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ORIGINAL ARTICLE

Multi-criteria group decision-making methods based on new intuitionistic fuzzy Einstein hybrid weighted aggregation operators

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Abstract Intuitionistic fuzzy sets (IFSs) are a very efficient tool to depict uncertain or fuzzy information. In the course of decision making with IFSs, intuitionistic fuzzy aggregation operators play a very important role which has received more and more attention in recent years. This paper proposes a family of intuitionistic fuzzy Einstein hybrid weighted operators, including the intuitionistic fuzzy Einstein hybrid weighted averaging operator, the intuitionistic fuzzy Einstein hybrid weighted geometric operator, the quasi-intuitionistic fuzzy Einstein hybrid weighted averaging operator, and the quasi-intuitionistic fuzzy Einstein hybrid weighted geometric operator. All these newly developed operators not only can weight both the intuitionistic fuzzy arguments and their ordered positions simultaneously but also have some desirable properties, such as idempotency, boundedness, and monotonicity. Based on these proposed operators, two algorithms are given to solve multi-criteria single-person decision making and multi-criteria group decision making with intuitionistic fuzzy information, respectively. Two numerical examples are provided to illustrate the practicality and validity of the proposed methods and aggregation operators.

Keywords Multi-criteria single-person decision making · Multi-criteria group decision making · Intuitionistic fuzzy set · Intuitionistic fuzzy Einstein hybrid weighted aggregation operator · Quasi-intuitionistic fuzzy Einstein hybrid weighted aggregation operator

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1 Introduction

Intuitionistic fuzzy set (IFS), introduced by Atanassov [1], is the generalization of Zadeh's fuzzy set [38]. IFS is characterized by a membership function and a non-membership function and thus can depict the fuzzy character of data more detailedly and comprehensively than Zadeh's fuzzy set which is only characterized by a membership function. The core of an IFS is intuitionistic fuzzy number (IFN) [25, 30], which is composed of the membership degree and non-membership degree. Intuitionistic fuzzy numbers (IFNs) are a very useful tool to express a decision-maker preference information under uncertain or vague environments. Until now, different kinds of intuitionistic fuzzy aggregation operators, which are suitable for different situations, have been given to aggregate IFNs. With the help of the algebraic operational laws on IFNs, Xu [25] developed some basic arithmetic aggregation operators, such as the intuitionistic fuzzy weighted averaging (IFWA) operator, the intuitionistic fuzzy ordered weighted averaging (IFOWA) operator, and the intuitionistic fuzzy hybrid averaging (IFHA) operator, for aggregating IFNs. Xu and Yager [30] developed some basic geometric aggregation operators, such as the intuitionistic fuzzy weighted geometric (IFWG) operator, the intuitionistic fuzzy ordered weighted geometric (IFOWG) operator, and the intuitionistic fuzzy hybrid geometric (IFHG) operator, and applied them to multiple attribute decision making (MADM) based on IFNs. These basic aggregation operators proposed in [25, 30] have been further generalized by using generalized means [6] and order inducing variables [34]. Zhao et al. [42] extended the IFWA, IFOWA, and IFHA operators and proposed a family of generalized aggregation operators, such as the generalized IFWA (GIFWA) operator, the generalized IFOWA (GIFOWA) operator, and the generalized IFHA (GIFHA) operator. Wei [20] proposed some induced geometric aggregation operators with intuitionistic fuzzy information and applied them to group decision making. Xia and Xu [21] proposed a series of intuitionistic fuzzy point operations, and then they developed various generalized intuitionistic fuzzy point aggregation operators. These basic operators proposed in [25, 30] had also been further generalized by combining the knowledge of dynamic programming, Choquet integral [3], and Dempster–Shafer theory of evidence [4]. Xu and Yager [31] defined dynamic IFWA operator and developed a procedure to solve the dynamic intuitionistic fuzzy MADM problems. Tan and Chen [15] and Xu [26] used the Choquet integral to propose some intuitionistic fuzzy aggregation operators. Xu and Xia [29] applied Choquet integral and Dempster–Shafer theory of evidence to aggregate intuitionistic fuzzy information and developed the induced generalized aggregation operators under intuitionistic fuzzy environments. The intuitionistic fuzzy Bonferroni means were proposed by Xu and Yager [32], based on which the generalized intuitionistic fuzzy Bonferroni means and the geometric Bonferroni means were established by Xia et al. [23, 24]. Yu and Xu [37] established a collection of prioritized intuitionistic fuzzy aggregation operators. Xu [27] proposed a class of intuitionistic fuzzy power aggregation operators. Zhang [39] developed a family of generalized intuitionistic fuzzy power geometric operators and applied them to multiple attribute group decision making (MAGDM) with intuitionistic fuzzy information. Recently, Xia et al. [22] developed some new aggregation operators for IFNs based on Archimedean t-conorm and t-norm. Xu and Cai [28] have provided a survey of the aggregation techniques of intuitionistic fuzzy information and their applications in various fields. Yu [36] developed some confidence intuitionistic fuzzy weighted aggregation operators, such as the confidence intuitionistic fuzzy weighted averaging (CIFWA) operator and the confidence intuitionistic fuzzy weighted geometric (CIFWG) operator. Yu [35] proposed the intuitionistic fuzzy geometric Heronian mean (IFGHM) operator and the intuitionistic fuzzy geometric weighed Heronian mean (IFGWHM) operator. Qin and Liu [14] developed the intuitionistic fuzzy Maclaurin symmetric mean (IFMSM) and the weighted intuitionistic fuzzy Maclaurin symmetric mean (WIFMSM). Liao and Xu [12] proposed a family of intuitionistic fuzzy hybrid weighted aggregation operators, such as the intuitionistic fuzzy hybrid weighted averaging operator, the intuitionistic fuzzy hybrid weighted geometric operator, the generalized intuitionistic fuzzy hybrid weighted averaging operator, and the generalized intuitionistic fuzzy hybrid weighted geometric operator. Zhao et al. [40] developed some heavy aggregation operators for aggregating intuitionistic fuzzy information and then applied them to develop some models for decision-making problems. Zhao et al. [43] developed some intuitionistic fuzzy density-based aggregation operators and investigated their applications to group decision making with intuitionistic preference relations.

It is noted that the above aggregation operators are developed based on the basic algebraic product and algebraic sum of IFSs, which are not the unique operations to model the intersection and union of IFSs. Recently, Deschrijver and Kerre [5] have constructed a generalized union and a generalized intersection of IFSs from a general t-norm and t-conorm. It is well known that the product and Einstein t-norms are two prototypical examples of the class of strict Archimedean t-norms [11]. Thus, for an intersection of IFSs, a good alternative to the algebraic product is the Einstein product. Equivalently, for an union of IFSs, a good alternative to the algebraic sum is the Einstein sum. Recently, Wang and Liu [16] developed the intuitionistic fuzzy Einstein weighted averaging (IFEWA) operator and the intuitionistic fuzzy Einstein ordered weighted averaging (IFEOWA) operator. Yu [36] proposed the confidence intuitionistic fuzzy Einstein weighted averaging (CIFEWA) operator and the confidence intuitionistic fuzzy Einstein weighted geometric (CIFEWG) operator. Wang and Liu [17] further developed the intuitionistic fuzzy Einstein weighted geometric (IFEWG) operator and the intuitionistic fuzzy Einstein ordered weighted geometric (IFEOWG) operator. Wang and Liu [16, 17] have also proven that the IFEWA, IFEWG, IFEOWA, and IFEOWG operators have the following properties: idempotency, boundedness, and monotonicity. It is noted that the IFEWA and IFEWG operators can be used to weight the intuitionistic fuzzy arguments, but ignore the importance degrees of the ordered positions of the arguments, whereas the IFOWA and IFOWG operators only weight the ordered position of each given argument, but ignore the importance degrees of the given arguments. To solve this drawback, Zhao and Wei [41] proposed the intuitionistic fuzzy Einstein hybrid averaging (IFEHA) operator and the intuitionistic fuzzy Einstein hybrid geometric (IFEHG) operator to aggregate intuitionistic fuzzy arguments, which weight both the given arguments and their ordered positions simultaneously. However, these two operators have a flaw that they do not satisfy some basic properties such as idempotency and boundedness, which are desirable for aggregating a finite collection of IFNs. To circumvent this issue, motivated by the hybrid weighted arithmetical averaging (HWAA) operator proposed by Lin and Jiang [13], we in this paper aim at developing some new intuitionistic fuzzy Einstein hybrid weighted aggregation operators, which not only weight the given arguments and their ordered positions simultaneously but also maintain those basic properties. The proposed intuitionistic fuzzy

Einstein hybrid weighted aggregation operators are generalizations of the HWAA operator within the context of IFSs. In addition, inspired by the quasi-arithmetical average [7, 9], we extend our proposed operator to more general forms and develop the quasi-intuitionistic fuzzy Einstein hybrid weighted averaging (QIFEHWA) operator and the quasi-intuitionistic fuzzy Einstein hybrid weighted geometric (QIFEHWG) operator. Moreover, we also give some procedures based on our proposed operators for multi-criteria single-person decision making and multi-criteria group decision making under intuitionistic fuzzy environments.

This paper is organized as follows. Section 2 gives some fundamental knowledge of IFS and the intuitionistic fuzzy Einstein aggregation operators. Section 3 develops the intuitionistic fuzzy Einstein hybrid weighted averaging (IFEHWA) operator and the intuitionistic fuzzy Einstein hybrid weighted geometric (IFEHWG) operator. Some desired properties of these two operators are also investigated in this section. Section 4 extends the IFEHWA and IFEHWG operators to the quasi-IFEHWA and the quasi-IFEHWG operators, respectively. In Sect. 5, we apply our proposed operators to develop two methods for multi-criteria single-person decision making and multi-criteria group decision making under intuitionistic fuzzy environments. Meanwhile, two practical examples are given to illustrate the validity and applicability of the proposed methods. The paper is concluded in Sect. 6.

2 Preliminaries

2.1 Intuitionistic fuzzy sets and intuitionistic fuzzy numbers

Definition 2.1 [1]. Let X be a fixed set, an intuitionistic fuzzy set (IFS) on X is defined as:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\} \tag{1}$$

which assigns to each element $x \in X$ a membership information $\mu_A(x)$ and a non-membership information $\nu_A(x)$, with the conditions that

$$0 \le \mu_A(x), \nu_A(x) \le 1, \quad \mu_A(x) + \nu_A(x) \le 1, \quad \forall x \in X. \tag{2}$$

Furthermore, $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ $(\forall x \in X)$ is called a hesitancy degree or intuitionistic index of x in A.

In the special case $\pi_A(x) = 0$, $\forall x \in X$, i.e., $\mu_A(x) + \nu_A(x) = 1$, $\forall x \in X$, the IFS A reduces to a fuzzy set [38].

Xu and Yager [30] called each pair $(\mu_A(x), \nu_A(x))$ an intuitionistic fuzzy number (IFN) and, for convenience, denoted an IFN by $\alpha = (\mu_\alpha, \nu_\alpha)$, where

$$0 \le \mu_\alpha, \nu_\alpha \le 1, \quad \mu_\alpha + \nu_\alpha \le 1. \tag{3}$$

For convenience, let M be the set of all the intuitionistic fuzzy numbers (IFNs).

For an IFN $\alpha = (\mu_\alpha, \nu_\alpha)$, Chen and Tan [2] introduced the score function $s(\alpha)$ to get the score of α. Later, Hong and Choi [10] defined the accuracy function $h(\alpha)$ to evaluate the accuracy degree of α:

$$s(\alpha) = \mu_\alpha - \nu_\alpha \tag{4}$$

$$h(\alpha) = \mu_\alpha + \nu_\alpha \tag{5}$$

Based on the score function and the accuracy function, Xu and Yager [30] gave an order relation between any two IFNs:

Definition 2.2 [30]. Let $\alpha_i = \left(\mu_{\alpha_i}, \nu_{\alpha_i}\right)$ $(i = 1, 2)$ be any two IFNs, $s(\alpha_i)$ and $h(\alpha_i)$ $(i = 1, 2)$ be the scores and accuracy degrees of α_i $(i = 1, 2)$, respectively, and then

1. If $s(\alpha_1) > s(\alpha_2)$, then $\alpha_1 > \alpha_2$;
2. If $s(\alpha_1) = s(\alpha_2)$, then

   If $h(\alpha_1) > h(\alpha_2)$, then $\alpha_1 > \alpha_2$;
   If $h(\alpha_1) = h(\alpha_2)$, then $\alpha_1 = \alpha_2$.

2.2 Einstein t-conorm and t-norm

The set theoretical operators have had an important role since in the beginning of fuzzy set (FS) theory. Starting from Zadeh's operators min and max, many other operators were introduced in the fuzzy set literature [38]. All types of the particular operators were included in the general concepts of the t-norms and t-conorms [8, 11], which satisfy the requirements of the conjunction and disjunction operators, respectively. The t-norms T and t-conorms S are the most general families of binary functions that map the unit square into the unit interval, i.e., $T : [0, 1]^2 \rightarrow [0, 1]$ and $S : [0, 1]^2 \rightarrow [0, 1]$, and they are related by the De Morgan duality, i.e., the t-conorm s can be defined as $S(a, b) = 1 - T(1 - a, 1 - b)$, $\forall a, b \in [0, 1]$.

Based on a t-norm and t-conorm, Deschrijver and Kerre [5] proposed a generalized intersection and a generalized union of intuitionistic fuzzy sets (IFSs).

Definition 2.3 [5]. Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle | x \in X\}$ be any two IFSs, and then the generalized intersection and generalized union between A and B are proposed as follows:

$$A \cap_{T,S} B = \{\langle x, T(\mu_A(x), \mu_B(x)), S(\nu_A(x), \nu_B(x))\rangle | x \in X\} \tag{6}$$

$$A \cup_{T,S} B = \{\langle x, S(\mu_A(x), \mu_B(x)), T(v_A(x), v_B(x))\rangle | x \in X\} \quad (7)$$

where any pair of dual t-norm T and t-conorm S can be used.

Definition 2.4 Let $\alpha_i = (\mu_{\alpha_i}, v_{\alpha_i})$ $(i = 1, 2)$ be any two IFNs, and then the generalized intersection and generalized union between α_1 and α_2 are defined as follows:

$$\alpha_1 \otimes_{T,S} \alpha_2 = \left(T(\mu_{\alpha_1}, \mu_{\alpha_2}), S(v_{\alpha_1}, v_{\alpha_2})\right) \quad (8)$$

$$\alpha_1 \oplus_{T,S} \alpha_2 = \left(S(\mu_{\alpha_1}, \mu_{\alpha_2}), T(v_{\alpha_1}, v_{\alpha_2})\right) \quad (9)$$

where any pair of dual t-norm T and t-conorm S can be used.

