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ORIGINAL ARTICLE

The multi-attribute group decision-making method based on the interval grey uncertain linguistic generalized hybrid averaging operator

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Abstract With respect to the multi-attribute group decision-making (MAGDM) problems with the interval grey uncertain linguistic variables (IGULVs), the MAGDM method based on the interval grey uncertain linguistic generalized hybrid averaging (IGULGHA) operator is proposed. Firstly, the operation rules, the properties and the comparing method of the IGUVs are introduced. Then, some aggregation operators such as interval grey uncertain linguistic generalized ordered weighted aggregation (IGULGOWA) operator and interval grey uncertain linguistic generalized hybrid aggregation (IGULGHA) operator are defined, and some properties such as idempotency, commutativity, monotonicity and boundedness are proved, and then, the decision-making methods based on these operators are presented to solve the group decision-making problems. Finally, a numerical example is used to show the proposed method, and the influence of the different position weight vector w and the different parameter λ on decision-making is analysed. The result shows that the method is simple and effective.

Keywords Multi-attribute group decision-making · Fuzzy set · Grey fuzzy number · Interval grey uncertain linguistic variables · Aggregation operator

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1 Introduction

Multi-attribute decision-makings have broad applications in society, economics, military and engineering technology. As the complexity and uncertainty of decision problems and decision environment, most of the multi-attribute decision-making (MADM) problems are uncertain and fuzzy, so fuzziness is an important factor to be considered in actual decision-making. In addition, in dealing with the problems with poor information, the decision problems have also shown the characteristics of grey. Therefore, the actual decision-making problems are often fuzzy and grey, which is called the grey fuzzy multiple attribute decision-making (GFMADM) problems.

The researches on GFMADM problems have got rich achievements. First, the ranking methods of grey fuzzy number were studied [3, 7, 8, 15, 16, 25, 31, 47]. For the GFMADM problems in which the grey part and the fuzzy part took the form of real numbers, Bu and Zhang [3] converted the grey number to the interval number and then used the ranking method of interval numbers to select the best alternative; Jin and Lou [15] proposed a fuzzy compromised decision-making method by calculating the differences between the alternatives and the fuzzy positive ideal solution/the negative ideal solution based on Hamming distance; Jin and Lou [16] ranked the alternatives by the distance between each alternative and the ideal solution. Luo and Liu [25] utilized the maximum entropy method to determine attribute weight for the GFMADM problems with unknown weights and selected the best alternative based on the linear combination of grey part and fuzzy part. Further, Zhu et al. [49] proposed the evaluation model by converting an interval grey fuzzy number to an interval number for the MADM problems in which the fuzzy part took the form of interval numbers and the grey part took the form of the real numbers. Meng et al. [27] expressed greyness and fuzziness of the GFMADM problems by interval numbers and proposed the mathematical model by converting the grey fuzzy information to the interval numbers. Wang and Wang [33] developed a decision-making method for the GFMADM problems in which the fuzzy part and the grey part were the interval numbers based on the OWA operator. Wang and Wang [34] presented a decision-making method for the GFMADM problems with incomplete information based on objective programming.

Because the linguistic variable is easier to express fuzzy information, the researches on the MADM problems based on the linguistic variables have achieved fruitful results [4, 6, 9–11, 19–24, 38, 39, 41, 42]. Based on the linguistic variable and the concept of grey fuzzy number, Jin and Liu [13] defined interval grey linguistic variable, as well as its operation rules and distance formula, and proposed an extended TOPSIS method for the MAGDM problems with the unknown attribute weight.

The aggregation operators are the important research topics, which are attracting more and more attention. The ordered weighted averaging (OWA) operator proposed by Yager is a very common aggregation method [45]. Further, Yager [46] introduced the generalized ordered weighted averaging (GOWA) operator which is an extension of the OWA operator by adding an additional parameter. In addition, it is also regarded as a generalization of the generalized mean operator and the OWA operators. Since appearance of GOWA operator, it has come into wide use [1, 2, 17, 30, 48]. However, because OWA operator only weights the position of each attribute value and cannot deal with the weighting each attribute value. In order to overcome the weaknesses, Xu and Da [44] proposed the hybrid averaging (HA) operator which combined the weighted average (WA) and the OWA operator. Merigó and Casanovas [29] proposed the generalized hybrid averaging (GHA) operator which can generalize a wide range of aggregation operators, including the HA, the hybrid geometric averaging operator, the hybrid quadratic averaging operator and so on. Merigó and Casanovas [28] proposed the fuzzy generalized hybrid averaging (FGHA) operator and applied it to the MADM problems with the fuzzy information. Wang and Wu [35] defined interval grey uncertain linguistic, as well as its operation principles and distance formula, and proposed an interval grey uncertain linguistic ordered weighted C-OWA (IGULOWC-OWA) operator and discussed its characteristics. For the MADM problems in which the attribute weights are completely known and the attribute values of alternatives take the form of interval grey uncertain linguistic, a multicriteria decision method based on the IGULOWC-OWA operator is proposed. In the method, the criteria values of alternatives are converted into uncertain linguistics. Then, the collective values of alternatives can be obtained by using the aggregation operator. Jin and Liu [14] proposed interval grey linguistic variables weighted harmonic aggregation operators and then presented the MAGDM method based on these operators. Ma et al. [26] proposed some induced correlated aggregation operators for the interval grey uncertain linguistic variables and then developed some methods for the MAGDM problems.

Based on the interval grey uncertain linguistic variable, this paper proposed some new aggregation operators, such as interval grey uncertain linguistic variable generalized ordered weighted aggregation (IGULGOWA) operator and interval grey uncertain linguistic variable generalized hybrid aggregation (IGULGHA) operator, and presented the some new decision-making methods based on these aggregation operators. In order to do so, this paper is organized as follows. In Sect. 2, we briefly review some concepts of grey fuzzy set and uncertain linguistic variable, as well as traditional generalized hybrid averaging (GHA) operator. In Sect. 3, we present the concept, the operation rules, the properties, and the comparing method of interval grey uncertain linguistic variables and then present some aggregation operators, such as interval grey uncertain linguistic variable generalized ordered weighted aggregation (IGULGOWA) operator and interval grey uncertain linguistic variable generalized hybrid aggregation (IGULGHA) operator, and study some properties of these operators. In Sect. 4, we give the decision-making methods based on the IGULGHA operator. In Sect. 5, an illustrative example is pointed out. In Sect. 6, we summarize the main conclusions found in the paper.

