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ORIGINAL ARTICLE

Difference kernel iterative method for linear and nonlinear partial differential equations

Yasir Khan[1] · Naeem Faraz[2] · Zdenek Smarda[3]

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Abstract The purpose of this paper was to propose a new method to solve partial differential equations arising in the field of science and engineering. In this new method, we have reduced the multiple integrals into a single integral and expressed it in terms of a difference kernel. To make the calculation easy and convenient, we have used the Laplace transformation to solve the difference kernel. The method is very simple, easy to understand and calculation minimizing as compared to the Adomian decomposition method and the variational iteration method. Some examples are given to verify the reliability and efficiency of the method.

Keywords Difference kernel · Laplace transformation · Convolution theorem

1 Introduction

The concept of iterative methods has gained considerable popularity and importance, due to its demonstrated applications in numerous seemingly diverse and widespread fields of engineering and applied sciences. There are various iterative techniques to solve nonlinear differential equations arising in applied sciences and engineering, and for example, iterative methods have successfully applied to problems in physics [1–5], biology [6, 7], chemistry [8, 9], electrical engineering [10–12], civil engineering [13, 14] and mechanical engineering [15–18]. However, all these techniques have their limitations in applications. Inspired and motivated by the ongoing research in this area, we suggest a novel difference kernel iterative method (DKIM) for linear and nonlinear differential equations in this paper. The method does not require that the nonlinearities be differentiable with respect to the dependent variable and its derivatives. This method makes use of the difference kernel in the iteration formula. The method has linked up variational iteration method [19] and Adomian decomposition method [20] through a new iterative scheme called difference kernel iterative method. The strength of new iterative scheme is: there is no need to integrate the differential equation again and again as we do in Adomian decomposition method. Benefits of new iterative scheme over the variational iteration method are: It avoids the unnecessary calculations and no need to calculate Lagrange multiplier and construction of correctional functional. In this new method, we have introduced the difference kernel instead of Lagrange multiplier which is very easy to calculate. Through this difference kernel, we solve the integral by means of Laplace transform [20–26], which makes calculation very simple and easy to understand. The method is very simple, easy to understand, decreases the calculations as compared to so-called Adomian decomposition method and variational iteration method. Several examples are given to verify the reliability and efficiency of the method. To the best of our knowledge, difference kernel iteration method for solving differential equations is presented for the first time in the literature.

✉ Yasir Khan
yasirmath@yahoo.com

1 Department of Mathematics, University of Hafr Al-Batin, Hafr Al-Batin 31991, Saudi Arabia

2 Department of Mathematics, Shanghai University, 200444 Shanghai, China

3 Department of Mathematics, Faculty of Electrical Engineering and Communication, Brno University of Technology, Technicka 8, 616 00 Brno, Czech Republic

2 Method description

In order to elucidate the solution procedure, we consider the following nth order partial differential equation:

$$L^n f(x,t) = Rf(x,t) + Nf(x,t) + g(x,t), \qquad t > 0,\ x \in R, \tag{1}$$

where $L^n = \frac{\partial^n}{\partial t^n}$, $n \geq 1$, R is linear differential operator, N is the nonlinear differential operator, R and N are free of partial derivative with respect to variable t, g is the source term. As we are familiar with the fact that in all kinds of iterative techniques, except operator rest of terms are treated as a known function on the behalf of initial guess. In this present newly proposed idea, we have used same concept. We have bound all terms in one function except operator:

$$g + Nf + Rf = F\left(t, x, g, f, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}, \ldots\right). \tag{2}$$

Incorporating Eq. (2) in Eq. (1), we get

$$L^n f = F\left(t, x, g, f, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}, \ldots\right), \tag{3}$$

On integrating Eq. (3) with respect to t, one can obtain

$$L^{(n-1)} f = \int_0^t F\left(\xi, x, g, f, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}, \ldots\right) \mathrm{d}\xi + c_1(x), \tag{4}$$

again integrate Eq. (4)

$$L^{(n-2)} f = \int_0^t \int_0^\xi F\left(\tau, x, g, f, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}, \ldots\right) \times \mathrm{d}\tau \mathrm{d}\xi + c_1(x)t + c_2(x), \tag{5}$$

now we will convert this double integral in Eq. (5) into single integral by doing integration by parts:

$$\begin{aligned} L^{(n-2)} f &= \left[\xi \int_0^\xi F\left(\tau, x, g, f, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}, \ldots\right) \mathrm{d}\tau \Big|_0^t - \int_0^t \xi F\left(\xi, x, g, f, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}, \ldots\right) \mathrm{d}\xi \right] + c_1(x)t + c_2(x) \\ &= t \int_0^t F\left(\tau, x, g, f, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}, \ldots\right) \mathrm{d}\tau - \int_0^t \xi F\left(\xi, x, g, f, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}, \ldots\right) \mathrm{d}\xi + c_1(x)t + c_2(x) \\ &= \int_0^t (t-\xi) F\left(\xi, x, g, f, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}, \ldots\right) \mathrm{d}\xi + c_1(x)t + c_2(x), \end{aligned} \tag{6}$$

where we replaced τ by ξ since they are only dummy variables of integration. If we continue this process of integration, we can easy get final form as follows:

$$f(x,t) = \int_0^t \frac{(t-\xi)^{n-1}}{(n-1)!} F\left(t, x, g, f, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}, \ldots\right) \mathrm{d}\xi + \frac{c_1(x)t^{n-1}}{(n-1)!} + \frac{c_2(x)t^{n-2}}{(n-2)!} + \cdots_n (x). \tag{7}$$

By writing the constant of integration in the form $c_k(x) = \frac{\partial f^{n-k}(x,0^+)}{\partial t^{n-k}}$, $k = 1, \ldots, n$ and substituting Eq. (2) in Eq. (7), we obtain

$$f(x,t) = \sum_{k=0}^{n-1} \frac{\partial^k f(x,0^+)}{\partial t^k} \frac{t^k}{k!} + \int_0^t \frac{(t-\xi)^{n-1}}{(n-1)!} (Rf + Nf + g) \mathrm{d}\xi. \tag{8}$$

Equation (8) can be written in the following form:

$$f(x,t) = \sum_{k=0}^{n-1} \frac{\partial^k f(x,0^+)}{\partial t^k} \frac{t^k}{k!} + \int_0^t \frac{(t-\xi)^{n-1}}{(n-1)!} g(x,\xi) \mathrm{d}\xi + \int_0^t \frac{(t-\xi)^{n-1}}{(n-1)!} (Rf + Nf) \mathrm{d}\xi. \tag{9}$$

We define iteration form of Eq. (9) as follows:

$$f_p(x,t) = f_0(x,t) + \int_0^t \frac{(t-\xi)^{n-1}}{(n-1)!} (Rf_{p-1} + Nf_{p-1}) \mathrm{d}\xi, \quad p \geq 1, \tag{10}$$

where

$$f_0(x,t) = \sum_{k=0}^{n-1} \frac{\partial^k f(x,0^+)}{\partial t^k} \frac{t^k}{k!} + \int_0^t \frac{(t-\xi)^{n-1}}{(n-1)!} g(x,\xi) \mathrm{d}\xi$$

and $\frac{(t-\xi)^{n-1}}{(n-1)!}$ is difference kernel which we will denote by $K(t-\xi)$. The more refined form of Eq. (10) is

$$f_p(x,t) = f_0(x,t) + \int_0^t K(t-\xi)(Rf_{p-1} + Nf_{p-1}) \mathrm{d}\xi \tag{11}$$

The most important Laplace transform pair used for solving integral equations with difference kernel is

$$L\left\{ \int_0^t K(t-\xi) f_p(t,\xi) \mathrm{d}\xi \right\} = L\{K(t) * f_p(t)\} = K(s) f_p(s), \tag{12}$$

where $K(s) = L\{K(t)\}$ and $f_p(s) = L\{f_p(t,\xi)\}$. The integral in Eq. (12) is called the convolution product of the two functions. It is apparent now that the convolution product has the same integration limits (0, t) as that of Eq. (11). By using the result illustrated in Eq. (12) in Eq. (11), we get resulting equation for iteration method in the form

$$f_p(s) = f_0(s) + K(s)H_{p-1}(s) = f_0(s) + M_{p-1}(s), \quad p \geq 1, \tag{13}$$

where $H_{p-1}(s)$ is the Laplace transform of $(Rf_{p-1} + Nf_{p-1})$ and $M_{p-1}(s) = K(s)H_{p-1}(s)$. Inverse Laplace will give us solution

$$f_p(x,t) = f_0(x,t) + M_{p-1}(x,t). \tag{14}$$

Then, we get required solution

$$f(x,t) = \lim_{p\to\infty} f_p(x,t). \tag{15}$$

3 Application of method

Example 1 Consider the partial differential equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = 4\frac{\partial^2 u(x,t)}{\partial x^2}, \tag{16}$$

subject to the initial conditions

$$u(x,0) = \sin \pi x, \quad u_t(x,0) = 0. \tag{17}$$

The exact solution is

$$u(x,t) = \sin \pi x \cos 2\pi t. \tag{18}$$

To solve the Eq. (16), we follow the formulation, given in Sect. 2.