Various t-norms and t-conorms families can be used to perform the corresponding intersections and unions of IFNs. As examples of t-norms and t-conorms, Einstein product T_E and Einstein sum S_E are defined as follows [11]:

$$T_E(a, b) = \frac{a \cdot b}{1 + (1 - a) \cdot (1 - b)}, \quad S_E(a, b) = \frac{a + b}{1 + a \cdot b}, \quad \forall a, b \in [0, 1] \quad (10)$$

2.3 Einstein operations of intuitionistic fuzzy numbers

Motivated by Eq. (10), the Einstein product $\alpha_1 \otimes_E \alpha_2$ and Einstein sum $\alpha_1 \oplus_E \alpha_2$ on two IFNs $\alpha_1 = (\mu_{\alpha_1}, v_{\alpha_1})$ and $\alpha_2 = (\mu_{\alpha_2}, v_{\alpha_2})$ are defined as follows [16, 17]:

$$\alpha_1 \otimes_E \alpha_2 = \left(\frac{\mu_{\alpha_1}\mu_{\alpha_2}}{1 + (1 - \mu_{\alpha_1})(1 - \mu_{\alpha_2})}, \frac{v_{\alpha_1} + v_{\alpha_2}}{1 + v_{\alpha_1}v_{\alpha_2}}\right) \quad (11)$$

$$\alpha_1 \oplus_E \alpha_2 = \left(\frac{\mu_{\alpha_1} + \mu_{\alpha_2}}{1 + \mu_{\alpha_1}\mu_{\alpha_2}}, \frac{v_{\alpha_1}v_{\alpha_2}}{1 + (1 - v_{\alpha_1})(1 - v_{\alpha_2})}\right) \quad (12)$$

$$\lambda\alpha = \left(\frac{(1 + \mu_\alpha)^\lambda - (1 - \mu_\alpha)^\lambda}{(1 + \mu_\alpha)^\lambda + (1 - \mu_\alpha)^\lambda}, \frac{2v_\alpha^\lambda}{(2 - v_\alpha)^\lambda + v_\alpha^\lambda}\right), \quad \lambda > 0 \quad (13)$$

$$\alpha^\lambda = \left(\frac{2\mu_\alpha^\lambda}{(2 - \mu_\alpha)^\lambda + \mu_\alpha^\lambda}, \frac{(1 + v_\alpha)^\lambda - (1 - v_\alpha)^\lambda}{(1 + v_\alpha)^\lambda + (1 - v_\alpha)^\lambda}\right), \quad \lambda > 0 \quad (14)$$

Based on the above Einstein operations of IFNs, a series of intuitionistic fuzzy Einstein aggregation operators were developed to aggregate intuitionistic fuzzy information:

Definition 2.5 [16, 17]. Let $\alpha_i = (\mu_{\alpha_i}, v_{\alpha_i})$ $(i = 1, 2, \ldots, n)$ be a collection of IFNs, and let $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of α_i $(i = 1, 2, \ldots, n)$ with $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$, and then

1. An intuitionistic fuzzy Einstein weighted averaging (IFEWA) operator is a mapping $\text{IFEWA} : M^n \to M$, such that

$$\text{IFEWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \mathop{\oplus}_{i=1}^{n} (w_i\alpha_i) = \left(\frac{\prod_{i=1}^n (1 + \mu_{\alpha_i})^{w_i} - \prod_{i=1}^n (1 - \mu_{\alpha_i})^{w_i}}{\prod_{i=1}^n (1 + \mu_{\alpha_i})^{w_i} + \prod_{i=1}^n (1 - \mu_{\alpha_i})^{w_i}}, \frac{2\prod_{i=1}^n v_{\alpha_i}^{w_i}}{\prod_{i=1}^n (2 - v_{\alpha_i})^{w_i} + \prod_{i=1}^n v_{\alpha_i}^{w_i}}\right) \quad (15)$$

2. An intuitionistic fuzzy Einstein weighted geometric (IFEWG) operator is a mapping $\text{IFEWG} : M^n \to M$, where

$$\text{IFEWG}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \mathop{\otimes}_{i=1}^{n} (\alpha_i^{w_i}) = \left(\frac{2\prod_{i=1}^n \mu_{\alpha_i}^{w_i}}{\prod_{i=1}^n (2 - \mu_{\alpha_i})^{w_i} + \prod_{i=1}^n \mu_{\alpha_i}^{w_i}}, \frac{\prod_{i=1}^n (1 + v_{\alpha_i})^{w_i} - \prod_{i=1}^n (1 - v_{\alpha_i})^{w_i}}{\prod_{i=1}^n (1 + v_{\alpha_i})^{w_i} + \prod_{i=1}^n (1 - v_{\alpha_i})^{w_i}}\right) \quad (16)$$

Based on the idea of the ordered weighted averaging (OWA) operator [33], the following operators can be defined:

Definition 2.6 [16, 17]. Let $\alpha_i = (\mu_{\alpha_i}, v_{\alpha_i})$ $(i = 1, 2, \ldots, n)$ be a collection of IFNs, $\alpha_{\sigma(i)}$ be the ith largest of them, and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the aggregation-associated vector such that $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$, and then

1. An intuitionistic fuzzy Einstein ordered weighted averaging (IFEOWA) operator $\text{IFEOWA} : M^n \to M$, where

$$\text{IFEOWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \mathop{\oplus}_{i=1}^{n} (\omega_i\alpha_{\sigma(i)}) = \left(\frac{\prod_{i=1}^n \left(1 + \mu_{\alpha_{\sigma(i)}}\right)^{\omega_i} - \prod_{i=1}^n \left(1 - \mu_{\alpha_{\sigma(i)}}\right)^{\omega_i}}{\prod_{i=1}^n \left(1 + \mu_{\alpha_{\sigma(i)}}\right)^{\omega_i} + \prod_{i=1}^n \left(1 - \mu_{\alpha_{\sigma(i)}}\right)^{\omega_i}}, \frac{2\prod_{i=1}^n v_{\alpha_{\sigma(i)}}^{\omega_i}}{\prod_{i=1}^n \left(2 - v_{\alpha_{\sigma(i)}}\right)^{\omega_i} + \prod_{i=1}^n v_{\alpha_{\sigma(i)}}^{\omega_i}}\right) \quad (17)$$

2. An intuitionistic fuzzy Einstein ordered weighted geometric (IFEOWG) operator $\text{IFEOWG}: M^n \to M$, where

$$\text{IFEOWG}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \underset{i=1}{\overset{n}{\otimes}} \left(\alpha_{\sigma(i)}^{\omega_i}\right) = \left(\frac{2\prod_{i=1}^n \mu_{\alpha_{\sigma(i)}}^{\omega_i}}{\prod_{i=1}^n \left(2-\mu_{\alpha_{\sigma(i)}}\right)^{\omega_i} + \prod_{i=1}^n \mu_{\alpha_{\sigma(i)}}^{\omega_i}}, \frac{\prod_{i=1}^n \left(1+v_{\alpha_{\sigma(i)}}\right)^{\omega_i} - \prod_{i=1}^n \left(1-v_{\alpha_{\sigma(i)}}\right)^{\omega_i}}{\prod_{i=1}^n \left(1+v_{\alpha_{\sigma(i)}}\right)^{\omega_i} + \prod_{i=1}^n \left(1-v_{\alpha_{\sigma(i)}}\right)^{\omega_i}} \right) \tag{18}$$

It is noted that the IFEWA and IFEWG operators only weight the intuitionistic fuzzy arguments themselves, but ignore the importance of the ordered position of the arguments, while the IFEOWA and IFEOWG operators only weight the ordered position of each given arguments, but ignore the importance of the arguments. To solve this drawback, Zhao and Wei [41] introduced some hybrid aggregation operators for intuitionistic fuzzy arguments, which weight all the given arguments and their ordered positions.

Definition 2.7 [41]. For a collection of IFNs $\alpha_i = \left(\mu_{\alpha_i}, v_{\alpha_i}\right)$ $(i = 1, 2, \ldots, n)$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ is the weight vector of them with $\lambda_i \in [0, 1]$ and $\sum_{i=1}^n \lambda_i = 1$, where n is the balancing coefficient which plays a role of balance, and then we define the following aggregation operators, which are all based on the mapping $M^n \to M$ with an aggregation-associated vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$:

1. The intuitionistic fuzzy Einstein hybrid averaging (IFEHA) operator:

$$\text{IFEHA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \underset{i=1}{\overset{n}{\oplus}} \left(\omega_i \dot{\alpha}_{\sigma(i)}\right) = \left(\frac{\prod_{i=1}^n \left(1+\dot{\mu}_{\alpha_{\sigma(i)}}\right)^{\omega_i} - \prod_{i=1}^n \left(1-\dot{\mu}_{\alpha_{\sigma(i)}}\right)^{\omega_i}}{\prod_{i=1}^n \left(1+\dot{\mu}_{\alpha_{\sigma(i)}}\right)^{\omega_i} + \prod_{i=1}^n \left(1-\dot{\mu}_{\alpha_{\sigma(i)}}\right)^{\omega_i}}, \frac{2\prod_{i=1}^n \dot{v}_{\alpha_{\sigma(i)}}^{\omega_i}}{\prod_{i=1}^n \left(2-\dot{v}_{\alpha_{\sigma(i)}}\right)^{\omega_i} + \prod_{i=1}^n \dot{v}_{\alpha_{\sigma(i)}}^{\omega_i}} \right) \tag{19}$$

where $\dot{\alpha}_{\sigma(i)} = \left(\dot{\mu}_{\alpha_{\sigma(i)}}, \dot{v}_{\alpha_{\sigma(i)}}\right)$ is the ith largest of $\dot{\alpha}_k = \left(\dot{\mu}_{\alpha_k}, \dot{v}_{\alpha_k}\right) = n\lambda_k\alpha_k$ $(k = 1, 2, \ldots, n)$.

2. The intuitionistic fuzzy Einstein hybrid geometric (IFEHG) operator:

$$\text{IFEHG}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \underset{i=1}{\overset{n}{\otimes}} \left(\ddot{\alpha}_{\sigma(i)}^{\omega_i}\right) = \left(\frac{2\prod_{i=1}^n \ddot{\mu}_{\alpha_{\sigma(i)}}^{\omega_i}}{\prod_{i=1}^n \left(2-\ddot{\mu}_{\alpha_{\sigma(i)}}\right)^{\omega_i} + \prod_{i=1}^n \ddot{\mu}_{\alpha_{\sigma(i)}}^{\omega_i}}, \frac{\prod_{i=1}^n \left(1+\ddot{v}_{\alpha_{\sigma(i)}}\right)^{\omega_i} - \prod_{i=1}^n \left(1-\ddot{v}_{\alpha_{\sigma(i)}}\right)^{\omega_i}}{\prod_{i=1}^n \left(1+\ddot{v}_{\alpha_{\sigma(i)}}\right)^{\omega_i} + \prod_{i=1}^n \left(1-\ddot{v}_{\alpha_{\sigma(i)}}\right)^{\omega_i}} \right) \tag{20}$$

where $\ddot{\alpha}_{\sigma(i)} = \left(\ddot{\mu}_{\alpha_{\sigma(i)}}, \ddot{v}_{\alpha_{\sigma(i)}}\right)$ is the ith largest of $\ddot{\alpha}_k = \left(\ddot{\mu}_{\alpha_k}, \ddot{v}_{\alpha_k}\right) = \alpha_k^{n\lambda_k}$, $(k = 1, 2, \ldots, n)$.

Particularly, if $\omega = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, then the IFEHA and IFEHG operators reduce to the IFEWA and IFEWG operators, respectively; if $\lambda = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, then the IFEHA and IFEHG operators reduce to the IFEOWA and IFEOWG operators, respectively.

3 Some new intuitionistic fuzzy Einstein hybrid weighted aggregation operators

3.1 Intuitionistic fuzzy Einstein hybrid weighted averaging operators

Although the IFEHA (IFEHG) operator generalizes both the IFEWA (IFEWG) and IFEOWA (IFEOWG) operators and reflects both the given importance and the ordered position of the arguments, there is a flaw that the IFEHA (IFEHG) operator does not satisfy some desirable properties, such as boundedness and idempotency. An example can be used to illustrate this drawback.

Example 3.1 Assume $\alpha_1 = (0.7, 0.3)$, $\alpha_2 = (0.7, 0.3)$, and $\alpha_3 = (0.7, 0.3)$ are three IFNs, whose weight vector is $\lambda = (1, 0, 0)^T$ and the aggregation-associated vector is also $\omega = (1, 0, 0)^T$. Then,

$$\dot{\alpha}_1 = 3 \times 1 \otimes \alpha_1 = 3\alpha_1 = \left(\frac{(1+0.7)^3 - (1-0.7)^3}{(1+0.7)^3 + (1-0.7)^3}, \frac{2 \times 0.3^3}{(2-0.3)^3 + 0.3^3} \right) = (0.9891, 0.0109)$$

$$\dot{\alpha}_2 = 3 \times 0 \otimes \alpha_2 = 0\alpha_2 = \left(\frac{(1+0.7)^0 - (1-0.7)^0}{(1+0.7)^0 + (1-0.7)^0}, \frac{2 \times 0.3^0}{(2-0.3)^0 + 0.3^0} \right) = (0, 1)$$

$$\dot{\alpha}_3 = 3\times 0 \otimes \alpha_3 = 0\alpha_3 = \left(\frac{(1+0.7)^0-(1-0.7)^0}{(1+0.7)^0+(1-0.7)^0}, \frac{2\times 0.3^0}{(2-0.3)^0+0.3^0}\right) = (0,1)$$

Since $s(\dot{\alpha}_1) > s(\dot{\alpha}_2) = s(\dot{\alpha}_3)$, thus $\dot{\alpha}_{\sigma(1)} = \dot{\alpha}_1$, $\dot{\alpha}_{\sigma(2)} = \dot{\alpha}_2$, and $\dot{\alpha}_{\sigma(3)} = \dot{\alpha}_3$. By using Eq. (19), we have

$$\mathrm{IFEHA}(\alpha_1,\alpha_2,\alpha_3) = \mathop{\oplus}_{i=1}^{3}\left(\omega_i\dot{\alpha}_{\sigma(i)}\right) = \left(\frac{\prod_{i=1}^3\left(1+\dot{\mu}_{\alpha_{\sigma(i)}}\right)^{\omega_i}-\prod_{i=1}^3\left(1-\dot{\mu}_{\alpha_{\sigma(i)}}\right)^{\omega_i}}{\prod_{i=1}^3\left(1+\dot{\mu}_{\alpha_{\sigma(i)}}\right)^{\omega_i}+\prod_{i=1}^3\left(1-\dot{\mu}_{\alpha_{\sigma(i)}}\right)^{\omega_i}}, \frac{2\prod_{i=1}^3\dot{v}_{\alpha_{\sigma(i)}}^{\omega_i}}{\prod_{i=1}^3\left(2-\dot{v}_{\alpha_{\sigma(i)}}\right)^{\omega_i}+\prod_{i=1}^3\dot{v}_{\alpha_{\sigma(i)}}^{\omega_i}}\right) = (0.9891, 0.0109)$$

Obviously, $\mathrm{IFEHA}(\alpha_1,\alpha_2,\alpha_3) \neq (0.7,0.3)$ and $\mathrm{IFEHA}(\alpha_1,\alpha_2,\alpha_3) > (0.7,0.3) = \max_{1\le i\le 3}\{\alpha_i\}$.

Analogously,

$$\ddot{\alpha}_1 = \alpha_1^{3\times 1} = \alpha_1^3 = \left(\frac{2\times 0.7^3}{(2-0.7)^3+0.7^3}, \frac{(1+0.3)^3-(1-0.3)^3}{(1+0.3)^3+(1-0.3)^3}\right) = (0.2701, 0.7299)$$

$$\ddot{\alpha}_2 = \alpha_2^{3\times 0} = \alpha_2^0 = \left(\frac{2\times 0.7^0}{(2-0.7)^0+0.7^0}, \frac{(1+0.3)^0-(1-0.3)^0}{(1+0.3)^0+(1-0.3)^0}\right) = (1,0)$$

$$\ddot{\alpha}_3 = \alpha_3^{3\times 0} = \alpha_3^0 = \left(\frac{2\times 0.7^0}{(2-0.7)^0+0.7^0}, \frac{(1+0.3)^0-(1-0.3)^0}{(1+0.3)^0+(1-0.3)^0}\right) = (1,0)$$

Since $s(\ddot{\alpha}_2) = s(\ddot{\alpha}_3) > s(\ddot{\alpha}_1)$, thus $\ddot{\alpha}_{\sigma(1)} = \alpha_2$, $\ddot{\alpha}_{\sigma(2)} = \alpha_3$, and $\ddot{\alpha}_{\sigma(3)} = \alpha_1$. By using Eq. (20), we have

$$\mathrm{IFEHG}(\alpha_1,\alpha_2,\alpha_3) = \mathop{\otimes}_{i=1}^{3}\left(\ddot{\alpha}_{\sigma(i)}^{\omega_i}\right) = \left(\frac{2\prod_{i=1}^3\ddot{\mu}_{\alpha_{\sigma(i)}}^{\omega_i}}{\prod_{i=1}^3\left(2-\ddot{\mu}_{\alpha_{\sigma(i)}}\right)^{\omega_i}+\prod_{i=1}^3\ddot{\mu}_{\alpha_{\sigma(i)}}^{\omega_i}}, \frac{\prod_{i=1}^3\left(1+\ddot{v}_{\alpha_{\sigma(i)}}\right)^{\omega_i}-\prod_{i=1}^3\left(1-\ddot{v}_{\alpha_{\sigma(i)}}\right)^{\omega_i}}{\prod_{i=1}^3\left(1+\ddot{v}_{\alpha_{\sigma(i)}}\right)^{\omega_i}+\prod_{i=1}^3\left(1-\ddot{v}_{\alpha_{\sigma(i)}}\right)^{\omega_i}}\right) = (1,0)$$

Obviously, $\mathrm{IFEHG}(\alpha_1,\alpha_2,\alpha_3) \neq (0.7,0.3)$ and $\mathrm{IFEHG}(\alpha_1,\alpha_2,\alpha_3) > (0.7,0.3) = \max_{1\le i\le 3}\{\alpha_i\}$.

Since boundedness and idempotency are the most important properties for every aggregation operators [13], but the IFEHA and IFEHG operators do not meet these basic properties, we need to develop some new hybrid aggregation operators which also weight the importance of each argument and its ordered position simultaneously. In this section below, we focus on solving this problem and try to develop some new hybrid operators for IFNs.

Consider the IFEOWA operator given as Eq. (17), it can be equivalently written as:

$$\mathrm{IFEOWA}(\alpha_1,\alpha_2,\ldots,\alpha_n) = \mathop{\oplus}_{i=1}^{n}\left(\omega_{\sigma^{-1}(i)}\alpha_i\right) \tag{21}$$

where $\sigma^{-1}:\{1,2,\ldots,n\}\to\{1,2,\ldots,n\}$ is the inverse permutation of σ. α_i is the $\sigma^{-1}(i)$th largest element of the collection of IFNs $\alpha_i = \left(\mu_{\alpha_i}, v_{\alpha_i}\right)$ $(i=1,2,\ldots,n)$. Let $\varepsilon = \sigma^{-1}$, and then Eq. (21) can also be written as

$$\mathrm{IFEOWA}(\alpha_1,\alpha_2,\ldots,\alpha_n) = \mathop{\oplus}_{i=1}^{n}\left(\omega_{\varepsilon(i)}\alpha_i\right) \tag{22}$$

It is clear that α_i is the $\varepsilon(i)$th largest element of the collection of IFNs $\alpha_i = \left(\mu_{\alpha_i}, v_{\alpha_i}\right)$ $(i=1,2,\ldots,n)$. Motivated by this, supposing the weighting vector of the elements is $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_n)^T$, in order to weight the position and the element simultaneously, we can use such a form as $\oplus_{i=1}^n\left(\lambda_i\omega_{\varepsilon(i)}\alpha_i\right)$, which weights both the position and the element. After normalization, a new intuitionistic fuzzy Einstein hybrid weighted averaging operator can be generated.