2 Preliminaries

2.1 Grey fuzzy math [5, 18, 32, 36]

Definition 1 Suppose $\tilde{A}$ is the fuzzy subset in the space $X = \{x\}$, if the membership degree $\mu_A(x)$ of x to $\tilde{A}$ is the grey number in the interval [0, 1], and its grey degree is $v_A(x)$, then $\tilde{A}$ is called the grey fuzzy set in space X:

$$\underset{\otimes}{\tilde{A}} = \{(x, \mu_A(x), v_A(x))|x \in X\} \quad (1)$$

The grey fuzzy set is also expressed by $\underset{\otimes}{\tilde{A}} = \left(\tilde{A}, \underset{\otimes}{A}\right)$, where $\tilde{A} = \{(x, \mu_A(x))|x \in X\}$ is called the fuzzy part of grey fuzzy set $\underset{\otimes}{\tilde{A}}$ and $A_{\otimes} = \{(x, v_A(x))|x \in X\}$ is called the grey part of $\underset{\otimes}{\tilde{A}}$. So the grey fuzzy set is a generalization of the grey set and the fuzzy set.

Definition 2 Let $X = \{x\}$ and $Y = \{y\}$ be the given spaces; if $v_R(x, y)$ is the grey of the membership function

$\mu_R(x, y)$ of $\tilde{R}$ which is the fuzzy relationship between x and y, then grey fuzzy set $\underset{\otimes}{\tilde{R}} = \{((x,y), \mu_R(x,y), v_R(x,y))|x \in X, y \in Y\}$ is called the grey fuzzy relationship in direct product space $X \times Y$, which is shown as the grey fuzzy matrix.

$$\underset{\otimes}{\tilde{R}} = \begin{bmatrix} (\mu_{11}, v_{11}) & (\mu_{12}, v_{12}) & \cdots & (\mu_{1n}, v_{1n}) \\ (\mu_{21}, v_{21}) & (\mu_{22}, v_{22}) & \cdots & (\mu_{2n}, v_{2n}) \\ \vdots & \vdots & \vdots & \vdots \\ (\mu_{m1}, v_{m1}) & (\mu_{m2}, v_{m2}) & \cdots & (\mu_{mn}, v_{mn}) \end{bmatrix} \quad (2)$$

where $\underset{\otimes}{\tilde{R}} = \left(\tilde{R}, \underset{\otimes}{R}\right)$ represents the grey fuzzy relationship in space $X \times Y$. $\tilde{R} = \{((x,y), \mu_A(x,y))|x \in X, y \in Y\}$ represents the fuzzy relationship in space $X \times Y$, and $\underset{\otimes}{\tilde{R}} = \{((x, y), v_A(x, y))|x \in X, y \in Y\}$ represents the grey relationship in direct product space $X \times Y$.

2.2 Uncertain linguistic variables

Suppose $S = (s_0, s_1, \ldots, s_{l-1})$ is a finite and ordered discrete linguistic term set, where l is an odd number. In real situation, l can be assigned to 3, 5, 7, etc. In this paper, $l = 7$, the set S could be given as follows [10]:

$$\begin{aligned} S &= (s_0, s_1, s_2, s_3, s_4, s_5, s_6) \\ &= \{\text{very poor, poor, slightly poor, fair, slightly good,} \\ &\quad \text{good, very good}\}. \end{aligned}$$

For any linguistic set $S = (s_0, s_1, \ldots, s_{l-1})$, because the relationship between the element s_i and its subscript i is strictly monotone increasing [12, 41], we can define the function f:$s_i = f(i)$ which is the strictly monotone increasing function for subscript i. In order to preserve all the given information, the discrete linguistic label $S = (s_0, s_1, \ldots, s_{l-1})$ is extended to a continuous linguistic set $\bar{S} = \{s_\alpha | \alpha \in R\}$

Definition 3 [40, 43] Suppose $\tilde{s} = [s_a, s_b], s_a, s_b \in \bar{S}$ and $a \le b$, s_a and s_b are the lower limit and upper limit of $\tilde{s}$, respectively, and then, $\tilde{s}$ is called an uncertain linguistic variable.

Let $\tilde{S}$ be the set of all uncertain linguistic variables, the operational rules for two uncertain linguistic variables $\tilde{s}_1 = [s_{a1}, s_{b1}]$ and $\tilde{s}_2 = [s_{a2}, s_{b2}]$ are defined as follows [40, 43]:

1. $\tilde{s}_1 \oplus \tilde{s}_2 = [s_{a1}, s_{b1}] \oplus [s_{a2}, s_{b2}] = [s_{a1+a2}, s_{b1+b2}]$ (3)
2. $\tilde{s}_1 \otimes \tilde{s}_2 = [s_{a1}, s_{b1}] \otimes [s_{a2}, s_{b2}] = [s_{a1\times a2}, s_{b1\times b2}]$ (4)
3. $\tilde{s}_1 / \tilde{s}_2 = [s_{a1}, s_{b1}]/[s_{a2}, s_{b2}] = [s_{a1/b2}, s_{b1/a2}]$ if $a2 \neq 0, b2 \neq 0$ (5)
4. $\lambda\tilde{s}_1 = \lambda[s_{a1}, s_{b1}] = [s_{\lambda * a1}, s_{\lambda * b1}]$ (6)
5. $\lambda(\tilde{s}_1 \oplus \tilde{s}_2) = \lambda\tilde{s}_1 \oplus \lambda\tilde{s}_2$ (7)
6. $(\lambda_1 + \lambda_2)\tilde{s}_1 = \lambda_1\tilde{s}_1 \oplus \lambda_2\tilde{s}_1$ (8)

2.3 The generalized OWA operator

The GOWA operator [46] is a generalization of the OWA operator [45] by using generalized means, which can be defined as follows.

Definition 4 [46] A GOWA operator of dimension n is a mapping GOWA: $R^n \to R$ that has an associated weight vector $w = (w_1, w_2, \ldots, w_n)^T$ with $w_j \ge 0$ and $\sum_{j=1}^{n} w_j = 1$. Such that

$$\mathrm{GOWA}(a_1, a_2, \ldots, a_n) = \left(\sum_{j=1}^{n} w_j a_{\sigma(j)}^{\lambda}\right)^{1/\lambda} \quad (9)$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$, such that $a_{\sigma(j-1)} \ge a_{\sigma(j)}$ for all $j = 2, \ldots, n$. In addition, λ is a parameter such that $\lambda \in (-\infty, 0) \cup (0, +\infty)$.

Some properties of the GOWA operators are shown as follows.

