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ORIGINAL ARTICLE

Bifurcation study of neuron firing activity of the modified Hindmarsh–Rose model

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Abstract In this paper, the effects of different parameters on the dynamic behavior of the nonlinear dynamical system are investigated based on modified Hindmarsh–Rose neural nonlinear dynamical system model. We have calculated and analyzed dynamic characteristics of the model under different parameters by using single parameter bifurcation diagram, time response diagram and two parameter bifurcation diagram. The results show that the period-adding bifurcation (with or without chaos), period-doubling bifurcation and intermittent chaos phenomenon (periodic and intermittent chaotic) can be observed more clearly and directly from the two parameter bifurcation diagram, and the optimal parameters matching interval can also be found easily.

Keywords Neural model · Modified Hindmarsh–Rose Model · Bifurcation · Two-parameter dynamic characteristics

1 Introduction

The basic structural unit of biological nerve system is neuron, the activities of which are mainly manifested in the formation, change and transmission of bio-electricity signals. Biological nervous system is a complex multi-level information neural network, which is formed through the connections of large amount of nerve cells. Therefore, in the process of neuron firing and information coding, complex nonlinear dynamical behaviors are involved. Since neuron-dynamics is an inter-discipline between bio-physiology and nonlinear dynamics, it has biology- and dynamics-significance to study neuron firing [1]. With the continuous theoretical development in bio-physiology and nonlinear dynamics, various biological neuron models are built based on the data obtained from different experiments. The application of nonlinear dynamics theories and methods in the numerical calculation and analysis of these neuron models makes it possible to observe some phenomena hardly discovered in neuron-physiological experiments. It promotes the rapid development of medical science and neurology by providing guidance for practical physiological experiments, as well as theoretical basis for studies in medical science.

The neuron mathematical models mainly include: based on the equivalent circuit theory, British biologist Hodgkin and Huxley conducted the experiment of squid giant axon stimulation and recorded the resting potential as well as the action potential of the cells in 1950s. After rigorous analysis of these experimental data recorded, in 1952, they built the dynamic equation of HH model, which described the electric potential activity of cell membrane accurately [2–4]. The establishment of this model makes it possible to study the generation mechanism of neuron firing activity through a mathematical model, which has guiding significance to understand the generation mechanism of neuron action potential in early times. Moreover, the HH model is the prototype of all subsequent excitable cell models, and provides a realistic basis for the description of the electrophysiological property of all excitable cells. FitHugh and Nagumo constructed the FHN model by simplifying the HH model and adding a recovery variable to ensure the slow-time variability of action potential. This

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two-dimension model accurately presents the action potential generation for further analyses in future [5]. Based on the HH model established by Hodgkin and Huxley, biologists has put forward many characterized neuron models or simplified ones, such as, the Morris–Lecar (ML) [6], Chay model [7], Hindmarsh–Rose (HR) model [8], the Plant model and the pre-Botzinger model [9], etc. Through theoretical justification or numerical simulation, many experts and scholars have studied the firing patterns and the dynamics properties of these models [10–16].

In these researches, many complex firing patterns, such as the periodic or chaotic behavior of bursting or spiking, have been discovered [17–23]. Analyses of the HR models show that both external impulse current and different parameters will influence the dynamic behavior of the model [24–27]. In the ML model and Chay model, value changes of single parameter in the system, such as the reversal voltage of the system, the time scale factor which controls the slow variable, and conductance, will have different influence on firing patterns of the system [28, 29]. The analysis of inter-spike interval reveals that period-doubling bifurcation and period-adding bifurcation are common bifurcation structure types in neuron models. But most existing studies about these models are about influences of single-parameter variations on the dynamic behaviors of the system. As for studies of two-parameter bifurcation, different bifurcations are obtained by varying the bifurcation parameter with the condition parameter fixed correspondingly on two different levels [30]. Unfortunately, this is not an intuitive method that can reflect the changes that occur with different parameters varying within a certain range at the same time. In this paper, the modified HR model is studied. As the parameter varies within certain range, the bifurcation structure and dynamic features of the system are analyzed from aspects of the inter-spike bifurcation diagram, time-response diagram, etc. And the two-dimension colored diagram of bifurcation with two parameters varying within a certain range at the same time, is employed to represent the dynamic features and the rich dynamic phenomena.

2 Modified HR model

The neuron model is based on the numerical expression constructed by Hindmarsh and Rose from the snail neuron cells data derived from the voltage clamp experiments. For its simple numerical expression, this model is regarded by many scholars as the idealistic one in the study of actual neuron firing. The expression of this model is:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= y - ax^3 + bx^2 - z\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= c - \mathrm{d}x^2 - y\\ \frac{\mathrm{d}z}{\mathrm{d}t} &= r[s(x - X - z)]\end{aligned} \tag{1}$$

x represents membrane potential of the cell; y represents internal current-related recovery variable; z represents slow varying regulating current; all other variables are parameters of the system.