Definition 3.1 For a collection of IFNs $\alpha_i = \left(\mu_{\alpha_i}, v_{\alpha_i}\right)$ $(i=1,2,\ldots,n)$, an intuitionistic fuzzy Einstein hybrid weighted averaging (IFEHWA) operator is a mapping $\mathrm{IFEHWA}: M^n \to M$, defined by an associated weighting vector $\omega = (\omega_1,\omega_2,\ldots,\omega_n)^T$ with $\omega_i \in [0,1]$ and $\sum_{i=1}^n\omega_i = 1$, such that

$$\mathrm{IFEHWA}(\alpha_1,\alpha_2,\ldots,\alpha_n) = \mathop{\oplus}_{i=1}^{n}\left(\frac{\lambda_i\omega_{\varepsilon(i)}}{\sum_{i=1}^n\lambda_i\omega_{\varepsilon(i)}}\alpha_i\right) \tag{23}$$

where $\varepsilon:\{1,2,\ldots,n\}\to\{1,2,\ldots,n\}$ is the permutation such that α_i is the $\varepsilon(i)$th largest element of the collection of IFNs α_i $(i=1,2,\ldots,n)$, and $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_n)^T$ is the weighting vector of the IFNs α_i $(i=1,2,\ldots,n)$, with $\lambda_i \in [0,1]$ and $\sum_{i=1}^n\lambda_i = 1$.

By using the different manifestation of weighting vector, the IFEHWA operator can be reduced into some special cases. For example, if the associated weighting vector $\omega = \left(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right)^T$, then the IFEHWA operator reduces to the IFEWA operator (Eq. 15); if $\lambda = \left(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right)^T$, then the IFEHWA operator reduces to the IFEOWA operator (Eq. 17). It must be pointed out that the weighting operation of the ordered position can be synchronized with the weighting operation of the given importance by the

IFEHWA operator. This characteristic is different from the IFEHA operator.

Based on Eq. (15), we can easily obtain the following result.

Theorem 3.1 *For a collection of IFNs* $\alpha_i = \left(\mu_{\alpha_i}, v_{\alpha_i}\right)$ $(i = 1, 2, \ldots, n)$, *the aggregated value by using the IFEHWA operator is also an IFN, and*

$$\mathrm{IFEHWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(\frac{\prod_{i=1}^n (1+\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}} - \prod_{i=1}^n (1-\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}}{\prod_{i=1}^n (1+\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}} + \prod_{i=1}^n (1-\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}}, \frac{2\prod_{i=1}^n v_{\alpha_i}^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}}{\prod_{i=1}^n (2-v_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}} + \prod_{i=1}^n v_{\alpha_i}^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}} \right) \tag{24}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is an associated weighting vector with $\omega_i \in [0,1]$ and $\sum_{i=1}^n \omega_i = 1$, $\varepsilon : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is the permutation such that α_i is the $\varepsilon(i)$th largest element of the collection of IFNs α_i $(i = 1, 2, \ldots, n)$, and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ is the weighting vector of the IFNs α_i $(i = 1, 2, \ldots, n)$, with $\lambda_i \in [0,1]$ and $\sum_{i=1}^n \lambda_i = 1$.

Example 3.2 Let $\alpha_1 = (0.4, 0.5)$, $\alpha_2 = (0.7, 0.1)$, and $\alpha_3 = (0.6, 0.3)$ be three IFNs, whose weight vector is $\lambda = (0.2, 0.5, 0.3)^T$ and the aggregation-associated vector is $\omega = (0.1, 0.7, 0.2)^T$.

At first, comparing α_1, α_2, and α_3 by using the score function given as Eq. (4), we have $s(\alpha_1) = -0.1$, $s(\alpha_2) = 0.6$, and $s(\alpha_3) = 0.3$. Since $s(\alpha_2) > s(\alpha_3) > s(\alpha_1)$, we obtain $\alpha_2 > \alpha_3 > \alpha_1$ and hence $\varepsilon(1) = 3$, $\varepsilon(2) = 1$, and $\varepsilon(3) = 2$. Then,

$$\frac{\lambda_1\omega_{\varepsilon(1)}}{\sum_{i=1}^3 \lambda_i\omega_{\varepsilon(i)}} = \frac{0.2 \times 0.2}{0.2 \times 0.2 + 0.5 \times 0.1 + 0.3 \times 0.7} = 0.1333,$$

$$\frac{\lambda_2\omega_{\varepsilon(2)}}{\sum_{i=1}^3 \lambda_i\omega_{\varepsilon(i)}} = 0.1667, \quad \frac{\lambda_1\omega_{\varepsilon(1)}}{\sum_{i=1}^3 \lambda_i\omega_{\varepsilon(i)}} = 0.7000$$

Then, by using Eq. (24), we can calculate that $\mathrm{IFEHWA}(\alpha_1, \alpha_2, \alpha_3) = (0.5956, 0.2714)$.

Theorem 3.2 (Idempotency). *Let* α_i $(i = 1, 2, \ldots, n)$ *be a collection of IFNs, and if all* α_i $(i = 1, 2, \ldots, n)$ *are equal, i.e.,* $\alpha_i = \alpha = (\mu_\alpha, v_\alpha)$, *for all i, then*

$$\mathrm{IFEHWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \mathrm{IFEHWA}(\alpha, \alpha, \ldots, \alpha) = \alpha \tag{25}$$

Proof According to Definition 3.1 and Theorem 3.1, we have

$$\mathrm{IFEHWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \mathrm{IFEHWA}(\alpha, \alpha, \ldots, \alpha)$$
$$= \left(\frac{\prod_{i=1}^n (1+\mu_\alpha)^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}} - \prod_{i=1}^n (1-\mu_\alpha)^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}}{\prod_{i=1}^n (1+\mu_\alpha)^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}} + \prod_{i=1}^n (1-\mu_\alpha)^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}}, \frac{2\prod_{i=1}^n v_\alpha^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}}{\prod_{i=1}^n (2-v_\alpha)^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}} + \prod_{i=1}^n v_\alpha^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}} \right)$$
$$= (\mu_\alpha, v_\alpha) = \alpha$$

This completes the proof. □

Theorem 3.3 (Boundedness). *Let* $\alpha_i = \left(\mu_{\alpha_i}, v_{\alpha_i}\right)$ $(i = 1, 2, \ldots, n)$ *be a collection of IFNs, and*

$$\alpha^- = \left(\min_i\{\mu_{\alpha_i}\}, \max_i\{v_{\alpha_i}\}\right), \quad \alpha^+ = \left(\max_i\{\mu_{\alpha_i}\}, \min_i\{v_{\alpha_i}\}\right)$$

and then

$$\alpha^- \leq \mathrm{IFEHWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \alpha^+ \tag{26}$$

Proof Let $\mathrm{IFEHWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha = (\mu_\alpha, v_\alpha)$. According to Theorem 3.1, we have

$$\mu_\alpha = \frac{\prod_{i=1}^n (1+\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}} - \prod_{i=1}^n (1-\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}}{\prod_{i=1}^n (1+\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}} + \prod_{i=1}^n (1-\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}}$$

$$v_\alpha = \frac{2\prod_{i=1}^n v_{\alpha_i}^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}}{\prod_{i=1}^n (2-v_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}} + \prod_{i=1}^n v_{\alpha_i}^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_v}}$$

Firstly, μ_α can be equivalently written as:

$$\mu_\alpha = 1 - \frac{2}{1 + \prod_{i=1}^n \left(\frac{1+\mu_{\alpha_i}}{1-\mu_{\alpha_i}}\right)^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}}$$

Let $f(x) = \frac{1+x}{1-x}$, $x \in [0,1]$; then, $f'(x) = \frac{2}{(1-x)^2} > 0$; thus, $f(x)$ is an increasing function. Since $\min_i\{\mu_{\alpha_i}\} \leq \mu_{\alpha_i} \leq \max_i\{\mu_{\alpha_i}\}$, for all i, then $f\left(\min_i\{\mu_{\alpha_i}\}\right) \leq f\left(\mu_{\alpha_i}\right) \leq f\left(\max_i\{\mu_{\alpha_i}\}\right)$, for all i, i.e., $\frac{1+\min_i\{\mu_{\alpha_i}\}}{1-\min_i\{\mu_{\alpha_i}\}} \leq \frac{1+\mu_{\alpha_i}}{1-\mu_{\alpha_i}} \leq \frac{1+\max_i\{\mu_{\alpha_i}\}}{1-\max_i\{\mu_{\alpha_i}\}}$, for all i. Therefore, we have

$$\begin{aligned}
\min_i\{\mu_{\alpha_i}\} &= 1 - \frac{2}{1 + \prod_{i=1}^n \left(\frac{1+\min_i\{\mu_{\alpha_i}\}}{1-\min_i\{\mu_{\alpha_i}\}}\right)^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}} \\
&\leq 1 - \frac{2}{1 + \prod_{i=1}^n \left(\frac{1+\mu_{\alpha_i}}{1-\mu_{\alpha_i}}\right)^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}} = \mu_\alpha \\
&\leq 1 - \frac{2}{1 + \prod_{i=1}^n \left(\frac{1+\max_i\{\mu_{\alpha_i}\}}{1-\max_i\{\mu_{\alpha_i}\}}\right)^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}} = \max_i\{\mu_{\alpha_i}\}
\end{aligned} \tag{27}$$

Secondly, v_α can be equivalently written as:

$$v_\alpha = \frac{2}{1 + \prod_{i=1}^n \left(\frac{2 - v_{\alpha_i}}{v_{\alpha_i}}\right)^{(\lambda_i \omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i \omega_{\varepsilon(i)}}}$$

Let $g(y) = \frac{2-y}{y}$, $y \in [0,1]$; then, $g'(y) = \frac{-2}{y^2} < 0$; thus, $g(y)$ is an decreasing function. Since $\min_i\{v_{\alpha_i}\} \le v_{\alpha_i} \le \max_i\{v_{\alpha_i}\}$, for all i, then $g(\max_i\{v_{\alpha_i}\}) \le g(v_{\alpha_i}) \le g(\min_i\{v_{\alpha_i}\})$, for all i, i.e., $\frac{2-\max_i\{v_{\alpha_i}\}}{\max_i\{v_{\alpha_i}\}} \le \frac{2-v_{\alpha_i}}{v_{\alpha_i}} \le \frac{2-\min_i\{v_{\alpha_i}\}}{\min_i\{v_{\alpha_i}\}}$, for all i. Therefore, we have

$$\begin{aligned}\min_i\{v_{\alpha_i}\} &= \frac{2}{1 + \prod_{i=1}^n \left(\frac{2-\min_i\{v_{\alpha_i}\}}{\min_i\{v_{\alpha_i}\}}\right)^{(\lambda_i \omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i \omega_{\varepsilon(i)}}} \\ &\le \frac{2}{1 + \prod_{i=1}^n \left(\frac{2-v_{\alpha_i}}{v_{\alpha_i}}\right)^{(\lambda_i \omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i \omega_{\varepsilon(i)}}} = v_\alpha \\ &\le 1 - \frac{2}{1 + \prod_{i=1}^n \left(\frac{2-\max_i\{v_{\alpha_i}\}}{\max_i\{v_{\alpha_i}\}}\right)^{(\lambda_i \omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i \omega_{\varepsilon(i)}}} = \max_i\{v_{\alpha_i}\}\end{aligned} \tag{28}$$

Then, according to Eq. (4), we obtain

$$s(\alpha) = \mu_\alpha - v_\alpha \le \max_i\{\mu_{\alpha_i}\} - \min_i\{v_{\alpha_i}\} = s(\alpha^+),$$
$$s(\alpha) = \mu_\alpha - v_\alpha \ge \min_i\{\mu_{\alpha_i}\} - \max_i\{v_{\alpha_i}\} = s(\alpha^-)$$

If $s(\alpha) = s(\alpha^+)$ and $s(\alpha) > s(\alpha^-)$, then by Definition 2.2, $\alpha^- < \text{IFEHWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) < \alpha^+$.

If $s(\alpha) = s(\alpha^+)$, i.e., $\mu_\alpha - v_\alpha = \max_i\{\mu_{\alpha_i}\} - \min_i\{v_{\alpha_i}\}$, then by Eqs. (27) and (28), it follows that $\mu_\alpha = \max_i\{\mu_{\alpha_i}\}$ and $v_\alpha = \min_i\{v_{\alpha_i}\}$; thus, $h(\alpha) = \mu_\alpha + v_\alpha = \max_i\{\mu_{\alpha_i}\} + \min_i\{v_{\alpha_i}\} = h(\alpha^+)$, which implies that IFEHWA $(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha^+$.

If $s(\alpha) = s(\alpha^+)$, i.e., $\mu_\alpha - v_\alpha = \min_i\{\mu_{\alpha_i}\} - \max_i\{v_{\alpha_i}\}$, then by Eqs. (39) and (42), we have $\mu_\alpha = \min_i\{\mu_{\alpha_i}\}$ and $v_\alpha = \max_i\{v_{\alpha_i}\}$; thus, $h(\alpha) = \mu_\alpha + v_\alpha = \min_i\{\mu_{\alpha_i}\} + \max_i\{v_{\alpha_i}\} = h(\alpha^-)$, which implies that $\alpha^- =$ IFEHWA $(\alpha_1, \alpha_2, \ldots, \alpha_n)$.

From the above analysis, we can conclude that Eq. (26) always holds. □

Theorem 3.4 (Monotonicity). *Let $\alpha_i = (\mu_{\alpha_i}, v_{\alpha_i})$ $(i = 1, 2, \ldots, n)$ and $\beta_i = (\mu_{\beta_i}, v_{\beta_i})$ $(i = 1, 2, \ldots, n)$ be two collections of IFNs. Assume that $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is an associated weighting vector with $\omega_i \in [0,1]$ and $\sum_{i=1}^n \omega_i = 1$, $\varepsilon: \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is the permutation such that α_i is the $\varepsilon(i)$th largest element of the collection of IFNs α_i $(i = 1, 2, \ldots, n)$, $\delta: \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is the permutation such that β_i is the $\delta(i)$th largest element of the collection of IFNs β_i $(i = 1, 2, \ldots, n)$,* and *$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ is the weighting vector of the IFNs α_i and β_i $(i = 1, 2, \ldots, n)$, with $\lambda_i \in [0,1]$ and $\sum_{i=1}^n \lambda_i = 1$. If $\mu_{\alpha_i} \le \mu_{\beta_i}$, $v_{\alpha_i} \ge v_{\beta_i}$, and $\varepsilon(i) = \delta(i)$, for all i, then*

$$\text{IFEHWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) \le \text{IFEHWA}(\beta_1, \beta_2, \ldots, \beta_n) \tag{29}$$

Proof Let $\text{IFEHWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha$ and IFEHWA $(\beta_1, \beta_2, \ldots, \beta_n) = \beta$. Let $f(x) = \frac{1+x}{1-x}$, $x \in [0,1]$; then, it is an increasing function. If $\mu_{\alpha_i} \le \mu_{\beta_i}$, for all i, then $f(\mu_{\alpha_i}) \le f(\mu_{\beta_i})$, i.e., $\frac{1+\mu_{\alpha_i}}{1-\mu_{\alpha_i}} \le \frac{1+\mu_{\beta_i}}{1-\mu_{\beta_i}}$, for all i. Therefore, we have

$$\begin{aligned}\mu_\alpha &= \frac{\prod_{i=1}^n (1+\mu_{\alpha_i})^{(\lambda_i \omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i \omega_{\varepsilon(i)}} - \prod_{i=1}^n (1-\mu_{\alpha_i})^{(\lambda_i \omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i \omega_{\varepsilon(i)}}}{\prod_{i=1}^n (1+\mu_{\alpha_i})^{(\lambda_i \omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i \omega_{\varepsilon(i)}} + \prod_{i=1}^n (1-\mu_{\alpha_i})^{(\lambda_i \omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i \omega_{\varepsilon(i)}}} \\ &= 1 - \frac{2}{1 + \prod_{i=1}^n \left(\frac{1+\mu_{\alpha_i}}{1-\mu_{\alpha_i}}\right)^{(\lambda_i \omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i \omega_{\varepsilon(i)}}} \\ &\le 1 - \frac{2}{1 + \prod_{i=1}^n \left(\frac{1+\mu_{\beta_i}}{1-\mu_{\beta_i}}\right)^{(\lambda_i \omega_{\delta(i)})/\sum_{i=1}^n \lambda_i \omega_{\delta(i)}}} \\ &= \frac{\prod_{i=1}^n (1+\mu_{\beta_i})^{(\lambda_i \omega_{\delta(i)})/\sum_{i=1}^n \lambda_i \omega_{\delta(i)}} - \prod_{i=1}^n (1-\mu_{\beta_i})^{(\lambda_i \omega_{\delta(i)})/\sum_{i=1}^n \lambda_i \omega_{\delta(i)}}}{\prod_{i=1}^n (1+\mu_{\beta_i})^{(\lambda_i \omega_{\delta(i)})/\sum_{i=1}^n \lambda_i \omega_{\delta(i)}} + \prod_{i=1}^n (1-\mu_{\beta_i})^{(\lambda_i \omega_{\delta(i)})/\sum_{i=1}^n \lambda_i \omega_{\delta(i)}}} = \mu_\beta\end{aligned} \tag{30}$$