$$u_p(x,t) = u_0(x,t) + \int_0^t K(t-\xi)\left(4\frac{\partial^2 u_{p-1}}{\partial x^2}\right)\mathrm{d}\xi, \quad p \geq 1. \tag{19}$$

Hence

$$u_0 = \sum_{k=0}^{n-1} \frac{\partial^k u(x,0^+)}{\partial t^k}\frac{t^k}{k!} = \sin \pi x, \quad K(t-\xi) = (t-\xi), \tag{20}$$

$$u_p(x,t) = u_0(x,t) + \int_0^t (t-\xi)\left(4\frac{\partial^2 u_{p-1}}{\partial x^2}\right)\mathrm{d}\xi, \quad p \geq 1. \tag{21}$$

For $p = 1$, we have

$$\begin{aligned} u_1(x,t) &= \sin \pi x + \int_0^t (t-\xi)\left(-4\pi^2 \sin \pi x\right)\mathrm{d}\xi \\ &= \sin \pi x - 4\pi^2 \sin \pi x \int_0^t (t-\xi)\,\mathrm{d}\xi. \end{aligned} \tag{22}$$

$$u_1(x,s) = \sin \pi x\frac{1}{s} - 4\pi^2 \sin \pi x\frac{1}{s^3}.$$

Thus

$$u_1(x,t) = \sin \pi x - \frac{4\pi^2 t^2 \sin \pi x}{2!} = \sin \pi x - \frac{(2\pi)^2 t^2 \sin \pi x}{2!}.$$

In the same way, rest of iterations can be calculated as follows:

$$\begin{aligned} u_2(x,t) &= \sin\pi x - \frac{(2\pi)^2}{2!}t^2 \sin\pi x + \frac{(2\pi)^4}{4!}t^4 \sin\pi x \\ &= \sin\pi x\left(1 - \frac{(2\pi)^2}{2!}t^2 + \frac{(2\pi)^4}{4!}t^4\right), u_3(x,t) \\ &= \sin\pi x\left(1 - \frac{(2\pi)^2}{2!}t^2 + \frac{(2\pi)^4}{4!}t^4 - \frac{(2\pi)^6}{6!}t^6\right), u_4(x,t) \\ &= \sin\pi x\left(1 - \frac{(2\pi)^2}{2!}t^2 + \frac{(2\pi)^4}{4!}t^4 - \frac{(2\pi)^6}{6!}t^6 + \frac{(2\pi)^8}{8!}t^8\right), \\ &\vdots \\ u_k(x,t) &= \sin\pi x \sum_{i=0}^{k} (-1)^{2i}\frac{(2\pi)^{2i}}{(2i)!}t^{2i}, \\ &\vdots \end{aligned} \tag{23}$$

Then, we obtain the required solution in the form

$$\begin{aligned} u(x,t) &= \lim_{k\to\infty} u_k(x,t) = \sin \pi x \sum_{i=0}^{\infty} (-1)^{2i}\frac{(2\pi)^{2i}}{(2i)!}t^{2i} \\ &= \sin \pi x \cos 2\pi t. \end{aligned} \tag{24}$$

Example 2 Consider the partial differential equation with the initial conditions

$$\begin{aligned} &\frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t) = 0, \\ &u(x,0) = 1 + \sin x,\ u_t(x,0) = 0. \end{aligned} \tag{25}$$

By applying the aforesaid method, we have

$$u_p(x,t) = u_0(x,t) - \int_0^t K(t-\xi)\left(\frac{\partial^2 u_{p-1}}{\partial x^2} + u_{p-1}\right)\mathrm{d}\xi$$

with $n = 2$ difference kernel become $K(t-\xi) = (t-\xi)$, and from initial conditions, we get

$$u_0(x,t) = \sum_{k=0}^{n-1} \frac{\partial^k u(x,0^+)}{\partial t^k}\frac{t^k}{k!} = 1 + \sin x \tag{26}$$