Based on the HR model, a similar neuron model is analyzed in this paper. Since its structure is completely modeled on the HR model, it is collectively known as modified HR model. Its expression is [31]:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= -s\left(-ax^3 + x^2\right) - y - bz\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \phi(x^2 - y)\\ \frac{\mathrm{d}z}{\mathrm{d}t} &= \varepsilon(sa_1x_1 + b_1 - kz)\end{aligned} \tag{2}$$

a, b, a_1, b_1, k, s are parameters of the system. Their corresponding values are: $a = 0.5$, $b = 1$, $a_1 = -0.1$, $k = 0.2$. There are detailed descriptions about them in [31].

3 Dynamic characteristics of the modified HR model with single parameter as variable

3.1 Dynamic characteristics with parameter s as variable

With $\varepsilon = 0.02$, $b_1 = -0.045$ unchanged, and the parameter s as a variable, Fig. 1 is the corresponding inter-spike bifurcation diagram. And Fig. 2 is the corresponding

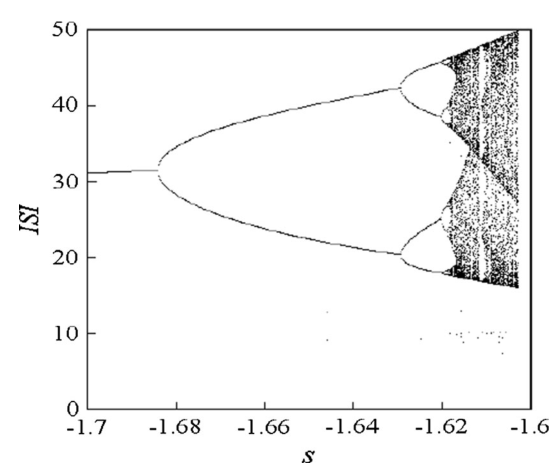

Fig. 1 Inter-spike interval bifurcation diagram corresponding to s

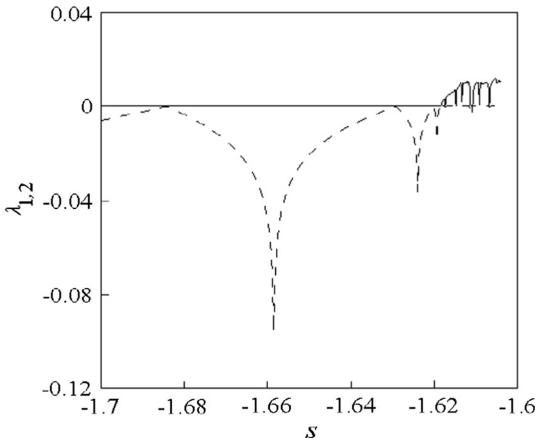

Fig. 2 Lyapunov exponent diagram corresponding to s

Lyapunov exponent diagram. When s value varies within $[-1.7, -1.6]$, it can be seen clearly from Fig. 1 that: from simple single period activity, the system enters into the ultimate chaotic state after undergoing period-doubling bifurcation process, in which it goes through patterns of the period-2, period-4, period-8, etc. Figure 2 (the Lyapunov exponent diagram) indicates that: when the s value is -1.684, the system enters into period-2 after period-doubling bifurcation; and when s value is -1.629, it switches from period-2 to period-4 after bifurcation. As the s value increases, the system goes through period-8 and period-16; when the maximum of Lyapunov exponent corresponding to s value that varies within $[-1.62, -1.6]$ is above zero, it enters into the chaotic state. Therefore, period-doubling bifurcation is a means by which a non-linear system transits from periodic to chaotic stage.

3.2 Dynamic characteristics with parameter ε as variable

In addition to the typical period doubling bifurcation, the period-adding bifurcation will also occur in the neuron model. In period-adding bifurcation, the firing period of the system increases by one each time. With other parameters unchanged, and the parameter ε taken as a variable, the inter-spike interval bifurcation diagram is shown in Fig. 3. With the decrease of ε value, the system transforms from period-1 firing pattern into the period-2 spiking; and when ε value is 0.02, the system enters into the chaotic bursting. With the further decrease of ε value, the chaotic bursting pattern degenerates into period-3 spiking which will enter into chaotic firing interval after period-doubling bifurcation. Then, the chaotic firing pattern degenerates into period-4 spiking. The cycle of periodic-chaotic-periodic firing pattern keeps repeating. After each chaotic firing pattern, compared with that of its previous pattern, the