Let $g(y) = \frac{2-y}{y}$, $y \in [0,1]$; then, it is an decreasing function. If $v_{\alpha_i} \ge v_{\beta_i}$, for all i, then $g(v_{\alpha_i}) \le g(v_{\beta_i})$, i.e., $\frac{2-v_{\alpha_i}}{v_{\alpha_i}} \le \frac{2-v_{\beta_i}}{v_{\beta_i}}$, for all i. Therefore, we have

$$\begin{aligned}v_\alpha &= \frac{2\prod_{i=1}^n v_{\alpha_i}^{(\lambda_i \omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i \omega_{\varepsilon(i)}}}{\prod_{i=1}^n (2 - v_{\alpha_i})^{(\lambda_i \omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i \omega_{\varepsilon(i)}} + \prod_{i=1}^n v_{\alpha_i}^{(\lambda_i \omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i \omega_{\varepsilon(i)}}} \\ &= \frac{2}{1 + \prod_{i=1}^n \left(\frac{2-v_{\alpha_i}}{v_{\alpha_i}}\right)^{(\lambda_i \omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i \omega_{\varepsilon(i)}}} \\ &\ge \frac{2}{1 + \prod_{i=1}^n \left(\frac{2-v_{\beta_i}}{v_{\beta_i}}\right)^{(\lambda_i \omega_{\delta(i)})/\sum_{i=1}^n \lambda_i \omega_{\delta(i)}}} \\ &= \frac{2\prod_{i=1}^n v_{\beta_i}^{(\lambda_i \omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i \omega_{\varepsilon(i)}}}{\prod_{i=1}^n (2 - v_{\beta_i})^{(\lambda_i \omega_{\delta(i)})/\sum_{i=1}^n \lambda_i \omega_{\delta(i)}} + \prod_{i=1}^n v_{\beta_i}^{(\lambda_i \omega_{\delta(i)})/\sum_{i=1}^n \lambda_i \omega_{\delta(i)}}} = v_\beta\end{aligned} \tag{31}$$

By Eq. (4), we have $s(\alpha) = \mu_\alpha - v_\alpha \le \mu_\beta - v_\beta = s(\beta)$.

If $s(\alpha) < s(\beta)$, then by Definition 2.2, we have $\alpha < \beta$.

If $s(\alpha) = s(\beta)$, i.e., $s(\alpha) = \mu_\alpha - v_\alpha = \mu_\beta - v_\beta = s(\beta)$, then, by Eqs. (30) and (31), we have $\mu_\alpha = \mu_\beta$ and $v_\alpha = v_\beta$. Thus, $h(\alpha) = \mu_\alpha + v_\alpha = \mu_\beta + v_\beta = h(\beta)$, which implies that $\alpha = \beta$.

Based on the above analysis, we can conclude that Eq. (29) always holds. □

Theorems 3.2, 3.3, and 3.4 reveal that the IFEHWA operator has the idempotency, the boundedness, and the monotonicity, just as the IFEWA and IFEOWA operators have. Meanwhile, it can also weight both the given arguments and their ordered positions simultaneously just as the

IFEHA operator does. From this point of view, the IFEHWA operator is more reasonable and powerful than the IFEWA, IFEOWA, and IFEHA operators.

Example 3.3 Let us use our developed IFEHWA operator to revisit Example 3.1. We have

$$\mathrm{IFEHWA}(\alpha_1,\alpha_2,\alpha_3) = (0.7, 0.3) = \alpha_1 = \alpha_2 = \alpha_3$$

which satisfies the properties of idempotency and boundedness. This is also consistent with our intuition. From this example, we can see that our proposed IFEHWA operator is more reasonable than the IFEHA operator developed by Zhao and Wei [41].

Moreover, we investigate some other desirable properties of the IFEHWA operator.

Theorem 3.5 *Let $\alpha_i = (\mu_{\alpha_i}, v_{\alpha_i})$ $(i = 1, 2, \ldots, n)$ be a collection of IFNs. Suppose that $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is an associated weighting vector with $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$, $\varepsilon : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is the permutation such that α_i is the $\varepsilon(i)$th largest element of the collection of IFNs α_i $(i = 1, 2, \ldots, n)$, and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ is the weighting vector of the IFNs α_i $(i = 1, 2, \ldots, n)$, with $\lambda_i \in [0, 1]$ and $\sum_{i=1}^n \lambda_i = 1$. If $r > 0$ is a real number, then*

$$\mathrm{IFEHWA}(r\alpha_1, r\alpha_2, \ldots, r\alpha_n) = r\mathrm{IFEHWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) \tag{32}$$

Proof

1. Let $\beta_i = r\alpha_i$, and then, by Eq. (13), we have

$$\beta_i = \left(\frac{(1+\mu_{\alpha_i})^r - (1-\mu_{\alpha_i})^r}{(1+\mu_{\alpha_i})^r + (1-\mu_{\alpha_i})^r}, \frac{2v_{\alpha_i}^r}{(2-v_{\alpha_i})^r + v_{\alpha_i}^r}\right)$$

Let $\delta : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ be the permutation such that β_i is the $\delta(i)$th largest element of the collection of IFNs β_i $(i = 1, 2, \ldots, n)$. For any $i, j = 1, 2, \ldots, n$, $i \neq j$, without loss of generality, let $\alpha_i \leq \alpha_j$, i.e., $\mu_{\alpha_i} \leq \mu_{\alpha_j}$ and $v_{\alpha_i} \geq v_{\alpha_j}$, and then,

$$\begin{aligned}\mu_{\beta_i} &= \frac{(1+\mu_{\alpha_i})^r - (1-\mu_{\alpha_i})^r}{(1+\mu_{\alpha_i})^r + (1-\mu_{\alpha_i})^r} \\ &= 1 - \frac{2}{\left(\frac{1+\mu_{\alpha_i}}{1-\mu_{\alpha_i}}\right)^r + 1} \leq 1 - \frac{2}{\left(\frac{1+\mu_{\alpha_j}}{1-\mu_{\alpha_j}}\right)^r + 1} \\ &= \frac{\left(1+\mu_{\alpha_j}\right)^r - \left(1-\mu_{\alpha_j}\right)^r}{\left(1+\mu_{\alpha_j}\right)^r + \left(1-\mu_{\alpha_j}\right)^r} = \mu_{\beta_j}\end{aligned}$$

$$\begin{aligned}v_{\beta_i} &= \frac{2v_{\alpha_i}^r}{(2-v_{\alpha_i})^r + v_{\alpha_i}^r} = \frac{2}{\left(\frac{2-v_{\alpha_i}}{v_{\alpha_i}}\right)^r + 1} \geq \frac{2}{\left(\frac{2-v_{\alpha_j}}{v_{\alpha_j}}\right)^r + 1} \\ &= \frac{2v_{\alpha_j}^r}{\left(2-v_{\alpha_j}\right)^r + v_{\alpha_j}^r} = v_{\beta_j},\end{aligned}$$

and thus, we have $\beta_i \leq \beta_j$, which implies that $\delta(i) = \varepsilon(i)$, for all $i = 1, 2, \ldots, n$.
For the left-hand side of Eq. (32), we have

$$\begin{aligned}&\mathrm{IFEHWA}(r\alpha_1, r\alpha_2, \ldots, r\alpha_n) = \mathrm{IFEHWA}(\beta_1, \beta_2, \ldots, \beta_n) \\ &= \left(\begin{array}{c}\dfrac{\prod_{i=1}^n \left(1 + \frac{(1+\mu_{\alpha_i})^r - (1-\mu_{\alpha_i})^r}{(1+\mu_{\alpha_i})^r + (1-\mu_{\alpha_i})^r}\right)^{(\lambda_i\omega_{\delta(i)})/\sum_{i=1}^n \lambda_i\omega_{\delta(i)}} - \prod_{i=1}^n \left(1 - \frac{(1+\mu_{\alpha_i})^r - (1-\mu_{\alpha_i})^r}{(1+\mu_{\alpha_i})^r + (1-\mu_{\alpha_i})^r}\right)^{(\lambda_i\omega_{\delta(i)})/\sum_{i=1}^n \lambda_i\omega_{\delta(i)}}}{\prod_{i=1}^n \left(1 + \frac{(1+\mu_{\alpha_i})^r - (1-\mu_{\alpha_i})^r}{(1+\mu_{\alpha_i})^r + (1-\mu_{\alpha_i})^r}\right)^{(\lambda_i\omega_{\delta(i)})/\sum_{i=1}^n \lambda_i\omega_{\delta(i)}} + \prod_{i=1}^n \left(1 - \frac{(1+\mu_{\alpha_i})^r - (1-\mu_{\alpha_i})^r}{(1+\mu_{\alpha_i})^r + (1-\mu_{\alpha_i})^r}\right)^{(\lambda_i\omega_{\delta(i)})/\sum_{i=1}^n \lambda_i\omega_{\delta(i)}}}, \\ \dfrac{2\prod_{i=1}^n \left(\frac{2v_{\alpha_i}^r}{(2-v_{\alpha_i})^r + v_{\alpha_i}^r}\right)^{(\lambda_i\omega_{\delta(i)})/\sum_{i=1}^n \lambda_i\omega_{\delta(i)}}}{\prod_{i=1}^n \left(2 - \frac{2v_{\alpha_i}^r}{(2-v_{\alpha_i})^r + v_{\alpha_i}^r}\right)^{(\lambda_i\omega_{\delta(i)})/\sum_{i=1}^n \lambda_i\omega_{\delta(i)}} + \prod_{i=1}^n \left(\frac{2v_{\alpha_i}^r}{(2-v_{\alpha_i})^r + v_{\alpha_i}^r}\right)^{(\lambda_i\omega_{\delta(i)})/\sum_{i=1}^n \lambda_i\omega_{\delta(i)}}}\end{array}\right) \\ &= \left(\begin{array}{c}\dfrac{\prod_{i=1}^n (1+\mu_{\alpha_i})^{(r\lambda_i\omega_{\delta(i)})/\sum_{i=1}^n \lambda_i\omega_{\delta(i)}} - \prod_{i=1}^n (1-\mu_{\alpha_i})^{(r\lambda_i\omega_{\delta(i)})/\sum_{i=1}^n \lambda_i\omega_{\delta(i)}}}{\prod_{i=1}^n (1+\mu_{\alpha_i})^{(r\lambda_i\omega_{\delta(i)})/\sum_{i=1}^n \lambda_i\omega_{\delta(i)}} + \prod_{i=1}^n (1-\mu_{\alpha_i})^{(r\lambda_i\omega_{\delta(i)})/\sum_{i=1}^n \lambda_i\omega_{\delta(i)}}}, \\ \dfrac{2\prod_{i=1}^n v_{\alpha_i}^{(r\lambda_i\omega_{\delta(i)})/\sum_{i=1}^n \lambda_i\omega_{\delta(i)}}}{\prod_{i=1}^n (2-v_{\alpha_i})^{(r\lambda_i\omega_{\delta(i)})/\sum_{i=1}^n \lambda_i\omega_{\delta(i)}} + \prod_{i=1}^n v_{\alpha_i}^{(r\lambda_i\omega_{\delta(i)})/\sum_{i=1}^n \lambda_i\omega_{\delta(i)}}}\end{array}\right)\end{aligned}$$

2. For the right-hand side of Eq. (32), we can obtain

$$
\begin{aligned}
&r\text{IFEHWA}(\alpha_1, \alpha_2, \ldots, \alpha_n)\\
&= \left(\frac{\left(1 + \frac{\prod_{i=1}^n (1+\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}} - \prod_{i=1}^n (1-\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}}{\prod_{i=1}^n (1+\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}} + \prod_{i=1}^n (1-\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}}\right)^r - \left(1 - \frac{\prod_{i=1}^n (1+\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}} - \prod_{i=1}^n (1-\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}}{\prod_{i=1}^n (1+\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}} + \prod_{i=1}^n (1-\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}}\right)^r}{\left(1 + \frac{\prod_{i=1}^n (1+\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}} - \prod_{i=1}^n (1-\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}}{\prod_{i=1}^n (1+\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}} + \prod_{i=1}^n (1-\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}}\right)^r + \left(1 - \frac{\prod_{i=1}^n (1+\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}} - \prod_{i=1}^n (1-\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}}{\prod_{i=1}^n (1+\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}} + \prod_{i=1}^n (1-\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}}\right)^r}, \right.\\
&\qquad \left. \frac{2\left(\frac{2\prod_{i=1}^n v_{\alpha_i}^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}}{\prod_{i=1}^n (2-v_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}} + \prod_{i=1}^n v_{\alpha_i}^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}}\right)^r}{\left(2 - \frac{2\prod_{i=1}^n v_{\alpha_i}^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}}{\prod_{i=1}^n (2-v_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}} + \prod_{i=1}^n v_{\alpha_i}^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}}\right)^r + \left(\frac{2\prod_{i=1}^n v_{\alpha_i}^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}}{\prod_{i=1}^n (2-v_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}} + \prod_{i=1}^n v_{\alpha_i}^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}}\right)^r} \right)\\
&= \left(\frac{\prod_{i=1}^n (1+\mu_{\alpha_i})^{(r\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}} - \prod_{i=1}^n (1-\mu_{\alpha_i})^{(r\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}}{\prod_{i=1}^n (1+\mu_{\alpha_i})^{(r\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}} + \prod_{i=1}^n (1-\mu_{\alpha_i})^{(r\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}}, \frac{2\prod_{i=1}^n v_{\alpha_i}^{(r\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}}{\prod_{i=1}^n (2-v_{\alpha_i})^{(r\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}} + \prod_{i=1}^n v_{\alpha_i}^{(r\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n \lambda_i\omega_{\varepsilon(i)}}} \right)
\end{aligned}
$$

Therefore, we have $\text{IFEHWA}(r\alpha_1, r\alpha_2, \ldots, r\alpha_n) = r\text{IFEHWA}(\alpha_1, \alpha_2, \ldots, \alpha_n)$, which completes the proof. □

Theorem 3.6 *Let* $\alpha_i = (\mu_{\alpha_i}, v_{\alpha_i})$ $(i = 1, 2, \ldots, n)$ *be a collection of IFNs. Suppose that* $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ *is an associated weighting vector with* $\omega_i \in [0, 1]$ *and* $\sum_{i=1}^n \omega_i = 1$, $\varepsilon : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ *is the permutation such that* α_i *is the* $\varepsilon(i)$th *largest element of the collection of IFNs* α_i (α_i), *and* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ *is the weighting vector of the IFNs* α_i (α_i), *with* $\lambda_i \in [0, 1]$ *and* $\sum_{i=1}^n \lambda_i = 1$, *If* $\alpha = (\mu_\alpha, v_\alpha)$ *is an IFN, then*

$$\text{IFEHWA}(\alpha_1 \oplus \alpha, \alpha_2 \oplus \alpha, \ldots, \alpha_n \oplus \alpha) = \text{IFEHWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) \oplus \alpha \tag{33}$$

Proof

1. Let $\beta_i = \alpha_i \oplus \alpha$, and then, by Eq. (12), we have

$$\beta_i = \alpha_i \oplus \alpha = \left(\frac{\mu_{\alpha_i} + \mu_\alpha}{1 + \mu_{\alpha_i}\mu_\alpha}, \frac{v_{\alpha_i} v_\alpha}{1 + (1 - v_{\alpha_i})(1 - v_\alpha)} \right)$$

Let $\delta : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ be the permutation such that β_i is the $\delta(i)$th largest element of the collection of IFNs β_i $(i = 1, 2, \ldots, n)$. For any $i, j = 1, 2, \ldots, n$, $i \neq j$, without loss of generality, let $\alpha_i \leq \alpha_j$, i.e., $\mu_{\alpha_i} \leq \mu_{\alpha_j}$ and $v_{\alpha_i} \geq v_{\alpha_j}$. Let $f(x) = \frac{x+a}{1+ax}$, $x, a \in [0, 1]$; then, $f'(x) = \frac{1-a^2}{(1+ax)^2} > 0$; thus, $f(x)$ is an increasing function. Thus,

$$\mu_{\beta_i} = \frac{\mu_{\alpha_i} + \mu_\alpha}{1 + \mu_{\alpha_i}\mu_\alpha} \leq \frac{\mu_{\alpha_j} + \mu_\alpha}{1 + \mu_{\alpha_j}\mu_\alpha} = \mu_{\beta_j}$$

