multiplication by a nonnegative ordinary number defined on nonnegative IT2 TrFNs will obtain additional nonnegative IT2 TrFNs. Clearly, the optimal degrees of membership $\Gamma(A_{ij_1})$, $\Gamma(A_{ij_2}), \ldots$, and $\Gamma(A_{ij_{f_{ik}^0}})$ are nonnegative ordinary numbers. Thus, the computation result of the Π_{ik}^0 value is a nonnegative IT2 TrFN. For brevity, denote $h_{\Pi_{ik}^0}^- = \min_{\vartheta=1}^{f_{ik}^0} h_{W_{j_\vartheta}}^-$, $h_{\Pi_{ik}^0}^+ = \min_{\vartheta=1}^{f_{ik}^0} h_{W_{j_\vartheta}}^+$, $\pi_{\eta ik}^{0-} = \sum_{\vartheta=1}^{f_{ik}^0} \Gamma(A_{ij_\vartheta}) \cdot w_{\eta j_\vartheta}^-$, and $\pi_{\eta ik}^{0+} = \sum_{\vartheta=1}^{f_{ik}^0} \Gamma(A_{ij_\vartheta}) \cdot w_{\eta j_\vartheta}^+$ for $\eta = 1, 2, 3, 4$. Then, the entry Π_{ik}^0 in the rank contribution matrix Π^0 is expressed as the following:

$$\Pi_{ik}^0 = \left[\left(\pi_{1ik}^{0-}, \pi_{2ik}^{0-}, \pi_{3ik}^{0-}, \pi_{4ik}^{0-}; h_{\Pi_{ik}^0}^-\right), \left(\pi_{1ik}^{0+}, \pi_{2ik}^{0+}, \pi_{3ik}^{0+}, \pi_{4ik}^{0+}; h_{\Pi_{ik}^0}^+\right)\right], \tag{31}$$

where $0 \le \pi_{1ik}^{0-} \le \pi_{2ik}^{0-} \le \pi_{3ik}^{0-} \le \pi_{4ik}^{0-}$, $0 \le \pi_{1ik}^{0+} \le \pi_{2ik}^{0+} \le \pi_{3ik}^{0+} \le \pi_{4ik}^{0+}$, $\pi_{1ik}^{0+} \le \pi_{1ik}^{0-}$, $\pi_{4ik}^{0-} \le \pi_{4ik}^{0+}$, and $0 \le h_{\Pi_{ik}^0}^- \le h_{\Pi_{ik}^0}^+ \le 1$.

The rank contribution matrix Π^0 provides a simple approach for effectively measuring the contribution of each alternative to the overall performance (i.e., overall agreement with the criterion-wise rankings) if it is assigned a particular rank. Nevertheless, a troublesome issue might occur in situations where some alternatives are tied with respect to a criterion, resulting in confusing computations of the Π_{ik}^0 values. To overcome this difficulty, the initial ranking must be separated into $\varphi!$ equalized rankings when the φ alternatives are tied in terms of a specific criterion

[11, 13]. Additionally, each of these equalized rankings is weighted $1/\varphi!$.

Let φ_j represent the number of tied alternatives regarding a specific criterion c_j. Note that, if no tied alternatives are found with respect to c_j, we assume that $\varphi_j = 1$. Let f_{ik} denote the frequency with which the alternative z_i is ranked as the k-th criterion-wise ranking after separating the initial criterion-wise ranking into $\varphi_j!$ equalized rankings for all $c_j \in C$. Then, we obtain the adjusted rank frequency matrix F as follows:

$$F = \begin{array}{c} \\ z_1 \\ z_2 \\ \vdots \\ z_m \end{array} \begin{array}{c} \begin{array}{cccc} \text{1st} & \text{2nd} & \cdots & m\text{-th} \end{array} \\ \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1m} \\ f_{21} & f_{22} & \cdots & f_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m1} & f_{m2} & \cdots & f_{mm} \end{bmatrix} \end{array}. \quad (32)$$

Theorem 8 *The entry* f_{ik} $(i, k = 1, 2, \ldots, m)$ *in the adjusted rank frequency matrix* F *satisfies the following properties:*

(T8.1) $\sum_{k=1}^{m} f_{ik} = n$ *for* $i = 1, 2, \ldots, m$;

(T8.2) $\sum_{i=1}^{m} f_{ik} = n$ *for* $k = 1, 2, \ldots, m$.

Proof (T8.1) and (T8.2) are obvious. □

Let $c_{j_1}, c_{j_2}, \ldots,$ and $c_{j_{f_{ik}}}$ be the criteria for which the alternative z_i is ranked k-th. The adjusted rank contribution matrix Π is expressed as follows:

$$\Pi = \begin{array}{c} \\ z_1 \\ z_2 \\ \vdots \\ z_m \end{array} \begin{array}{c} \begin{array}{cccc} \text{1st} & \text{2nd} & \cdots & m\text{-th} \end{array} \\ \begin{bmatrix} \Pi_{11} & \Pi_{12} & \cdots & \Pi_{1m} \\ \Pi_{21} & \Pi_{22} & \cdots & \Pi_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \Pi_{m1} & \Pi_{m2} & \cdots & \Pi_{mm} \end{bmatrix} \end{array}, \quad (33)$$

where

$$\Pi_{ik} = \frac{\Gamma(A_{ij_1})}{\varphi_{j_1}!} \cdot W_{j_1} \oplus \frac{\Gamma(A_{ij_2})}{\varphi_{j_2}!} \cdot W_{j_2} \oplus \cdots \oplus \frac{\Gamma(A_{ij_{f_{ik}}})}{\varphi_{j_{f_{ik}}}!} \cdot W_{j_{f_{ik}}} \quad (34)$$

for $i, k = 1, 2, \ldots, m$. Similarly, applying Zadeh's extension principle to the IT2 TrFN environment, we calculate Π_{ik} in the following manner:

$$\Pi_{ik} = \left[\left(\sum_{\vartheta=1}^{f_{ik}} \frac{\Gamma(A_{ij_\vartheta}) \cdot w_{1j_\vartheta}^-}{\varphi_{j_\vartheta}!}, \sum_{\vartheta=1}^{f_{ik}} \frac{\Gamma(A_{ij_\vartheta}) \cdot w_{2j_\vartheta}^-}{\varphi_{j_\vartheta}!}, \sum_{\vartheta=1}^{f_{ik}} \frac{\Gamma(A_{ij_\vartheta}) \cdot w_{3j_\vartheta}^-}{\varphi_{j_\vartheta}!}, \sum_{\vartheta=1}^{f_{ik}} \frac{\Gamma(A_{ij_\vartheta}) \cdot w_{4j_\vartheta}^-}{\varphi_{j_\vartheta}!}; \min_{\vartheta=1}^{f_{ik}} h_{W_{j_\vartheta}}^- \right), \right.$$
$$\left. \left(\sum_{\vartheta=1}^{f_{ik}} \frac{\Gamma(A_{ij_\vartheta}) \cdot w_{1j_\vartheta}^+}{\varphi_{j_\vartheta}!}, \sum_{\vartheta=1}^{f_{ik}} \frac{\Gamma(A_{ij_\vartheta}) \cdot w_{2j_\vartheta}^+}{\varphi_{j_\vartheta}!}, \sum_{\vartheta=1}^{f_{ik}} \frac{\Gamma(A_{ij_\vartheta}) \cdot w_{3j_\vartheta}^+}{\varphi_{j_\vartheta}!}, \sum_{\vartheta=1}^{f_{ik}} \frac{\Gamma(A_{ij_\vartheta}) \cdot w_{4j_\vartheta}^+}{\varphi_{j_\vartheta}!}; \min_{\vartheta=1}^{f_{ik}} h_{W_{j_\vartheta}}^+ \right) \right]. \quad (35)$$

