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ORIGINAL ARTICLE

Consensus problems for multi-agent systems with nonlinear algorithms

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Abstract The paper investigates consensus problems for multi-agent systems with nonlinear algorithms. Group consensus algorithms with actuator saturation for the first-order and second-order multi-agent systems are proposed. In addition, the adaptive consensus algorithm with nonlinear dynamic is also given. By applying the graph theory, Lyapunov function, and LaSalle's invariance principle, consensus conditions for multi-agent systems are derived. Finally, three simulation examples are provided to denote the effectiveness of obtained theoretical results.

Keywords Multi-agent · Consensus · Actuator saturation · Group consensus

1 Introduction

In recent years, consensus problem for multi-agent systems has been a hot topic in the control field, owing to widely applications of multi-agent systems to numerous areas [1–26] such as vehicle traffic control, robot formation control, communication network of automatic weather stations. The key problem of consensus is that how to design consensus algorithms makes all agents reach the same constant as time goes on. Moreover, consensus algorithms of multi-agent systems are usually adopted in distributed control, that is, every agent only uses the neighbor's information, which doesn't need the global information.

So far, results of consensus problems for multi-agent system have been obtained. In general, researchers have studied linear consensus algorithms and nonlinear cases. Owing to accelerate the convergence rate or limited ability of physical machines, nonlinear consensus algorithms for multi-agent systems are considered. Consensus problem of multi-agent systems with time delays was discussed in [4]. And consensus tracking algorithms were proposed in [5, 6]. Since every machine has the physical limitations in practice, consensus algorithms with input saturation are significant and important to study. Consensus problem for the first-order multi-agent systems with input saturation was discussed in [7], where authors pointed out that the consensus algorithm which was widely applied in the first-order multi-agent systems was efficient to the ones with input saturation. Leader-following consensus problem of the second-order multi-agent systems with actuator saturation was investigated in [8]. Moreover, authors in [8] extended the results under the fixed topology to that under switching topologies. In jointly connected topologies, nonlinear consensus algorithms for leader-following consensus problems for multi-agent systems were proposed in [9], where finite-time consensus conditions were given by applying LaSalle's invariance principle. Finite-time consensus problem was extended the results of the first-order multi-agent systems in [10] to that of the second-order multi-agent systems in [9], where the leader was jointly reachable under switching topologies. Based on algebraic Riccati equality and Lyapunov function, semi-global consensus problem for the high-order multi-agent system with input saturation was investigated in [11], where small gains were introduced. By using the low-gain method, leader-

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following consensus algorithms for high-order multi-agent systems with input saturation were discussed under the undirected topology in [12]. Consensus problem for high-order discrete-time multi-agent systems with input saturation was studied under fixed topology in [13]. Output regulation of multi-agent systems with input saturation was discussed under switching topologies in [14]. In addition, finite-time consensus was investigated in [15].

Since there exists external uncertainty, stochastic noises, and disturbance, it is important and meaningful to study adaptive consensus problems, which has attracted increasing attention. Adaptive consensus problem for leader-following multi-agent systems with nonlinear dynamics was discussed in [16]. For high-order multi-agent systems with unknown nonlinear uncertainty, adaptive finite-time consensus algorithm was given in [17]. By applying the Lyapunov stability theory, adaptive consensus problem for the second-order multi-agent systems with time delays and nonlinear uncertainty was investigated in [18]. And two adaptive distributed consensus algorithms for high-order multi-agent systems were given in [19], where they are the ones with time-varying weight for each edge and for each node in the communication topology, respectively. And by parameterizing the acceleration and nonlinear disturbance, adaptive leader-following consensus problem of the second-order multi-agent systems was investigated in [20]. In addition, other adaptive consensus problems were studied in [21, 22]. Moreover, agents in multi-agent systems may achieve several consensus according to different tasks or unexpected environment. Then, group consensus of multi-agent systems was introduced in [23, 24]. Furthermore, group consensus algorithms for the continuous- and discrete-time second-order multi-agent systems were given in [25] and [26], respectively.

To the best of our knowledge, few results about group consensus for multi-agent systems with input saturation have been investigated, which is our motivation. In this paper, group consensus algorithms with input saturation for the first-order and second-order multi-agent systems will be given. In addition, adaptive leader-following consensus algorithm will be also proposed, which has been not investigated fully.

The paper is organized as follows. In Sect. 2, graph theory needed in this paper is given. Main results of our contribution are in Sect. 3. In Sect. 4, three examples are given to denote the effectiveness of proposed methods in this paper. In Sect. 5, concluding remarks are presented.

Notations: The matrix $P > (\geq)0$ means that P is positive (semi) definite matrix. And P^T is the transpose of the matrix P. The symbol $\| * \|$ denotes the Euclidean norm. $\mathbf{1}_n$ represents a column vector with every element 1.

2 Graph theory

In this section, we give some preliminaries about graph theory in this paper. Assume that $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ is a directed graph corresponding to the fixed communication topology, where $\mathcal{V} = \{1, \ldots, i, \ldots, n\}$ is the set of nodes with i representing the ith agent, $\mathcal{E}$ is a set of edges, $\mathcal{A} = [a_{ij}]_{n\times n}$ is a weighted adjacent matrix. The Laplacian matrix $L = [l_{ij}]_{n\times n}$ associated with the graph $\mathcal{G}$ is defined with $l_{ij} = -a_{ij}$ for $i \neq j$ and $l_{ii} = \sum_{j=1, j\neq i}^{n} a_{ij}$ for $i = j$. From the above definition, L has at least a simple zero eigenvalue. In the directed graph $\mathcal{G}$, if there exist some positive constants $w_i > 0$ for $i = 1, \ldots, n$, $w_i a_{ij} = a_{ji} w_j$ holds for $i, j = 1, \ldots, n$. Then, we say the graph $\mathcal{G}$ is detailed balanced.

In some situations, graph $\bar{\mathcal{G}} = \{\bar{\mathcal{V}}, \bar{\mathcal{E}}, \bar{\mathcal{A}}\}$ including $n + m$ agents is divided into two subgroups $\mathcal{G}_1 = \{\mathcal{V}_1, \mathcal{E}_1, \mathcal{A}_1\}$ with $\mathcal{V}_1 = \{1, \ldots, n\}$ and $\mathcal{G}_2 = \{\mathcal{V}_2, \mathcal{E}_2, \mathcal{A}_2\}$ with $\mathcal{V}_2 = \{n+1, \ldots, n+m\}$, where $\bar{\mathcal{V}} = \mathcal{V}_1 \cup \mathcal{V}_2$ and $\bar{\mathcal{E}} = \mathcal{E}_1 \cup \mathcal{E}_2$. If $\sum_{j=n}^{n+m} a_{ij} = 0$ for $i \in \mathcal{G}_1, j \in \mathcal{G}_2$ and $\sum_{j=1}^{n} a_{ij} = 0$ for $i \in \mathcal{G}_2, j \in \mathcal{G}_1$ are satisfied, we say there is a balance between $\mathcal{G}_1$ and $\mathcal{G}_2$.