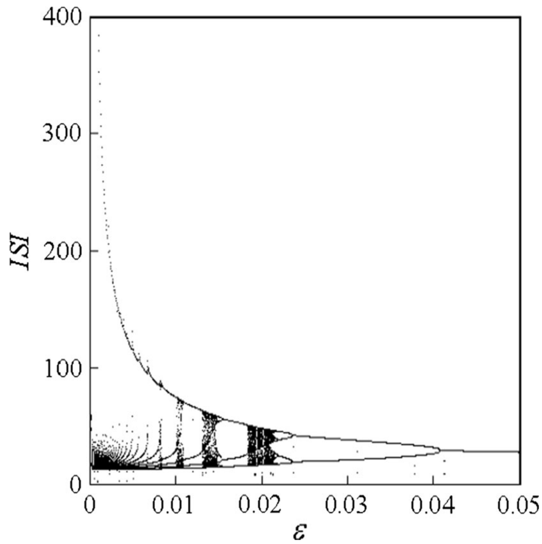

Fig. 3 Inter-spike interval bifurcation diagram corresponding to ε

period number will increase by one. This period adding firing activity with regular chaos is a common phenomenon in the excitable neuron model.

3.3 Dynamic characteristics with parameter b_1 as variable

With other parameter unchanged, and variable b_1 value varying within $[-0.1, -0.02]$, the corresponding inter-spike interval bifurcation diagram is shown in Fig. 4. It can be seen that firing pattern of the system is period-1 pattern with relatively small b_1 values. When b_1 increases to the point of -0.54, there will be bifurcation in the system. From period-1 firing pattern, undergoing period-doubling

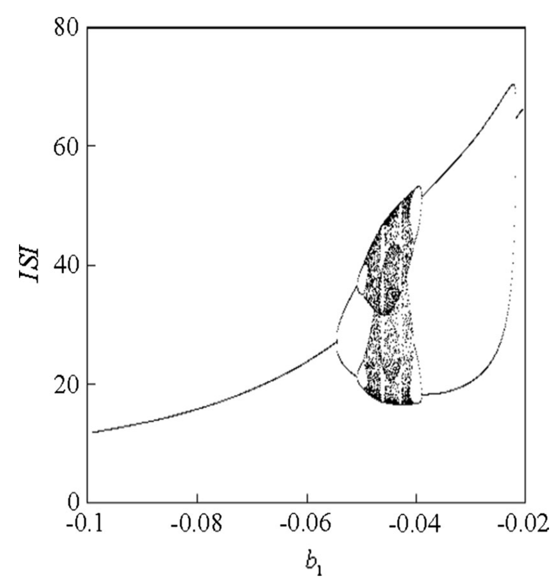

Fig. 4 Inter-spike interval bifurcation diagram corresponding to b_1

bifurcation, the system goes through period-2 pattern, period-4 pattern, etc, and enters into chaotic firing state, where there is interior crisis in it. As b_1 value increases to a certain point, there will be inverse period-doubling bifurcation in the system, and from period-4 pattern, the chaotic firing pattern switches back to the period-1 spiking pattern.

4 The effect of two-parameter variation on dynamic behavior of the system

Infinitesimal disturbance of one or more variables values will alter the effect of these parameters on dynamic behavior of the system. In the neuron model experiment, it is hard to keep one parameter as variable and the values of all others unchanged. It is more common that the values of several parameters vary at the same time and within a certain range. Therefore, it is practically meaningful to study the influence of several changing parameters on the dynamic features of the neuron system.

In this section, there are two parameters varying within a certain range simultaneously, and the corresponding dynamic behavior of the system is analyzed.

4.1 Parameter ε and b_1 as variables

First of all, in the system where ε and b_1 are taken as variables, with ε value varying within [0, 0.04], and b_1 value within [−0.06, −0.02], calculate and draw simulation bifurcation diagrams of the model correspondingly, as is shown in Fig. 5; different colors represent different cycles in the figures; the right numbers in the figure represent discharge cycles; greater than or equal to 16 big cycle discharge activity or chaos phenomenon is labeled with sixteen colors. When b_1 value varies within [−0.06, −0.055], it can be seen from a longitudinal viewpoint that part of the diagram is colored red completely, which indicates that the system is in a periodic spiking state,and small changes of time scale value does not affect the whole bifurcation structure of system. When b_1 value varies within [−0.055, −0.02], the firing pattern becomes complicated. And burst firing of period-2, period-3, period-4, period-5, period-6, period-8, period-12, period-16 pattern, etc., can be observed in the system.