Because $1/(\varphi_{j_1}!), 1/(\varphi_{j_2}!), \ldots,$ and $1/(\varphi_{j_{f_{ik}}}!)$ are nonnegative ordinary numbers, the computation results of the Π_{ik} values are also IT2 TrFNs. For brevity, denote $h_{\Pi_{ik}}^- = \min_{\vartheta=1}^{f_{ik}} h_{W_{j_\vartheta}}^-$, $h_{\Pi_{ik}}^+ = \min_{\vartheta=1}^{f_{ik}} h_{W_{j_\vartheta}}^+$, $\pi_{\eta ik}^- = \sum_{\vartheta=1}^{f_{ik}} \Gamma(A_{ij_\vartheta}) \cdot w_{\eta j_\vartheta}^- \big/ \varphi_{j_\vartheta}!$, and $\pi_{1ik}^+ = \sum_{\vartheta=1}^{f_{ik}} \Gamma(A_{ij_\vartheta}) \cdot w_{1j_\vartheta}^+ \big/ \varphi_{j_\vartheta}!$ for $\eta = 1, 2, 3, 4$. Then, the entry Π_{ik} in the adjusted rank contribution matrix Π is expressed as the following:

$$\Pi_{ik} = \left[\left(\pi_{1ik}^-, \pi_{2ik}^-, \pi_{3ik}^-, \pi_{4ik}^-; h_{\Pi_{ik}}^- \right), \left(\pi_{1ik}^+, \pi_{2ik}^+, \pi_{3ik}^+, \pi_{4ik}^+; h_{\Pi_{ik}}^+ \right) \right], \quad (36)$$

where $0 \le \pi_{1ik}^- \le \pi_{2ik}^- \le \pi_{3ik}^- \le \pi_{4ik}^-$, $0 \le \pi_{1ik}^+ \le \pi_{2ik}^+ \le \pi_{3ik}^+ \le \pi_{4ik}^+$, $\pi_{1ik}^+ \le \pi_{1ik}^-$, $\pi_{4ik}^- \le \pi_{4ik}^+$, and $0 \le h_{\Pi_{ik}}^- \le h_{\Pi_{ik}}^+ \le 1$.

Based on the rank contribution matrix Π^0 or the adjusted rank contribution matrix Π, we attempt to obtain an aggregate ranking that can effectively combine the relative performance of each alternative with respect to each criterion. In other words, this aggregate ranking is an overall ranking that is in the closest agreement with the criterion-wise rankings. Therefore, when the alternative z_i is assigned to the k-th overall rank, we can employ Π_{ik}^0 or Π_{ik} to measure the contributions of z_i to the overall ranking. Additionally, the larger the contribution indicated by Π_{ik}^0 or Π_{ik}, the more concordance from assigning z_i to the k-th overall rank will result.

However, the values of Π_{ik}^0 and Π_{ik} are IT2 TrFNs and cannot be easily compared. Thus, we use a signed distance-based method presented by Chen [10, 12, 14] and Chen et al. [20] to obtain comparable values of Π_{ik}^0 (or Π_{ik}) for each $i, k = 1, 2, \ldots, m$. A signed distance (i.e., an oriented distance or directed distance) has often been employed to determine the rankings of fuzzy numbers [14]. The concept of signed distances has been extended to the decision environment of IT2 TrFNs, and the signed distance-based method has been successfully employed to develop several useful MCDA methods [10, 12–15, 20]. The signed distance-based method can use both positive and negative values to define the ordering of IT2 TrFNs, an approach that considerably differs from ordinary distance measures. Signed distances are real numbers; thus, they satisfy linear ordering. It follows that the signed distance based on IT2 TrFNs satisfies the law of trichotomy. Therefore, the concept of signed distances is suggested for acquiring comparable values of Π_{ik}^0 in the rank contribution matrix Π^0 or Π_{ik} in the adjusted rank contribution matrix Π.

In reference to Chen [10, 12, 15] and Chen et al. [20], the signed distance d_{ik}^0 from Π_{ik}^0 to the level-1 fuzzy number map onto the vertical axis at the origin of the coordinates is expressed as follows:

$$d_{ik}^0 = \frac{1}{8}\left[\pi_{1ik}^{0-} + \pi_{2ik}^{0-} + \pi_{3ik}^{0-} + \pi_{4ik}^{0-} + 4\cdot\pi_{1ik}^{0+} + 2\cdot\pi_{2ik}^{0+} \right. \\ \left. +2\cdot\pi_{3ik}^{0+} + 4\cdot\pi_{4ik}^{0+} + 3\left(\pi_{2ik}^{0+} + \pi_{3ik}^{0+} - \pi_{1ik}^{0+} - \pi_{4ik}^{0+}\right)\frac{h_{\Pi_{ik}^0}^-}{h_{\Pi_{ik}^0}^+}\right], \tag{37}$$

for all $i, k = 1, 2, \ldots, m$. The signed distance d_{ik} from Π_{ik} to the level-1 fuzzy number map onto the vertical axis at the origin of the coordinates is expressed as follows:

$$d_{ik} = \frac{1}{8}\left[\pi_{1ik}^- + \pi_{2ik}^- + \pi_{3ik}^- + \pi_{4ik}^- + 4\cdot\pi_{1ik}^+ + 2\cdot\pi_{2ik}^+ \right. \\ +2\cdot\pi_{3ik}^+ + 4\cdot\pi_{4ik}^+ \\ \left. +3\left(\pi_{2ik}^+ + \pi_{3ik}^+ - \pi_{1ik}^+ - \pi_{4ik}^+\right)\frac{h_{\Pi_{ik}}^-}{h_{\Pi_{ik}}^+}\right], \tag{38}$$

for all $i, k = 1, 2, \ldots, m$.

Computing the signed distances d_{ik}^0 or d_{ik} for all $i, k = 1, 2, \ldots, m$, we can then find the alternative z_i for each k-th rank that maximizes $\sum_{k=1}^m d_{ik}^0$ or $\sum_{k=1}^m d_{ik}$. This problem is an $m!$ comparison problem. A pure integer linear programming model can be established in the case of a large m to solve the $m!$ comparison problem. Let a permutation matrix P denote an $m \times m$ square matrix whose element $P_{ik} = 1$ if the alternative z_i is assigned to the overall rank k; otherwise, $P_{ik} = 0$. In general, we prefer the overall ranking in which the value of $d_{ik}^0 \cdot P_{ik}$ or $d_{ik} \cdot P_{ik}$ is the largest. This is because the overall ranking with the highest $d_{ik}^0 \cdot P_{ik}$ or $d_{ik} \cdot P_{ik}$ produces the best compromise among all of the criterion-wise rankings of the alternatives. Accordingly, all of the conceivable rankings must be examined, and the one that yields the largest value of $d_{ik}^0 \cdot P_{ik}$ or $d_{ik} \cdot P_{ik}$ will be chosen. Because the alternative z_i can be assigned to only one rank in the overall ranking, we know that $\sum_{k=1}^m P_{ik} = 1$. Similarly, a given rank k can only have one alternative assigned to it, and thus $\sum_{i=1}^m P_{ik} = 1$.