3 Main results

3.1 Group consensus for the first-order multi-agent system

Consider the ith agent's dynamics of the first-order multi-agent system, which is described as follows

$$\dot{x}_i(t) = sat(u_i(t)), \tag{1}$$

where $sat(u(t)) = [sat(u_1(t))^T \quad \cdots \quad sat(u_{n+m}(t))^T]^T$,

$$sat(u_i(t)) = \begin{cases} u_i(t), & \text{if} \quad -1 \leq u_i(t) \leq 1, \\ 1, & \text{if} \quad u_i(t) > 1, \\ -1, & \text{if} \quad u_i(t) < -1. \end{cases} \tag{2}$$

In the paper, the group consensus protocol is considered as follows

$$u_i(t) = \begin{cases} \sum_{j=1}^{n} a_{ij}(x_j(t) - x_i(t)) + \sum_{j=n+1}^{n+m} a_{ij} x_j(t), i \in \mathcal{G}_1, \\ \sum_{j=n+1}^{n+m} a_{ij}(x_j(t) - x_i(t)) + \sum_{j=1}^{n} a_{ij} x_j(t), i \in \mathcal{G}_2, \end{cases} \tag{3}$$

where $\mathcal{G}_1 = \{\mathcal{V}_1, \mathcal{E}_1, \mathcal{A}_1\}$, $\quad \mathcal{G}_2 = \{\mathcal{V}_2, \mathcal{E}_2, \mathcal{A}_2\}$.

Assumption 1

(1) $\sum_{j=n+1}^{n+m} a_{ij} = 0$ with $a_{i,n+i} < 0$ and $a_{ij} > 0$ for $i \neq j$ and $j \neq n+i$, $i \in \mathcal{G}_1, j \in \mathcal{G}_2$;

(2) $\sum_{j=1}^{n} a_{ij} = 0$ with $a_{i,n+i} < 0$ and $a_{ij} > 0$ for $i \neq j$ and $j \neq n+i$, for $i \in \mathcal{G}_2, j \in \mathcal{G}_1$.

Together (1) with (3), we have

$$\dot{X}(t) = sat(-HX(t)), \tag{4}$$

where $X(t)=[x_1^T(t)\, x_2^T(t) \cdots x_{n+m}^T(t)]^T$, $H=[h_{ij}]_{(n+m)\times(n+m)}$ with $h_{ij}=-a_{ij}$ for $i\neq j$ and $h_{ii}=\sum_{j=1,j\neq i}^{n+m} a_{ij}$ for $i=j$.

Set $\xi(t) = HX(t)$, system (4) can be rewritten as follows

$$\dot{\xi}(t) = -Hsat(\xi(t)). \tag{5}$$

Thus, stability of system (4) is converted to stability of system (5).

Assumption 2 In the directed topology, suppose that H has only two simple zero eigenvalues and the others have positive real parts.

Definition 1 ([23, 24]) System (1) is said to achieve group consensus if the following conditions are satisfied:

(1) $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0$, for $i,j \in \mathcal{G}_1$;
(2) $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0$, for $i,j \in \mathcal{G}_2$;
(3) $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| \neq 0$, for $i \in \mathcal{G}_1, j \in \mathcal{G}_2$.

Theorem 1 *Suppose that Assumptions 1 and 2 hold and there is a directed topology $\bar{\mathcal{G}}$. Under the consensus protocol (3), system (1) can achieve group consensus in finite time.*

Proof Firstly, a set M is introduced in the following

$$M = \{|\xi_i(t)| \le 1, |\xi_j(t)| \le 1, \quad \text{for} \quad i \in \mathcal{G}_1, j \in \mathcal{G}_2\}. \tag{6}$$

In the next, we will show that $\xi_i(t) \in M$ holds for any the initial condition $\xi_i(0) \in M$. Following the method in [7], we obtain

$$\begin{aligned}
\dot{\xi}_1(t) &= -a_{11}sat(\xi_1(t)) + \sum_{j=2}^{n} a_{1j}sat(\xi_j(t)) + \sum_{j=n+1}^{n+m} a_{1j}\xi_j(t)\\
&= \sum_{j=2}^{n} a_{1j}(sat(\xi_j(t)) - sat(\xi_1(t))) + \sum_{j=n+1}^{n+m} a_{1j}\xi_j(t),\\
\dot{\xi}_{n+1}(t) &= \sum_{j=1}^{n} a_{n+1,j}sat(\xi_j(t)) - a_{n+1,n+1}sat(\xi_{n+1,n+1}(t))\\
&\quad + \sum_{j=n+2}^{n+m} a_{n+1,j}sat(\xi_j(t))\\
&= \sum_{j=n+2}^{n+m} a_{n+1,j}(sat(\xi_j(t)) - sat(\xi_{n+1}(t)))\\
&\quad + \sum_{j=1}^{n} a_{n+1,j}\xi_j(t).
\end{aligned} \tag{7}$$

In view of Assumption 1, (7) can be rewritten

$$\begin{aligned}
\dot{\xi}_1(t) &= \sum_{j=2}^{n} a_{1j}(sat(\xi_j(t)) - sat(\xi_1(t)))\\
&\quad + \sum_{j=n+2}^{n+m} a_{1j}(sat(\xi_j(t)) - sat(\xi_{n+1}(t))),\\
\dot{\xi}_{n+1}(t) &= \sum_{j=2}^{n} a_{n+1,j}(sat(\xi_j(t)) - sat(\xi_1(t)))\\
&\quad + \sum_{j=n+2}^{n+m} a_{n+1,j}(sat(\xi_j(t)) - sat(\xi_{n+1}(t))).
\end{aligned} \tag{8}$$

Taking $|\xi_1(t)| = 1$, $|\xi_{n+1}(t)| = 1$, $-1 \le \xi_j(t) \le 1$ for $j = 2,\ldots,n, n+2,\ldots,n+m$, we have

$$\begin{aligned}
\dot{\xi}_1(t) &\le \sum_{j=2}^{n} a_{1j}(\xi_j(t) - 1) + \sum_{j=n+2}^{n+m} a_{1j}(\xi_j(t) - 1)\\
&\le 0, if\, \xi_1(t) = 1, \xi_{n+1}(t) = 1,\\
\dot{\xi}_{n+1}(t) &\le \sum_{j=2}^{n} a_{n+1,j}(\xi_j(t) - 1) + \sum_{j=n+2}^{n+m} a_{n+1,j}(\xi_j(t) - 1)\\
&\le 0, if\, \xi_1(t) = 1, \xi_{n+1}(t) = 1,
\end{aligned} \tag{9}$$

$$\begin{aligned}
\dot{\xi}_1(t) &\ge \sum_{j=2}^{n} a_{1j}(\xi_j(t) - 1) + \sum_{j=n+2}^{n+m} a_{1j}(\xi_j(t) - 1)\\
&\ge 0, if\, \xi_1(t) = -1, \xi_{n+1}(t) = -1,\\
\dot{\xi}_{n+1}(t) &\ge \sum_{j=2}^{n} a_{n+1,j}(\xi_j(t) - 1) + \sum_{j=n+2}^{n+m} a_{n+1j}(\xi_j(t) - 1)\\
&\ge 0, if\, \xi_1(t) = -1, \xi_{n+1}(t) = -1.
\end{aligned} \tag{10}$$

From the above analysis, if $|\xi_1(0)| \le 1$ and $|\xi_{n+1}(0)| \le 1$, we have $|\xi_1(t)| \le 1$ and $|\xi_{n+1}(t)| \le 1$. Similarly, $|\xi_i(t)| \le 1$ is obtained if $|\xi_i(0)| \le 1$. Then, we obtain M is an invariant set.