From Fig. 5, it can be observed easily and clearly that period-doubling bifurcation occurs in the system. In the diagram,from top to bottom, it can be seen intuitively that period-1 is followed by period-2; and period-2 is followed by period-4; period-4 is followed by period-8; period-8 is followed by period-16; and period-3 is followed by period-6; period-6 is followed by period-12. Many such period-doubling can be observed if there are sufficient refinements in the calculation, these period-doubling bifurcations can

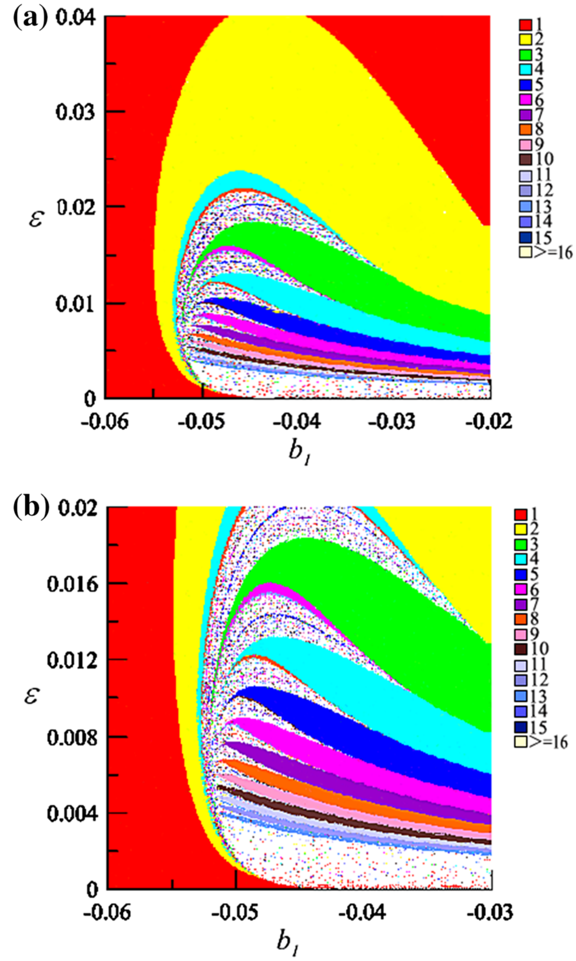

Fig. 5 Bifurcation diagram corresponding to ε and b_1. **a** Bifurcation diagram of $\varepsilon \in [0,\ 0.04]$ and $b_1 \in [-0.06,\ -0.02]$. **b** Bifurcation diagram of $\varepsilon \in [0, 0.02]$ and $b_1 \in [-0.06, -0.03]$ (color figure online)

be illustrated with different colored period numbers. Period-doubling is one path of neuronal discharge model leading to chaos; we can clearly observe in Fig. 5 that the bursting process of system from period-doubling gradually enters into chaotic discharge.

Moreover, Fig. 5 also shows clearly that there will be periodic firing after intermittent chaos in the system. With b_1 value varying within(−0.055, −0.03),and ε value within [0.01, 0.025], from period-2 burst firing pattern, undergoing period-doubling bifurcation, the system goes through period-4, period-8, period-16, and then enters into intermittent chaotic firing interval, during which there are complex dynamic phenomena, as well as the occurrences of periodic windows in the system. Along with the decrease of ε value, the chaotic firing pattern of the system degenerates into period-3 burst firing pattern; then through bifurcation, the system enters intermittent chaotic firing which latter degenerates into period-4 burst firing pattern. With the above process being repeated, the system enters into a new periodic firing stage. It also illustrates that period window region of chaotic firing changes with the parameter: with the decrease of ε, the chaotic firing period window also decreases, and then ultimately disappears.

From Fig. 5, it is easily observed the process of periodic discharge with chaos and periodic discharge without chaos. In Fig. 5, period-4, period-5, period-6, period-7, period-8 and period-9 these gradual growing periodic discharge increase their numbers of periods is through period-adding pattern in a certain parameter range. For example, when $b_1 \in [-0.03, -0.02]$, the system will present on the period-adding discharge without chaos phenomenon. And it can also be observed in the figure, with increasing in the number of periods, the periodic range gradually decreases, which is shown in the picture that as the parameters change the color band gradually narrows. From Fig. 5, it is not difficult to see that when the parameters b_1 change in the $[-0.055, -0.03]$ interval range, with the parameter ε changes, the system experiences period-adding discharge with intermittent chaotic activities. In the process with chaotic period-adding bifurcation, the chaotic window gradually decreases as the number of period increases. And when the number of periods reaches a certain value, the chaotic window disappears and the system changes into period-adding process without chaos. From Fig. 5b it can be more clearly observed that the system comes into chaotic discharges through period-doubling, and then the saddle-node bifurcation will end the chaotic discharge and enter a new period discharge process. From the Fig. 5a, b, we may observe that the small perturbations of a single parameter will not affect system discharge mode, and only when two parameters simultaneously changes, the system's discharge state will change. This fully shows that compared to the single parameter, the two-parameter dynamic characteristics of neuron discharge model has more guiding significance to the reality.

4.2 Parameter ε and s as variables

Secondly, in the system where the parameter ε and the parameter s are taken as variables, calculate and draw simulation bifurcation diagrams of the model correspondingly, as is shown in Fig. 6. From Fig. 6a, it can be seen that the system presents abundant dynamic phenomena with ε value varying within [0, 0.04] and s value varying within $[-1.7, -1.5]$.

