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ORIGINAL ARTICLE

Chaos control of a bounded 4D chaotic system

Hassan Saberi Nik · Mahin Golchaman

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Abstract This paper is concerned with the problem of optimal and adaptive control for controlling chaos in a novel bounded four-dimensional (4D) chaotic system. This system can display hyperchaos, chaos, quasiperiodic and periodic behaviors, and may have a unique equilibrium, three equilibria and five equilibria for the different system parameters. An optimal control law is designed for the novel bounded chaotic system, based on the Pontryagin minimum principle. Furthermore, we propose Lyapunov stability conditions to control the new bounded 4D chaotic system with unknown parameters by a feedback control approach. Numerical simulations are presented to show the effectiveness of the proposed chaos control scheme.

Keywords Optimal control · New bounded four-dimensional (4D) chaotic system · Lyapunov function · Pontryagin minimum principle · Legendre spectral method

1 Introduction

Chaotic dynamical systems are very complex nonlinear systems that exhibit unpredictable and irregular behaviors. A chaotic system has several particular features such as extreme sensitivity to initial conditions and system parameter variations, broad spectra of Fourier transform, fractal properties of the motion in phase space, and strange attractors. Chaotic dynamics has been studied in many fields of science and engineering such as physics, biology, electronic circuits, chemistry and mechanical engineering [1–5].

Recently a considerable amount of research is devoted to study the chaotic behavior of nonlinear dynamical systems. In some applications, it is required to control a system in order to eliminate chaos. The first attempt to control a chaotic dynamical system with an analytical method was made in 1990s by Ott, Grebogy and Yorke (OGY method) [6, 7]. After that, the methods for stabilizing unstable periodic orbits (UPOs) embedded in chaotic attractors have been extensively studied in the field of nonlinear dynamics. For instance, adaptive control, adaptive fuzzy control, sliding mode control, robust control, time-delayed feedback control, etc. [8–11].

Ultimate bound sets have important applications in chaos control and its synchronization [12–15]. It can also be applied in estimating the fractal dimensions of chaotic attractors, such as the Hausdorff dimension and the Lyapunov dimension of chaotic attractors [16–18]. Recently, a novel bounded 4D chaotic system with the nonlinear terms in the form of quadratic function was presented by Zhang and Tang [19]. It is shown that the new system can display hyperchaos, chaos, quasiperiodic and periodic behaviors, and may have a unique equilibrium, three equilibria and five equilibria respectively corresponding to the different parameters. The authors have investigated the ultimate bound and positively invariant set for the chaotic system based on the Lyapunov function method, and obtained a hyperelliptic estimate of it for the system with certain parameters.

Designing optimal controllers for chaotic dynamical systems have been investigated by many researchers [20–22]. There are many applications for optimal control of chaotic dynamics in mechanical systems [23], medical and drug systems [25], tumor and cancer models [26, 27] and

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so on. In the present paper, we use a strategy for optimal control of Zhang and Tang chaotic system. For this purpose, we will apply the Pontryagin minimum principle (PMP). Furthermore, the design of the feedback controller is achieved through an application of the optimal control and Lyapunov stability theory which guarantee the global stability of the nonlinear error system. In addition, most real-world engineering systems can be described by complex dynamical networks composed of many different subsystems with connections in and among them [28]. Thus, it is important to further explore the networked systems via various coupling. Due to the significance of scientific and engineering background of collective dynamics, one could start with studying the optimal and adaptive synchronization of complex dynamical networks.

Dynamical systems modeled by nonlinear differential equations. The exact solution for the chaotic systems in the general form does not exist therefore numerical methods are needed for simulating dynamical systems and computing their Lyapunov characteristic exponents (LCE). Several numerical methods have been applied to solve the chaotic systems. Spectral methods are one of the principal methods of discretization for the numerical solution of differential equations. The main advantage of these methods lies in their accuracy for a given number of unknowns. For smooth problems in simple geometries, they offer exponential rates of convergence/spectral accuracy. In contrast, finite-difference and finite-element methods yield only algebraic convergence rates. The three most widely used spectral versions are the Galerkin, collocation, and tau methods [29–31]. Collocation methods [31, 32] have become increasingly popular for solving differential equations, also they are very useful in providing highly accurate solutions to chaotic systems.