In addition, let $g(y) = \frac{by}{1+(1-y)(1-b)}$, $y, b \in [0, 1]$; then, $g'(y) = \frac{b(2-b)}{(1+(1-y)(1-b))^2} > 0$; thus, $g(y)$ is an increasing function. Thus,

$$v_{\beta_i} = \frac{v_{\alpha_i} v_\alpha}{1 + (1 - v_{\alpha_i})(1 - v_\alpha)} \geq \frac{v_{\alpha_j} v_\alpha}{1 + (1 - v_{\alpha_j})(1 - v_\alpha)} = v_{\beta_j}$$

and thus, we have $\beta_i \leq \beta_j$, which implies that $\delta(i) = \varepsilon(i)$, for all $i = 1, 2, \ldots, n$.
For the left-hand side of Eq. (33), we have

$$
\begin{aligned}
&\mathrm{IFEHWA}(\alpha_1 \oplus \alpha, \alpha_2 \oplus \alpha, \ldots, \alpha_n \oplus \alpha) = \mathrm{IFEHWA}(\beta_1, \beta_2, \ldots, \beta_n) \\
&= \left(\begin{array}{l}
\dfrac{\prod_{i=1}^{n}\left(1+\frac{\mu_{\alpha_i}+\mu_\alpha}{1+\mu_{\alpha_i}\mu_\alpha}\right)^{\left(\lambda_i\omega_{\delta(i)}\right)/\sum_{i=1}^{n}\lambda_i\omega_{\delta(i)}} - \prod_{i=1}^{n}\left(1-\frac{\mu_{\alpha_i}+\mu_\alpha}{1+\mu_{\alpha_i}\mu_\alpha}\right)^{\left(\lambda_i\omega_{\delta(i)}\right)/\sum_{i=1}^{n}\lambda_i\omega_{\delta(i)}}}{\prod_{i=1}^{n}\left(1+\frac{\mu_{\alpha_i}+\mu_\alpha}{1+\mu_{\alpha_i}\mu_\alpha}\right)^{\left(\lambda_i\omega_{\delta(i)}\right)/\sum_{i=1}^{n}\lambda_i\omega_{\delta(i)}} + \prod_{i=1}^{n}\left(1-\frac{\mu_{\alpha_i}+\mu_\alpha}{1+\mu_{\alpha_i}\mu_\alpha}\right)^{\left(\lambda_i\omega_{\delta(i)}\right)/\sum_{i=1}^{n}\lambda_i\omega_{\delta(i)}}}, \\
\dfrac{2\prod_{i=1}^{n}\left(\frac{v_{\alpha_i}v_\alpha}{1+\left(1-v_{\alpha_i}\right)(1-v_\alpha)}\right)^{\left(\lambda_i\omega_{\delta(i)}\right)/\sum_{i=1}^{n}\lambda_i\omega_{\delta(i)}}}{\prod_{i=1}^{n}\left(2-\frac{v_{\alpha_i}v_\alpha}{1+\left(1-v_{\alpha_i}\right)(1-v_\alpha)}\right)^{\left(\lambda_i\omega_{\delta(i)}\right)/\sum_{i=1}^{n}\lambda_i\omega_{\delta(i)}} + \prod_{i=1}^{n}\left(\frac{v_{\alpha_i}v_\alpha}{1+\left(1-v_{\alpha_i}\right)(1-v_\alpha)}\right)^{\left(\lambda_i\omega_{\delta(i)}\right)/\sum_{i=1}^{n}\lambda_i\omega_{\delta(i)}}}
\end{array} \right) \\
&= \left(\begin{array}{l}
\dfrac{(1+\mu_\alpha)\prod_{i=1}^{n}\left(1+\mu_{\alpha_i}\right)^{\left(\lambda_i\omega_{\delta(i)}\right)/\sum_{i=1}^{n}\lambda_i\omega_{\delta(i)}} - (1-\mu_\alpha)\prod_{i=1}^{n}\left(1-\mu_{\alpha_i}\right)^{\left(\lambda_i\omega_{\delta(i)}\right)/\sum_{i=1}^{n}\lambda_i\omega_{\delta(i)}}}{(1+\mu_\alpha)\prod_{i=1}^{n}\left(1+\mu_{\alpha_i}\right)^{\left(\lambda_i\omega_{\delta(i)}\right)/\sum_{i=1}^{n}\lambda_i\omega_{\delta(i)}} + (1-\mu_\alpha)\prod_{i=1}^{n}\left(1-\mu_{\alpha_i}\right)^{\left(\lambda_i\omega_{\delta(i)}\right)/\sum_{i=1}^{n}\lambda_i\omega_{\delta(i)}}}, \\
\dfrac{2v_\alpha\prod_{i=1}^{n}v_{\alpha_i}^{\left(\lambda_i\omega_{\delta(i)}\right)/\sum_{i=1}^{n}\lambda_i\omega_{\delta(i)}}}{(2-v_\alpha)\prod_{i=1}^{n}\left(2-v_{\alpha_i}\right)^{\left(\lambda_i\omega_{\delta(i)}\right)/\sum_{i=1}^{n}\lambda_i\omega_{\delta(i)}} + v_\alpha\prod_{i=1}^{n}v_{\alpha_i}^{\left(\lambda_i\omega_{\delta(i)}\right)/\sum_{i=1}^{n}\lambda_i\omega_{\delta(i)}}}
\end{array} \right)
\end{aligned}
$$

For the right-hand side of Eq. (33), we can obtain

$$
\begin{aligned}
&\mathrm{IFEHWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) \oplus \alpha \\
&= \left(\dfrac{\prod_{i=1}^{n}\left(1+\mu_{\alpha_i}\right)^{\left(\lambda_i\omega_{\varepsilon(i)}\right)/\sum_{i=1}^{n}\lambda_i\omega_{\varepsilon(i)}} - \prod_{i=1}^{n}\left(1-\mu_{\alpha_i}\right)^{\left(\lambda_i\omega_{\varepsilon(i)}\right)/\sum_{i=1}^{n}\lambda_i\omega_{\varepsilon(i)}}}{\prod_{i=1}^{n}\left(1+\mu_{\alpha_i}\right)^{\left(\lambda_i\omega_{\varepsilon(i)}\right)/\sum_{i=1}^{n}\lambda_i\omega_{\varepsilon(i)}} + \prod_{i=1}^{n}\left(1-\mu_{\alpha_i}\right)^{\left(\lambda_i\omega_{\varepsilon(i)}\right)/\sum_{i=1}^{n}\lambda_i\omega_{\varepsilon(i)}}}, \dfrac{2\prod_{i=1}^{n}v_{\alpha_i}^{\left(\lambda_i\omega_{\varepsilon(i)}\right)/\sum_{i=1}^{n}\lambda_i\omega_{\varepsilon(i)}}}{\prod_{i=1}^{n}\left(2-v_{\alpha_i}\right)^{\left(\lambda_i\omega_{\varepsilon(i)}\right)/\sum_{i=1}^{n}\lambda_i\omega_{\varepsilon(i)}} + \prod_{i=1}^{n}v_{\alpha_i}^{\left(\lambda_i\omega_{\varepsilon(i)}\right)/\sum_{i=1}^{n}\lambda_i\omega_{\varepsilon(i)}}} \right) \oplus \alpha \\
&= \left(\begin{array}{l}
\dfrac{(1+\mu_\alpha)\prod_{i=1}^{n}\left(1+\mu_{\alpha_i}\right)^{\left(\lambda_i\omega_{\varepsilon(i)}\right)/\sum_{i=1}^{n}\lambda_i\omega_{\varepsilon(i)}} - (1-\mu_\alpha)\prod_{i=1}^{n}\left(1-\mu_{\alpha_i}\right)^{\left(\lambda_i\omega_{\varepsilon(i)}\right)/\sum_{i=1}^{n}\lambda_i\omega_{\varepsilon(i)}}}{(1+\mu_\alpha)\prod_{i=1}^{n}\left(1+\mu_{\alpha_i}\right)^{\left(\lambda_i\omega_{\varepsilon(i)}\right)/\sum_{i=1}^{n}\lambda_i\omega_{\varepsilon(i)}} + (1-\mu_\alpha)\prod_{i=1}^{n}\left(1-\mu_{\alpha_i}\right)^{\left(\lambda_i\omega_{\varepsilon(i)}\right)/\sum_{i=1}^{n}\lambda_i\omega_{\varepsilon(i)}}}, \\
\dfrac{2v_\alpha\prod_{i=1}^{n}v_{\alpha_i}^{\left(\lambda_i\omega_{\varepsilon(i)}\right)/\sum_{i=1}^{n}\lambda_i\omega_{\varepsilon(i)}}}{(2-v_\alpha)\prod_{i=1}^{n}\left(2-v_{\alpha_i}\right)^{\left(\lambda_i\omega_{\varepsilon(i)}\right)/\sum_{i=1}^{n}\lambda_i\omega_{\varepsilon(i)}} + v_\alpha\prod_{i=1}^{n}v_{\alpha_i}^{\left(\lambda_i\omega_{\varepsilon(i)}\right)/\sum_{i=1}^{n}\lambda_i\omega_{\varepsilon(i)}}}
\end{array} \right)
\end{aligned}
$$

So, we have $\mathrm{IFEHWA}(\alpha_1 \oplus \alpha, \alpha_2 \oplus \alpha, \ldots, \alpha_n \oplus \alpha) = \mathrm{IFEHWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) \oplus \alpha$, which completes the proof. □

According to Theorems 3.5 and 3.6, we can easily obtain Theorem 3.7:

Theorem 3.7 *Let $\alpha_i = \left(\mu_{\alpha_i}, v_{\alpha_i}\right)$ $(i = 1, 2, \ldots, n)$ be a collection of IFNs. Suppose that $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is an associated weighting vector with $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n} \omega_i = 1$, $\varepsilon : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is the permutation such that α_i is the $\varepsilon(i)$th largest element of the collection of IFNs α_i $(i = 1, 2, \ldots, n)$, and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ is the weighting vector of the IFNs α_i $(i = 1, 2, \ldots, n)$, with $\lambda_i \in [0, 1]$ and $\sum_{i=1}^{n} \lambda_i = 1$. If $r > 0$ is a real number and α is an IFN, then*

$$\mathrm{IFEHWA}(r\alpha_1 \oplus \alpha, r\alpha_2 \oplus \alpha, \ldots, r\alpha_n \oplus \tilde{\alpha}) = r\mathrm{IFEHWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) \oplus \alpha \quad (34)$$

Theorem 3.8 *Let* $\alpha_i = (\mu_{\alpha_i}, v_{\alpha_i})$ *and* $\beta_i = (\mu_{\beta_i}, v_{\beta_i})$ $(i = 1, 2, \ldots, n)$ *be two collections of IFNs. Suppose that* $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ *is an associated weighting vector with* $\omega_i \in [0, 1]$ *and* $\sum_{i=1}^n \omega_i = 1$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ *is the weighting vector of the IFNs* α_i, β_i, *and* $\alpha_i + \beta_i$ $(i = 1, 2, \ldots, n)$, *with* $\lambda_i \in [0, 1]$ *and* $\sum_{i=1}^n \lambda_i = 1$, $\varepsilon : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ *is the permutation such that* α_i *is the* $\varepsilon(i)$th *largest element of the collection of IFNs* α_i $(i = 1, 2, \ldots, n)$, $\delta : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ *is the permutation such that* β_i *is the* $\beta(i)$th *largest element of the collection of IFNs* β_i $(i = 1, 2, \ldots, n)$, *and* $\theta : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ *is the permutation such that* $\alpha_i + \beta_i$ *is the* $\theta(i)$th *largest element of the collection of IFNs* $\alpha_i + \beta_i$ $(i = 1, 2, \ldots, n)$, then

$$\mathrm{IFEHWA}(\alpha_1 \oplus \beta_1, \alpha_2 \oplus \beta_2, \ldots, \alpha_n \oplus \beta_n) = \mathrm{IFEHWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) \oplus \mathrm{IFEHWA}(\beta_1, \beta_2, \ldots, \beta_n) \quad (35)$$

Proof For the left-hand side of Eq. (35), according to Eq. (12), we have

$$\alpha_i + \beta_i = \left(\frac{\mu_{\alpha_i} + \mu_{\beta_i}}{1 + \mu_{\alpha_i}\mu_{\beta_i}}, \frac{v_{\alpha_i} v_{\beta_i}}{1 + (1 - v_{\alpha_i})(1 - v_{\beta_i})} \right)$$

and then, we have

$$\mathrm{IFEHWA}(\alpha_1 \oplus \beta_1, \alpha_2 \oplus \beta_2, \ldots, \alpha_n \oplus \beta_n)$$

$$= \left(\frac{\prod_{i=1}^n \left(1 + \frac{\mu_{\alpha_i}+\mu_{\beta_i}}{1+\mu_{\alpha_i}\mu_{\beta_i}}\right)^{(\lambda_i\omega_{\theta(i)})/\sum_{i=1}^n \lambda_i\omega_{\theta(i)}} - \prod_{i=1}^n \left(1 - \frac{\mu_{\alpha_i}+\mu_{\beta_i}}{1+\mu_{\alpha_i}\mu_{\beta_i}}\right)^{(\lambda_i\omega_{\theta(i)})/\sum_{i=1}^n \lambda_i\omega_{\theta(i)}}}{\prod_{i=1}^n \left(1 + \frac{\mu_{\alpha_i}+\mu_{\beta_i}}{1+\mu_{\alpha_i}\mu_{\beta_i}}\right)^{(\lambda_i\omega_{\theta(i)})/\sum_{i=1}^n \lambda_i\omega_{\theta(i)}} + \prod_{i=1}^n \left(1 - \frac{\mu_{\alpha_i}+\mu_{\beta_i}}{1+\mu_{\alpha_i}\mu_{\beta_i}}\right)^{(\lambda_i\omega_{\theta(i)})/\sum_{i=1}^n \lambda_i\omega_{\theta(i)}}}, \frac{2\prod_{i=1}^n \left(\frac{v_{\alpha_i}v_{\beta_i}}{1+(1-v_{\alpha_i})(1-v_{\beta_i})}\right)^{(\lambda_i\omega_{\theta(i)})/\sum_{i=1}^n \lambda_i\omega_{\theta(i)}}}{\prod_{i=1}^n \left(2 - \frac{v_{\alpha_i}v_{\beta_i}}{1+(1-v_{\alpha_i})(1-v_{\beta_i})}\right)^{(\lambda_i\omega_{\theta(i)})/\sum_{i=1}^n \lambda_i\omega_{\theta(i)}} + \prod_{i=1}^n \left(\frac{v_{\alpha_i}v_{\beta_i}}{1+(1-v_{\alpha_i})(1-v_{\beta_i})}\right)^{(\lambda_i\omega_{\theta(i)})/\sum_{i=1}^n \lambda_i\omega_{\theta(i)}}} \right)$$

$$= \left(\frac{\prod_{i=1}^n \left(1 + \mu_{\alpha_i}\mu_{\beta_i} + \mu_{\alpha_i} + \mu_{\beta_i}\right)^{(\lambda_i\omega_{\theta(i)})/\sum_{i=1}^n \lambda_i\omega_{\theta(i)}} - \prod_{i=1}^n \left(1 + \mu_{\alpha_i}\mu_{\beta_i} - \mu_{\alpha_i} - \mu_{\beta_i}\right)^{(\lambda_i\omega_{\theta(i)})/\sum_{i=1}^n \lambda_i\omega_{\theta(i)}}}{\prod_{i=1}^n \left(1 + \mu_{\alpha_i}\mu_{\beta_i} + \mu_{\alpha_i} + \mu_{\beta_i}\right)^{(\lambda_i\omega_{\theta(i)})/\sum_{i=1}^n \lambda_i\omega_{\theta(i)}} + \prod_{i=1}^n \left(1 + \mu_{\alpha_i}\mu_{\beta_i} - \mu_{\alpha_i} - \mu_{\beta_i}\right)^{(\lambda_i\omega_{\theta(i)})/\sum_{i=1}^n \lambda_i\omega_{\theta(i)}}}, \frac{2\prod_{i=1}^n \left(v_{\alpha_i}v_{\beta_i}\right)^{(\lambda_i\omega_{\theta(i)})/\sum_{i=1}^n \lambda_i\omega_{\theta(i)}}}{\prod_{i=1}^n \left((2 - v_{\alpha_i})(2 - v_{\beta_i})\right)^{(\lambda_i\omega_{\theta(i)})/\sum_{i=1}^n \lambda_i\omega_{\theta(i)}} + \prod_{i=1}^n \left(v_{\alpha_i}v_{\beta_i}\right)^{(\lambda_i\omega_{\theta(i)})/\sum_{i=1}^n \lambda_i\omega_{\theta(i)}}} \right)$$