In the next section, we will show all solutions of system (5) converge to the origin in the invariant set M. There exists an invertible matrix $\bar{P}$, such that

$$\bar{P}^{-1}H\bar{P} = \Lambda, \tag{11}$$

where $\Lambda = \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & J_1 & & \\ & & & \ddots & \\ & & & & J_r \end{bmatrix}$, J_i is the Jordan canonical block for $i = 1,\ldots,r$. Observing Assumption 2, we know that 0 is simple eigenvalues with geometric multiplicity 2. In the set M, system (5) is changed to

$$\dot{\xi}(t) = -H\xi(t). \tag{12}$$

From (11), which implies

$$\dot{\xi}(t) = -\bar{P}\Lambda\bar{P}^{-1}\xi(t). \tag{13}$$

The first and second columns of the matrix $\bar{P}$ are $p_1 = [\underbrace{1 \ \cdots \ 1}_{n} \quad 0\cdots 0]^T$, $p_2 = [0\cdots 0 \quad \underbrace{1 \ \cdots \ 1}_{m}]^T$, w_1 and w_2 denote the first and second rows of the matrix $\bar{P}^{-1}$.

Noting $\xi(0) = Hx(0)$, we have

$$\begin{aligned} \lim_{t\to\infty} \xi(t) &= \lim_{t\to\infty} \bar{P}exp(-\Lambda t)\bar{P}^{-1}\xi(0) \\ &= \lim_{t\to\infty}[p_1 \quad p_2][w_1^T \quad w_2^T]^T(Hx(0)). \end{aligned} \tag{14}$$

Since w_1 and w_2 are left eigenvectors of H corresponding to zero eigenvalue, $[w_1^T \quad w_2^T]^T H = 0$ is derived. Then,

$$\lim_{t\to\infty} \xi(t) = 0. \tag{15}$$

Noting $\xi(t) = Hx(t)$, $\lim_{t\to\infty}(Hx(t)) = 0$ is implied in (15). Therefore, as time t goes to infinity, $x(t)$ will converge null space of matrix H, which is generated by the vector p_1 and p_2. Thus,

$$\lim_{t\to\infty} x(t) \to \begin{bmatrix} \alpha \mathbf{1}_n \\ \beta \mathbf{1}_m \end{bmatrix}, \tag{16}$$

where α is a constant, and β is a constant.

Following a similar line to that in [7], we obtain that system (5) converges to the invariant set M in finite time. Then, system (1) can achieve group consensus under the consensus protocol with saturation (3) in finite time. The proof is completed. □

Remark 1 Theorem 1 has extended the consensus problem for the first-order multi-agent system in [7] to group consensus problems. In the paper, the fixed topology is directed. Finite-time consensus protocols with saturation were proposed in [15], while we give another method for group consensus of multi-agent systems with saturation.

Remark 2 The linear group consensus algorithms were proposed in [23–26]. However, the nonlinear group consensus algorithm with actuator saturation for the first-order multi-agent system is given in this paper. Group consensus was investigated in an undirected topology in [23], while group consensus is studied in the directed topology in this paper.

In the following, we will discuss partly the saturation of the actuators in the consensus protocol. Consider the ith dynamic behavior of multi-agent system is described as follows

$$\dot{x}_i(t) = u_i(t), \tag{17}$$

where $x_i(t)$ and $u_i(t)$ represent the position and input control of the ith agent, respectively. The group consensus algorithm is proposed

$$u_i(t) = \begin{cases} \sum_{j=1}^{n} a_{ij}sat(x_j(t) - x_i(t)) + \sum_{j=n+1}^{n+m} a_{ij}(x_j(t)), i \in \mathcal{G}_1, \\ -\sum_{j=1}^{n} a_{ij}(x_j(t)) + \sum_{j=n+1}^{n+m} a_{ij}sat(x_j(t) - x_i(t)), i \in \mathcal{G}_2. \end{cases} \tag{18}$$

Set $e_1 = \frac{1}{n}\sum_{i=1}^{n} x_i(0)$, $e_2 = \frac{1}{m}\sum_{i=n+1}^{n+m} x_i(0)$. Take $\tilde{x}_i(t) = x_i(t) - e_1$ for $i = 1, \ldots, n$ and $\tilde{x}_j(t) = x_j(t) - e_2$ for $j = n+1, \ldots, n+m$, $\tilde{X}(t) = [\tilde{x}_1^T(t) \quad \cdots \quad \tilde{x}_{n+m}^T(t)]^T$.

Combining (17) and (18), we have

$$\dot{\tilde{x}}_i(t) = \begin{cases} \sum_{j=1}^{n} a_{ij}sat(\tilde{x}_j(t) - \tilde{x}_i(t)) + \sum_{j=n+1}^{n+m} a_{ij}\tilde{x}_j(t), i \in \mathcal{G}_1, \\ -\sum_{j=1}^{n} a_{ij}\tilde{x}_j(t) + \sum_{j=n+1}^{n+m} a_{ij}sat(\tilde{x}_j(t) - \tilde{x}_i(t)), i \in \mathcal{G}_2. \end{cases} \tag{19}$$

Lemma 1 ([9]) *Suppose that the topology $\bar{\mathcal{G}}$ is undirected and connected, $a_{ij} = a_{ji}$ for $i, j = 1, \ldots, n+m$. If $f : R \to R$ is an odd function, for $\tilde{x}(t) \in R^n$,*

$$\begin{aligned} \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\tilde{x}_i(t)f(\tilde{x}_i(t) - \tilde{x}_j(t)) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\tilde{x}_i(t) \\ - \tilde{x}_j(t))f(\tilde{x}_i(t) - \tilde{x}_j(t)). \end{aligned} \tag{20}$$

Corollary 1 Suppose that the topology $\bar{\mathcal{G}}$ *is connected and undirected. Under the group consensus protocol (18), system (17) can achieve group average consensus.*