This paper is organized as follows. In Sect. 2, we discuss the chaotic Zhang–Tang system and it's stability analysis. We first show the existence of an attractor, and then discuss the existence of equilibria and their stability. In Sect. 3, we discuss the control of chaos for the Zhang–Tang system. In Sect. 4, the dynamic estimators of uncertain parameters in the Zhang–Tang system is investigated based on the Lyapunov stability theory from the conditions on the asymptotic stability of this system about its steady states. In Sect. 5, we summarize the main results obtained in this paper.

2 The novel four-dimensional chaotic system and stability analysis

In this section, we discuss on equilibrium and stability of the chaotic Zhang–Tang system [19]. Consider the nonlinear system of the form

$$\begin{aligned}
\dot{x}_1 &= a_1x_1 + a_2x_4 - x_2x_3,\\
\dot{x}_2 &= -a_3x_1 + a_4x_2 + b_1x_1x_3,\\
\dot{x}_3 &= a_5x_3 + b_2x_1x_2 + b_3x_1x_4,\\
\dot{x}_4 &= a_6x_2 + a_7x_4 - b_4x_1x_3,
\end{aligned} \tag{1}$$

where x_i $(i = 1, 2, 3, 4)$ are system state variables. $a_i < 0, i = 1, 2, \ldots, 7, b_j > 0, (j = 1, 2, 3, 4)$ are constant parameters of the system. Note that when $a_2 = a_3 = b_4 = 0$, system (1) would reduce to the form of the chaotic system proposed in [33]. Nevertheless, system (1) will display a completely different dynamics with that of the chaotic system in [33].

With different parameters a_i and b_j, it is shown that system (1) can display hyperchaos, chaos, quasiperiodic and periodic behaviors. Figures 1 and 2 show the time response and the strange attractors such as hyperchaos and chaos, respectively.

2.1 Dissipation

The differential coefficient of the system (1) stream can be obtained as

$$\nabla \cdot F = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} + \frac{\partial F_3}{\partial x_4} = a_1 + a_4 + a_5 + a_7, \tag{2}$$

while

$$\begin{aligned}
F &= (F_1, F_2, F_3, F_4)\\
&= (a_1x_1 + a_2x_4 - x_2x_3, -a_3x_1 + a_4x_2 + b_1x_1x_3, a_5x_3\\
&\quad + b_2x_1x_2 + b_3x_1x_4, a_6x_2 + a_7x_4 - b_4x_1x_3).
\end{aligned}$$

Therefore, to ensure that system (1) being dissipative, it is required that $a_1 + a_4 + a_5 + a_7 < 0$. Under this condition, system (1) converges exponentially

$$\frac{\mathrm{dF}}{\mathrm{d}t} = (a_1 + a_4 + a_5 + a_7)F \Rightarrow F = F_0\mathrm{e}^{(a_1+a_4+a_5+a_7)t}. \tag{3}$$

Therefore, it can be concluded from (3) that in the case of $a_1 = -0.16$; $a_4 = -0.15$; $a_5 = -0.45$; $a_7 = -0.4$, the exponential contraction rate of the forced dissipative system is calculated as

$$F = F_0\mathrm{e}^{-1.16t}. \tag{4}$$

This implies that each volume containing the system trajectory shrinks to zero as $t \to \infty$ at an exponential rate, $a_1 + a_4 + a_5 + a_7$.

Remark 2.1 In [19] based on the Lyapunov function method, a hyperelliptic estimate of the ultimate bound and positively invariant set for the system with certain parameters was obtained. In fact, with the parameters $a_1 = a_2 = -1.5$, $a_3 = -1.2$, $a_4 = -0.8$, $a_5 = -0.5$, $a_6 = -2$, $a_7 = -0.3$, $b_1 = 2$, $b_2 = 0.3$, $b_3 = 1$, $b_4 = 0.2$, the chaotic system (1) has the following hyperellipsoid:

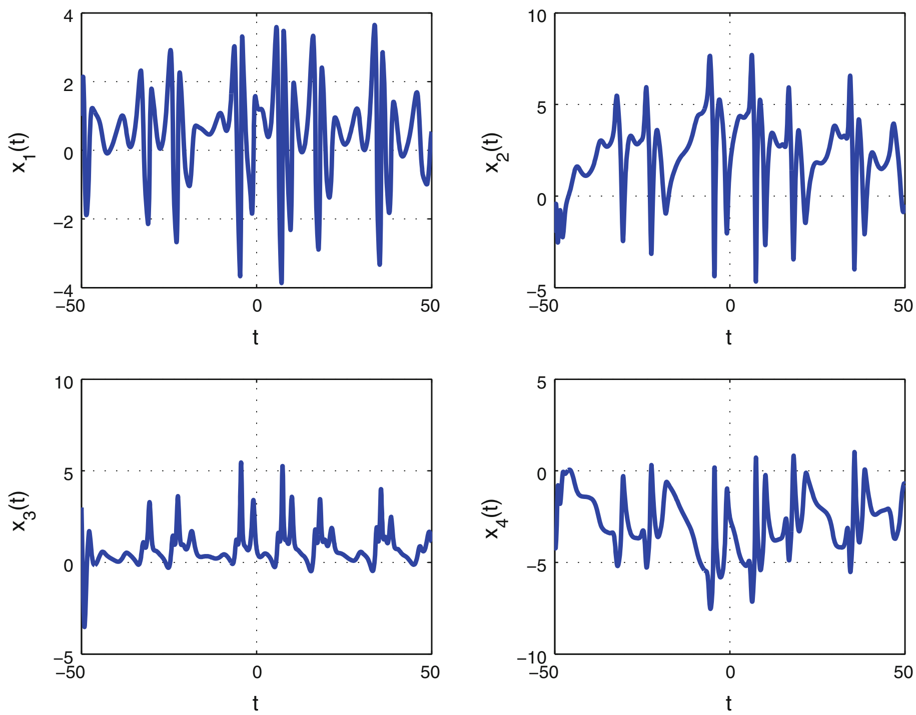

Fig. 1 Time responses of the system states with parameters $a_1 = -0.16$, $a_2 = -0.35$, $a_3 = -0.75$, $a_4 = -0.15$, $a_5 = -0.45$, $a_6 = -0.5$, $a_7 = -0.4$, $b_1 = 1.5$, $b_2 = 1.1$, $b_3 = 1$, $b_4 = 1.15$

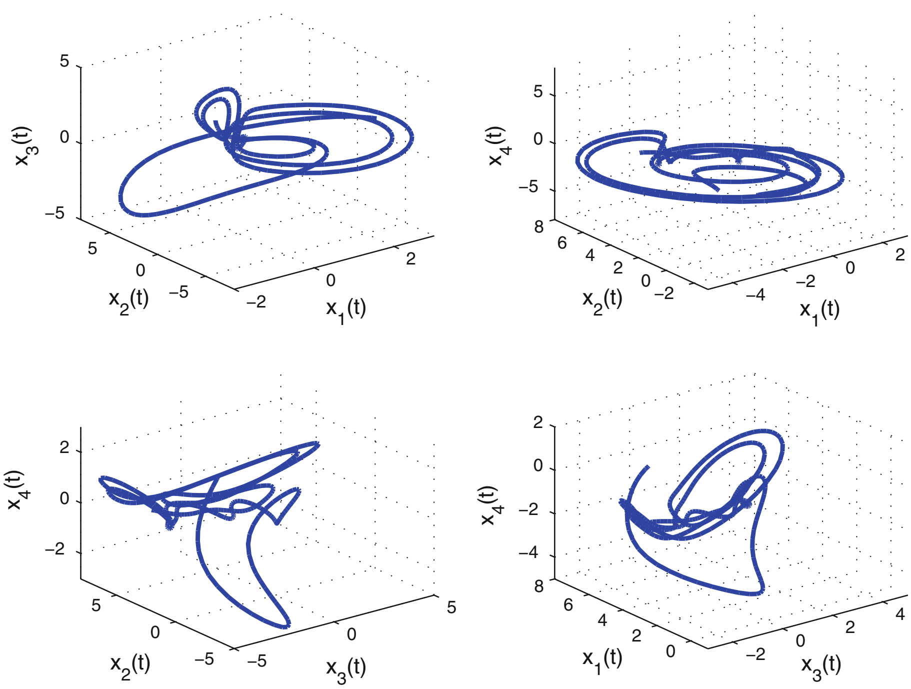

Fig. 2 Chaotic attractors of the system (1) with parameters $a_1 = -0.16$, $a_2 = -0.35$, $a_3 = -0.75$, $a_4 = -0.15$, $a_5 = -0.45$, $a_6 = -0.5$, $a_7 = -0.4$, $b_1 = 1.5$, $b_2 = 1.1$, $b_3 = 1$, $b_4 = 1.15$

$$\Omega = \{X | 30.06x_1^2 + 15x_2^2 + 0.2(x_3 + 225.45)^2 + x_4^2 \leq 17128\}.$$