1. When $\lambda \to -\infty$, if $w_j \neq 0$ for all j, then $\mathrm{GOWA}(a_1, a_2, \ldots, a_n) = a_{\sigma(n)} = \min(a_1, a_2, \ldots, a_n)$. The GOWA operator reduces to the min operator. However, if $w_1 = 1$ and $w_j = 0$ for all $j \neq 1$, then even though $\lambda \to -\infty$, we get $\mathrm{GOWA}(a_1, a_2, \ldots, a_n) = a_{\sigma(1)} = \max(a_1, a_2, \ldots, a_n)$.
2. When $\lambda \to 0$, $\mathrm{GOWA}(a_1, a_2, \ldots, a_n) = \prod_{j=1}^{n} a_{\sigma(j)}^{w_j}$. The GOWA operator reduces to the OWG operator.
3. When $\lambda = 1$, $\mathrm{GOWA}(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j a_{\sigma(j)}$. The GOWA operator reduces to the OWA operator.
4. When $\lambda \to +\infty$, if $w_j \neq 0$ for all j, then $\mathrm{GOWA}(a_1, a_2, \ldots, a_n) = a_{\sigma(1)} = \max(a_1, a_2, \ldots, a_n)$. The GOWA operator reduces to the max operator. However, if $w_n = 1$ and $w_j = 0$ for all $j \neq n$, then even though $\lambda \to +\infty$, we get $\mathrm{GOWA}(a_1, a_2, \ldots, a_n) = a_{\sigma(n)} = \min(a_1, a_2, \ldots, a_n)$.

3 Interval grey uncertain linguistic variables

3.1 The definition of interval grey uncertain linguistic variables (IGULVs)

Definition 5 Let $\underset{\otimes}{\tilde{A}} = \left(\tilde{A}, \underset{\otimes}{A}\right)$ be the grey fuzzy number, if its fuzzy part is an uncertain linguistic variable

$\tilde{s} = [s_a, s_b]$, $s_a, s_b \in \bar{S}$, and its grey part $\underset{\otimes}{A}$ is in an interval $[g_A^L, g_A^U]$, then $\underset{\otimes}{\tilde{A}}$ is called an interval grey uncertain linguistic variable (IGULV).

3.2 The operations of the IGULVs

Let $\underset{\otimes}{\tilde{A}} = \left([s_{a1}, s_{a2}], \left[g_A^L, g_A^U\right]\right)$, $\underset{\otimes}{\tilde{B}} = \left([s_{b1}, s_{b2}], \left[g_B^L, g_B^U\right]\right)$ and $\underset{\otimes}{\tilde{C}} = \left([s_{c1}, s_{c2}], \left[g_C^L, g_C^U\right]\right)$ be any three IGULVs, the operational rules are defined as follows:

1. $$\underset{\otimes}{\tilde{A}} + \underset{\otimes}{\tilde{B}} = \left([s_{a1+b1}, s_{a2+b2}], \left[\max\left(g_A^L, g_B^L\right), \max\left(g_A^U, g_B^U\right)\right]\right) \quad (10)$$
2. $$\underset{\otimes}{\tilde{A}} \times \underset{\otimes}{\tilde{B}} = \left([s_{a1\times b1}, s_{a2\times b2}], \left[\max\left(g_A^L, g_B^L\right), \max\left(g_A^U, g_B^U\right)\right]\right) \quad (11)$$
3. $$\underset{\otimes}{\tilde{A}} / \underset{\otimes}{\tilde{B}} = \left([s_{a1/b2}, s_{a2/b1}], \left[\max\left(g_A^L, g_B^L\right), \max\left(g_A^U, g_B^U\right)\right]\right) \text{ where, } b1, b2 \neq 0 \quad (12)$$
4. $$k\underset{\otimes}{\tilde{A}} = \left([s_{k\times a1}, s_{k\times a2}], \left[g_A^L, g_A^U\right]\right) \quad (13)$$
5. $$\left(\underset{\otimes}{\tilde{A}}\right)^k = \left([s_{a1^k}, s_{a2^k}], \left[g_A^L, g_A^U\right]\right) \quad (14)$$

It can be seen that the IGULVs have the following properties

1. $$\underset{\otimes}{\tilde{A}} + \underset{\otimes}{\tilde{B}} = \underset{\otimes}{\tilde{B}} + \underset{\otimes}{\tilde{A}} \quad (15)$$
2. $$\underset{\otimes}{\tilde{A}} \times \underset{\otimes}{\tilde{B}} = \underset{\otimes}{\tilde{B}} \times \underset{\otimes}{\tilde{A}} \quad (16)$$
3. $$\underset{\otimes}{\tilde{A}} + \underset{\otimes}{\tilde{B}} + \underset{\otimes}{\tilde{C}} = \underset{\otimes}{\tilde{A}} + \left(\underset{\otimes}{\tilde{B}} + \underset{\otimes}{\tilde{C}}\right) \quad (17)$$
4. $$\underset{\otimes}{\tilde{A}} \times \underset{\otimes}{\tilde{B}} \times \underset{\otimes}{\tilde{C}} = \underset{\otimes}{\tilde{A}} \times \left(\underset{\otimes}{\tilde{B}} \times \underset{\otimes}{\tilde{C}}\right) \quad (18)$$
5. $$\underset{\otimes}{\tilde{A}} \times \left(\underset{\otimes}{\tilde{B}} + \underset{\otimes}{\tilde{C}}\right) = \underset{\otimes}{\tilde{A}} \times \underset{\otimes}{\tilde{B}} + \underset{\otimes}{\tilde{A}} \times \underset{\otimes}{\tilde{C}} \quad (19)$$
6. $$(\lambda_1 + \lambda_2)\underset{\otimes}{\tilde{A}} = \lambda_1 \underset{\otimes}{\tilde{A}} + \lambda_2 \underset{\otimes}{\tilde{A}} \quad (20)$$

3.3 The comparing method for the IGULVs

1. The expectation value of an IGULV
   Definition 6 Let $\underset{\otimes}{\tilde{A}} = \left([s_{a1}, s_{a2}], \left[g_A^L, g_A^U\right]\right)$ be an IGULV, its expectation value is defined as follows
   $$E\left(\underset{\otimes}{\tilde{A}}\right) = \frac{s_{a1} + s_{a2}}{2} \times \left(1 - \frac{g_A^L + g_A^U}{2}\right) = s_{\left(1 - \frac{g_A^L + g_A^U}{2}\right) \times (a1 + a2)/2} \quad (21)$$
   Example 1 Assume that $\underset{\otimes}{\tilde{A}} = ([s_3, s_5], [0.2, 0.5])$. According to Definition 8, the expectation value of the IGULV $\underset{\otimes}{\tilde{A}}$ is calculated as follows:
   $$E\left(\underset{\otimes}{\tilde{A}}\right) = s_{\left(1 - \frac{0.2+0.5}{2}\right) \times (3+5)/2} = s_{2.6}$$
2. The comparing method of the IGULVs
   Let $\underset{\otimes}{\tilde{A}} = \left(s_\alpha, \left[g_A^L, g_A^U\right]\right)$ and $\underset{\otimes}{\tilde{B}} = \left(s_\beta, \left[g_B^L, g_B^U\right]\right)$ be two IGULVs, if $E\left(\underset{\otimes}{\tilde{A}}\right) \geq E\left(\underset{\otimes}{\tilde{B}}\right)$, then $\underset{\otimes}{\tilde{A}} \geq \underset{\otimes}{\tilde{B}}$, and vice versa.