Proof Choose the Lyapunov function $V(\tilde{x}(t)) = \frac{1}{2}\sum_{i=1}^{n+m} \tilde{x}_i^2(t)$. Since the saturation function is odd, by Lemma 1 and calculating the derivation of $V(\tilde{x}(t))$, we obtain

$$\begin{aligned} \dot{V}(\tilde{x}(t)) &= \sum_{i=1}^{n+m} \tilde{x}_i(t)\dot{\tilde{x}}_i(t) \\ &= \sum_{i=1}^{n} \tilde{x}_i(t) \sum_{j=1}^{n} a_{ij}sat(\tilde{x}_j(t) - \tilde{x}_i(t)) \\ &\quad + \sum_{i=1}^{n} \tilde{x}_i(t) \sum_{j=n+1}^{n+m} a_{ij}\tilde{x}_j(t) + \sum_{i=n+1}^{n+m} \tilde{x}_i(t) \\ &\qquad \times \sum_{j=n+1}^{n+m} a_{ij}sat(\tilde{x}_j(t) - \tilde{x}_i(t)) \\ &\qquad - \sum_{i=n+1}^{n+m} \tilde{x}_i(t) \sum_{j=1}^{n} a_{ij}\tilde{x}_j(t) \\ &= -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\tilde{x}_i(t) - \tilde{x}_j(t))sat(\tilde{x}_i(t) - \tilde{x}_j(t)) \\ &\quad + \sum_{i=1}^{n} \tilde{x}_i(t) \sum_{j=n+1}^{n+m} a_{ij}\tilde{x}_j(t) \\ &\quad - \frac{1}{2}\sum_{i=n+1}^{n+m}\sum_{j=n+1}^{n+m} a_{ij}(\tilde{x}_i(t) - \tilde{x}_j(t))sat(\tilde{x}_i(t) - \tilde{x}_j(t)) \\ &\quad - \sum_{i=1}^{n}\sum_{j=n+1}^{n+m} a_{ji}\tilde{x}_i(t)\tilde{x}_j(t) \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\tilde{x}_i(t)-\tilde{x}_j(t))sat(\tilde{x}_i(t)-\tilde{x}_j(t)) \\
&\quad -\frac{1}{2}\sum_{i=n+1}^{n+m}\sum_{j=n+1}^{n+m} a_{ij}(\tilde{x}_i(t)-\tilde{x}_j(t))sat(\tilde{x}_i(t)-\tilde{x}_j(t)) \\
&\leq 0.
\end{aligned} \tag{21}
$$

From the above analysis, $\dot{V}(\tilde{x}(t)) = 0$ if and only if $\tilde{x}_i(t) = \tilde{x}_j(t)$ for $i,j \in \mathcal{G}_1$ and $\tilde{x}_i(t) = \tilde{x}_j(t)$ for $i,j \in \mathcal{G}_2$ hold. By using LaSalle's invariance principle, we know that system (17) can reach group average consensus. The proof is completed. □

3.2 Group consensus for the second-order multi-agent system

In this section, we will discuss group consensus for the second-order multi-agent system. The ith dynamic of the second-order multi-agent system is considered as follows

$$
\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = sat(u_i(t)), \end{cases} \tag{22}
$$

where $x_i(t)$ and $v_i(t)$ is the position and velocity of the ith agent, respectively. And the group consensus protocol is given as follows

$$
u_i(t) = \begin{cases} \sum_{j=1}^{n} a_{ij}\left[(x_j(t)-x_i(t)) + (v_j(t)-v_i(t))\right] \\ \quad -\sum_{j=n+1}^{n+m} a_{ij}\left[x_j(t)+v_j(t)\right], i \in \mathcal{G}_1, \\ \sum_{j=n+1}^{n+m} a_{ij}\left[(x_j(t)-x_i(t)) + (v_j(t)-v_i(t))\right] \\ \quad -\sum_{j=1}^{n} a_{ij}\left[x_j(t)+v_j(t)\right], i \in \mathcal{G}_2. \end{cases} \tag{23}
$$

Set $a = \frac{1}{n}\sum_{j=1}^{n} x_j(t)$, $b = \frac{1}{n}\sum_{j=1}^{n} v_j(t)$, $\bar{a} = \frac{1}{m}\sum_{j=n+1}^{n+m} x_j(t)$, $\bar{b} = \frac{1}{m}\sum_{j=n+1}^{n+m} v_j(t)$, $\tilde{x}_i(t) = x_i(t) - a$, $\tilde{v}_i(t) = v_i(t) - b$ for $i \in \mathcal{G}_1$. Set $\tilde{x}_i(t) = x_i(t) - \bar{a}$, $\tilde{v}_i(t) = v_i(t) - \bar{b}$ for $i \in \mathcal{G}_2$. Together (22) with (23), we have

$$
\begin{cases} \dot{\tilde{x}}_i(t) = \tilde{v}_i(t), \\ \dot{\tilde{v}}_i(t) = \tilde{u}_i(t), \end{cases} \tag{24}
$$

where $\tilde{u}_i(t) = \begin{cases} sat(u_i(t)) - \frac{1}{n}\sum_{j=1}^{n} sat(u_j(t)), i \in \mathcal{G}_1, \\ sat(u_i(t)) - \frac{1}{m}\sum_{j=n+1}^{n+m} sat(u_j(t)), i \in \mathcal{G}_2. \end{cases}$

Assumption 3 Suppose that the topology $\bar{\mathcal{G}}$ is strongly connected and detailed balanced. Then, there exist positive constants $w_i > 0$, such that $WH = H^T W$, where $W = diag\{w_1, \ldots, w_n, w_{n+1}, \ldots, w_{n+m}\}$, H is the same as that in Theorem 1.

Lemma 2 ([21, 27]) *If $V(t)$ satisfies the following conditions:*

(i) *$V(t)$ is lower bounded;*
(ii) *$\dot{V}(t)$ is negative semi-definite;*
(iii) *$\dot{V}(t)$ is uniformly continuous in time or $\ddot{V}(t)$ is bounded, then $\dot{V}(t) \to 0$ as $t \to \infty$.*

Theorem 2 *Suppose the directed topology $\bar{\mathcal{G}}$ is connected and detailed balance, Assumptions 1–3 holds. Under the consensus protocol (23), system (22) can achieve group consensus.*