Consider the situations in which no tied alternatives exist when comparing the $\Gamma(A_{ij})$ values with respect to each criterion $c_j \in C$, i.e., $\varphi_j = 1$ for all $j = 1, 2, \ldots, n$. The proposed likelihood-based assignment model can be expressed using the following pure integer linear programming format:

$$[M1] \quad \begin{aligned} &\max \sum_{i=1}^m \sum_{k=1}^m d_{ik}^0 \cdot P_{ik} \\ &\text{s.t.} \sum_{k=1}^m P_{ik} = 1,\ i = 1, 2, \cdots, m, \\ &\sum_{i=1}^m P_{ik} = 1,\ k = 1, 2, \cdots, m, \\ &P_{ik} = 0 \text{ or } 1 \text{ for all } i \text{ and } k. \end{aligned} \tag{39}$$

Alternately, consider the situations in which some tied alternatives exist when comparing the $\Gamma(A_{ij})$ values, i.e., $\varphi_j \neq 1$ for at least one $j = 1, 2, \ldots, n$. Then, the proposed likelihood-based assignment model in tied situations can be expressed as follows:

$$[M2] \quad \begin{aligned} &\max \sum_{i=1}^m \sum_{k=1}^m d_{ik} \cdot P_{ik} \\ &\text{s.t.} \sum_{k=1}^m P_{ik} = 1,\ i = 1, 2, \cdots, m, \\ &\sum_{i=1}^m P_{ik} = 1,\ k = 1, 2, \cdots, m, \\ &P_{ik} = 0 \text{ or } 1 \text{ for all } i \text{ and } k. \end{aligned} \tag{40}$$

The signed distances d_{ik}^0 and d_{ik} can be obtained via Π_{ik}^0 in the rank contribution matrix Π^0 and Π_{ik} in the adjusted rank contribution matrix Π, respectively, from the criterion-wise rankings, whereas P_{ik} is unknown and has yet to be determined by the likelihood-based assignment model. Using a branch-and-bound algorithm, we can solve the pure integer linear programming problem [M1] or [M2] to acquire the optimal permutation matrix P^*. The optimal ordering can then be sequentially obtained by multiplying Z by P^*.

4.2 Proposed algorithm

The proposed likelihood-based assignment method for solving an MCDA problem within the IT2 TrFN decision environment comprises three phases: problem formulation and data input; likelihood-based comparisons and the ranking procedure; and construction of the linear assignment model. Figure 2 depicts a flowchart of the proposed likelihood-based assignment method.

The three phases can be summarized by the following series of steps.

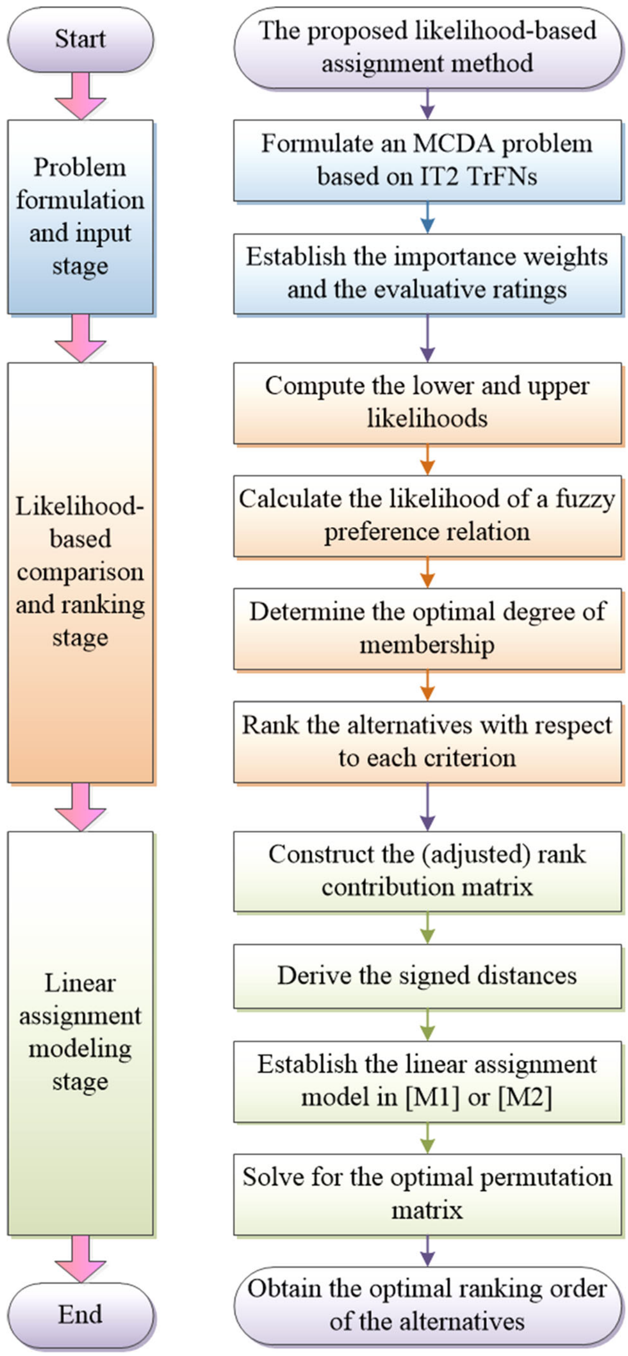

Fig. 2 Flowchart of the proposed likelihood-based assignment method

4.2.1 *Steps 1 and 2: problem formulation and input stage*

Step 1 Formulate an MCDA problem. Specify the criterion set $C = \{c_1, c_2, \ldots, c_n\}$ and the alternative set $Z = \{z_1, z_2, \ldots, z_m\}$.

Step 2 Select appropriate linguistic variables or other data collection tools to establish the importance weight W_j in (10) of the criterion $c_j \in C$ and the evaluative rating A_{ij} in (9) of the alternative $z_i \in Z$ with respect to the criterion $c_j \in C$ based on IT2 TrFNs.

4.2.2 *Steps 3–6: likelihood-based comparison and ranking stage*

Step 3 Apply (11) and (12) to compute the lower and upper likelihoods, $L^-(A_{ij} \geq A_{i'j})$ and $L^+(A_{ij} \geq A_{i'j})$, respectively, of a fuzzy preference relation $A_{ij} \geq A_{i'j}$ for each criterion $c_j \in C$ and each pair of alternatives $(z_i, z_{i'})$, where $z_i, z_{i'} \in Z$ and $i \neq i'$.

Step 4 Calculate the likelihood $L(A_{ij} \geq A_{i'j})$ using (13) for each criterion $c_j \in C$ and each pair of alternatives $(z_i, z_{i'})$, where $z_i, z_{i'} \in Z$ and $i \neq i'$. Moreover, $L(A_{ij} \geq A_{ij}) = 0.5$ for $i = 1, 2, \ldots, m$.

Step 5 Employ (24) to determine the optimal degree of membership $\Gamma(A_{ij})$ for $z_i \in Z$ and $c_j \in C$.

Step 6 Rank the m alternatives with respect to each criterion $c_j \in C_{\mathrm{I}}$ and $c_j \in C_{\mathrm{II}}$ according to the descending order and the ascending order, respectively, of the $\Gamma(A_{ij})$ values. If the φ_j alternatives are tied with respect to a specific criterion c_j, then the $\varphi_j!$ equalized rankings should be listed separately.

4.2.3 *Steps 7–10: linear assignment modeling stage*

Step 7 Compute Π^0_{ik} using (30) and construct the rank contribution matrix Π^0 when $\varphi_j = 1$ for all $j = 1, 2, \ldots, n$. Alternately, compute Π_{ik} using (35) and establish the adjusted rank contribution matrix Π when $\varphi_j \neq 1$ for at least one $j = 1, 2, \ldots, n$.

Step 8 Use (37) or (38) to derive the signed distances d^0_{ik} or d_{ik}, respectively, for each $i, k = 1, 2, \ldots, m$.

Step 9 Define the permutation matrix P as an $m \times m$ square matrix. Construct a linear assignment model in [M1] or [M2] according to the d^0_{ik} values or the d_{ik} values, respectively.