Figure 6b is the bifurcation diagram of the system with parameter ε value varying within [0, 0.02] and parameter s value varying within $[-1.7, -1.6]$. Figure 6c is bifurcation diagram of the system with parameter ε value varying within [0, 0.01] and parameter s value varying within $[-1.7, -1.6]$. From Fig. 6b, it can be observed easily and clearly that period-doubling bifurcation occurs in the system. In the diagram, from the upper left to lower right, it can be seen intuitively that period-1 is followed by period-2; and period-2 is followed by period-4; period-4 is followed by period-8; period-8 is followed by period-16; and period-3 is followed by period-6; period-6 is followed by period-12. Many such period-doubling can be observed if there are sufficient refinements in the calculation, these period-doubling bifurcations can be illustrated with different colored period numbers in the Fig. 6b. Moreover, Fig. 6b also shows clearly that there will be periodic firing after intermittent chaos in the system. From top to bottom to see Fig. 6b, undergoing period-doubling bifurcation, the system goes through period-4, period-8, period-16, and then enters into intermittent chaotic firing interval from period-2 burst firing pattern, during which there are complex dynamic phenomena, as well as the occurrences of periodic windows in the system. Along with the decrease of ε value, the chaotic firing pattern of the system degenerates into period-3 burst firing pattern; then through bifurcation, the system enters intermittent chaotic firing which latter degenerates into period-4 burst firing pattern. With the above process being repeated, the system enters into a new periodic firing stage. It also illustrates that period window region of chaotic firing changes with the parameter: with the decrease of the parameter ε, the chaotic firing period window also decreases, and then ultimately disappears. Figure 6c is the lower part of the Fig. 6b, it can be seen this phenomenon more clearly from Fig. 6c.

Figure 6d is bifurcation diagram of the system with parameter ε value varying within [0.01, 0.04], and parameter s value varying within $[-1.7, -1.6]$. Figure 6e is bifurcation diagram of the system with parameter ε value varying within [0, 0.02] and parameter s value varying within $[-1.62, -1.58]$.

Figure 6d indicates that the system will present various complicated rhythmic firing pattern. When ε is above 0.026, the firing of the system takes on period-1 and period-2 pattern, and minor change of the time scale will not affect the bifurcation structure of the whole system. With the decrease of ε value, the bifurcation can be observed clearly. The adjacent two colors represent different period numbers, for example: period-1 is adjacent to period-2, which is adjacent to period-4; period-3 is adjacent to period-6; period-4 is adjacent to period-8, etc. Refined calculation can lead to the observation of many such period-doubling bifurcations. For the firing pattern of neuron system, one of the routes to chaos is via period-doubling bifurcation.

Figure 6d shows clearly the process, in which the system enters into chaotic burst firing via period-doubling bifurcation. From careful observation, we know that chaotic firing is not actually disordered, but accompanied with small periodical windows. The chaotic firing interval