For the right-hand side of Eq. (35), we have

$$\begin{aligned}
&\mathrm{IFEHWA}(\alpha_1,\alpha_2,\ldots,\alpha_n)\oplus \mathrm{IFEHWA}(\beta_1,\beta_2,\ldots,\beta_n)\\
&=\left(\frac{\prod_{i=1}^n(1+\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n\lambda_i\omega_{\varepsilon(i)}}-\prod_{i=1}^n(1-\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n\lambda_i\omega_{\varepsilon(i)}}}{\prod_{i=1}^n(1+\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n\lambda_i\omega_{\varepsilon(i)}}+\prod_{i=1}^n(1-\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n\lambda_i\omega_{\varepsilon(i)}}},\ \frac{2\prod_{i=1}^n v_{\alpha_i}^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n\lambda_i\omega_{\varepsilon(i)}}}{\prod_{i=1}^n(2-v_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n\lambda_i\omega_{\varepsilon(i)}}+\prod_{i=1}^n v_{\alpha_i}^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n\lambda_i\omega_{\varepsilon(i)}}}\right)\\
&\oplus\left(\frac{\prod_{i=1}^n\left(1+\mu_{\beta_i}\right)^{(\lambda_i\omega_{\delta(i)})/\sum_{i=1}^n\lambda_i\omega_{\delta(i)}}-\prod_{i=1}^n\left(1-\mu_{\beta_i}\right)^{(\lambda_i\omega_{\delta(i)})/\sum_{i=1}^n\lambda_i\omega_{\delta(i)}}}{\prod_{i=1}^n\left(1+\mu_{\beta_i}\right)^{(\lambda_i\omega_{\delta(i)})/\sum_{i=1}^n\lambda_i\omega_{\delta(i)}}+\prod_{i=1}^n\left(1-\mu_{\beta_i}\right)^{(\lambda_i\omega_{\delta(i)})/\sum_{i=1}^n\lambda_i\omega_{\delta(i)}}},\ \frac{2\prod_{i=1}^n v_{\beta_i}^{(\lambda_i\omega_{\delta(i)})/\sum_{i=1}^n\lambda_i\omega_{\delta(i)}}}{\prod_{i=1}^n(2-v_{\beta_i})^{(\lambda_i\omega_{\delta(i)})/\sum_{i=1}^n\lambda_i\omega_{\delta(i)}}+\prod_{i=1}^n v_{\beta_i}^{(\lambda_i\omega_{\delta(i)})/\sum_{i=1}^n\lambda_i\omega_{\delta(i)}}}\right)\\
&=\left(\frac{\prod_{i=1}^n(1+\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n\lambda_i\omega_{\varepsilon(i)}}\prod_{i=1}^n\left(1+\mu_{\beta_i}\right)^{(\lambda_i\omega_{\delta(i)})/\sum_{i=1}^n\lambda_i\omega_{\delta(i)}}-\prod_{i=1}^n(1-\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n\lambda_i\omega_{\varepsilon(i)}}\prod_{i=1}^n\left(1-\mu_{\beta_i}\right)^{(\lambda_i\omega_{\delta(i)})/\sum_{i=1}^n\lambda_i\omega_{\delta(i)}}}{\prod_{i=1}^n(1+\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n\lambda_i\omega_{\varepsilon(i)}}\prod_{i=1}^n\left(1+\mu_{\beta_i}\right)^{(\lambda_i\omega_{\delta(i)})/\sum_{i=1}^n\lambda_i\omega_{\delta(i)}}+\prod_{i=1}^n(1-\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n\lambda_i\omega_{\varepsilon(i)}}\prod_{i=1}^n\left(1-\mu_{\beta_i}\right)^{(\lambda_i\omega_{\delta(i)})/\sum_{i=1}^n\lambda_i\omega_{\delta(i)}}},\right.\\
&\left.\frac{2\prod_{i=1}^n v_{\alpha_i}^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n\lambda_i\omega_{\varepsilon(i)}}\prod_{i=1}^n v_{\beta_i}^{(\lambda_i\omega_{\delta(i)})/\sum_{i=1}^n\lambda_i\omega_{\delta(i)}}}{\prod_{i=1}^n(2-v_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n\lambda_i\omega_{\varepsilon(i)}}\prod_{i=1}^n(2-v_{\beta_i})^{(\lambda_i\omega_{\delta(i)})/\sum_{i=1}^n\lambda_i\omega_{\delta(i)}}+\prod_{i=1}^n v_{\alpha_i}^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n\lambda_i\omega_{\varepsilon(i)}}\prod_{i=1}^n v_{\beta_i}^{(\lambda_i\omega_{\delta(i)})/\sum_{i=1}^n\lambda_i\omega_{\delta(i)}}}\right)
\end{aligned}$$

Therefore, we can obtain

$$\begin{aligned}
&\mathrm{IFEHWA}(\alpha_1\oplus\beta_1,\alpha_2\oplus\beta_2,\ldots,\alpha_n\oplus\beta_n)\\
&=\mathrm{IFEHWA}(\alpha_1,\alpha_2,\ldots,\alpha_n)\oplus\mathrm{IFEHWA}(\beta_1,\beta_2,\ldots,\beta_n)
\end{aligned}$$

which completes the proof. □

3.2 Intuitionistic fuzzy Einstein hybrid weighted geometric operators

Analogously, we also can develop the IFEHWG operator for IFNs:

Definition 3.2 For a collection of IFNs $\alpha_i=\left(\mu_{\alpha_i},v_{\alpha_i}\right)$ $(i=1,2,\ldots,n)$, an intuitionistic fuzzy Einstein hybrid weighted geometric (IFEHWG) operator is a mapping $\mathrm{IFEHWG}: M^n\to M$, defined by an associated weighting vector $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$ with $\omega_i\in[0,1]$ and $\sum_{i=1}^n\omega_i=1$, such that

$$\mathrm{IFEHWG}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\underset{i=1}{\overset{n}{\otimes}}\alpha_i^{\frac{\lambda_i\omega_{\varepsilon(i)}}{\sum_{i=1}^n\lambda_i\omega_{\varepsilon(i)}}}\tag{36}$$

where $\varepsilon:\{1,2,\ldots,n\}\to\{1,2,\ldots,n\}$ is the permutation such that α_i is the $\varepsilon(i)$th largest element of the collection of IFNs α_i $(i=1,2,\ldots,n)$, and $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n)^T$ is the weighting vector of the IFNs α_i $(i=1,2,\ldots,n)$, with $\lambda_i\in[0,1]$ and $\sum_{i=1}^n\lambda_i=1$. Particularly, if the associated weighting vector $\omega=\left(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right)^T$, then the IFEHWG operator reduces to the IFEWG operator; if $\lambda=\left(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right)^T$, then the IFEHWG operator reduces to the IFEOWG operator. With the IFEHWG operator, the weighting operation of the ordered position also can be synchronized with the weighting operation of the given importance, while the IFEHG operator does not have this characteristic.

Based on Eq. (16), we can easily obtain the following result.

Theorem 3.9 *For a collection of IFNs $\alpha_i=\left(\mu_{\alpha_i},v_{\alpha_i}\right)$ $(i=1,2,\ldots,n)$, the aggregated value by using the IFEHWG operator is also an IFN, and*

$$\begin{aligned}
&\mathrm{IFEHWG}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\
&=\left(\frac{2\prod_{i=1}^n\mu_{\alpha_i}^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n\lambda_i\omega_{\varepsilon(i)}}}{\prod_{i=1}^n(2-\mu_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n\lambda_i\omega_{\varepsilon(i)}}+\prod_{i=1}^n\mu_{\alpha_i}^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n\lambda_i\omega_{\varepsilon(i)}}},\right.\\
&\left.\frac{\prod_{i=1}^n(1+v_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n\lambda_i\omega_{\varepsilon(i)}}-\prod_{i=1}^n(1-v_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n\lambda_i\omega_{\varepsilon(i)}}}{\prod_{i=1}^n(1+v_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n\lambda_i\omega_{\varepsilon(i)}}+\prod_{i=1}^n(1-v_{\alpha_i})^{(\lambda_i\omega_{\varepsilon(i)})/\sum_{i=1}^n\lambda_i\omega_{\varepsilon(i)}}}\right)
\end{aligned}\tag{37}$$

where $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$ is an associated weighting vector with $\omega_i\in[0,1]$ and $\sum_{i=1}^n\omega_i=1$, $\varepsilon:\{1,2,\ldots,n\}\to\{1,2,\ldots,n\}$ is the permutation such that α_i is the $\varepsilon(i)$th largest element of the collection of IFNs α_i $(i=1,2,\ldots,n)$, and $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n)^T$ is the weighting vector of the IFNs α_i $(i=1,2,\ldots,n)$, with $\lambda_i\in[0,1]$ and $\sum_{i=1}^n\lambda_i=1$.

Example 3.4 Let us use the IFEHWG operator to fuse the IFNs α_1, α_2, and α_3 in Example 3.2. According to Theorem 3.9, we have $\mathrm{IFEHWG}(\alpha_1,\alpha_2,\alpha_3)=(0.5859, 0.2974)$.

Theorem 3.10 (Idempotency). *Let α_i $(i = 1, 2, ldots, n)$ be a collection of IFNs, and if all α_i $(i = 1, 2, ldots, n)$ are equal, i.e., $\alpha_i = \alpha = (\mu_\alpha, v_\alpha)$, for all i, then*

$$\text{IFEHWG}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \text{IFEHWG}(\alpha, \alpha, \ldots, \alpha) = \alpha \tag{38}$$

Theorem 3.11 (Boundedness). *Let $\alpha_i = (\mu_{\alpha_i}, v_{\alpha_i})$ $(i = 1, 2, \ldots, n)$ be a collection of IFNs, and*

$$\alpha^- = (\min_i\{\mu_{\alpha_i}\}, \max_i\{v_{\alpha_i}\}),$$
$$\alpha^+ = (\max_i\{\mu_{\alpha_i}\}, \min_i\{v_{\alpha_i}\})$$

and then

$$\alpha^- \leq \text{IFEHWG}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \alpha^+ \tag{39}$$

Theorem 3.12 (Monotonicity). *Let $\alpha_i = (\mu_{\alpha_i}, v_{\alpha_i})$ $(i = 1, 2, \ldots, n)$ and $\beta_i = (\mu_{\beta_i}, v_{\beta_i})$ $(i = 1, 2, \ldots, n)$ be two collections of IFNs. Assume that $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is an associated weighting vector with $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$, $\varepsilon : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is the permutation such that α_i* is the *$\varepsilon(i)$*th *largest element of the collection of IFNs α_i $(i = 1, 2, \ldots, n)$, $\delta : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is the permutation such that β_i is the $\delta(i)$*th *largest element of the collection of IFNs β_i $(i = 1, 2, \ldots, n)$, and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ is the weighting vector of the IFNs α_i* and *β_i $(i = 1, 2, \ldots, n)$, with $\lambda_i \in [0, 1]$ and $\sum_{i=1}^n \lambda_i = 1$. If $\mu_{\alpha_i} \leq \mu_{\beta_i}$, $v_{\alpha_i} \geq v_{\beta_i}$, and $\varepsilon(i) = \delta(i)$, for all i, then*

$$\text{IFEHWG}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \text{IFEHWG}(\beta_1, \beta_2, \ldots, \beta_n) \tag{40}$$

Since the IFEHWG operator can not only weight both the given arguments and their ordered positions simultaneously but also maintain those ideal properties, idempotency, boundedness, and monotonicity, just as the IFEWG and IFEOWG operators have, it is more powerful and efficient in fusing intuitionistic fuzzy information. It takes in both the advantages of IFEWG, IFEOWG, and IFEHG operators and, meanwhile, circumvents their disadvantages. Thus, the proposed IFEHWG operator has more wide applications in the practical decision-making process.

Example 3.5 Let us use our proposed IFEHWG operator to calculate Example 3.1. We have

$$\text{IFEHWG}(\alpha_1, \alpha_2, \alpha_3) = (0.7, 0.3) = \alpha_1 = \alpha_2 = \alpha_3$$

which means the IFEHWG operator satisfies idempotency and boundedness, which in other words is more reasonable than Zhao and Wei's IFEHG operator [41].

Moreover, we investigate some other desirable properties of the IFEHWG operator.

Theorem 3.13 *Let $\alpha_i = (\mu_{\alpha_i}, v_{\alpha_i})$ $(i = 1, 2, \ldots, n)$ be a collection of IFNs. Suppose that $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is an associated weighting vector with $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$, $\varepsilon : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is the permutation such that α_i is the $\varepsilon(i)$*th *largest element of the collection of IFNs α_i $(i = 1, 2, \ldots, n)$, and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ is the weighting vector of the IFNs α_i $(i = 1, 2, \ldots, n)$, with $\lambda_i \in [0, 1]$ and $\sum_{i=1}^n \lambda_i = 1$. If $r > 0$ is a real number, then*

$$\text{IFEHWG}(\alpha_1^r, \alpha_2^r, \ldots, \alpha_n^r) = (\text{IFEHWG}(\alpha_1, \alpha_2, \ldots, \alpha_n))^r \tag{41}$$

Theorem 3.14 *Let $\alpha_i = (\mu_{\alpha_i}, v_{\alpha_i})$ $(i = 1, 2, \ldots, n)$ be a collection of IFNs. Suppose that $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is an associated weighting vector with $\omega_i \in [0, 1]$* and *$\sum_{i=1}^n \omega_i = 1$, $\varepsilon : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is the permutation such that α_i is the $\varepsilon(i)$*th *largest element of the collection of IFNs α_i $(i = 1, 2, \ldots, n)$, and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ is the weighting vector of the IFNs α_i $(i = 1, 2, \ldots, n)$, with $\lambda_i \in [0, 1]$ and $\sum_{i=1}^n \lambda_i = 1$, If $\alpha = (\mu_\alpha, v_\alpha)$ is an IFN, then*

$$\text{IFEHWG}(\alpha_1 \otimes \alpha, \alpha_2 \otimes \alpha, \ldots, \alpha_n \otimes \alpha) = \text{IFEHWG}(\alpha_1, \alpha_2, \ldots, \alpha_n) \otimes \alpha \tag{42}$$

Theorem 3.15 *Let $\alpha_i = (\mu_{\alpha_i}, v_{\alpha_i})$ $(i = 1, 2, \ldots, n)$ be a collection of IFNs. Suppose that $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is an associated weighting vector with $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$, $\varepsilon : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is the permutation such that α_i is the $\varepsilon(i)$*th *largest element of the collection of IFNs α_i $(i = 1, 2, \ldots, n)$, and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ is the weighting vector of the IFNs α_i $(i = 1, 2, \ldots, n)$, with $\lambda_i \in [0, 1]$ and $\sum_{i=1}^n \lambda_i = 1$. If $r > 0$ is a real number and α is an IFN, then*

$$\text{IFEHWG}(\alpha_1^r \otimes \alpha, \alpha_2^r \otimes \alpha, \ldots, \alpha_n^r \otimes \alpha) = (\text{IFEHWG}(\alpha_1, \alpha_2, \ldots, \alpha_n))^r \otimes \alpha \tag{43}$$

Theorem 3.16 *Let $\alpha_i = (\mu_{\alpha_i}, v_{\alpha_i})$ and $\beta_i = (\mu_{\beta_i}, v_{\beta_i})$ ($i = 1, 2, \ldots, n$) be two collections of IFNs. Suppose that $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is an associated weighting vector with $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ is the weighting vector of the IFNs α_i, β_i, and $\alpha_i + \beta_i$ ($i = 1, 2, \ldots, n$), with $\lambda_i \in [0, 1]$ and $\sum_{i=1}^n \lambda_i = 1$, $\varepsilon : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is the permutation such that α_i is the $\varepsilon(i)$*th *largest element of the collection of IFNs α_i ($i = 1, 2, \ldots, n$), $\delta : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is the permutation such that β_i is the $\beta(i)$*th *largest element of the collection of IFNs β_i ($i = 1, 2, \ldots, n$), and $\theta : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is the permutation such that $\alpha_i + \beta_i$ is the $\theta(i)$*th *largest element of the collection of IFNs $\alpha_i + \beta_i$ ($i = 1, 2, \ldots, n$), then*

$$\mathrm{IFEHWG}(\alpha_1 \otimes \beta_1, \alpha_2 \otimes \beta_2, \ldots, \alpha_n \otimes \beta_n) = \mathrm{IFEHWG}(\alpha_1, \alpha_2, \ldots, \alpha_n) \otimes \mathrm{IFEHWG}(\beta_1, \beta_2, \ldots, \beta_n) \quad (44)$$

4 Quasi-intuitionistic fuzzy Einstein hybrid weighted aggregation operators

If we replace the arithmetical average and the arithmetical geometric average in Definitions 3.1 and 3.2 with the quasi-arithmetical average [7, 9], respectively, then the QIFEHWA and QIFEHWG operators will be obtained, which are in mathematical forms as below:

Definition 4.1 For a collection of IFNs $\alpha_i = (\mu_{\alpha_i}, v_{\alpha_i})$ $(i = 1, 2, \ldots, n)$, let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ be the weight vector of them with $\lambda_i \in [0, 1]$ and $\sum_{i=1}^{n} \lambda_i = 1$. Then, the following aggregation operators are defined, which are all based on the mapping $M^n \to M$ with an aggregation-associated vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n} \omega_i = 1$, and a continuous strictly monotonic function $g(x)$:

1. The quasi-intuitionistic fuzzy Einstein hybrid weighted averaging (QIFEHWA) operator:

$$\mathrm{QIFEHWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = g^{-1}\left(\oplus_{i=1}^{n}\left(\frac{\lambda_i \omega_{\varepsilon(i)}}{\sum_{i=1}^{n} \lambda_i \omega_{\varepsilon(i)}} g(\alpha_i)\right)\right) \quad (45)$$

2. The quasi-intuitionistic fuzzy Einstein hybrid weighted geometric (QIFEHWG) operator:

$$\mathrm{QIFEHWG}(\alpha_1, \alpha_2, \ldots, \alpha_n) = g^{-1}\left(\otimes_{i=1}^{n} (g(\alpha_i))^{(\lambda_i \omega_{\varepsilon(i)})/\sum_{i=1}^{n} \lambda_i \omega_{\varepsilon(i)}}\right) \quad (46)$$

where $\varepsilon : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is the permutation such that α_i is the $\varepsilon(i)$th largest element of the collection of IFNs α_i $(i = 1, 2, \ldots, n)$.