Step 10 Solve [M1] or [M2] using a branch-and-bound algorithm and obtain the optimal permutation matrix P^*. Next, apply the permutation matrix P^* to Z to obtain the optimal ranking order of the m alternatives.

Table 1 Linguistic variables and their corresponding IT2 TrFNs

Linguistic terms	Corresponding IT2 TrFNs
Absolutely low (AL)	[(0.0000, 0.0000, 0.0000, 0.0000; 1.0), (0.0000, 0.0000, 0.0000, 0.0000; 1.0)]
Very low (VL)	[(0.0075, 0.0075, 0.0150, 0.0525; 0.8), (0.0000, 0.0000, 0.0200, 0.0700; 1.0)]
Low (L)	[(0.0875, 0.1200, 0.1600, 0.1825; 0.8), (0.0400, 0.1000, 0.1800, 0.2300; 1.0)]
Medium low (ML)	[(0.2325, 0.2550, 0.3250, 0.3575; 0.8), (0.1700, 0.2200, 0.3600, 0.4200; 1.0)]
Medium (M)	[(0.4025, 0.4525, 0.5375, 0.5675; 0.8), (0.3200, 0.4100, 0.5800, 0.6500; 1.0)]
Medium high (MH)	[(0.6500, 0.6725, 0.7575, 0.7900; 0.8), (0.5800, 0.6300, 0.8000, 0.8600; 1.0)]
High (H)	[(0.7825, 0.8150, 0.8850, 0.9075; 0.8), (0.7200, 0.7800, 0.9200, 0.9700; 1.0)]
Very high (VH)	[(0.9475, 0.9850, 0.9925, 0.9925; 0.8), (0.9300, 0.9800, 1.0000, 1.0000; 1.0)]
Absolutely high (AH)	[(1.0000, 1.0000, 1.0000, 1.0000; 1.0), (1.0000, 1.0000, 1.0000, 1.0000; 1.0)]

5 Applications and comparative analyses

This section illustrates and discusses the proposed likelihood-based assignment method by applying it to two practical MCDA applications: the selection problem of landfill sites introduced by Chen [13] and the medical decision-making problem presented by Chen et al. [20]. Furthermore, comparative discussions are subsequently devoted to validate the results of the proposed method with the results from relevant outranking methods. Finally, additional comparisons are conducted to investigate the effectiveness and advantages of the proposed method relative to a widely used scoring method.

5.1 Illustrative application

Only a small number of attempts to extend the linear assignment method to the interval type-2 fuzzy decision environment or allow it to cope with IT2 TrFN data have thus far been conducted. Chen [13] developed an interval type-2 fuzzy linear assignment method based on signed distances to handle multiple criteria decisions and implemented the proposed method to a practical example involving a landfill site selection problem in Kaohsiung City. Chen's interval type-2 fuzzy linear assignment method and our proposed method belong to the linear assignment methodology. Thus, Chen's illustrative application of landfill site selection can provide a common basis for implementing a comparative analysis.

The illustrative application presented by Chen [13] assesses the characteristics of candidate landfill sites to determine whether the candidate sites are qualified in regard to the interests and the rights of the stakeholders and general public using seven criteria: transportation convenience (c_1), terrain suitability (c_2), community equity (c_3), environmental impact (c_4), ecological impact (c_5), construction cost (c_6), and historic impact (c_7). Among the criteria for evaluating candidate sites, c_1, c_2, and c_3 are the benefit criteria, whereas all of the others are cost criteria. Given the dense population of Kaohsiung City, the number of suitable landfill locations is limited. There are four candidate locations: z_1 (located in the Cijin district), z_2 (located at the side of an embankment in the Nanzih district), z_3 (located on a hillside by the sea in the Gushan district), and z_4 (located on a hillside in the Zuoying district).

Table 2 Linguistic data of the importance weights and the evaluative ratings

Criteria	Importance weights	Ratings of the candidate sites			
		z_1	z_2	z_3	z_4
c_1	H	L	AL	AH	H
c_2	MH	AH	VH	AL	L
c_3	H	AL	H	MH	H
c_4	MH	L	M	AH	VH
c_5	MH	L	ML	AH	MH
c_6	M	AH	ML	M	M
c_7	ML	AH	L	VH	H

The computational procedure of the proposed likelihood-based assignment method is illustrated as follows. In Step 1, there are four candidate landfill sites in the MCDA problem; the set of all candidate sites is denoted as $Z = \{z_1, z_2, z_3, z_4\}$. The set of criteria is denoted as $C = \{c_1, c_2, \ldots, c_7\}$, with $C_{\mathrm{I}} = \{c_1, c_2, c_3\}$ and $C_{\mathrm{II}} = \{c_4, c_5, c_6, c_7\}$. In Step 2, the city government evaluated the four landfill sites based on the seven criteria and the importance weights of the criteria using the linguistic rating system in Table 1 [13]. The details of the linguistic evaluations are outlined in Table 2. These linguistic terms were subsequently converted into the IT2 TrFN evaluative ratings A_{ij} of $z_i \in Z$ with respect to $c_j \in C$ and the IT2 TrFN importance weights W_j of $c_j \in C$.

In Step 3, we calculated $L^-(A_{ij} \geq A_{i'j})$ and $L^+(A_{ij} \geq A_{i'j})$ for each pair of alternatives $(z_i, z_{i'})$ (z_i, $z_{i'} \in Z$ and $i \neq i'$) with respect to each criterion $c_j \in C$. Then, we computed the likelihood $L(A_{ij} \geq A_{i'j})$ according to Step 4. Note that $L(A_{ij} \geq A_{ij}) = 0.5$ for $i = 1, 2, 3, 4$. The computation results of the likelihoods for all the fuzzy preference relations between IT2 TrFNs are presented in Table 3.

In Step 5, we applied (24) to obtain the optimal degree of membership $\Gamma(A_{ij})$ for $z_i \in Z$ and $c_j \in C$. For example, $\Gamma(A_{23})$ is computed as follows:

$$\begin{aligned}\Gamma(A_{23}) &= \frac{1}{4(4-1)}\Big(L(A_{23} \geq A_{13}) + L(A_{23} \geq A_{23}) \\ &\quad + L(A_{23} \geq A_{33}) + L(A_{23} \geq A_{43}) + \frac{4}{2} - 1\Big) \\ &= \frac{1}{4(4-1)}\left(1.00 + 0.50 + 0.79 + 0.50 + \frac{4}{2} - 1\right) \\ &= 0.32.\end{aligned}$$

Next, we applied Step 6 to rank the four alternatives in terms of each benefit criterion and each cost criterion according to the decreasing order and the increasing order, respectively, of the $\Gamma(A_{ij})$ values. Because the alternatives z_2 and z_4 are tied with respect to the criterion c_3, $\varphi_3 = 2$, and the original ranking $z_2 \sim z_4 \succ z_3 \succ z_1$ is equalized as two corresponding rankings, $z_2 \succ z_4 \succ z_3 \succ z_1$ and $z_4 \succ z_2 \succ z_3 \succ z_1$. Additionally, the alternatives z_3 and z_4 are tied with respect to the criterion c_6; thus, $\varphi_6 = 2$, and the original ranking $z_2 \succ z_3 \sim z_4 \succ z_1$ is equalized as two corresponding rankings, $z_2 \succ z_3 \succ z_4 \succ z_1$ and $z_2 \succ z_4 \succ z_3 \succ z_1$. The other $\varphi_j = 1$ for $j = 1,2,4,5,7$. The results of the optimal degrees of membership and criterion-wise rankings are indicated in Table 4.