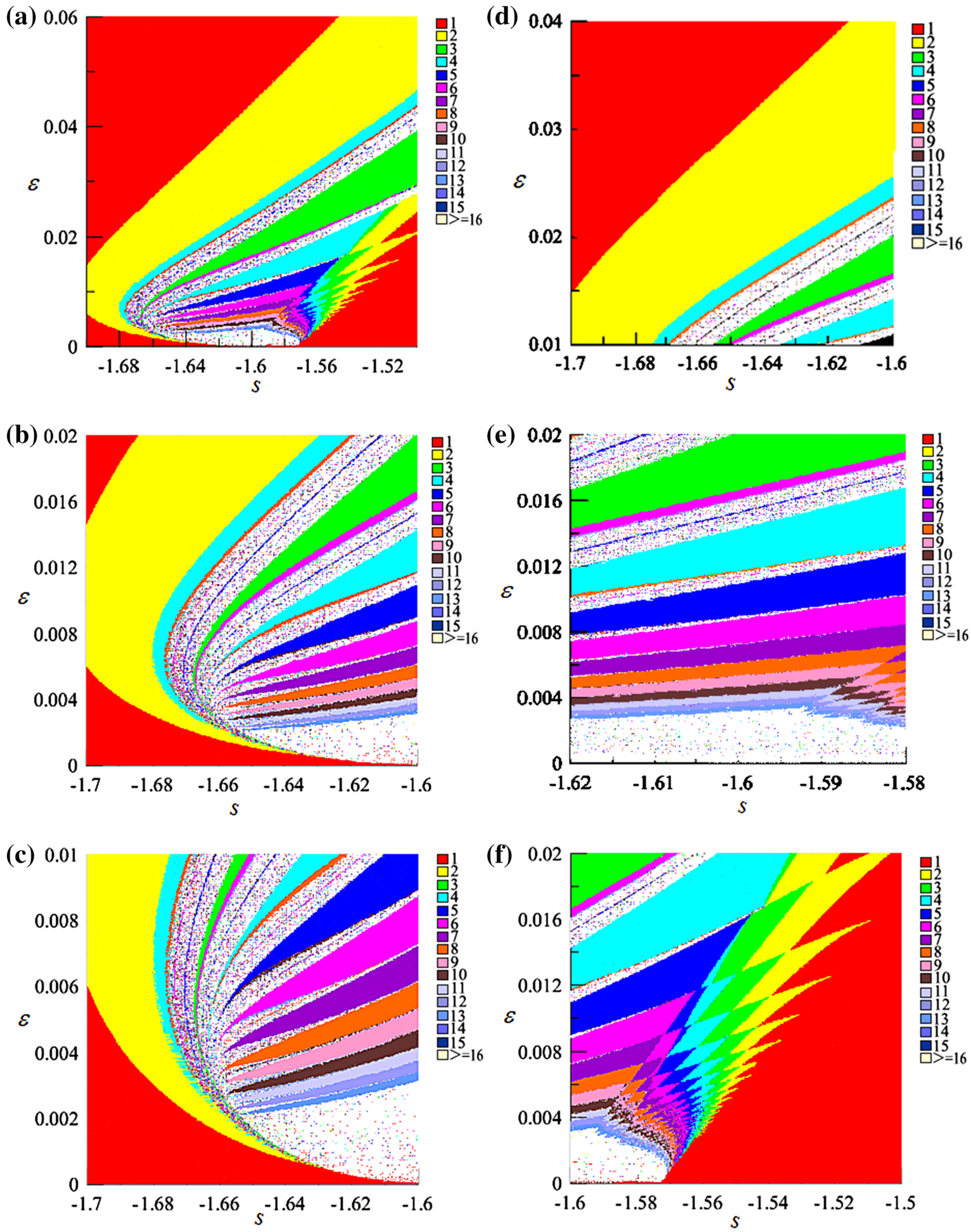

Fig. 6 Bifurcation diagram corresponding to ε and s. **a** Bifurcation diagram of $\varepsilon \in [0, 0.06]$ and $s \in [-1.7, -1.5]$. **b** Bifurcation diagram of $\varepsilon \in [0.01, 0.02]$ and $s \in [-1.7, -1.6]$. **c** Bifurcation diagram of $\varepsilon \in [0, 0.01]$ and $s \in [-1.7, -1.6]$. **d** Bifurcation diagram of $\varepsilon \in [0.01, 0.04]$ and $s \in [-1.7, -1.6]$. **e** Bifurcation diagram of $\varepsilon \in [0, 0.02]$ and $s \in [-1.62, -1.58]$. **f** Bifurcation diagram of $\varepsilon \in [0, 0.02]$ and $s \in [-1.6, -1.5]$ (color figure online)

can be seen clearly, when there are two parameters varying at the same time in the system. For example, as soon as saddle node bifurcation quickly ends, the system switches into a periodic burst firing pattern with a new period number. Along with the value changes of the parameters with a certain range, the system repeats the periodic firing to chaotic burst firing via period-doubling bifurcation. Then via saddle node bifurcation, the chaotic burst firing pattern shifts into a new periodic one. It can be more clearly seen the process from Fig. 6e.

Figure 6d, e also show that: the firing activity of the system switches from period-2 pattern to period-4 pattern via period-doubling bifurcation; after undergoing another period-doubling bifurcation, period-4 pattern shifts into period-8 pattern, and then the system takes on the intermittent chaotic firing activity. As the parameter value changes, the intermittent chaotic firing activity shifts into period-3 firing pattern via saddle node bifurcation. Through period-doubling bifurcation, period-3 pattern is replaced by period-6, after which the system enters into chaotic firing. Then the system will repeat the same chaotic firing patterns, after which it degenerate into new periodical firing pattern with the new period number increasing by one each time. This is the period-adding bifurcation with chaos. With the increase of period number, the chaotic firing periodic window decreases between two periodic windows, and then ultimately disappears.

From Fig. 6f, it is easily observed the process of periodic discharge with chaos and periodic discharge without chaos. From the upper left of Fig. 6f it can be seen that the system experiences period-adding discharge with intermittent chaotic activities with the parameter ε changes. In the process with chaotic period-adding bifurcation, the chaotic window gradually decreases as the number of period increases. And when the number of periods reaches a certain value, the chaotic window disappears and the system changes into period-adding process without chaos. From Fig. 6e it can be more clearly observed that the system comes into chaotic discharges through period-doubling, and then the saddle-node bifurcation will end the chaotic discharge and enter a new period discharge process.

In the lower right part of Fig. 6f, from right to left, it can seen that period-1 is adjacent to period-2, which is adjacent to period-3; period-3 is adjacent to period-4; period-4 is adjacent to period-5, etc, these gradual growing periodic discharge increase their numbers of periods is through period-adding pattern in a certain parameter range. And it can also be observed in the figure, with increasing in the number of periods, the periodic range gradually decreases, which is shown in the figure that as the parameters change the color band gradually narrows.