Note that when assigning different weighting vector of ω or λ or choosing different types of function $g(x)$, the QIFEHWA and QIFEHWG operators will reduce to many special cases, which can be set out as follows:

1. If the associated weighting vector $\omega = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, then the QIFEHWA operator reduces to the quasi-intuitionistic fuzzy Einstein weighted averaging (QIFEWA) operator shown as:

$$\mathrm{QIFEWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = g^{-1}\left(\oplus_{i=1}^{n} \lambda_i g(\alpha_i)\right) \quad (47)$$

while the QIFEHWG operator reduces to the quasi-intuitionistic fuzzy Einstein weighted geometric (QIFEWG) operator shown as:

$$\mathrm{QIFEWG}(\alpha_1, \alpha_2, \ldots, \alpha_n) = g^{-1}\left(\mathop{\otimes}_{i=1}^{n} (g(\alpha_i))^{\lambda_i}\right) \quad (48)$$

2. If the arguments' weight vector $\lambda = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, then the QIFEHWA operator reduces to the quasi-intuitionistic fuzzy Einstein ordered weighted averaging (QIFEOWA) operator shown as:

$$\mathrm{QIFEOWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = g^{-1}\left(\mathop{\oplus}_{i=1}^{n} \omega_i g\left(\alpha_{\sigma(i)}\right)\right) \quad (49)$$

while the QIFEHWG operator reduces to the quasi-intuitionistic fuzzy Einstein ordered weighted geometric (QIFEOWG) operator shown as:

$$\mathrm{QIFEOWG}(\alpha_1, \alpha_2, \ldots, \alpha_n) = g^{-1}\left(\mathop{\otimes}_{i=1}^{n} \left(g\left(\alpha_{\sigma(i)}\right)\right)^{\omega_i}\right) \quad (50)$$

3. If $g(x) = x$, then the QIFEHWA operator reduces to the IFEHWA operator given as Definition 3.1, while the QIFEHWG operator reduces to the IFEHWG operator given as Definition 3.2. It is obvious and herein we do not show some proofs.
4. If $g(x) = \ln(x)$, then the QIFEHWA operator reduces to the IFEHWG operator given as Definition 3.2, while the QIFEHWG operator reduces to the IFEHWA operator given as Definition 3.1. The derivation can be shown as below:

$$\begin{aligned}\mathrm{QIFEHWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) &= g^{-1}\left(\frac{\oplus_{i=1}^{n}\left(\lambda_i \omega_{\varepsilon(i)} g(\alpha_i)\right)}{\sum_{i=1}^{n} \lambda_i \omega_{\varepsilon(i)}}\right) \\ &= e^{\frac{\oplus_{i=1}^{n}\left(\lambda_i \omega_{\varepsilon(i)} \ln(\alpha_i)\right)}{\sum_{i=1}^{n} \lambda_i \omega_{\varepsilon(i)}}} \\ &= \left(e^{\mathop{\oplus}_{i=1}^{n}\left(\lambda_i \omega_{\varepsilon(i)} \ln(\alpha_i)\right)}\right)^{1/\sum_{i=1}^{n} \lambda_i \omega_{\varepsilon(i)}} \\ &= \mathop{\otimes}_{i=1}^{n} \alpha_i^{(\lambda_i \omega_{\varepsilon(i)})/\sum_{i=1}^{n} \lambda_i \omega_{\varepsilon(i)}} \\ &= \mathrm{IFEHWG}(\alpha_1, \alpha_2, \ldots, \alpha_n)\end{aligned}$$

while

$$\begin{aligned}\mathrm{QIFEHWG}(\alpha_1, \alpha_2, \ldots, \alpha_n) &= e^{\mathop{\otimes}_{i=1}^{n} (\ln(\alpha_i))^{(\lambda_i \omega_{\varepsilon(i)})/\sum_{i=1}^{n} \lambda_i \omega_{\varepsilon(i)}}} \\ &= \frac{e^{\mathop{\otimes}_{i=1}^{n} (\ln(\alpha_i))^{\lambda_i \omega_{\varepsilon(i)}}}}{\sum_{i=1}^{n} \lambda_i \omega_{\varepsilon(i)}} \\ &= \frac{\oplus_{i=1}^{n} \lambda_i \omega_{\varepsilon(i)} \alpha_i}{\sum_{i=1}^{n} \lambda_i \omega_{\varepsilon(i)}} \\ &= \mathrm{IFEHWA}(\alpha_1, \alpha_2, \ldots, \alpha_n)\end{aligned}$$

It must be pointed out that the QIFEHWA and QIFEHWG operators are not just these special situations above, and some other special cases can also be constructed choosing different types of the function $g(x)$ for the QIFEHWA and QIFEHWG operators, such as $g(x)=x^t$, $g(x)=1-(1-x)^t$, $g(x)=\sin\left(\frac{\pi x}{2}\right)$, $g(x)=1-\sin\left(\frac{\pi(1-x)}{2}\right)$, $g(x)=\cos\left(\frac{\pi x}{2}\right)$, $g(x)=1-\cos\left(\frac{\pi(1-x)}{2}\right)$, $g(x)=\tan\left(\frac{\pi x}{2}\right)$, $g(x)=1-\tan\left(\frac{\pi(1-x)}{2}\right)$, and $g(x)=t^x$.

The QIFEHWA and QIFEHWG operators have some desirable properties similar to the IFEHWA and IFEHWG operators. In should be noted that the proofs of these properties are also similar to the IFEHWA and IFEHWG operators. Therefore, we will not list out these properties here due to space limitations.

5 Two approaches to multi-criteria single-person decision making and multi-criteria group decision making under intuitionistic fuzzy environments based on the proposed operators

In this section, we will apply the developed operators to multi-criteria single-person decision making and multi-criteria group decision making, respectively.

5.1 Multi-criteria decision making with intuitionistic fuzzy information

When a decision maker intends to evaluate a collection of m alternatives $X=\{x_1,x_2,\ldots,x_m\}$ with respect to the predetermined n criteria $C=\{c_1,c_2,\ldots,c_n\}$, he/she may find it is hard to give a single value or a single interval for the membership degree of an element to a given set but an IFN due to the complexity of the problem and the incomplete information. For example, suppose that the decision maker uses an IFN $\alpha_{ij}=\left(\mu_{\alpha_{ij}},v_{\alpha_{ij}}\right)$ to express his/her preference information about the alternatives x_i under the criterion c_j, where $\mu_{\alpha_{ij}^{(k)}}$ indicates the degree that the alternative x_i satisfies the criterion c_j given by the decision maker and $v_{\alpha_{ij}^{(k)}}$ indicates the degree that the alternative x_i does not satisfy the attribute c_j given by the decision maker, with the conditions: $\mu_{\alpha_{ij}^{(k)}},v_{\alpha_{ij}^{(k)}}\in[0,1]$ and $\mu_{\alpha_{ij}^{(k)}}+v_{\alpha_{ij}^{(k)}}\leq 1$. All the IFNs α_{ij} $(i,j=1,2,\ldots,n)$ construct the intuitionistic fuzzy decision matrix $A=\left(\alpha_{ij}\right)_{n\times n}$. He/she also determines the importance degrees λ_j $(j=1,2,\ldots,n)$ for the relevant criteria according to his/her preferences, where $\lambda_j\in[0,1]$, $j=1,2,\ldots,n$, and $\sum_{j=1}^{n}\lambda_j=1$. Meanwhile, since different alternatives may have different focuses and advantages, to reflect this issue, the decision maker also gives the ordering weights ω_j $(j=1,2,\ldots,n)$ for different criteria, where $\omega_j\in[0,1]$, $j=1,2,\ldots,n$, and $\sum_{j=1}^{n}\omega_j=1$.

Based on the developed aggregation operators, we can propose a procedure for the decision maker to select the best choice with intuitionistic fuzzy information, which involves the following steps:

Algorithm 1

Step 1. Utilize the QIFEHWA operator

$$\begin{aligned}\alpha_i&=\text{QIFEHWA}(\alpha_{i1},\alpha_{i2},\ldots,\alpha_{in})\\&=g^{-1}\left(\underset{j=1}{\overset{n}{\oplus}}\left(\frac{\lambda_j\omega_{\varepsilon(j)}}{\sum_{j=1}^{n}\lambda_j\omega_{\varepsilon(j)}}g\left(\alpha_{ij}\right)\right)\right),\\ i&=1,2,\ldots,m\end{aligned}\tag{51}$$

or the QIFEHWG operator

$$\begin{aligned}\alpha_i&=\text{QIFEHWA}(\alpha_{i1},\alpha_{i2},\ldots,\alpha_{in})\\&=g^{-1}\left(\underset{j=1}{\overset{n}{\oplus}}\left(\frac{\lambda_j\omega_{\varepsilon(j)}}{\sum_{j=1}^{n}\lambda_j\omega_{\varepsilon(j)}}g\left(\alpha_{ij}\right)\right)\right),\\ i&=1,2,\ldots,m\end{aligned}\tag{52}$$

to obtain the overall preference values α_i $(i=1,2,\ldots,m)$ with respect to the alternative x_i,where $\varepsilon:\{1,2,\ldots,n\}\to\{1,2,\ldots,n\}$ is the permutation such that α_{ij} is the $\varepsilon(ij)$th largest element of the collection of IFNs α_{ij} $(j=1,2,\ldots,n)$, and g is a continuous strictly monotonic function.

Step 2. Compute the score functions $s(\alpha_i)$ $(i=1,2,\ldots,m)$ of α_i $(i=1,2,\ldots,m)$ by Eq. (4) and the accuracy degree $h(\alpha_i)$ $(i=1,2,\ldots,m)$ of α_i $(i=1,2,\ldots,m)$ by Eq. (5), respectively, and then rank α_i $(i=1,2,\ldots,m)$ by Definition 2.2.

Step 3. Rank all the alternatives x_i $(i=1,2,\ldots,m)$ and then select the optimal one(s).

Step 4. End.

We next use a numerical example (adapted from Wang et al. [19]) to implement our method:

Example 5.1 Consider a person is interested in investing his money to any one of the four portfolios: bank deposit (BD, x_1), debentures (DB, x_2), government bonds (GB, x_3), and shares (SH, x_4). Out of these portfolios, he has to choose only one based on four criteria: return (c_1), risk (c_2), tax benefits (c_3), and liquidity (c_4). The four possible portfolios x_i $(i=1,2,3,4)$ are to be evaluated using the intuitionistic fuzzy information by the decision maker under the above four attributes, as listed in the intuitionistic fuzzy decision matrix $A=\left(\alpha_{ij}\right)_{4\times 4}$ (see Table 1).

The weight information of these four criteria is also determined by the decision maker as $\lambda=(0.3,0.4,0.2,0.1)^T$. In addition, since different portfolios may focus on different points, the person gives another weight vector $\omega=(0.5,0.2,0.2,0.1)^T$ for each

Table 1 Intuitionistic fuzzy decision matrix A

	c_1	c_2	c_3	c_4
x_1	(0.7, 0.3)	(0.4, 0.5)	(0.5, 0.4)	(0.3, 0.6)
x_2	(0.2, 0.5)	(0.3, 0.5)	(0.8, 0.1)	(0.7, 0.1)
x_3	(0.8, 0.2)	(0.2, 0.3)	(0.6, 0.3)	(0.2, 0.7)
x_4	(0.9, 0.1)	(0.8, 0.1)	(0.2, 0.7)	(0.2, 0.6)

criterion, which denotes that the most prominent feature of the portfolio assigns more weight while the remainders assign less weight. In the following, we use Algorithm 1 to select the most desirable portfolio, which involves the following steps:

Step 1. Utilize the QIFEHWA operator (without the loss of generality, let $g(x) = x$) to obtain the overall IFNs α_i for the portfolios x_1, x_2, x_3, x_4. Take x_1 as an example. Since $s(\alpha_{11}) = 0.4$, $s(\alpha_{12}) = -0.1$, $s(\alpha_{13}) = 0.1$, $s(\alpha_{14}) = -0.3$, then $\alpha_{11} > \alpha_{13} > \alpha_{12} > \alpha_{14}$. Thus, $\varepsilon(11) = 1$, $\varepsilon(12) = 3$, $\varepsilon(13) = 2$, $\varepsilon(14) = 4$. It follows that

$$\frac{\lambda_1\omega_{\varepsilon(11)}}{\sum_{j=1}^{4}\lambda_j\omega_{\varepsilon(1j)}} = \frac{0.3\times 0.5}{0.3\times 0.5 + 0.4\times 0.2 + 0.2\times 0.2 + 0.1\times 0.1} = 0.5357,$$
$$\frac{\lambda_2\omega_{\varepsilon(12)}}{\sum_{j=1}^{4}\lambda_j\omega_{\varepsilon(1j)}} = 0.2857,$$
$$\frac{\lambda_3\omega_{\varepsilon(13)}}{\sum_{j=1}^{4}\lambda_j\omega_{\varepsilon(1j)}} = 0.1429,$$
$$\frac{\lambda_4\omega_{\varepsilon(14)}}{\sum_{j=1}^{4}\lambda_j\omega_{\varepsilon(1j)}} = 0.0357.$$

Thus, using Eq. (24), we can calculate that $\alpha_1 = \text{IFEHWA}(\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}) = \text{IFEHWA}((0.7, 0.3), (0.4, 0.5), (0.5, 0.4), (0.3, 0.6)) = (0.5884, 0.3734)$
Similarly, the results for alternatives x_2, x_3, and x_4 can be calculated by the IFEHWA operator.

$$\alpha_2 = \text{IFEHWA}(\alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{24}) = (0.5962, 0.2258)$$

$$\alpha_3 = \text{IFEHWA}(\alpha_{31}, \alpha_{32}, \alpha_{33}, \alpha_{34}) = (0.6368, 0.2507)$$

$$\alpha_4 = \text{IFEHWA}(\alpha_{41}, \alpha_{42}, \alpha_{43}, \alpha_{44}) = (0.8254, 0.1361)$$

Step 2: Calculate the scores $s(\alpha_i)$ $(i = 1, 2, 3, 4)$ of α_i $(i = 1, 2, 3, 4)$: $s(\alpha_1) = 0.2149$, $s(\alpha_2) = 0.3704$, $s(\alpha_3) = 0.3860$, $s(\alpha_4) = 0.6893$
Since $s(\alpha_4) > s(\alpha_3) > s(\alpha_2) > s(\alpha_1)$, we get $\alpha_4 > \alpha_3 > \alpha_2 > \alpha_1$ and then $x_4 \succ x_3 \succ x_2 \succ x_1$, i.e., the portfolio x_4: shares (SH) are the most desirable choice for the decision maker.
If we use the QIFEHWG operator (let $g(x) = x$) instead of the QIFEHWA operator to aggregate the decision information, then we can obtain the overall IFNs α_i for the portfolios x_1, x_2, x_3, x_4 as follows:

$$\alpha_1 = (0.5591, 0.3868), \quad \alpha_2 = (0.4920, 0.3050),$$
$$\alpha_3 = (0.5206, 0.2657), \quad \alpha_4 = (0.7284, 0.1984)$$

Finally, we can compute the score values $s(\alpha_i)$ $(i = 1, 2, 3, 4)$ and the variance values $h(\alpha_i)$ $(i = 1, 2, 3, 4)$ of α_i $(i = 1, 2, 3, 4)$. By ranking $s(\alpha_i)$ $(i = 1, 2, 3, 4)$, we can get the priorities of the alternatives x_i $(i = 1, 2, 3, 4)$. Since $s(\alpha_1) = 0.1723$, $s(\alpha_2) = 0.1870$, $s(\alpha_3) = 0.2549$, and $s(\alpha_4) = 0.5300$, we get $s(\alpha_4) > s(\alpha_3) > s(\alpha_2) > s(\alpha_1)$, then $\alpha_4 > \alpha_3 > \alpha_2 > \alpha_1$ and $x_4 \succ x_3 \succ x_2 \succ x_1$, i.e., the portfolio x_4: shares (SH) are the most desirable choice for the decision maker, which is the same as that obtained by the QIFEHWA operator.
If we use Zhao and Wei's IFEHA operator (Eq. 19) to solve this problem, then we have

$$\alpha_1 = \text{IFEHA}(\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}) = (0.6388, 0.3228),$$

$$\alpha_2 = \text{IFEHA}(\alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{24}) = (0.5596, 0.2610)$$

$$\alpha_3 = \text{IFEHA}(\alpha_{31}, \alpha_{32}, \alpha_{33}, \alpha_{34}) = (0.6874, 0.2039)$$

$$\alpha_4 = \text{IFEHA}(\alpha_{41}, \alpha_{42}, \alpha_{43}, \alpha_{44}) = (0.8545, 0.0787)$$

Since $s(\alpha_1) = 0.3161$, $s(\alpha_2) = 0.2986$, $s(\alpha_3) = 0.4835$, and $s(\alpha_4) = 0.7758$, we get $s(\alpha_4) > s(\alpha_3) > s(\alpha_1) > s(\alpha_2)$, then $\alpha_4 > \alpha_3 > \alpha_1 > \alpha_2$, and $x_4 \succ x_3 \succ x_1 \succ x_2$, which is slightly different from the results derived by our approach as the positions of the portfolios x_1 and x_2 are changed. With Zhao and Wei's IFEHA operator, the portfolio x_4: shares (SH) turn out to be the most desirable choice for the decision maker. The result is the same as ours which explains the validity of our method. Meanwhile, when using Zhao and Wei's IFEHA operator, we need to calculate $\dot{\alpha}_k = n\lambda_k\alpha_k$ first and compare them, and then calculate $\omega_i\dot{\alpha}_{\sigma(i)}$, after which, we shall compute the aggregation values $\oplus_{i=1}^{n}\left(\omega_i\dot{\alpha}_{\sigma(i)}\right)$. Obviously, the computation process with Zhao and Wei's IFEHA operator is very complex. As for our proposed IFEHWA operator, the weighting operation of the ordered position is synchronized with the weighting operation of the given importance, which is in the mathematical form as $\lambda_i\omega_{\varepsilon(i)}$. Since both λ_i and $\omega_{\varepsilon(i)}$ are crisp numbers, we only need to calculate $\frac{\oplus_{i=1}^{n}\left(\lambda_i\omega_{\varepsilon(i)}\alpha_i\right)}{\sum_{i=1}^{n}\lambda_i\omega_{\varepsilon(i)}}$, which makes our proposed IFEHWA operator is easier to calculate than Zhao and Wei's IFEHA operator.