2.2 Equilibrium and stability

In the following, we consider the equilibrium of the system (1). Obviously the origin $S_0(0, 0, 0, 0)$ is an equilibrium of the system (1). For the nonzero equilibria, we have

$$\begin{cases} x_1 = \pm\sqrt{\frac{(k_i a_3 - a_4) a_5 a_7}{(a_6 b_3 - a_7 b_2) b_1 - (k_i a_3 - a_4) b_3 b_4}}, \\ x_2 = \frac{1}{k_i} x_1, \\ x_3 = \frac{-a_6 b_1 b_3 - a_7 b_1 b_2 - (k_i a_3 - a_4) b_3 b_4}{k_i a_5 a_7 b_1} x_1^2, \\ x_4 = \frac{(k_i a_3 - a_4) b_4 - a_6 b_1}{k_i a_7 b_1} x_1, \quad i = 1, 2, \end{cases} \tag{5}$$

with

$$k_1 = \frac{B + \sqrt{B^2 - 4a_4a_7A}}{2A},$$
$$k_2 = \frac{B - \sqrt{B^2 - 4a_4a_7A}}{2A},$$

and

$$A = a_1a_7b_1 + a_2a_3b_4, \quad B = a_2a_4b_4 + a_2a_6b_1 + a_3a_7.$$

Therefore, it can be concluded from (5) that in the case of $B^2 < 4a_4a_7A$ the system (1) has a unique zero equilibrium S_0, and in the case of $B^2 = 4a_4a_7A$ the system (1) has two nonzero equilibria when the inequality (6) holds. In the case of $B^2 > 4a_4a_7A$, the system (1) has four nonzero equilibria when the inequality (6) holds for both $i = 1$, 2, and the system (1) has two nonzero equilibria when the inequality (6) holds only for $i = 1$ or $i = 2$, and the system (1) has a unique zero equilibria when the inequality (6) holds neither for $i = 1$ nor for $i = 2$.

$$(k_ia_3 - a_4)((a_6b_3 - a_7b_2)b_1 - (k_ia_3 - a_4)b_3b_4) > 0. \quad (6)$$

It can be verified that the system with $a_1 = -0.16$, $a_2 = -0.35$, $a_3 = -0.75$, $a_4 = -0.15$, $a_5 = -0.45$, $a_6 = -0.5$, $a_7 = -0.4$, $b_1 = 1.5$, $b_2 = 1.1$, $b_3 = 1$, $b_4 = 1.15$. has a unique equilibrium zero, and the system corresponding with $a_1 = -0.3$, $a_2 = -0.5$, $a_3 = -0.6$, $a_4 = -0.1$, $a_5 = -0.1$, $a_6 = -0.6$, $a_7 = -0.15$, $b_1 = 1.2$, $b_2 = 1.5$, $b_3 = 2.5$, $b_4 = 0.4$ has three equilibria as:

$$E_1 = (0,0,0,0), E_2 = (0.8859, 0.3320, 58.3507, -0.2203), E_3 = (-0.8859, -0.3320, 58.3507, 0.2203).$$

Proposition 2.1 *The equilibrium points of the system (1) with the parameters of hyperchaos and chaos are unstable.*

Proof The Jacobian matrix of the system (1) is given by

$$J = \begin{bmatrix} a_1 & -x_3 & -x_2 & a_2 \\ -a_3 + b_1x_3 & a_4 & b_1x_1 & 0 \\ b_2x_2 + b_3x_4 & b_2x_1 & a_5 & b_3x_1 \\ -b_4x_3 & a_6 & -b_4x_1 & a_7 \end{bmatrix}. \quad (7)$$