3.4 The GHA operator based on the IGULVs

Definition 7 Let $\underset{\otimes j}{\tilde{A}} = \left([s_{a_j}, s_{b_j}], \left[g_j^L, g_j^U\right]\right)$ be a set of IGULVs, then an interval grey uncertain linguistic generalized ordered weighted averaging (IGULGOWA) operator of dimension n is a mapping IGULGOWA: $\Omega^n \to \Omega$ that has an associated weight vector $w = (w_1, w_2, \ldots, w_n)^T$ with $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$. Such that

$$\text{IGULGOWA}\left(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}\right) = \left(\sum_{j=1}^{n} w_j \underset{\otimes\sigma(j)}{\tilde{A}^\lambda}\right)^{1/\lambda} \quad (22)$$

where Ω is the set of all IGULVs and $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is any permutation of $(1, 2, \ldots, n)$, such that $\underset{\otimes\sigma(j-1)}{\tilde{A}} \geq \underset{\otimes\sigma(j)}{\tilde{A}}$ for all $j = 2, \ldots, n$. In addition, λi s a parameter such that $\lambda \in (-\infty, 0) \cup (0, +\infty)$.

The characteristic of the IGULGOWA operator is that the IGULVs $\underset{\otimes j}{\tilde{A}} (j = 1, 2, \ldots, n)$ are ranked in descending order and aggregated with weights. w_j is associated with the *jth* position of the aggregation process. So w is called the position weighted vector.

Now, we consider some special cases of the IGULGOWA operator:

1. When $\lambda \to -\infty$ and $w_j \neq 0$ for all j, then the IGULGOWA operator reduces to the Min operator.

$$\text{IGULGOWA}(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}) = \underset{\otimes \sigma(n)}{\tilde{A}} = \min(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}).$$

However, if $w_1 = 1$ and $w_j = 0$ for all $j \neq 1$, then even though $\lambda \to -\infty$, we get IGULGOWA $(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}) = \underset{\otimes \sigma(1)}{\tilde{A}} = \max(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}})$.

2. When $\lambda = -1$, then the IGULGOWA operator reduces to an interval grey uncertain linguistic variable ordered weighted harmonic averaging operator (IGULOWHA):

$$IGULOWHA\left(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}\right) = \frac{1}{\sum_{j=1}^{n} \frac{w_j}{\underset{\otimes \sigma(j)}{\tilde{A}}}}$$

3. When $\lambda \to 0$, then the IGULGOWA operator reduces to an interval grey uncertain linguistic variable ordered weighted geometric operator (IGULOWG)

$$IGULOWG(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}) = \prod_{j=1}^{n} \underset{\otimes \sigma(j)}{\tilde{A}}^{w_j}.$$

4. When $\lambda = 1$, then the IGULGOWA operator reduces to an interval grey uncertain linguistic variable ordered weighted averaging operator (IGULOWA)

$$IGULOWA(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}) = \sum_{j=1}^{n} w_j \underset{\otimes \sigma(j)}{\tilde{A}}$$

5. When $\lambda = 2$, then the IGULGOWA operator reduces to an interval grey uncertain linguistic variable ordered weighted quadratic averaging operator (IGULOWQA).

$$IGULOWQA(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}) = \left(\sum_{j=1}^{n} w_j \tilde{A}^2_{\otimes \sigma(j)}\right)^{1/2}$$

6. When $\lambda \to +\infty$, and $w_j \neq 0$ for all j, then the IGULGOWA operator reduces to the Max operator.

$$\text{IGULGOWA}(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}) = \underset{\otimes \sigma(1)}{\tilde{A}} = \max(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}).$$

However, if $w_n = 1$ and $w_j = 0$ for all $j \neq n$, then even though $\lambda \to +\infty$, we get

$$\text{IGULGOWA}(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}) = \underset{\otimes \sigma(n)}{\tilde{A}} = \min(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}).$$

Based on the operation rules of the IGULVs, the formula (22) will be deduced to

$$\begin{aligned}
&\text{IGULGOWA}(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}) \\
&= \left(\sum_{j=1}^{n} w_j \tilde{A}^{\lambda}_{\otimes \sigma(j)}\right)^{1/\lambda} \\
&= \left(\left[s_{\left(\sum_{j=1}^{n}\left(a^{\lambda}_{\sigma(j)} \times w_j\right)\right)^{1/\lambda}}, s_{\left(\sum_{j=1}^{n}\left(b^{\lambda}_{\sigma(j)} \times w_j\right)\right)^{1/\lambda}}\right],\right. \\
&\quad \left. \times \left[\max_j\left(g_j^L\right), \max_j\left(g_j^U\right)\right]\right)
\end{aligned} \tag{23}$$

Example 2 Let us consider the aggregation of three IGULVs.Let $\underset{\otimes 1}{\tilde{A}} = ([s_3, s_5], [0.2, 0.5])$, $\underset{\otimes 2}{\tilde{A}} = ([s_3, s_3], [0.4, 0.5])$ and $\underset{\otimes 3}{\tilde{A}} = ([s_4, s_6], [0.3, 0.5])$, according to Definition 6, the expectation values of $\underset{\otimes 1}{\tilde{A}}$, $\underset{\otimes 2}{\tilde{A}}$ and $\underset{\otimes 3}{\tilde{A}}$ are calculated as

$$E\left(\underset{\otimes 1}{\tilde{A}}\right) = s_{2.6}, \quad E\left(\underset{\otimes 2}{\tilde{A}}\right) = s_{1.65} \quad \text{and} \quad E\left(\underset{\otimes 3}{\tilde{A}}\right) = s_{3.0},$$

so, we can get $\sigma(1) = 3$, $\sigma(2) = 1$, $\sigma(3) = 2$.