Because $\varphi_3, \varphi_6 \neq 1$, we employed (35) to determine Π_{ik} for i, $k = 1, 2, 3, 4$ according to Step 7. Take the argument π^-_{142} of Π_{42} $(= [(\pi^-_{142}, \pi^-_{242}, \pi^-_{342}, \pi^-_{442}; h^-_{\Pi_{42}}), (\pi^+_{142}, \pi^+_{242}, \pi^+_{342}, \pi^+_{442}; h^+_{\Pi_{42}})])$ for example. As indicated in (36), we have

$$\begin{aligned}\pi^-_{142} &= \sum_{\vartheta=1}^{f_{42}} \frac{\Gamma(A_{ij_\vartheta}) \cdot w^-_{1j_\vartheta}}{\varphi_{j_\vartheta}!} \\ &= \frac{\Gamma(A_{41}) \cdot w^-_{11}}{1!} + \frac{\Gamma(A_{43}) \cdot w^-_{13}}{2!} + \frac{\Gamma(A_{46}) \cdot w^-_{16}}{2!} \\ &\quad + \frac{\Gamma(A_{47}) \cdot w^-_{17}}{1!} = 0.29 \cdot 0.7825 + \frac{0.32 \cdot 0.7825}{2} \\ &\quad + \frac{0.24 \cdot 0.4025}{2} + 0.22 \cdot 0.2325 = 0.45.\end{aligned}$$

Then, in Step 8, we used (38) to calculate the signed distance d_{ik} for $i, k = 1,2,3,4$. Consider Π_{31} $(= [(0.30, 0.31, 0.34, 0.34; 0.80), (0.27, 0.30, 0.35, 0.37; 1.00)])$ for example. The signed distance d_{31} of Π_{31} is computed as follows:

$$\begin{aligned}d_{31} &= \frac{1}{8}\Big[0.30 + 0.31 + 0.34 + 0.34 + 4 \cdot 0.27 + 2 \cdot 0.30 \\ &\quad + 2 \cdot 0.35 + 4 \cdot 0.37 + 3(0.30 + 0.35 - 0.27 - 0.37)\frac{0.80}{1.00}\Big] \\ &= 0.65.\end{aligned}$$

The computation results of the adjusted rank contribution matrix Π and signed distances are revealed in Table 5.

In Step 9, we defined the permutation matrix P as a 4×4 square matrix, with element P_{ik} for i, $k = 1, 2, 3, 4$. According to [M2], we established a linear assignment model with d_{ik} values as follows:

$$\begin{aligned}&\max 0.93 \cdot P_{11} + 0.00 \cdot P_{12} + 0.34 \cdot P_{13} + 0.82 \cdot P_{14} \\ &\quad + 0.49 \cdot P_{21} + 1.28 \cdot P_{22} \\ &\quad + 0.00 \cdot P_{23} + 0.24 \cdot P_{24} + 0.65 \cdot P_{31} + 0.12 \cdot P_{32} \\ &\quad + 0.69 \cdot P_{33} + 1.29 \cdot P_{34} \\ &\quad + 0.27 \cdot P_{41} + 1.01 \cdot P_{42} + 1.23 \cdot P_{43} + 0.00 \cdot P_{44} \\ &\text{s.t.}\ P_{11} + P_{12} + P_{13} + P_{14} = 1,\ P_{21} + P_{22} + P_{23} + P_{24} = 1, \\ &\quad P_{31} + P_{32} + P_{33} + P_{34} = 1,\ P_{41} + P_{42} + P_{43} + P_{44} = 1, \\ &\quad P_{11} + P_{21} + P_{31} + P_{41} = 1,\ P_{12} + P_{22} + P_{32} + P_{42} = 1, \\ &\quad P_{13} + P_{23} + P_{33} + P_{43} = 1,\ P_{14} + P_{24} + P_{34} + P_{44} = 1, \\ &\quad P_{ik} = 0 \text{ or } 1 \text{ for all } i \text{ and } k.\end{aligned} \tag{41}$$

Finally, in Step 10, we solved the model in (41) using a branch-and-bound algorithm. The optimal objective value is 4.73. Additionally, we obtained the optimal permutation matrix P^* as follows:

$$\mathrm{P}^* = \begin{array}{c} \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{array}\begin{array}{c} \begin{array}{cccc} \text{1st} & \text{2nd} & \text{3rd} & \text{4th} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}.$$

Next, we determined the ordering (z_1, z_2, z_3, z_4) by multiplying Z by P^*. That is, $(z_1, z_2, z_3, z_4) \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = (z_1, z_2, z_4, z_3)$.

The optimal ranking order of the four candidate sites is $z_1 \succ z_2 \succ z_4 \succ z_3$. Thus, z_1 (located in the Cijin district) is the best choice. This ranking result is the same as that obtained using the interval type-2 fuzzy linear assignment method based on signed distances [13].

5.2 Comparative discussions with outranking methods

The linear assignment methodology belongs to outranking models in the MCDA field. Accordingly, this paper further selected other well-known outranking methods (e.g., ELECTRE and QUALIFLEX) to conduct a comparative analysis and examine the applicability and advantages of the proposed method.

Table 3 Results of the likelihoods of $A_{ij} \geq A_{i'j}$ (z_i, $z_{i'} \in Z$)

	$L^-(A_{1j} \geq A_{2j})$	$L^+(A_{1j} \geq A_{2j})$	$L(A_{1j} \geq A_{2j})$		$L^-(A_{3j} \geq A_{1j})$	$L^+(A_{3j} \geq A_{1j})$	$L(A_{3j} \geq A_{1j})$
c_1	0.70	1.00	0.85	c_1	1.00	1.00	1.00
c_2	1.00	1.00	1.00	c_2	0.00	0.00	0.00
c_3	0.00	0.00	0.00	c_3	1.00	1.00	1.00
c_4	0.00	0.11	0.05	c_4	1.00	1.00	1.00
c_5	0.01	0.30	0.15	c_5	1.00	1.00	1.00
c_6	1.00	1.00	1.00	c_6	0.00	0.00	0.00
c_7	1.00	1.00	1.00	c_9	0.00	0.00	0.00

	$L^-(A_{1j} \geq A_{3j})$	$L^+(A_{1j} \geq A_{3j})$	$L(A_{1j} \geq A_{3j})$		$L^-(A_{3j} \geq A_{2j})$	$L^+(A_{3j} \geq A_{2j})$	$L(A_{3j} \geq A_{2j})$
c_1	0.00	0.00	0.00	c_1	1.00	1.00	1.00
c_2	1.00	1.00	1.00	c_2	0.00	0.00	0.00
c_3	0.00	0.00	0.00	c_3	0.05	0.36	0.21
c_4	0.00	0.00	0.00	c_4	1.00	1.00	1.00
c_5	0.00	0.00	0.00	c_5	1.00	1.00	1.00
c_6	1.00	1.00	1.00	c_6	0.74	0.98	0.86
c_7	1.00	1.00	1.00	c_7	1.00	1.00	1.00

	$L^-(A_{1j} \geq A_{4j})$	$L^+(A_{1j} \geq A_{4j})$	$L(A_{1j} \geq A_{4j})$		$L^-(A_{3j} \geq A_{4j})$	$L^+(A_{3j} \geq A_{4j})$	$L(A_{3j} \geq A_{4j})$
c_1	0.00	0.00	0.00	c_1	1.00	1.00	1.00
c_2	1.00	1.00	1.00	c_2	0.00	0.30	0.15
c_3	0.00	0.00	0.00	c_3	0.05	0.36	0.21
c_4	0.00	0.00	0.00	c_4	1.00	1.00	1.00
c_5	0.00	0.00	0.00	c_5	1.00	1.00	1.00
c_6	1.00	1.00	1.00	c_6	0.33	0.67	0.50
c_7	1.00	1.00	1.00	c_7	0.65	1.00	0.83