Proof Set $\tilde{v}(t) = [\tilde{v}_1^T(t) \quad \cdots \quad \tilde{v}_{n+m}^T(t)]^T$ and $\tilde{x}(t) = [\tilde{x}_1^T(t) \quad \cdots \quad \tilde{x}_{n+m}^T(t)]^T$. Choose the Lyapunov function candidate in the following

$$
\begin{aligned}
V(\tilde{x}(t),\tilde{v}(t)) &= \frac{1}{2}\tilde{v}^T(t)WH\tilde{v}(t) + \sum_{i=1}^{n} w_i\Phi \\
&\times \left(-\sum_{j=1}^{n} a_{ij}(\tilde{x}_j(s)+\tilde{v}_j(s)) - \sum_{j=n+1}^{n+m} a_{ij}\tilde{x}_j(s) - \sum_{j=n+1}^{n+m} a_{ij}\tilde{v}_j(s)\right) \\
&+ \sum_{i=n+1}^{n+m} w_i\Phi\left(-\sum_{j=n+1}^{n+m} a_{ij}(\tilde{x}_j(s)+\tilde{v}_j(s)) - \sum_{j=1}^{n} a_{ij}\tilde{x}_j(s) - \sum_{j=1}^{n} a_{ij}\tilde{v}_j(s)\right),
\end{aligned} \tag{25}
$$

where $\Phi(s) = \int_0^s sat(t)dt$. From Assumption 3, we know that WH is a positive semi-definite matrix. Then, $V(t) \geq 0$ is obtained in (25). Set $\alpha = [\mathbf{1}_n^T \frac{1}{n}\sum_{j=1}^{n} sat(u_j(t)) \quad \mathbf{1}_m^T \frac{1}{m}\sum_{j=n+1}^{n+m} sat(u_j(t))]^T$. In view of $H\alpha = 0$, it implies

$$
\begin{aligned}
\tilde{v}^T(t)H\dot{\tilde{v}}(t) &= 2\tilde{v}^T(t)H\tilde{u}(t) \\
&= 2\tilde{v}^T(t)Hsat(u(t)),
\end{aligned} \tag{26}
$$

where $\tilde{u}(t) = [\tilde{u}_1^T(t) \quad \cdots \quad \tilde{u}_{n+m}^T(t)]^T$, $u(t) = [u_1(t) \quad \cdots \quad u_{n+m}^T(t)]^T$. Take the derivation of $V(t)$ in (25), which implies

$$
\begin{aligned}
\dot{V}(t) &= \sum_{i=1}^{n} \tilde{v}_i^T(t)\left(\sum_{j=1}^{n} w_i a_{ij} sat(u_j(t)) + \sum_{j=n+1}^{n+m} w_i a_{ij} sat(u_j(t))\right) \\
&\quad + \sum_{i=n+1}^{n+m} \tilde{v}_i(t)\sum_{j=n+1}^{n+m} w_i a_{ij} sat(u_j(t)) \\
&\quad + \sum_{i=1}^{n} w_i sat(u_i(t))\left[-\sum_{j=1}^{n} a_{ij}\tilde{v}_j(t) - \sum_{j=1}^{n} a_{ij} sat(u_j(t))\right. \\
&\quad \left. -\sum_{j=n+1}^{n+m} a_{ij}\tilde{v}_j(t) - \sum_{j=n+1}^{n+m} a_{ij} sat(u_j(t))\right] \\
&\quad + \sum_{i=n+1}^{n+m} w_i sat(u_i(t))\left[-\sum_{j=n+1}^{n+m} a_{ij}\tilde{v}_j(t)\right. \\
&\quad \left. -\sum_{j=n+1}^{n+m} a_{ij} sat(u_j(t)) - \sum_{j=1}^{n} a_{ij}\tilde{v}_j(t) - \sum_{j=1}^{n} a_{ij} sat(u_j(t))\right] \\
&\quad + \sum_{i=n+1}^{n+m} \tilde{v}_i(t)\sum_{j=1}^{n} w_i a_{ij} sat(u_j(t)).
\end{aligned} \tag{27}
$$

Observing $w_i a_{ij} = a_{ji} w_j$,

$$\dot{V}(t) = -\bar{\Psi}^T(t) W H \bar{\Psi}(t), \tag{28}$$

where $W = diag\{w_1, \ldots, w_{n+m}\}$, $\bar{\Psi}(t) = [sat(u_1(t))^T \cdots sat(u_n(t))^T \; sat(u_{n+1}(t))^T \cdots sat(u_{n+m}(t))^T]^T$. W and H is the same as that in (25). Since the matrix WH is the positive semi-definite matrix, in view of (28), we obtain

$$\dot{V}(t) \leq 0. \tag{29}$$

Noting $sat(u_i(t))$ is uniformly continuous in time, $\dot{V}(t)$ is uniformly continuous in time. By applying Lemma 2, $\lim_{t\to\infty} \dot{V}(t) = 0$ is derived. Then, we have $\lim_{t\to\infty} \tilde{x}_1(t) = \cdots = \lim_{t\to\infty} \tilde{x}_n(t)$, $\lim_{t\to\infty} \tilde{v}_1(t) = \cdots = \lim_{t\to\infty} \tilde{v}_n(t)$, $\lim_{t\to\infty} \tilde{x}_{n+1}(t) = \cdots = \lim_{t\to\infty} \tilde{x}_{n+m}(t)$, $\lim_{t\to\infty} \tilde{v}_{n+1}(t) = \cdots = \lim_{t\to\infty} \tilde{v}_{n+m}(t)$. Observing (24), we have $\lim_{t\to\infty} x_1(t) = \cdots = \lim_{t\to\infty} x_n(t)$, $\lim_{t\to\infty} v_1(t) = \cdots = \lim_{t\to\infty} v_n(t)$, $\lim_{t\to\infty} x_{n+1}(t) = \cdots = \lim_{t\to\infty} x_{n+m}(t)$, $\lim_{t\to\infty} v_{n+1}(t) = \cdots = \lim_{t\to\infty} v_{n+m}(t)$. Thus, system (22) can reach group consensus under the consensus protocol (23). The proof is completed. □

Remark 3 In [8], authors investigated single consensus for the second-order for multi-agent systems subjected to actuator saturation. In this paper, group consensus of the second-order multi-agent system is discussed in the directed topology. Furthermore, we can extend the above results to group average consensus. In order to save space, we omit it.

3.3 Adaptive consensus for the second-order multi-agent system

In this section, adaptive leader-following consensus for multi-agent systems is considered. The dynamic of the leader is denoted as follows

$$\begin{cases} \dot{x}_0(t) = v_0(t), \\ \dot{v}_0(t) = u_0(t), \end{cases} \tag{30}$$

where $x_0(t)$, $v_0(t)$, $u_0(t)$ denote the position, velocity, and input control of the leader, respectively. Suppose $u_0(t)$ is the nonlinear function, which is parameterized as follows

$$u_0(t) = \phi_0^T(t)\theta_0(t), \tag{31}$$

where $\phi_0(t)$ is the basis function, $\theta_0(t)$ is unknown parameter.

The dynamic behavior of the ith agent for the second-order multi-agent systems can be expressed as follows

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = u_i(t), \end{cases} \tag{32}$$

where $x_i(t)$, $v_i(t)$, $u_i(t)$ denote the ith agent's position, velocity, and input control, respectively.