Assume that the position weight vector is $w = (0.5, 0.3, 0.2)$ and $\lambda = 2$, we have the aggregated value by formula (23) as follows

$$\begin{aligned}
&\text{IGULGOWA}(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \underset{\otimes 3}{\tilde{A}}) \\
&= \left(\left[s_{(0.5\times 4^2+0.3\times 3^2+0.2\times 3^2)^{1/2}}, s_{(0.5\times 6^2+0.3\times 5^2+0.2\times 3^2)^{1/2}}\right],\right. \\
&\quad \left. \times \left[\max(0.2, 0.4, 0.3), \max(0.5, 0.5, 0.5)\right]\right) \\
&= ([s_{3.536}, s_{5.225}], [0.4, 0.5])
\end{aligned}$$

When the parameter λ gets the other specific values, we have the aggregated values shown as Table 1.

In addition, it is easy to see from Table 1 that the aggregated values are monotonic increasing with respect to the parameter λ.

The IGULGOWA operator has the following properties similar to those of the GOWA operator [46].

1. **Theorem 1** (Commutativity)

If $\left(\underset{\otimes 1}{\tilde{A}'}, \underset{\otimes 2}{\tilde{A}'}, \ldots, \underset{\otimes n}{\tilde{A}'}\right)$ *is any permutation of* $\left(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}\right)$, *then*

$$\text{IGULGOWA}\left(\underset{\otimes 1}{\tilde{A}'}, \underset{\otimes 2}{\tilde{A}'}, \ldots, \underset{\otimes n}{\tilde{A}'}\right) = \text{IGULGOWA}\left(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}\right)$$

Table 1 The aggregated value of three IGULVs with the specific parameter value λ

λ	$\text{IGULGOWA}\left(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \underset{\otimes 3}{\tilde{A}}\right)$
$\lambda \to -\infty$	$([s_3, s_3], [0.4, 0.5])$
$\lambda = -1$	$([s_{3.429}, s_{4.762}], [0.4, 0.5])$
$\lambda \to 0$	$([s_{3.464}, s_{4.945}], [0.4, 0.5])$
$\lambda = 1$	$([s_{3.5}, s_{5.1}], [0.4, 0.5])$
$\lambda = 2$	$([s_{3.536}, s_{5.225}], [0.4, 0.5])$
$\lambda \to +\infty$	$([s_4, s_6], [0.3, 0.5])$

Proof

Let $\text{IGULGOWA}(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}) = \left(\sum_{j=1}^{n} w_j \underset{\otimes\sigma(j)}{\tilde{A}^{\lambda}}\right)^{1/\lambda}$

$$\text{IGULGOWA}\left(\underset{\otimes 1}{\tilde{A}'}, \underset{\otimes 2}{\tilde{A}'}, \ldots, \underset{\otimes n}{\tilde{A}'}\right) = \left(\sum_{j=1}^{n} w_j \underset{\otimes\sigma(j)}{\tilde{A}'^{\lambda}}\right)^{1/\lambda}$$

Since $\left(\underset{\otimes 1}{\tilde{A}'}, \underset{\otimes 2}{\tilde{A}'}, \ldots, \underset{\otimes n}{\tilde{A}'}\right)$ is any permutation of $\left(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}\right)$, we have $\underset{\otimes\sigma(j)}{\tilde{A}} = \underset{\otimes\sigma(j)}{\tilde{A}'}$, for all j, and then

$$\text{IGULGOWA}\left(\underset{\otimes 1}{\tilde{A}'}, \underset{\otimes 2}{\tilde{A}'}, \ldots, \underset{\otimes n}{\tilde{A}'}\right) = \text{IGULGOWA}\left(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}\right)$$

2. **Theorem 2** (Idempotency)
If $\underset{\otimes j}{\tilde{A}} = \underset{\otimes}{\tilde{A}}(\underset{\otimes}{\tilde{A}} = ([s_a, s_b], [g^L, g^U]))$ *for all* $\underset{\otimes j}{\tilde{A}}$, *then*

$$\text{IGULGOWA}\left(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}\right) = \underset{\otimes}{\tilde{A}}$$

Proof
Since $\underset{\otimes j}{\tilde{A}} = \underset{\otimes}{\tilde{A}}$, for all $\underset{\otimes j}{\tilde{A}}$, we have

$$\text{IGULGOWA}(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}) = \left(\sum_{j=1}^{n} w_j \underset{\otimes\sigma(j)}{\tilde{A}^{\lambda}}\right)^{1/\lambda} = \left(\sum_{j=1}^{n} w_j \underset{\otimes}{\tilde{A}^{\lambda}}\right)^{1/\lambda} = \left(\underset{\otimes}{\tilde{A}^{\lambda}} \sum_{j=1}^{n} w_j\right)^{1/\lambda} = \left(\underset{\otimes}{\tilde{A}^{\lambda}}\right)^{1/\lambda} = \underset{\otimes}{\tilde{A}}$$

3. **Theorem 3** (Monotonicity)
If $\underset{\otimes j}{\tilde{A}} \le \underset{\otimes j}{\tilde{A}'}$ *for all j, then*

$$\text{IGULGOWA}\left(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}\right) \le \text{IGULGOWA}\left(\underset{\otimes 1}{\tilde{A}'}, \underset{\otimes 2}{\tilde{A}'}, \ldots, \underset{\otimes n}{\tilde{A}'}\right)$$

Proof

Let $\text{IGULGOWA}(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}) = \left(\sum_{j=1}^{n} w_j \underset{\otimes\sigma(j)}{\tilde{A}^{\lambda}}\right)^{1/\lambda}$

$$\text{IGULGOWA}\left(\underset{\otimes 1}{\tilde{A}'}, \underset{\otimes 2}{\tilde{A}'}, \ldots, \underset{\otimes n}{\tilde{A}'}\right) = \left(\sum_{j=1}^{n} w_j \underset{\otimes\sigma(j)}{\tilde{A}'^{\lambda}}\right)^{1/\lambda}$$

Since $\underset{\otimes j}{\tilde{A}} \le \underset{\otimes j}{\tilde{A}'}$ for all j, we have $\underset{\otimes\sigma(j)}{\tilde{A}} \le \underset{\otimes\sigma(j)}{\tilde{A}'}$ and then $\text{IGULGOWA}\left(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}\right) \le \text{IGULGOWA}\left(\underset{\otimes 1}{\tilde{A}'}, \underset{\otimes 2}{\tilde{A}'}, \ldots, \underset{\otimes n}{\tilde{A}'}\right)$

4. **Theorem 4** (Boundedness)

$$\min\left(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}\right) \le \text{IGULGOWA}\left(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}\right) \le \max\left(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}\right)$$