4.3 Parameter b_1 and s as variables

In the system where the parameter b_1 and the parameter s are taken as variables, calculate and draw simulation bifurcation diagrams of the model correspondingly, as is shown in Fig. 7. From Fig. 7a, it can be seen that the system presents abundant dynamic phenomena with b_1 value varying within $[-0.06, -0.02]$ and s value varying within $[-1.8, -1.5]$.

Figure 7b is the bifurcation diagram of the system with parameter b_1 value varying within $[-0.06, -0.02]$ and parameter s value varying within $[-1.7, -1.6]$. Figure 7b shows that: When ε is under -1.63, the firing pattern of the system is in period-1 and period-2 pattern, and minor change of parameter ε will not affect the bifurcation structure of the whole system. With the increase of s value, the period-doubling bifurcation can be observed clearly. The adjacent two colors represent different period numbers, for example: period-1 is adjacent to period-2 which is adjacent to period-4; and period-4 is adjacent to period-8, etc. Refined calculation can lead to the observation of many such period-doubling bifurcations. Moreover, it also indicates that: with the increase of the period number, the period range decreases, which is reflected in the figure as the narrowing down of colored region with the parameter variation.

Figure 7c is bifurcation diagram of the system with parameter b_1 value varying within $[-0.06, -0.04]$ and parameter s value varying within $[-1.6, -1.5]$. Figure 7d is bifurcation diagram of the system with parameter b_1 value varying within $[-0.05, -0.04]$ and parameter s value varying within $[-1.62, -1.58]$. Figure 7c shows that: the firing activity of the system switches from period-2 pattern to period-4 pattern via period-doubling bifurcation; after undergoing another period-doubling bifurcation, period-4 pattern shifts into period-8 pattern, and then the system takes on the intermittent chaotic firing activity. As the parameter value changes, the intermittent chaotic firing activity shifts into period-3 firing pattern via saddle node bifurcation. Through period-doubling bifurcation, period-3 pattern is replaced by period-6, after which the system enters into chaotic firing. Then the system will repeat the same chaotic firing patterns, after which it degenerate into new periodical firing pattern with the new period number increasing by one each time. This is the period-adding bifurcation with chaos. With the increase of period number, the chaotic firing periodic window decreases between two periodic windows, it can be clearly seen this process from Fig. 7d.

Therefore, compared with the condition where there is only one parameter as variable, the effect of parameter minor change on the system is more obvious, when two parameters taken as variables in the system. From Fig. 5, we know that the firing pattern of the system will not alter with minor changes of a single parameter. Generally, the following three aspects can be observed from bifurcation diagram of two variables as control parameters:

(1) When two parameters are taken as control parameters, the bifurcation diagram is an integrated diagram of single parameter bifurcation diagram, that is, the cross or vertical section of the diagram is the bifurcation diagram with one parameter being unchanged and the other as control parameter in the system;