For $p = 1$, we get

$$\begin{aligned} u_1(x,t) &= 1 + \sin x - \int_0^t (t-\xi)\left(\frac{\partial^2 u_0}{\partial x^2} + u_0\right)\mathrm{d}\xi \\ &= 1 + \sin x - \int_0^t (t-\xi)\mathrm{d}\xi. \end{aligned} \tag{27}$$

By taking Laplace transform of Eq. (27), we get

$$u_1(x,s) = \frac{1}{s} + \frac{1}{s}\sin x - \frac{1}{s^3}. \quad (28)$$

Inverse Laplace will give us first iteration, i.e.,

$$u_1(x,t) = 1 + \sin x - \frac{t^2}{2!}. \quad (29)$$

In the same way, rest of iterations have the form

$$\begin{aligned} u_2(x,t) &= 1 + \sin x - \frac{t^2}{2!} + \frac{t^4}{4!}, \\ u_3(x,t) &= 1 + \sin x - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!}, \\ &\vdots \\ u_k(x,t) &= \sin x + \sum_{i=0}^{k} (-1)^i \frac{t^{2i}}{2i!}, \\ &\vdots \end{aligned} \quad (30)$$

Therefore, the solution is given by

$$\begin{aligned} u(x,t) &= \lim_{k\to\infty} u_k(x,t) = \lim_{k\to\infty}\left(\sin x + \sum_{i=0}^{k} (-1)^i \frac{t^{2i}}{2i!}\right) \\ &= \sin x + \cos t. \end{aligned} \quad (31)$$

Example 3 Consider the homogeneous nonlinear gas dynamic equation

$$\frac{\partial u(x,t)}{\partial t} + \frac{1}{2}\frac{\partial (u^2(x,t))}{\partial x} - u(x,t)(1 - u(x,t)) = 0, \quad (32)$$

subject to the initial condition

$$u(x,0) = e^{-x}. \quad (33)$$

From the initial condition, we have

$$u_0 = \sum_{k=0}^{n-1} \frac{\partial^k u(x,0^+)}{\partial t^k}\frac{t^k}{k!} = e^{-x}. \quad (34)$$

Then, iteration formula has the form

$$\begin{aligned} u_p(x,t) &= u_0(x,t) + \int_0^t K(t-\xi) \\ &\times \left(-\frac{1}{2}\frac{\partial (u_{p-1}^2(x,t))}{\partial x} + u_{p-1}(x,t)(1 - u_{p-1}(x,t))\right) \\ &\times d\xi, \quad p \geq 1. \end{aligned} \quad (35)$$

For $p = 1$, we obtain

$$u_1(x,t) = e^{-x} + \int_0^t \left(-\frac{1}{2}(-2e^{-2x}) + e^{-x}(1 - e^{-x})\right) d\xi. \quad (36)$$

The same way as above, we get

$$\begin{aligned} u_1(x,t) &= e^{-x}(1+t), \\ u_2(x,t) &= e^{-x}\left(1 + t + \frac{t^2}{2!}\right), \\ u_3(x,t) &= e^{-x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}\right), \\ &\vdots \\ u_k(x,t) &= e^{-x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^k}{k!}\right) = e^{-x}\sum_{i=0}^{k}\frac{t^i}{i!}, \\ &\vdots \end{aligned} \quad (37)$$

Hence, the solution of Eq. (32) has the form

$$u(x,t) = \lim_{k\to\infty} e^{-x}\sum_{i=0}^{k}\frac{t^i}{i!} = e^{t-x}. \quad (38)$$

4 Conclusion

In this paper, we study the insightful idea of Adomian decomposition method and variational iteration method. We find some unnecessary calculations and a lengthy process of integration. To cope with this difficulty, we proposed another method called difference kernel iterative method, which gives equivalent results, the results obtained by ADM. The following observations have been made:

- DKIM eliminates the need to do repeated integrations as in Adomian decomposition method, and one can get the same results of Adomian method;
- Through this method, there is no need to calculate Lagrange multiplier of He's variational iteration method. We introduce difference kernel instead of Lagrange multiplier;
- This method avoids the unnecessary calculations in He's variational iteration method;
- By using the convolution property, as we have difference kernel in integral, we can avoid unnecessary calculation;
- We can easily convert Lagrange multiplier of He's VIM into difference kernel and then apply DKIM to avoid the unnecessary calculations appearing in VIM-I and VIM-II [27].

So in final conclusion, we can say that the present method is more useful, easy to understand and more effective as compared to ADM and VIM.

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