Proof
Suppose $\underset{\otimes}{\tilde{A}} = \min\left(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}\right)$, and $\underset{\otimes}{\tilde{B}} = \max\left(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}\right)$. According to the monotonicity of the IGULGOWA operator, we have

$$\text{IGULGOWA}\left(\underset{\otimes}{\tilde{A}}, \underset{\otimes}{\tilde{A}}, \ldots, \underset{\otimes}{\tilde{A}}\right) \le \text{IGULGOWA}\left(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}\right) \le \text{IGULGOWA}\left(\underset{\otimes}{\tilde{B}}, \underset{\otimes}{\tilde{B}}, \ldots, \underset{\otimes}{\tilde{B}}\right)$$

And according to the idempotency of the IGULGOWA operator, we have

$$\text{IGULGOWA}\left(\underset{\otimes}{\tilde{A}}, \underset{\otimes}{\tilde{A}}, \ldots, \underset{\otimes}{\tilde{A}}\right) = \underset{\otimes}{\tilde{A}},$$
$$\text{IGULGOWA}\left(\underset{\otimes}{\tilde{B}}, \underset{\otimes}{\tilde{B}}, \ldots, \underset{\otimes}{\tilde{B}}\right) = \underset{\otimes}{\tilde{B}}$$

So, we can get

$$\min\left(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}\right) \le \text{IGULGOWA}\left(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}\right) \le \max\left(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}\right).$$

In addition, about position weighted vector w, it can be determined according to the actual decision-making problems, it is also determined by the method proposed by Wang and Xu [37]. The formula is shown as follows:

$$w_{j+1} = \frac{C_{n-1}^{j}}{2^{n-1}}\ j = 0, 1, \ldots, n-1 \tag{24}$$

The IGULGOWA operator only weights the position of each IGULV and cannot consider the self-importance of the IGULV. Therefore, in order to overcome the weaknesses, the interval grey uncertain linguistic generalized hybrid aggregation operator is defined as follows.

Definition 8 Let $\underset{\otimes j}{\tilde{A}} = \left([s_{a_j}, s_{b_j}], \left[g_j^L, g_j^U\right]\right)$ be a set of IGULVs, then an interval grey uncertain linguistic generalized hybrid averaging (IGULGHA) operator of dimension n is a mapping IGULGHA: $\Omega^n \to \Omega$, that has an associated weight vector $w = (w_1, w_2, \ldots, w_n)^T$ with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$. Such that

$$IGULGHA(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}) = \left(\sum_{j=1}^{n} w_j \underset{\otimes \sigma(j)}{\tilde{B}^{\lambda}}\right)^{1/\lambda} \tag{25}$$

where Ω is the set of all IGULVs and $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is any permutation of $(1, 2, \ldots, n)$, such that $\underset{\otimes \sigma(j-1)}{\tilde{B}} \geq \underset{\otimes \sigma(j)}{\tilde{B}}$ for all $j = 2, \ldots, n$. $\underset{\otimes j}{\tilde{B}} = n\omega_j \underset{\otimes j}{\tilde{A}} (j = 1, \ldots, n)$, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is the weighting vector of the $\underset{\otimes j}{\tilde{A}}$, with $\omega_j \geq 0$ and $\sum_{j=1}^{n} \omega_j = 1$. n is the balance factor. In addition, λ is a parameter such that $\lambda \in (-\infty, 0) \cup (0, +\infty)$.

Based on the operation rules of the IGULVs, the formula (25) will be deduced to

$$\begin{aligned} & IGULGHA(\underset{\otimes 1}{\tilde{A}}, \underset{\otimes 2}{\tilde{A}}, \ldots, \underset{\otimes n}{\tilde{A}}) \\ & = \left(\sum_{j=1}^{n} w_j \underset{\otimes \sigma(j)}{\tilde{B}^{\lambda}}\right)^{1/\lambda} \\ & = \left(\left[s_{\left(\sum_{j=1}^{n}\left((n\omega_{\sigma(j)} a_{\sigma(j)})^{\lambda} \times w_j\right)\right)^{1/\lambda}}, s_{\left(\sum_{j=1}^{n}\left((n\omega_{\sigma(j)} b_{\sigma(j)})^{\lambda} \times w_j\right)\right)^{1/\lambda}}\right],\right. \\ & \left. \times \left[\max_j\left(g_j^L\right), \max_j\left(g_j^U\right)\right]\right) \end{aligned} \tag{26}$$

Obviously, the IGULGOWA operator is the special case of the IGULGHA operator. The IGULGHA operator not only takes the importance of the IGULVs itself into account, but also its position importance.

4 The MAGDM method based on the interval grey uncertain linguistic variable generalized hybrid aggregation operator

4.1 The description of the MAGDM problems based on the IGULVs

Let $E = \{e_1, e_2, \ldots, e_p\}$ be the experts set in the group decision-making, $A = \{A_1, A_2, \ldots, A_m\}$be the set of alternatives, and $C = \{C_1, C_2, \ldots, C_n\}$ be the attribute set with respect to the alternatives. Supposed that $\underset{\otimes ij}{\tilde{A}^k} = \left(\left[t_{ij}^{Lk}, t_{ij}^{Uk}\right], \left[g_{ijk}^L, g_{ijk}^U\right]\right)$ is the attribute value in the attribute set C_j with respect to the alternative A_i given by expert e_k, which is the form of the IGULV. $\underset{\otimes}{\tilde{A}^k} = \left[\underset{\otimes ij}{\tilde{A}^k}\right]_{m \times n}$ is the decision-making matrix given by the expert e_k, and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$is the attribute weight, with$\sum_{j=1}^{n} \omega_j = 1$, where $t_{ij}^{Lk}, t_{ij}^{Uk} \in S$, S is the uncertain linguistic set. Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_p)$ be the expert weight, with$\sum_{k=1}^{p} \gamma_k = 1$. We can rank the alternatives based on the given information.