	$L^-(A_{2j} \geq A_{1j})$	$L^+(A_{2j} \geq A_{1j})$	$L(A_{2j} \geq A_{1j})$		$L^-(A_{4j} \geq A_{1j})$	$L^+(A_{4j} \geq A_{1j})$	$L(A_{4j} \geq A_{1j})$
c_1	0.00	0.30	0.15	c_1	1.00	1.00	1.00
c_2	0.00	0.00	0.00	c_2	0.00	0.00	0.00
c_3	1.00	1.00	1.00	c_3	1.00	1.00	1.00
c_4	0.89	1.00	0.95	c_4	1.00	1.00	1.00
c_5	0.70	0.99	0.85	c_5	1.00	1.00	1.00
c_6	0.00	0.00	0.00	c_6	0.00	0.00	0.00
c_7	0.00	0.00	0.00	c_7	0.00	0.00	0.00

	$L^-(A_{2j} \geq A_{3j})$	$L^+(A_{2j} \geq A_{3j})$	$L(A_{2j} \geq A_{3j})$		$L^-(A_{4j} \geq A_{2j})$	$L^+(A_{4j} \geq A_{2j})$	$L(A_{4j} \geq A_{2j})$
c_1	0.00	0.00	0.00	c_1	1.00	1.00	1.00
c_2	1.00	1.00	1.00	c_2	0.00	0.00	0.00
c_3	0.64	0.95	0.79	c_3	0.29	0.71	0.50
c_4	0.00	0.00	0.00	c_4	0.96	1.00	0.98
c_5	0.00	0.00	0.00	c_5	0.93	1.00	0.97
c_6	0.02	0.26	0.14	c_6	0.74	0.98	0.86
c_7	0.00	0.00	0.00	c_7	1.00	1.00	1.00

	$L^-(A_{2j} \geq A_{4j})$	$L^+(A_{2j} \geq A_{4j})$	$L(A_{2j} \geq A_{4j})$		$L^-(A_{4j} \geq A_{3j})$	$L^+(A_{4j} \geq A_{3j})$	$L(A_{4j} \geq A_{3j})$
c_1	0.00	0.00	0.00	c_1	0.00	0.00	0.00
c_2	1.00	1.00	1.00	c_2	0.70	1.00	0.85
c_3	0.29	0.71	0.50	c_3	0.64	0.95	0.79
c_4	0.00	0.04	0.02	c_4	0.00	0.00	0.00

Table 3 continued

	$L^-(A_{2j} \geq A_{4j})$	$L^+(A_{2j} \geq A_{4j})$	$L(A_{2j} \geq A_{4j})$		$L^-(A_{4j} \geq A_{3j})$	$L^+(A_{4j} \geq A_{3j})$	$L(A_{4j} \geq A_{3j})$
c_5	0.00	0.07	0.03	c_5	0.00	0.00	0.00
c_6	0.02	0.26	0.14	c_6	0.33	0.67	0.50
c_7	0.00	0.00	0.00	c_7	0.00	0.35	0.17

Table 4 Results of the optimal degrees of membership and criterion-wise rankings

	Benefit criteria			Cost criteria			
	c_1	c_2	c_3	c_4	c_5	c_6	c_7
Results of the optimal degree of membership $\Gamma(A_{ij})$ for $z_i \in Z$ and $c_j \in C$							
z_1	0.20	0.38	0.13	0.13	0.14	0.38	0.38
z_2	0.14	0.29	0.32	0.21	0.20	0.15	0.13
z_3	0.38	0.14	0.24	0.38	0.38	0.24	0.28
z_4	0.29	0.20	0.32	0.29	0.29	0.24	0.22
Rank							
Results of criterion-wise rankings							
1st	z_3	z_1	z_2, z_4	z_1	z_1	z_2	z_2
2nd	z_4	z_2		z_2	z_2	z_3, z_4	z_4
3rd	z_1	z_4	z_3	z_4	z_4		z_3
4th	z_2	z_3	z_1	z_3	z_3	z_1	z_1

	Benefit criteria				Cost criteria				
	c_1	c_2	$c_{3(1)}$	$c_{3(2)}$	c_4	c_5	$c_{6(1)}$	$c_{6(2)}$	c_7
Rank									
Adjusted results of criterion-wise rankings									
1st	z_3	z_1	z_2	z_4	z_1	z_1	z_2	z_2	z_2
2nd	z_4	z_2	z_4	z_2	z_2	z_2	z_3	z_4	z_4
3rd	z_1	z_4	z_3	z_3	z_4	z_4	z_4	z_3	z_3
4th	z_2	z_3	z_1	z_1	z_3	z_3	z_1	z_1	z_1

Chen et al. [20] developed an extended QUALIFLEX method for handling MCDA problems based on IT2 FSs. They also extended the ELECTRE method to the decision environment of IT2 FSs. Chen et al. [20] presented a medical decision-making problem of selecting a suitable treatment method for acute inflammatory demyelinating disease. Their case comes from the Department of Neurology, Chang Gung Memorial Hospital in Taiwan. They applied the extended QUALIFLEX method and the extended ELECTRE method to the medical decision-making problem. Additionally, Chen and Tsui [25] developed the intuitionistic fuzzy QUALIFLEX method based on optimistic and pessimistic estimations. Although Chen and Tsui's proposed method is suitable for the intuitionistic fuzzy decision environment, they also applied it to the same medical decision-making problem.

In addition to a comparative discussion regarding the selection problem of landfill sites using the interval type-2 fuzzy linear assignment method [13], we further applied the proposed method to a medical decision-making problem introduced by Chen et al. [20] to facilitate comparative analyses with other outranking methods. With regard to the medical decision-making problem, this paper compared the solution result via the proposed likelihood-based assignment

Table 5 Results of the adjusted rank contribution matrix Π and signed distances

Π_{ik} in the adjusted rank contribution matrix Π		d_{ik}
Π_{11}	[(0.42, 0.44, 0.49, 0.51; 0.80), (0.38, 0.41, 0.52, 0.56; 1.00)]	0.93
Π_{12}	[(0.00, 0.00, 0.00, 0.00; 1.00), (0.00, 0.00, 0.00, 0.00; 1.00)]	0.00
Π_{13}	[(0.16, 0.16, 0.18, 0.18; 0.80), (0.14, 0.16, 0.18, 0.19; 1.00)]	0.34
Π_{14}	[(0.34, 0.37, 0.44, 0.47; 0.80), (0.28, 0.34, 0.48, 0.53; 1.00)]	0.82
Π_{21}	[(0.22, 0.23, 0.26, 0.28; 0.80), (0.19, 0.21, 0.28, 0.31; 1.00)]	0.49
Π_{22}	[(0.58, 0.60, 0.67, 0.70; 0.80), (0.52, 0.57, 0.71, 0.76; 1.00)]	1.28
Π_{23}	[(0.00, 0.00, 0.00, 0.00; 1.00), (0.00, 0.00, 0.00, 0.00; 1.00)]	0.00
Π_{24}	[(0.11, 0.11, 0.12, 0.13; 0.80), (0.10, 0.11, 0.13, 0.14; 1.00)]	0.24
Π_{31}	[(0.30, 0.31, 0.34, 0.34; 0.80), (0.27, 0.30, 0.35, 0.37; 1.00)]	0.65
Π_{32}	[(0.05, 0.05, 0.06, 0.07; 0.80), (0.04, 0.05, 0.07, 0.08; 1.00)]	0.12
Π_{33}	[(0.30, 0.32, 0.37, 0.39; 0.80), (0.26, 0.30, 0.39, 0.43; 1.00)]	0.69
Π_{34}	[(0.59, 0.61, 0.68, 0.71; 0.80), (0.52, 0.57, 0.72, 0.77; 1.00)]	1.29
Π_{41}	[(0.13, 0.13, 0.14, 0.15; 0.80), (0.12, 0.12, 0.15, 0.16; 1.00)]	0.27
Π_{42}	[(0.45, 0.48, 0.53, 0.56; 0.80), (0.40, 0.45, 0.56, 0.61; 1.00)]	1.01
Π_{43}	[(0.56, 0.58, 0.66, 0.68; 0.80), (0.49, 0.54, 0.69, 0.75; 1.00)]	1.23
Π_{44}	[(0.00, 0.00, 0.00, 0.00; 1.00), (0.00, 0.00, 0.00, 0.00; 1.00)]	0.00