The consensus protocol is proposed as follows

$$\begin{aligned} u_i(t) = \phi_0^T(t)\tilde{\theta}_0(t) + c(t)\Bigg[\Bigg(&\sum_{j=1}^n a_{ij}(x_j(t-\tau) - x_i(t-\tau)) \\ &- b_i(x_i(t-\tau) - x_0(t-\tau))\Bigg) \\ &+ \Bigg(\sum_{j=1}^n a_{ij}(v_j(t-\tau) - v_i(t-\tau)) \\ &- b_i(v_i(t-\tau) - v_0(t-\tau))\Bigg)\Bigg], \end{aligned} \tag{33}$$

where $\tilde{\theta}_0(t)$ is the estimation of $\theta_0(t)$ in (31). Set $\hat{x}_i(t) = x_i(t) - x_0(t)$, $\hat{v}_i(t) = v_i(t) - v_0(t)$, $\hat{\theta}_i(t) = \tilde{\theta}_i(t) - \theta_i(t)$ for $i = 1, \ldots, n$. Under the consensus protocol (33), (32) can be rewritten

$$\begin{aligned} \dot{\hat{x}}_i(t) &= \hat{v}_i(t), \\ \dot{\hat{v}}_i(t) &= \phi_0^T(t)\hat{\theta}_0(t) - c(t)\left(\sum_{j=1}^n a_{ij}\hat{x}_i(t-\tau) + b_i\hat{x}_i(t-\tau)\right) \\ &\quad - c(t)\left(\sum_{j=1}^n a_{ij}\hat{v}_j(t-\tau) + b_i\hat{v}_i(t-\tau)\right), \end{aligned} \tag{34}$$

where $\tau > 0$ is the time delay, $\hat{H} = L + \bar{B}$ with $\bar{B} = diag\{b_1, \ldots, b_n\}$, L is Laplacian matrix corresponding to the follower's topology. If there is a path from leader to the ith follower, we have $b_i = 1$; otherwise, $b_i = 0$. Set $\bar{\xi}(t) = [\hat{x}_1^T(t) \cdots \hat{x}_{n+m}^T(t) \; \hat{v}_1^T(t) \cdots \hat{v}_{n+m}^T(t)]^T$. And (34) is rewritten in the matrix form, which implies

$$\dot{\bar{\xi}}(t) = A\bar{\xi}(t) + B\bar{\xi}(t-\tau) + F(\Phi(t)\Psi(t)) \tag{35}$$

where

$$F = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I_n \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ -c(t)\hat{H} & -c(t)\hat{H} \end{bmatrix},$$

$\Psi(t) = [\hat{\theta}_1(t)^T \cdots \hat{\theta}_n(t)^T]^T$, $\Phi(t) = diag\{\phi_0(t) \cdots \phi_0(t)\}$. Adaptive laws of $\hat{\theta}_i(t)$ and $c(t)$ are designed as follows

$$\dot{\hat{\theta}}_i(t) = -2r_i\phi_0(t)(\hat{x}_i(t) + \hat{v}_i(t)), \quad i = 1, \ldots, n, \tag{36}$$

$$\dot{c}(t) = -\mu\bar{\xi}^T(t)P\tilde{B}\bar{\xi}(t-\tau), \tag{37}$$

where $r_i > 0$, $\mu > 0$ are positive constants to be determined, P is defined in (42), $\tilde{B} = \begin{bmatrix} 0 & 0 \\ -\hat{H} & -\hat{H} \end{bmatrix}$.

Assumption 4 Suppose that $\phi_0(t)$ in (31) is persistently exciting(PE), that is, there exist positive constants $T_0 > 0$ and $\epsilon > 0$,

$$\int_t^{t+T_0} \Phi(s)\Phi^T(s)ds \geq \epsilon I_n > 0, \forall t > 0. \tag{38}$$

Lemma 3 ([16, 22]) *Let $\dot{X}(t) = f(X, t)$, where $X(t) \in R^n$. If $f(X, t) \to 0$ and $S^T(t)X(t) \to 0$, where S is upper* *bounded and satisfies the persistently exciting(PE) property in* (38), *then* $X(t) \to 0$.

Theorem 3 *Suppose that there exists at least a path form the leader to one follower in undirected and connected topology, Assumption 4 holds and* $\phi_0(t)$ *is upper bounded. Under the consensus protocol* (33), *systems* (30) *and* (32) *can reach consensus, if the following conditions hold for positive numbers* $\rho > 0, \gamma > 0$,

$$\Omega = \begin{bmatrix} \Xi + \frac{1}{\rho} P\hat{B}\hat{B}^T P + \gamma Q + \rho\tau I_n & 0 \\ 0 & -\gamma Q \end{bmatrix} < 0, \tag{39}$$

where $\hat{B}$ *is the same as that in* (47), *the matrix* P *is the same as that in* (41),

$$\Xi = P(A + \hat{B}) + (A + \hat{B})^T P = \begin{bmatrix} -\hat{c}\hat{H} - \hat{H}\hat{c} & 0 \\ 0 & 2(I_n - \hat{c}H) \end{bmatrix}. \tag{40}$$

Proof We take the Lyapunov function

$$V(t) = V_1(t) + V_2(t) + V_3(t), \tag{41}$$

where

$$V_1(t) = \bar{\xi}^T(t)P\bar{\xi}(t), \tag{42}$$

$$V_2(t) = \gamma \int_{t-\tau}^{t} \bar{\xi}^T(s)Q\bar{\xi}(s)ds, \tag{43}$$

$$V_3(t) = \rho \int_{t-\tau}^{t} (s - t + \tau)|\dot{\bar{\xi}}(s)|^2 ds, \tag{44}$$

$$V_4(t) = \frac{(c(t) - \hat{c})^2}{\mu} + \frac{\hat{\theta}^T(t)\hat{\theta}(t)}{2r_i}, \tag{45}$$

where $\rho > 0$ and $\gamma > 0$ are constant numbers, $\hat{c} \geq \frac{2}{\lambda_{min}(\hat{H})}$, $P = \begin{bmatrix} 2\hat{c}\hat{H} & I_n \\ I_n & I_n \end{bmatrix}$. From above, we know P is a positive definite matrix. Taking the derivation of $V_1(t)$, we have

$$\begin{aligned}\dot{V}_1(t) = {} & 2\bar{\xi}^T(t)PA\bar{\xi}(t) + 2\bar{\xi}^T(t)PB\bar{\xi}(t-\tau) \\ & + 2\bar{\xi}^T(t)PF\Phi(t)\Psi(t)\end{aligned} \tag{46}$$

Noting the condition (36) and (37), we have

$$\begin{aligned}\dot{V}_4(t) = {} & -2\bar{\xi}^T(t)PB\bar{\xi}(t-\tau) - 2\bar{\xi}^T(t)PF\Phi(t)\Psi(t) \\ & + 2\bar{\xi}^T(t)P\hat{B}\bar{\xi}(t-\tau),\end{aligned} \tag{47}$$

where $\hat{B} = \begin{bmatrix} 0 & 0 \\ -\hat{c}\hat{H} & -\hat{c}\hat{H} \end{bmatrix}$. Therefore,

$$\dot{V}_1(t) = 2\bar{\xi}^T(t)[P(A + \hat{B})]\bar{\xi}(t) - 2\bar{\xi}^T(t)P\hat{B}(\bar{\xi}(t) - \bar{\xi}(t-\tau)). \tag{48}$$