4.2 Decision-making steps

1. Aggregate the evaluation information of each expert
   According to the decision-making matrix $\underset{\otimes}{\overset{k}{\tilde{A}}} = \left[\underset{\otimes ij}{\overset{k}{\tilde{A}}}\right]_{m \times n}$ given by the expert e_k, we can get the group decision-making matrix $\underset{\otimes}{\tilde{X}} = \left[\underset{\otimes ij}{\tilde{X}}\right]_{m \times n}$ based on the IGULGHA operator, where $\underset{\otimes ij}{\tilde{X}} = IGULGHA\left(\underset{\otimes ij}{\tilde{A}^1}, \underset{\otimes ij}{\tilde{A}^2}, \ldots, \underset{\otimes ij}{\tilde{A}^p}\right)$.
2. Calculate the comprehensive evaluation value of each alternative
   According to the IGULGHA operator, we can calculate the comprehensive evaluation value of each alternative $\underset{\otimes i}{\tilde{Z}} = \left(\left[c_i^L, c_i^U\right], \left[z_i^L, z_i^U\right]\right) = IGULGHA\left(\underset{\otimes i1}{\tilde{X}}, \underset{\otimes i2}{\tilde{X}}, \ldots, \underset{\otimes in}{\tilde{X}}\right)$.
3. Rank the alternatives
   Because $\underset{\otimes i}{\tilde{Z}}$ is an IGULV, according to the ranking method shown in the Sect. 3.3, we can get the ranking values $E\left(\underset{\otimes i}{\tilde{Z}}\right)$. The larger the value $E\left(\underset{\otimes i}{\tilde{Z}}\right)$ is, the better the alternative is.

5 Numerical example

A practical use of the proposed approach involves the evaluating the technological innovation ability of the four enterprises $\{A_1, A_2, A_3$ and $A_4\}$, and there are four evaluating attributes which are the ability of innovative resources investment (C_1), the ability of innovation management (C_2), the ability of innovation tendency (C_3) and the ability of research and development (C_4). Based on the four attributes, the three experts $\{e_1, e_2$ and $e_3\}$ evaluated the technological innovation ability of the four enterprises. Suppose that $\gamma = (0.4, 0.32, 0.28)$ is the expert weight vector, and $\omega = (0.3, 0.2, 0.2, 0.3)$ is attribute weight vector. The attribute values given by the experts take the form of the IGULVs shown in Tables 2, 3 and 4. Let $S = (s_0, s_1, s_2, s_3, s_4, s_5, s_6)$ be the linguistic set which meaning is defined in Sect. 2.2. We will rank the four enterprises for their technological innovation ability.

5.1 The evaluation steps by the method proposed in this paper

The evaluation steps are shown as follows:

1. Get the group decision-making matrix $\underset{\otimes}{\tilde{X}}$.

   Suppose the position weight vector of the experts $w = (1/3, 1/3, 1/3)$ and $\lambda = 1$, we can aggregate the evaluation information (shown in Tables 2, 3 and 4) given by the experts $\{e_1, e_2, e_3\}$ based on the formula (26) and then get

$$\underset{\otimes}{\tilde{X}} = \begin{bmatrix} ([s_{3.68}, s_{4.68}][0.2, 0.4])([s_{2.00}, s_{2.60}][0.4, 0.4])([s_{3.08}, s_{4.08}][0.5, 0.5])([s_{3.64}, s_{4.24}][0.4, 0.5]) \\ ([s_{3.32}, s_{4.32}][0.4, 0.5])([s_{3.36}, s_{4.36}][0.4, 0.5])([s_{2.44}, s_{3.04}][0.2, 0.4])([s_{2.40}, s_{3.40}][0.5, 0.5]) \\ ([s_{3.60}, s_{3.60}][0.2, 0.4])([s_{4.00}, s_{4.28}][0.3, 0.4])([s_{2.52}, s_{2.80}][0.4, 0.4])([s_{3.40}, s_{4.08}][0.3, 0.3]) \\ ([s_{3.84}, s_{4.84}][0.5, 0.6])([s_{2.32}, s_{2.92}][0.4, 0.5])([s_{2.28}, s_{2.96}][0.3, 0.4])([s_{2.88}, s_{3.88}][0.4, 0.5]) \end{bmatrix}$$

2. Calculate $\underset{\otimes ij}{\tilde{B}} = n\omega_j \underset{\otimes ij}{\tilde{X}}$

$$\underset{\otimes}{\tilde{B}} = \begin{bmatrix} ([s_{4.42}, s_{5.62}], [0.2, 0.4]), ([s_{1.60}, s_{2.08}], [0.4, 0.4]), ([s_{2.46}, s_{3.26}], [0.5, 0.5]), ([s_{4.37}, s_{5.09}], [0.4, 0.5]) \\ ([s_{3.98}, s_{5.18}], [0.4, 0.5]), ([s_{2.69}, s_{3.49}], [0.4, 0.5]), ([s_{1.95}, s_{2.43}], [0.2, 0.4]), ([s_{2.88}, s_{4.08}], [0.5, 0.5]) \\ ([s_{4.32}, s_{4.32}], [0.2, 0.4]), ([s_{3.20}, s_{3.42}], [0.3, 0.4]), ([s_{2.02}, s_{2.24}], [0.4, 0.4]), ([s_{4.08}, s_{4.90}], [0.3, 0.3]) \\ ([s_{4.61}, s_{5.81}], [0.5, 0.6]), ([s_{1.86}, s_{2.34}], [0.4, 0.5]), ([s_{1.82}, s_{2.37}], [0.3, 0.4]), ([s_{3.46}, s_{4.66}], [0.4, 0.5]) \end{bmatrix}$$

Table 2 The attribute values of each attribute with respect to four enterprises given by expert e_1

Enterprises	Attribute (C_1)	Attribute (C_2)	Attribute (C_3)	Attribute (C_4)
A1	([s_4, s_5], [0.2, 0.3])	([s_2, s_2], [0.4, 0.4])	([s_4, s_5], [0.5, 0.5])	([s_3, s_3], [0.2, 0.4])
A2	([s_3, s_4], [0.4, 0.4])	([s_4, s_5], [0.4, 0.5])	([s_3, s_3], [0.1, 0.2])	([s_3, s_4], [0.5, 0.5])
A3	([s_3, s_3], [0.2, 0.3])	([s_4, s_4], [0.2, 0.3])	([s_4, s_4], [0.3, 0.3])	([s_4, s_5], [0.2, 0.3])
A4	([s_5, s_6], [0.5, 0.6])	([s_2, s_2], [0.2, 0.2])	([s_2, s_3], [0.2, 0.4])	([s_2, s_3], [0.3, 0.4])

Table 3 The attribute values of each attribute with respect to four enterprises given by expert e_2

Enterprises	Attribute (C_1)	Attribute (C_2)	Attribute (C_3)	Attribute (C_4)
A1	([s_3, s_4], [0.1, 0.3])	([s_2, s_3], [0.2, 0.3])	([s_2, s_3], [0.2, 0.2])	([s_5, s_6], [0.4, 0.5])
A2	([s_4, s_5], [0.4, 0.5])	([s_2, s_3], [0.3, 0.4])	([s_3, s_4], [0.2, 0.4])	([s_2, s_3], [0.2, 0.3])
A3	([s_4, s_4], [0.2, 0.4])	([s_4, s_4], [0.2, 0.3])	([s_2, s_2], [0.4, 0.4])	([s_3, s_3], [0.3, 0.3])
A4	([s_4, s_5], [0.3, 0.4])	([s_3, s_4], [0.4, 0.5])	([s_2, s_2], [0.3, 0.4])	([s_3, s_4], [0.2, 0.4])