Table 6 Comparison analysis of the obtained results

Applications	Comparative methods	The optimal ranking order of the alternatives
The selection problem of landfill sites	The interval type-2 fuzzy linear assignment method [13]	$z_1 \succ z_2 \succ z_4 \succ z_3$
	Proposed likelihood-based assignment method	$z_1 \succ z_2 \succ z_4 \succ z_3$
The medical decision-making problem of treatment options	The extended QUALIFLEX method [20]	$z_2 \succ z_3 \succ z_1$
	The intuitionistic fuzzy QUALIFLEX method [25]	$z_2 \succ z_3 \succ z_1$
	The extended ELECTRE method [20]	$z_2 \succ z_1$, $z_3 \succ z_1$
	Proposed likelihood-based assignment method	$z_2 \succ z_3 \succ z_1$

model with the results yielded by the extended QUALIFLEX method [20], the intuitionistic fuzzy QUALIFLEX method [25], and the extended ELECTRE method [20].

In the medical decision-making problem presented by Chen et al. [20], the attending physician assessed the patient's medical history and her current physical conditions and provided the following treatment options: steroid therapy (z_1), plasmapheresis (z_2), and albumin immune therapy (z_3). Nine criteria were used to evaluate the alternatives, including survival rate (c_1), severity of the side effects (c_2), severity of the complications (c_3), probability of a cure (c_4), discomfort index of the treatment (c_5), cost (c_6), number of days of hospitalization (c_7), probability of a recurrence (c_8), and self-care capacity (c_9). According to the IT2 TrFN data in Chen et al. [20], we applied the proposed method to analyze the same medical decision-making problem and constructed the following linear assignment model:

$$\begin{aligned}
&\max 0.62 \cdot P_{11} + 0.08 \cdot P_{12} + 1.76 \cdot P_{13} + 2.27 \cdot P_{21} \\
&+ 2.03 \cdot P_{22} + 0.06 \cdot P_{23} \\
&+ 1.97 \cdot P_{31} + 1.93 \cdot P_{32} + 0.24 \cdot P_{33} \\
&\text{s.t. } P_{11} + P_{12} + P_{13} = 1,\ P_{21} + P_{22} + P_{23} = 1, \\
&P_{31} + P_{32} + P_{33} = 1, \\
&P_{11} + P_{21} + P_{31} = 1,\ P_{12} + P_{22} + P_{32} = 1, \\
&P_{13} + P_{23} + P_{33} = 1, \\
&P_{ik} = 0 \text{ or } 1 \text{ for all } i \text{ and } k.
\end{aligned} \tag{42}$$

We solved the model in (42) and obtained $P^*_{13} = P^*_{21} = P^*_{32} = 1$ and the remaining $P^*_{ik} = 0$ in the optimal permutation matrix P^*. Therefore, the optimal ranking order of the three treatment options is $z_2 \succ z_3 \succ z_1$, which is the same as those obtained using the extended QUALIFLEX method [20] and the intuitionistic fuzzy QUALIFLEX method [25]. Additionally, the extended ELECTRE method renders the outranking relationships $z_2 \succ z_1$ and

$z_3 \succ z_1$ [20], which are also in agreement with the result obtained via our proposed method.

A summary of the results obtained by applying the proposed method and the comparative methods to the selection problem of landfill sites and the medical decision-making problem of treatment options is presented in Table 6.

Compared with the interval type-2 fuzzy linear assignment method [13], the computations associated with our proposed method are simple and effective. More specifically, we have no need to normalize IT2 TrFN evaluative ratings and importance weights when using the proposed likelihood-based assignment method. However, Chen's [13] method must use a scale normalization method to establish the normalized decision matrix and the normalized importance weights of criteria. Consider the application of the landfill site selection problem. Although the procedure of the proposed method and the developed linear assignment model in (41) are easy to implement, the illustrative application still reveals a valid and reasonable result using the proposed method.

Conversely, when employing Chen's [13] interval type-2 fuzzy linear assignment method to solve MCDA problems, it can be observed that no attention has been paid to incorporating the IT2 TrFN evaluative ratings into the weighted rank frequency matrix. More specifically, the weighted rank frequency matrix presented by Chen [13] only contains the information of the rank frequency and the criterion importance. In contrast, our proposed method considers all of the useful information contained in the evaluative ratings and importance weights. To make the information content complementary, we determine the (adjusted) rank contribution matrix by aggregating the products of multiplying the IT2 TrFN importance weights by the optimal degrees of membership. This approach can not only consider the criterion importance but also fully utilize information contained in the evaluative ratings and fuzzy preference relations between IT2 TrFNs.

Consider the second application of the medical decision-making problem. Note that the priority orders of z_2 and z_3 cannot be differentiated via the extended ELECTRE results even though z_1 is dominated by z_2 and z_3. In contrast, the proposed likelihood-based assignment method can differentiate the priority orders of z_2 and z_3, rendering the outranking relationship of $z_2 \succ z_3$. Therefore, the obtained result yielded by the proposed method provides more important information than that of Chen et al. [20] extended ELECTRE method. Compared with the extended QUALIFLEX method, the intuitionistic fuzzy QUALIFLEX method, and the extended ELECTRE method, the computations associated with the proposed method are simple and effective, and the linear assignment model in (42) is easy to solve for the optimal solution.

As a whole, the comparative results indicate that the solution result using the proposed method is valid and credible even though the computation procedure of the proposed method is much simpler than that of the comparative methods (i.e., the interval type-2 fuzzy linear assignment method, the extended QUALIFLEX method, the intuitionistic fuzzy QUALIFLEX method, and the extended ELECTRE method). The proposed likelihood-based assignment method fully utilizes the extended concept of likelihoods regarding fuzzy preference relations between IT2 TrFNs. This method does not require a complicated computation procedure and produces actionable and valid results that aid the MCDA process.

5.3 Additional comparative discussions with scoring methods

This subsection compares the results of the current likelihood-based assignment method with those of the widely used scoring model. The scoring model selects the alternative that has the highest score (or the maximum utility/value). Over the past few decades, the most traditionally representative method of the scoring model is the SAW method. Thus, this paper conducted a comparative analysis with the well-known SAW method to further examine the distinct advantage of the proposed method.