Thus, (46) can be rewritten as follows

$$\begin{aligned}\dot{V}_1(t) &= 2\bar{\xi}^T(t)[P(A + \hat{B})]\bar{\xi}(t) - 2\bar{\xi}^T(t)P\hat{B}(\bar{\xi}(t) - \bar{\xi}(t-\tau)) \\ &= 2\bar{\xi}^T(t)[P(A + \hat{B})]\bar{\xi}(t) - 2\bar{\xi}^T(t)P\hat{B}\int_{t-\tau}^{t} \dot{\bar{\xi}}(s)ds \\ &\leq 2\bar{\xi}^T(t)[P(A + \hat{B})]\bar{\xi}(t) + \rho^{-1}\bar{\xi}^T(t)P\hat{B}\hat{B}^T P\bar{\xi}(t) \\ &\quad + \rho\int_{t-\tau}^{t} |\dot{\bar{\xi}}(s)|^2 ds.\end{aligned} \tag{49}$$

Taking the derivation of $V_2(t)$ and $V_3(t)$, we have

$$\dot{V}_2(t) = \gamma\bar{\xi}^T(t)Q\bar{\xi}(t) - \gamma\bar{\xi}^T(t-\tau)Q\bar{\xi}(t-\tau), \tag{50}$$

$$\dot{V}_3(t) = \rho\tau|\bar{\xi}(t)|^2 - \rho\int_{t-\tau}^{t} |\dot{\bar{\xi}}(s)|^2 ds. \tag{51}$$

Combining (41)-(51), we have

$$\begin{aligned}\dot{V}(t) \leq {} & \bar{\xi}^T(t)[P(A + \hat{B}) + (A + \hat{B})^T P]\bar{\xi}(t) + \rho^{-1}\bar{\xi}^T(t) \\ & P\hat{B}\hat{B}^T P\bar{\xi}(t) + \gamma\bar{\xi}^T(t)Q\bar{\xi}(t) - \gamma\bar{\xi}^T(t-\tau)Q\bar{\xi}(t-\tau) \\ & + \rho\tau|\bar{\xi}(t)|^2.\end{aligned} \tag{52}$$

In view of the condition (39), we have

$$\dot{V}(t) < 0. \tag{53}$$

And we know that $\lim_{t\to\infty} \bar{\xi}(t) = 0$ and $\lim_{t\to\infty} \bar{\xi}(t-\tau) = 0$. Then, $\lim_{t\to\infty}(x_i(t) - x_0(t)) = 0$ and $\lim_{t\to\infty}(v_i(t) - v_0(t)) = 0$ are obtained. From (35), we obtain $\Phi(t)\Psi(t) \to 0$. Noting (36), we know that $\dot{\hat{\theta}}_i(t) \to 0$. From the Lemma 3, we have that $\lim_{t\to\infty} \hat{\theta}_i(t) = 0$. Thus, the proof is completed. □

Remark 4 By linearly parameterized models, authors have studied adaptive consensus protocols for the second-order multi-agent systems without time delay in [20]. Moreover, adaptive consensus problems of the second-order multi-agent systems without time delay were considered in [16] and [17]. In this paper, we discuss adaptive consensus for the second-order multi-agent systems with time delay under the nonlinear protocols.

4 Simulation examples

Example 1 Consider the first-order multi-agent system in Theorem 1 has seven agents. The agents $\{1, 2, 3\}$ are included in the first group. The agents $\{4, 5, 6, 7\}$ are included in the second group. Assume that the matrix H in (4) is that

$$H = \begin{bmatrix} 1 & 0 & -1 & -0.5 & 0.5 & 0 & 0 \\ -1 & 1 & 0 & 0.5 & -0.5 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ -0.5 & 0 & 0.5 & 1 & -1 & 0 & 0 \\ 0.5 & 0 & -0.5 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}. \tag{54}$$

According to Theorem 1, we know that under the consensus protocol (3), system (1) can reach group consensus. Figure 1 shows the trajectories of system $\dot{\xi}(t) = -Hsat(\xi(t))$. Figure 2 shows the curves of $x(t)$.

Example 2 Assume that there are seven agents in the directed topology $\bar{\mathcal{G}}$ in Theorem 2. A set of the agents $\{1, 2, 3\}$ is in the first group $\mathcal{G}_1$. And the agents $\{4, 5, 6, 7\}$ are in the second group $\mathcal{G}_2$. The adjacent weighted elements are $a_{13} = 1$, $a_{23} = 2$, $a_{31} = 2$, $a_{32} = 1$, $a_{14} = -1$, $a_{15} = 1$, $a_{24} = 0.5$, $a_{25} = -0.5$, $a_{41} = -0.5$, $a_{42} = 0.5$, $a_{51} = 1$, $a_{52} = -1$, $a_{45} = 1$, $a_{54} = 1$, $a_{56} = 1$, $a_{65} = 2$, $a_{67} = 1$, $a_{76} = 1$. The directed topology $\bar{\mathcal{G}}$ is detail balance