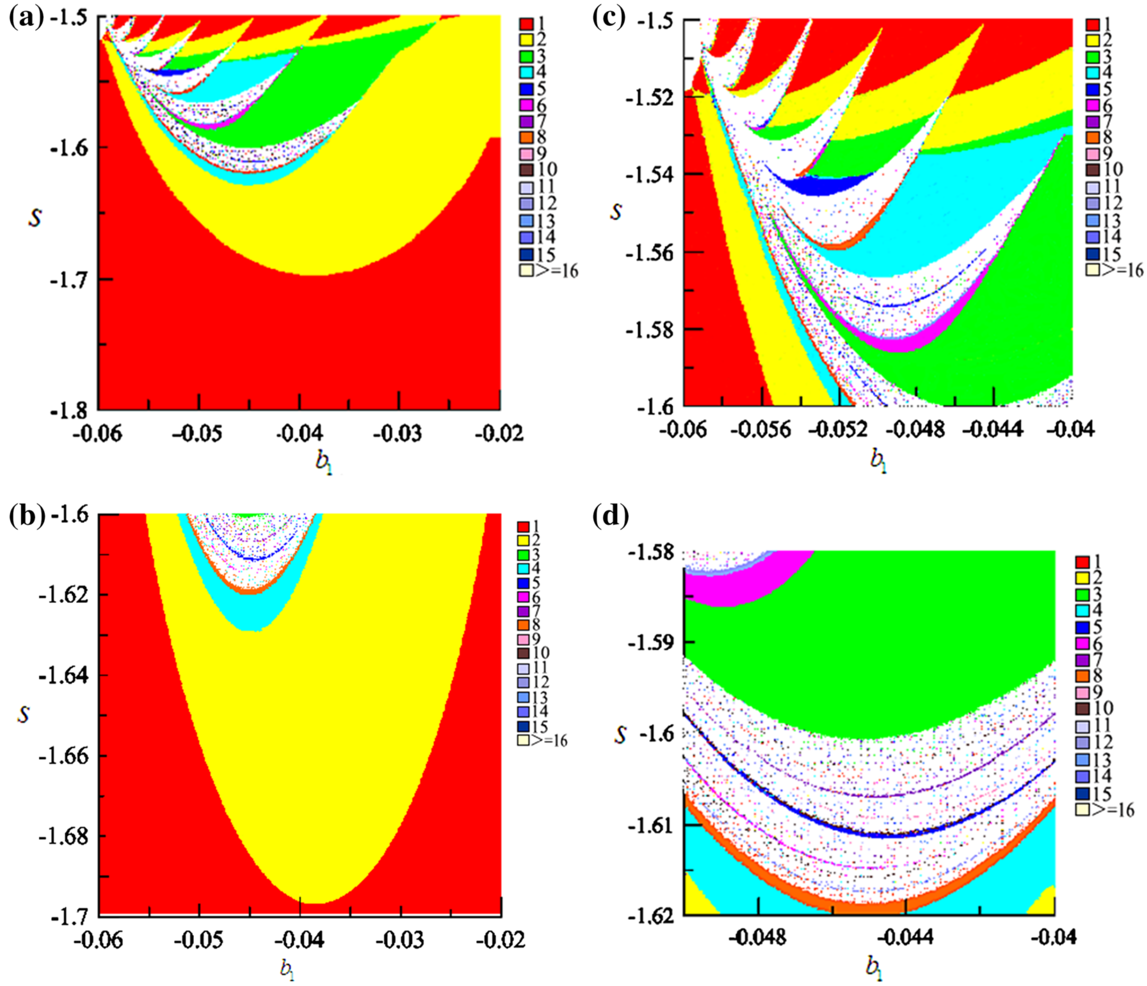

Fig. 7 Bifurcation diagram corresponding to s and b_1. **a** Bifurcation diagram of $s \in [-1.8, -1.5]$ and $b_1 \in [-0.06, -0.02]$. **b** Bifurcation diagram of $s \in [-1.7, -1.6]$ and $b_1 \in [-0.06, -0.02]$. **c** Bifurcation diagram of $s \in [-1.6, -1.5]$ and $b_1 \in [-0.06, -0.04]$. **d** Bifurcation diagram of $s \in [-1.62, -1.58]$ and $b_1 \in [-0.05, -0.04]$ (color figure online)

(2) When two parameters are taken as control parameters, period-adding bifurcation (with or without chaos), period-doubling bifurcation and intermittent chaos (periodic and intermittent chaotic) can be observed intuitively in the corresponding bifurcation diagram;

(3) In the bifurcation diagram, where two variables are control parameter, it is easy to determine the period number and the occurrence time point of the burst firing in the system, as well as the corresponding interval within which the parameter values vary.

5 Conclusions

Based on the modified HR model, the neuron model is calculated and simulated with C language programming and the Grapher, and then the effect of different parameters on the dynamic behavior of the modified HR neuron system are examine through analyzing the single parameter as well as two parameter bifurcation diagram.

The results show that: with other parameter unchanged, when s value increases, the system is on a stable periodic firing activity; and as s value decreases, the system enters into chaos via period-doubling bifurcation and there are interior crises during the chaotic interval. While other parameters remain unchanged, with the increase of ε value, the system undergoes inverse period-adding bifurcation with chaos occurring in this process. With other parameter unchanged, and b_1 value varying within $[-0.1, -0.02]$, via period-doubling bifurcation, the period-1 firing activity of the system switches into chaotic firing with interior crisis, and then through inverse period-doubling bifurcation, ultimately returns to the period-1 spiking pattern.

While other parameters remain unchanged, with the increase of ε value, the system undergoes inverse period-adding bifurcation with chaos occurring in this process. When ε value is under 0.02, the neuron system is in a chaotic firing state. With the increase of the parameter value, the chaotic firing state of the system switches into the steady period-1 spiking state.

With other parameter unchanged, and b_1 value varying within $[-0.1, -0.02]$, via period-doubling bifurcation, the period-1 firing activity of the system switches into chaotic firing with interior crisis, and then via inverse period-doubling bifurcation, period-4 pattern returns to the period-1 spiking pattern.

When two parameters are taken as variables, period-adding bifurcation (with or without chaos), period-doubling bifurcation and intermittent chaos (periodic and intermittent chaotic) can be observed intuitively in the corresponding bifurcation diagram. In the bifurcation diagram, where two variables are control parameters, it is easy to determine the period number and the occurrence time point of the burst firing in the system, as well as the corresponding interval within which the parameter values vary. Thus it has guiding significance for researchers in this field.

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Conflict of interest The authors declare that they have no conflict of interest.

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