5.2 Multi-criteria group decision making with intuitionistic fuzzy information

Consider a group decision-making problem with intuitionistic fuzzy information. Let $X = \{x_1, x_2, \ldots, x_m\}$ be a set of m alternatives, $C = \{c_1, c_2, \ldots, c_n\}$ be a collection of n criteria, and $D = \{d_1, d_2, \ldots, d_p\}$ be a set of p decision makers. Let $A^{(k)} = \left(\alpha_{ij}^{(k)}\right)_{m\times n}$ be an intuitionistic fuzzy decision matrix, where $\alpha_{ij}^{(k)} = \left(\mu_{\alpha_{ij}^{(k)}}, v_{\alpha_{ij}^{(k)}}\right)$ is an IFN provided by the decision maker $d_k \in D$; here, $\mu_{\alpha_{ij}^{(k)}}$ indicates the degree to which the alternative $x_i \in X$ satisfies the attribute $c_j \in C$ and $v_{\alpha_{ij}^{(k)}}$ indicates the degree to which the alternative $x_i \in X$ does not satisfy the attribute $c_j \in C$. The following conditions hold: $\mu_{\alpha_{ij}^{(k)}}, v_{\alpha_{ij}^{(k)}} \in [0,1]$, $\mu_{\alpha_{ij}^{(k)}} + v_{\alpha_{ij}^{(k)}} \leq 1$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$.

The decision maker d_k $(k = 1,2,\ldots,p)$ also determines the importance degrees $\lambda_j^{(k)}$ $(j = 1,2,\ldots,n)$ for the relevant criteria according to his/her preferences, where $\lambda_j^{(k)} \in [0,1]$, $j = 1,2,\ldots,n$, and $\sum_{j=1}^{n} \lambda_j^{(k)} = 1$. Meanwhile, since different alternatives may have different focuses and advantages, to reflect this issue, the decision maker d_k $(k = 1,2,\ldots,p)$ also gives the ordering weights $\omega_j^{(k)}$ $(j = 1,2,\ldots,n)$ for different criteria, where $\omega_j^{(k)} \in [0,1]$, $j = 1,2,\ldots,n$, and $\sum_{j=1}^{n} \omega_j^{(k)} = 1$. Suppose that the weight vector of the decision makers is $\eta = (\eta_1, \eta_2, \ldots, \eta_p)^T$, which satisfies $\eta_k \in [0,1]$, $k = 1,2,\ldots,p$, and $\sum_{k=1}^{p} \eta_k = 1$. Then, based on the developed aggregation operators, we give a method for GDM with intuitionistic fuzzy information, which consists of the following steps:

Algorithm 2 *Step 1.* Utilize the QIFEHWA (or QIFEHWG) operator to aggregate all $\alpha_{ij}^{(k)}$ $(j = 1,2,\ldots,n)$ corresponding to the alternative x_i, and then get the averaged IFN $\alpha_i^{(k)}$ of the alternative x_i over all the criteria for the decision maker d_k:

$$\alpha_i^{(k)} = \mathrm{QIFEHWA}\left(\alpha_{i1}^{(k)}, \alpha_{i2}^{(k)}, \ldots, \alpha_{in}^{(k)}\right) = g^{-1}\left(\frac{\oplus_{j=1}^{n}\left(\lambda_j^{(k)}\omega_{\varepsilon(ij)}^{(k)} g\left(\alpha_{ij}^{(k)}\right)\right)}{\sum_{j=1}^{n}\lambda_j^{(k)}\omega_{\varepsilon(ij)}^{(k)}}\right) \tag{53}$$

or

$$\alpha_i^{(k)} = \mathrm{QIFEHWG}\left(\alpha_{i1}^{(k)}, \alpha_{i2}^{(k)}, \ldots, \alpha_{in}^{(k)}\right) = g^{-1}\left(\otimes_{j=1}^{n}\left(g\left(\alpha_{ij}^{(k)}\right)\right)^{\left(\lambda_j^{(k)}\omega_{\varepsilon(ij)}^{(k)}\right)/\sum_{j=1}^{n}\lambda_j^{(k)}\omega_{\varepsilon(ij)}^{(k)}}\right) \tag{54}$$

where $\varepsilon : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ is the permutation such that $\alpha_{ij}^{(k)}$ is the $\varepsilon(ij)$th largest element of the collection of IFNs $\alpha_{ij}^{(k)}$ $(j = 1,2,\ldots,n)$, and g is a continuous strictly monotonic function.

Step 2. Utilize the IFEWA (or IFEWG) operator to aggregate all $\alpha_i^{(k)}$ $(k = 1,2,\ldots,p)$ into a collective IFN α_i of the alternative x_i:

$$\alpha_i = \mathrm{IFEWA}\left(\alpha_i^{(1)}, \alpha_i^{(2)}, \ldots, \alpha_i^{(p)}\right) = \overset{p}{\underset{k=1}{\oplus}}\left(\eta_k \alpha_i^{(k)}\right), \quad i = 1,2,\ldots,m \tag{55}$$

or

$$\alpha_i = \mathrm{IFEWG}\left(\alpha_i^{(1)}, \alpha_i^{(2)}, \ldots, \alpha_i^{(p)}\right) = \overset{p}{\underset{k=1}{\otimes}}\left(\alpha_i^{(k)}\right)^{\eta_k}, \quad i = 1,2,\ldots,m \tag{56}$$

Step 3. Compute the score functions $s(\alpha_i)$ $(i = 1,2,\ldots,m)$ of α_i $(i = 1,2,\ldots,m)$ by Eq. (4) and the accuracy degree $h(\alpha_i)$ $(i = 1,2,\ldots,m)$ of α_i $(i = 1,2,\ldots,m)$ by Eq. (5).

Step 4. Get the priority of the alternatives x_i $(i = 1,2,\ldots,m)$ by ranking $s(\alpha_i)$ and $h(\alpha_i)$ $(i = 1,2,\ldots,m)$ according to Definition 2.2.

We now use a numerical example (adapted from [18]) to illustrate our method:

Example 5.2 [18]. Suppose that a computer center in a university wishes to select a new information system to improve work productivity. After a preliminary screening, four alternatives x_i $(i = 1,2,3,4)$ remain in the candidate list. Three experts d_k $(k = 1,2,3)$ form a committee to act as decision makers; the decision-maker weight vector is $\eta = (0.2, 0.5, 0.3)^T$. There are four criteria that must be considered: (1) the costs of the hardware and software investment (c_1); (2) the contribution to organization performance (c_2); (3) the effort to transition from the current systems (c_3); and (4) the reliability of outsourcing software development (c_4). The experts d_k $(k = 1,2,3)$ evaluate the software packages x_i $(i = 1,2,3,4)$ with respect to the criteria c_j $(j = 1,2,3,4)$ and construct three intuitionistic fuzzy decision matrices $A^{(k)} = \left(\alpha_{ij}^{(k)}\right)_{4\times 4}$ $(k = 1,2,3)$ (see

Table 2 Intuitionistic fuzzy decision matrix $A^{(1)}$

	c_1	c_2	c_3	c_4
x_1	(0.5, 0.4)	(0.4, 0.3)	(0.5, 0.3)	(0.2, 0.6)
x_2	(0.5, 0.4)	(0.3, 0.7)	(0.2, 0.8)	(0.4, 0.5)
x_3	(0.2, 0.6)	(0.8, 0.1)	(0.6, 0.4)	(0.1, 0.7)
x_4	(0.1, 0.9)	(0.2, 0.8)	(0.7, 0.2)	(0.4, 0.6)

Table 3 Intuitionistic fuzzy decision matrix $A^{(2)}$

	c_1	c_2	c_3	c_4
x_1	(0.3, 0.6)	(0.2, 0.7)	(0.5, 0.5)	(0.5, 0.3)
x_2	(0.3, 0.7)	(0.6, 0.4)	(0.7, 0.2)	(0.4, 0.5)
x_3	(0.6, 0.3)	(0.4, 0.4)	(0.2, 0.7)	(0.3, 0.6)
x_4	(0.2, 0.5)	(0.5, 0.3)	(0.5, 0.4)	(0.3, 0.3)

Table 4 Intuitionistic fuzzy decision matrix $A^{(3)}$

	c_1	c_2	c_3	c_4
x_1	(0.7, 0.3)	(0.4, 0.5)	(0.5, 0.4)	(0.6, 0.2)
x_2	(0.5, 0.5)	(0.3, 0.5)	(0.8, 0.1)	(0.7, 0.1)
x_3	(0.8, 0.2)	(0.2, 0.3)	(0.6, 0.3)	(0.2, 0.7)
x_4	(0.9, 0.1)	(0.8, 0.1)	(0.2, 0.7)	(0.2, 0.6)

Tables 2, 3, 4). The decision maker d_k $(k = 1, 2, 3)$ determines the weight vector $\lambda^{(k)} = \left(\lambda_1^{(k)}, \lambda_2^{(k)}, \lambda_3^{(k)}, \lambda_4^{(k)}\right)$ of the four criteria according to his/her preferences, which are $\lambda^{(1)} = (0.4, 0.3, 0.1, 0.2)$, $\lambda^{(2)} = (0.1, 0.3, 0.5, 0.1)$, and $\lambda^{(3)} = (0.1, 0.2, 0.3, 0.4)$. Furthermore, considering the fact that different experts are familiar with different research fields, and meanwhile, different information systems may focus on different partitions, the experts may want to give more weights to the criterion which is more prominent. Hence, another weight vectors are determined by the experts according to their preferences, which are $\omega^{(1)} = (0.4, 0.3, 0.2, 0.1)$, $\omega^{(2)} = (0.3, 0.3, 0.2, 0.2)$, and $\omega^{(3)} = (0.5, 0.3, 0.1, 0.1)$.

To get the optimal information system, the following steps are given:

Step 1. Utilize the aggregation operator (such as the QIFEHWA or QIFEHWG operator) to aggregate all $\alpha_{ij}^{(k)}$ $(j = 1, 2, 3, 4)$ corresponding to the alternative x_i, and then get the averaged IFN $\alpha_i^{(k)}$ $(i = 1, 2, 3, 4)$ of the alternative x_i over all the criteria for the decision maker d_k $(k = 1, 2, 3)$. Here, we adopt the QIFEHWA operator and let $g(x) = x$, and then we can get

$$\alpha_1^{(1)} = (0.4382, 0.3541), \quad \alpha_2^{(1)} = (0.4315, 0.4868), \quad \alpha_3^{(1)} = (0.5937, 0.2601), \quad \alpha_4^{(1)} = (0.3640, 0.5966)$$

$$\alpha_1^{(2)} = (0.4226, 0.5215), \quad \alpha_2^{(2)} = (0.6290, 0.2973), \quad \alpha_3^{(2)} = (0.3415, 0.5126), \quad \alpha_4^{(2)} = (0.4679, 0.3641)$$

$$\alpha_1^{(3)} = (0.5897, 0.2421), \quad \alpha_2^{(3)} = (0.7330, 0.1191), \quad \alpha_3^{(3)} = (0.5699, 0.3284), \quad \alpha_4^{(3)} = (0.6932, 0.2200)$$

Step 2. Utilize the aggregation operator (such as the IFEWA or IFEWG operator) to aggregate all $\alpha_i^{(k)}$ $(k = 1, 2, 3)$ into a collective IFN α_i of the alternative x_i. Here, we use the IFEWA operator. Thus, we have

$$\alpha_1 = (0.4797, 0.3884), \quad \alpha_2 = (0.6308, 0.2541), \quad \alpha_3 = (0.4690, 0.3954), \quad \alpha_4 = (0.5272, 0.3503)$$

Step 3. Compute the score values $s(\alpha_i)$ $(i = 1, 2, 3, 4)$ of α_i $(i = 1, 2, 3, 4)$ by Eq. (4), and then we have $s(\alpha_1) = 0.0913$, $s(\alpha_2) = 0.3768$, $s(\alpha_3) = 0.0736$, and $s(\alpha_4) = 0.1769$.

Step 4. Since $s(\alpha_2) > s(\alpha_4) > s(\alpha_1) > s(\alpha_3)$, then we get $\alpha_2 > \alpha_4 > \alpha_1 > \alpha_3$ and $x_2 \succ x_4 \succ x_1 \succ x_3$, which means that x_2 is the most desirable information system.

If we use the QIFEHWG operator $(g(x) = x)$ instead of the QIFEHWA operator in Step 1 and the IFEWG operator instead of the IFEWA operator in Step 2, then we can obtain the averaged IFN $\alpha_i^{(k)}$ $(i = 1, 2, 3, 4, 5)$ of the alternative x_i over all the criteria for the decision maker d_k $(k = 1, 2, 3)$ and the collective IFN α_i of the alternative x_i as follows:

$$\alpha_1^{(1)} = (0.4261, 0.3649), \quad \alpha_2^{(1)} = (0.4191, 0.5115), \quad \alpha_3^{(1)} = (0.4506, 0.3715), \quad \alpha_4^{(1)} = (0.2867, 0.7027)$$

$$\alpha_1^{(2)} = (0.3953, 0.5414), \quad \alpha_2^{(2)} = (0.6086, 0.3328), \quad \alpha_3^{(2)} = (0.3126, 0.5488), \quad \alpha_4^{(2)} = (0.4550, 0.3695)$$

$$\alpha_1^{(3)} = (0.5829, 0.2567), \quad \alpha_2^{(3)} = (0.7075, 0.1442), \quad \alpha_3^{(3)} = (0.4840, 0.3751), \quad \alpha_4^{(3)} = (0.5202, 0.3451)$$

$$\alpha_1 = (0.4529, 0.4287), \quad \alpha_2 = (0.5948, 0.3181), \quad \alpha_3 = (0.3854, 0.4658), \quad \alpha_4 = (0.4343, 0.4433)$$

We further compute the scores $s(\alpha_i)$ $(i = 1, 2, 3, 4)$ of the collective IFN α_i $(i = 1, 2, 3, 4)$, and then we have $s(\alpha_1) = 0.0242$, $s(\alpha_2) = 0.2767$, $s(\alpha_3) = -0.0804$, and $s(\alpha_4) = -0.0091$, which indicates that $\alpha_2 > \alpha_1 > \alpha_4 > \alpha_3$ and thus $x_2 \succ x_1 \succ x_4 \succ x_3$, which is slightly different from the results derived by the QIFEHWG and IFEWG operators as the positions of the information systems x_1 and x_4 are changed. The main reason for this difference is that the QIFEHWA and IFEWA operators are developed based the usual arithmetic average which pays more attention to the group opinion and the number of arguments, while the QIFEHWG and IFEWG operators are developed based on the geometric mean which mainly focuses on the individual opinion and the average of arguments where the smaller deviation between arguments, the better the results by the QIFEHWG and IFEWG operators.

6 Concluding remarks

In this paper, we have pointed out the drawbacks of some existing aggregation operators for IFNs, and then some new intuitionistic fuzzy Einstein hybrid weighted aggregation operators, such as the IFEHWA operator, the IFEHWG operator, the QIFEHWA operator, and the QIFEHWG operator, have been introduced to overcome the drawbacks in the existed operators. The properties of these new operators have been clarified as well. To show the applications of our proposed intuitionistic fuzzy Einstein hybrid weighted aggregation operators, we have also proposed two simple procedures for multi-criteria single-person decision making and multi-criteria group decision making, respectively, and then used two numerical examples to illustrate the validity and applicability of the proposed procedures and also the hybrid aggregation operators.

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