In general, some tied alternatives possibly exist in the solution results when employing most of the existing decision-making methods to solve an MCDA problem. For example, an alternative ties with another alternative because of the same overall evaluation values if the SAW method is used or because of the same closeness coefficients if the TOPSIS method is used. In particular, the result of more than one best alternative is a sticky problem for the decision-maker because he/she must make a hard decision among the tied best alternatives. Consider a simple MCDA problem with two similar alternatives and three benefit criteria in the context of IT2 TrFNs. Assume that the three criteria are of equal importance. Because equal importance is assigned to the three criteria, let the IT2 TrFN importance weights $W_1 = W_2 = W_3 = [(1,1,1,1;1), (1,1,1,1;1)]$ for convenience. Suppose that the IT2 TrFN evaluative ratings are given as follows: $A_{11} = [(0.78, 0.82, 0.89, 0.91; 0.80), (0.74, 0.78, 0.92, 0.97; 1.00)]$, $A_{21} = [(0.78, 0.82, 0.89, 0.91; 0.80), (0.71, 0.78, 0.92, 0.97; 1.00)]$, $A_{12} = A_{22} = [(0.95, 0.99, 0.99, 0.99; 0.80), (0.93, 0.98, 1.00, 1.00; 1.00)]$, and $A_{13} = A_{23} = [(0.09, 0.12, 0.16, 0.18; 0.80), (0.04, 0.10, 0.18, 0.23; 1.00)]$. Note that the IT2 TrFN evaluative ratings of the alternatives z_1 and z_2 are the same except for a^{+}_{111} (=0.74) in A_{11} and a^{+}_{121} (=0.71) in A_{21}. It directly follows that the fuzzy preference relation

$A_{11} > A_{21}$ holds. Accordingly, $z_1 \succ z_2$ is the expected solution result.

Employing the SAW method, we calculated the weighted average value EV_i $(=[EV_i^-, EV_i^+])$ of each $z_i \in Z$ by multiplying A_{ij} by W_j and then summing these products over all of the criteria in the following manner:

$$EV_i = \left[(ev_{1i}^-, ev_{2i}^-, ev_{3i}^-, ev_{4i}^-; h_{EV_i}^-), (ev_{1i}^+, ev_{2i}^+, ev_{3i}^+, ev_{4i}^+; h_{EV_i}^+)\right]$$
$$= \left[\left(\sum_{j=1}^n a_{1ij}^- \cdot w_{1j}^-, \sum_{j=1}^n a_{2ij}^- \cdot w_{2j}^-, \sum_{j=1}^n a_{3ij}^- \cdot w_{3j}^-, \sum_{j=1}^n a_{4ij}^- \cdot w_{4j}^-; \min\left\{\min_j h_{A_{ij}}^-, \min_j h_{W_j}^-\right\}\right),\right.$$
$$\left.\left(\sum_{j=1}^n a_{1ij}^+ \cdot w_{1j}^+, \sum_{j=1}^n a_{2ij}^+ \cdot w_{2j}^+, \sum_{j=1}^n a_{3ij}^+ \cdot w_{3j}^+, \sum_{j=1}^n a_{4ij}^+ \cdot w_{4j}^+; \min\left\{\min_j h_{A_{ij}}^+, \min_j h_{W_j}^+\right\}\right)\right]. \tag{43}$$

The computation results in the example were as follows: $EV_1 =$ [(1.82, 1.93, 2.04, 2.08; 0.80), (1.71, 1.86, 2.10, 2.20; 1.00)] and $EV_2 =$ [(1.82, 1.93, 2.04, 2.08; 0.80), (1.68, 1.86, 2.10, 2.20; 1.00)].

Next, we employed the signed distance-based method to acquire the overall evaluation value of each alternative as follows:

$$d(EV_i) = \frac{1}{8}\left[ev_{1i}^- + ev_{2i}^- + ev_{3i}^- + ev_{4i}^- + 4 \cdot ev_{1i}^+ + 2 \cdot ev_{2i}^+ \right.$$
$$\left. + 2 \cdot ev_{3i}^+ + 4 \cdot ev_{4i}^+ + 3\left(ev_{2i}^+ + ev_{3i}^+ - ev_{1i}^+ - ev_{4i}^+\right)\frac{h_{EV_i}^-}{h_{EV_i}^+}\right]. \tag{44}$$

In the example, the overall evaluation values were derived as follows: $d(EV_1) = 3.94$ and $d(EV_2) = 3.94$. The result of $z_1 \sim z_2$ was acquired via the SAW method.

Instead, we applied the proposed method to the same decision-making problem. The following linear assignment model was constructed:

$$\max 2.02 \cdot P_{11} + 1.00 \cdot P_{12} + 1.00 \cdot P_{21} + 1.98 \cdot P_{22}$$
$$\text{s.t. } P_{11} + P_{12} = 1,\ P_{21} + P_{22} = 1,\ P_{11} + P_{21} = 1,$$
$$P_{12} + P_{22} = 1,$$
$$P_{ik} = 0 \text{ or } 1 \text{ for all } i \text{ and } k. \tag{45}$$

We solved the above model and acquired $P_{11}^* = P_{22}^* = 1$ and $P_{12}^* = P_{21}^* = 0$ in the optimal permutation matrix P^*. Thus, the ranking order of the alternatives is $z_1 \succ z_2$. Therefore, the priority orders of z_1 and z_2 can be clearly differentiated via the proposed method.

As revealed in the respective solution results using SAW and the proposed method, the likelihood-based assignment method can indeed differentiate the priority orders of similar alternatives and provide a definite complete ranking of all alternatives for decision support. This paper establishes the likelihood-based assignment model to determine the priority order of various alternatives. In general, the tied situation among alternatives hardly occurs by virtue of using the permutation matrix in the current method. Specifically, the proposed likelihood-based assignment method is expressed using a pure integer linear programming model. The solution result of the optimal permutation matrix can clearly differentiate the priority orders of all alternatives and render the outranking relationship of alternatives that have similar overall evaluation values (or closeness coefficients if the TOPSIS method is used). Therefore, the proposed method is appropriate for addressing the MCDA problem if the decision-maker cannot accept the final result of tied alternatives or would like to obtain a crisp complete ranking order of various alternatives. In short, differentiation of the priority orders among similar alternatives is a distinct advantage of the proposed method.

Combining the application results regarding the selection problem of landfill sites and the medical decision-making problem, the usefulness of the proposed likelihood-based assignment method for practical applications has been demonstrated. In brief, the proposed method has various significant advantages over the existing relevant methods, including simple and effective computations, full utilization of the information contained in the evaluative ratings and importance weights, a pure integer linear programming model that is easy to solve, and differentiation of the priority orders among all alternatives.

6 Conclusions

In this paper, we have proposed a new approach for evaluating and selecting the alternatives for solving MCDA problems in the context of IT2 TrFNs. This approach has been developed based on the linear assignment method with some significant modifications. First, we have established an effective ranking procedure using the concept of likelihoods to compare IT2 TrFN evaluative ratings. This paper has employed novel concepts of lower and upper likelihoods proposed by Chen [19] and Wang et al. [62] to determine the likelihood of a fuzzy preference relation in the context of IT2 TrFNs. Based on the obtained likelihoods, we have employed a ranking procedure using the optimal membership degree determination method to identify criterion-wise preference rankings of the decision alternatives. Consider the situation that no tied alternatives are found when comparing the optimal degrees of membership with respect to each criterion. We have established the concepts of a rank frequency matrix and a rank contribution matrix to combine the relative performances of the alternatives in terms of each criterion. Alternately, concerning the fact that some tied alternatives exist when comparing the optimal degrees of membership in regard to at least one criterion, we have proposed the concepts of an adjusted rank frequency matrix and an adjusted rank contribution matrix. To obtain an aggregate ranking of the alternatives, this paper has separately employed the rank contribution matrix and the adjusted rank contribution matrix to construct two likelihood-based assignment models using a signed distances approach. This aggregate ranking is an overall ranking that is in the closest agreement with the criterion-wise preferences of the alternatives. Furthermore, this paper has provided an algorithmic procedure consisting of three phases, including problem formulation, likelihood-based comparisons and ranking, and linear assignment modeling, to implement the proposed likelihood-based assignment method for handling MCDA problems in the context of IT2 TrFNs. The comparative results with other relevant methods also indicate that the effectiveness and the applicability of the proposed likelihood-based assignment method were validated within the decision environment of IT2 TrFNs.

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