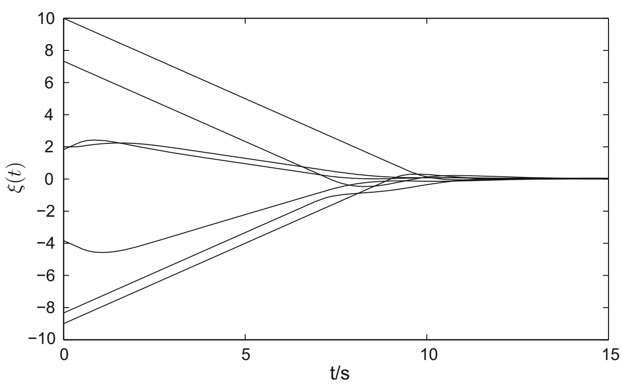

Fig. 1 Curves of $\xi(t)$ in system $\dot{\xi}(t) = -Hsat(\xi(t))$ in Theorem 1

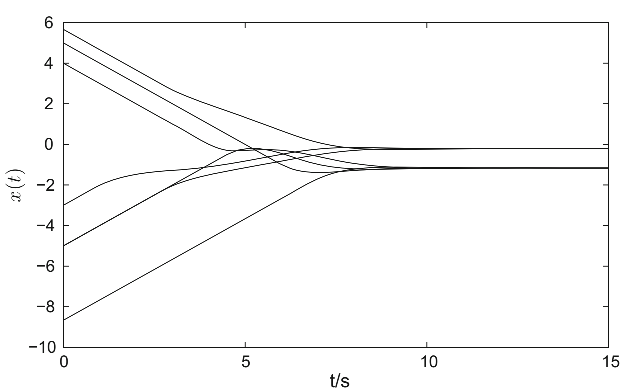

Fig. 2 Curves of $x(t)$ in Theorem 1

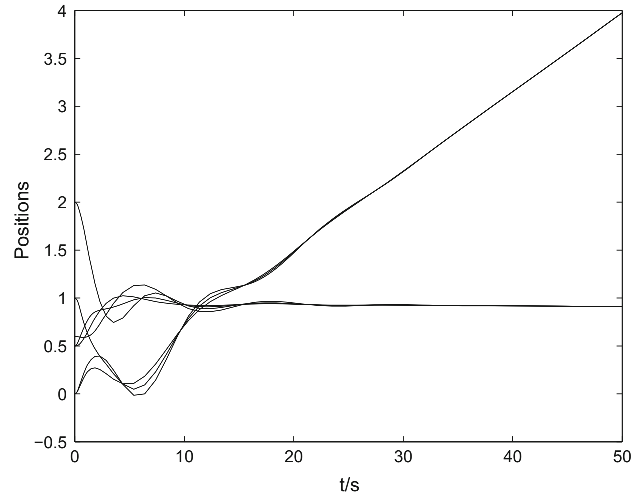

Fig. 3 Position trajectories of multi-agent systems in Theorem 2

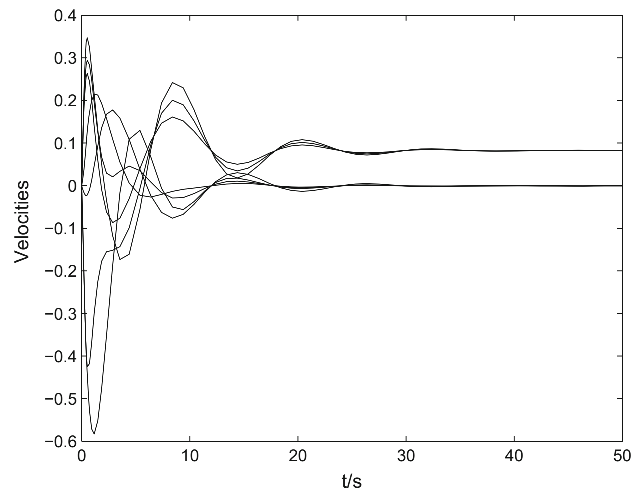

Fig. 4 Velocity trajectories of multi-agent systems in Theorem 2

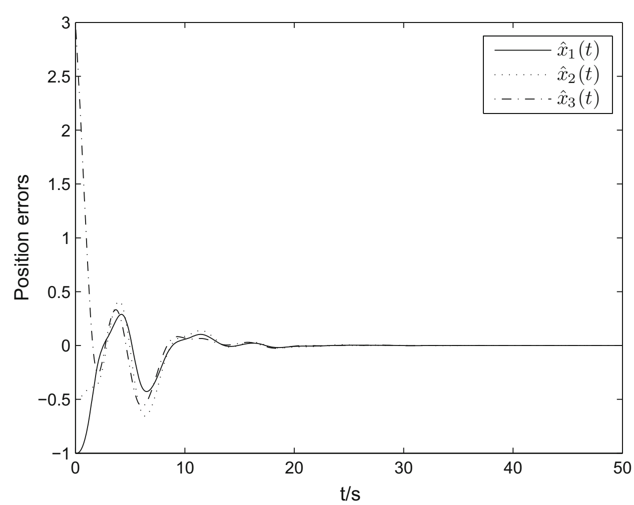

Fig. 5 Errors of positions between the leader and followers in Theorem 3

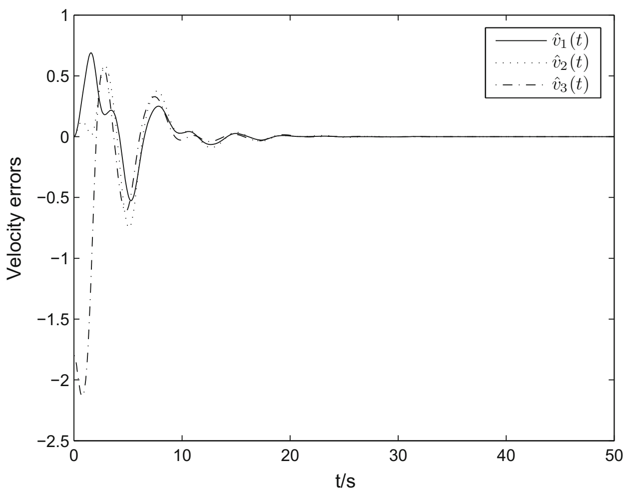

Fig. 6 Errors of velocities between the leader and followers in Theorem 3

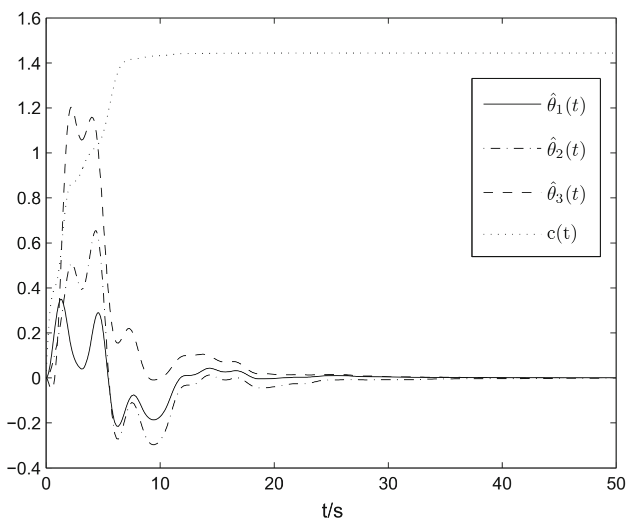

Fig. 7 Errors of the estimation $\hat{\theta}_i(t)$ for $i = 1, 2, 3$ and adaptive parameter $c(t)$ in Theorem 3

and Assumption 3 holds. According to Theorem 2, group consensus is derived. Then, Figs. 3 and 4 denote the curves of the positions and velocities in (22), respectively.

Example 3 Consider there are three followers and one leader in the multi-agent systems. Assume that the Laplacian matrix L associated with the followers' topology and the matrix $\bar{B}$ are

$$L = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{55}$$

Take the time delay $\tau = 0.2$, $\phi_0(t) = \sin t$, which is satisfies the conditions of Theorem 3. Under the consensus protocol (33), systems (30) and (32) can achieve adaptive leader-following consensus. Then, Fig. 5 is the picture of the position errors between the leader and the followers. Figure 6 denotes the curves of the velocity errors between the leader and the followers. Figure 7 shows the estimation errors $\theta_i(t)$ for $i = 1, 2, 3$ and adaptive parameter $c(t)$.

5 Conclusions

In this paper, group consensus algorithms subjected to actuator saturation have been given for the continuous-time multi-agent systems. Furthermore, adaptive consensus nonlinear algorithms are proposed. Based on the graph theory, LaSalle's invariance principle, and Lyapunov function, consensus conditions for multi-agent systems are derived. Finally, three simulation examples are provided to illustrate the obtained results in this paper. One direction of our future work is to study group consensus under stochastic topologies and noises.

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