Table 4 The attribute values of each attribute with respect to four enterprises given by expert e_3

Enterprises	Attribute (C_1)	Attribute (C_2)	Attribute (C_3)	Attribute (C_4)
A1	$([s_4, s_5], [0.2, 0.4])$	$([s_2, s_3], [0.3, 0.3])$	$([s_3, s_4], [0.4, 0.5])$	$([s_3, s_4], [0.2, 0.3])$
A2	$([s_3, s_4], [0.3, 0.3])$	$([s_4, s_5], [0.3, 0.4])$	$([s_1, s_2], [0.1, 0.2])$	$([s_2, s_3], [0.1, 0.2])$
A3	$([s_4, s_4], [0.2, 0.3])$	$([s_4, s_5], [0.3, 0.4])$	$([s_1, s_2], [0.1, 0.2])$	$([s_3, s_4], [0.2, 0.3])$
A4	$([s_2, s_3], [0.2, 0.3])$	$([s_2, s_3], [0.1, 0.3])$	$([s_3, s_4], [0.3, 0.4])$	$([s_4, s_5], [0.4, 0.5])$

3. Calculate the expectation values for all elements of $\underset{\otimes}{\tilde{B}}$

$$E(B_{11}) = 3.51,\ E(B_{12}) = 1.10, E(B_{13}) = 1.43,\ E(B_{14}) = 2.60$$
$$E(B_{21}) = 2.52,\ E(B_{22}) = 1.70,\ E(B_{23}) = 1.53,\ E(B_{24}) = 1.74$$
$$E(B_{31}) = 3.02,\ E(B_{32}) = 2.15,\ E(B_{33}) = 1.28,\ E(B_{34}) = 3.14$$
$$E(B_{41}) = 2.34,\ E(B_{42}) = 1.15,\ E(B_{43}) = 1.36,\ E(B_{44}) = 2.23$$

4. Calculate the comprehensive evaluation values of each alternative

   Suppose the position weight vector of the attributes $w = \left(1/4, 1/4, 1/4, 1/4\right)$ and $\lambda = 1$, we can get the comprehensive evaluation values of each alternative according to the formula (26).

$$\underset{\otimes}{\tilde{Z}} = \begin{pmatrix} [s_{3.21}, s_{4.01}], [0.5, 0.5] \\ [s_{2.88}, s_{3.80}], [0.5, 0.5] \\ [s_{3.40}, s_{3.72}], [0.4, 0.4] \\ [s_{2.94}, s_{3.79}], [0.5, 0.6] \end{pmatrix}$$

5. Calculate the expectation values

$$E(Z_1) = 1.81,\ E(Z_2) = 1.67, E(Z_3) = 2.14,\ E(Z_4) = 1.51$$

6. Rank the alternatives.

   According to the expectation values, we can get the orders of technological innovation ability of the four enterprises $\{A_1, A_2, A_3, A_4\}$: $A_3 \succ A_2 \succ A_1 \succ A_4$.

5.2 Discussion

In order to illustrate the influences of the position weight vector w and the parameter λ on decision-making results of this example, we use the different values w and λ in step (4) to rank the alternatives. The ranking results are shown in Table 5.

As we can see from Table 5, the ordering of the alternatives may be different for the different values w and λ in IGULGHA operator. Thus, the organization can properly select the desirable alternative according to his interest and the actual needs.

Table 5 Ordering of the alternatives by utilizing the different values w and λ in IGULGHA operator

w	λ	Ranking
$w = \left(1/4, 1/4, 1/4, 1/4\right)$	$\lambda = -2$	$A_3 \succ A_2 \succ A_1 \succ A_4$
	$\lambda = -1$	$A_3 \succ A_2 \succ A_1 \succ A_4$
	$\lambda \to 0$	$A_3 \succ A_1 \succ A_2 \succ A_4$
	$\lambda = 1$	$A_3 \succ A_1 \succ A_2 \succ A_4$
	$\lambda = 2$	$A_3 \succ A_1 \succ A_2 \succ A_4$
$w = \left(1/8, 3/8, 3/8, 1/8\right)$	$\lambda = -2$	$A_3 \succ A_2 \succ A_1 \succ A_4$
	$\lambda = -1$	$A_3 \succ A_1 \succ A_2 \succ A_4$
	$\lambda \to 0$	$A_3 \succ A_1 \succ A_2 \succ A_4$
	$\lambda = 1$	$A_3 \succ A_1 \succ A_2 \succ A_4$
	$\lambda = 2$	$A_3 \succ A_1 \succ A_2 \succ A_4$
$w = \left(3/8, 1/8, 1/8, 3/8\right)$	$\lambda = -2$	$A_3 \succ A_2 \succ A_1 \succ A_4$
	$\lambda = -1$	$A_3 \succ A_2 \succ A_1 \succ A_4$
	$\lambda \to 0$	$A_3 \succ A_2 \succ A_1 \succ A_4$
	$\lambda = 1$	$A_3 \succ A_1 \succ A_2 \succ A_4$
	$\lambda = 2$	$A_3 \succ A_1 \succ A_2 \succ A_4$

6 Conclusions

The fuzziness and the greyness are main characters in the real decision-making problems, so the research on grey fuzzy multiple attribute decision-making problems is very significant, while the generalized hybrid averaging (GHA) operator can generalize the hybrid averaging (HA) operator. But, the traditional generalized hybrid averaging (GHA) operator is generally suitable for aggregating the real numbers, and yet they will fail in dealing with grey fuzzy information. In this paper, we propose an interval grey uncertain linguistic variables generalized ordered weighted averaging (IGULGOWA) operator and generalized hybrid averaging (IGULGHA) operator and study some properties of these operators, such as commutativity, idempotency, monotonicity and boundedness. We have applied the IGULGHA operator to multiple attribute decision-making problems with the interval grey uncertain linguistic information. Finally, an illustrative example is given to verify the proposed method and to demonstrate its practicality and effectiveness, and the influence of the different position weight vector w and the different parameter λ on decision-making of this example is analysed. This proposed method in this paper enriched and developed the theory and method of grey fuzzy multiple attribute group decision-making and provided the new idea to solve the GFMADM problems. In the future, we shall continue working in the extension and application of the developed operators to other domains.

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