The eigenvalues of the Jacobian matrix J_{S_0} with $a_1 = -0.16$, $a_2 = -0.35$, $a_3 = -0.75$, $a_4 = -0.15$, $a_5 = -0.45$, $a_6 = -0.5$, $a_7 = -0.4$, $b_1 = 1.5$, $b_2 = 1.1$, $b_3 = 1$, $b_4 = 1.15$. can be calculated as

$$\lambda_1 = 0.2833, \lambda_2 = -0.4967 + j0.4275, \lambda_3 = -0.4967 - j0.4275, \lambda_1 = -0.45, \quad (8)$$

It is observed that according to Lyapanov stability theory the equilibrium point S_0 is unstable. Similarly, it can be verified that the nonzero equilibrium points of the system with the parameters $a_1 = -0.3$, $a_2 = -0.5$, $a_3 = -0.6$, $a_4 = -0.1$, $a_5 = -0.1$, $a_6 = -0.6$, $a_7 = -0.15$, $b_1 = 1.2$, $b_2 = 1.5$, $b_3 = 2.5$, $b_4 = 0.4$, are also unstable [19].

3 Optimal control of the novel bounded 4D chaotic system

3.1 Design of the optimal controller

In this subsection, optimal control problem of the chaotic Zhang–Tang system (1) is discussed. For this purpose, we will apply the PMP. First, we add the controls u_1, u_2, u_3 and u_4 to the equations in system (1):

$$\begin{aligned} \dot{x}_1 &= a_1x_1 + a_2x_4 - x_2x_3 + u_1, \\ \dot{x}_2 &= -a_3x_1 + a_4x_2 + b_1x_1x_3 + u_2, \\ \dot{x}_3 &= a_5x_3 + b_2x_1x_2 + b_3x_1x_4 + u_3, \\ \dot{x}_4 &= a_6x_2 + a_7x_4 - b_4x_1x_3 + u_4, \end{aligned} \quad (9)$$

where u_i, $i = 1, 2, 3, 4$ are control inputs which will be satisfied the optimality conditions, obtained via the PMP. The proposed control strategy is to design the optimal control inputs u_1, u_2, u_3 and u_4 such that the state trajectories tend to an unstable equilibrium point in a given finite time interval $[0, t_f]$. The initial and final conditions are

$$\begin{cases} x_1(0) = x_{1,0}, x_1(t_f) = \bar{x}_1, \\ x_2(0) = x_{2,0}, x_2(t_f) = \bar{x}_2, \\ x_3(0) = x_{3,0}, x_3(t_f) = \bar{x}_3, \\ x_4(0) = x_{4,0}, x_4(t_f) = \bar{x}_4, \end{cases} \quad (10)$$

where $\bar{x}_i, (i = 1,2,3,4)$, denote the coordinates of the equilibrium points.

The objective functional to be minimized is defined as

$$J = \frac{1}{2}\int_0^{t_f} \sum_{i=1}^{4} (\alpha_i(\phi_i - \overline{\phi}_i)^2 + \beta_i {u_i}^2)\mathrm{d}t, \quad (11)$$

where α_i, β_i, $(i = 1, 2, 3, 4)$ are positive constants, $\phi_i = x_i$, $(i = 1, 2, 3, 4)$ and $\overline{\phi}_i = \bar{x}_i, (i = 1,2,3,4)$. It is note that, the cost function is a positive definite function of the variables ϕ_i, and u_i, $i = 1, \ldots, 4$. In particular, we will derive the fundamental nonlinear two-point boundary value problem arising in PMP. The corresponding Hamiltonian function will be

$$\begin{aligned} H = &-\frac{1}{2}[\alpha_1(x_1 - \bar{x}_1)^2 + \alpha_2(x_2 - \bar{x}_2)^2 + \alpha_3(x_3 - \bar{x}_3)^2 \\ &+ \alpha_4(x_4 - \bar{x}_4)^2 + \beta_1 {u_1}^2 + \beta_2 {u_2}^2 + \beta_3 {u_3}^2 + \beta_4 {u_4}^2] \\ &+ \lambda_1[a_1x_1 + a_2x_4 - x_2x_3 + u_1] + \lambda_2[-a_3x_1 + a_4x_2 \\ &+ b_1x_1x_3 + u_2] + \lambda_3[a_5x_3 + b_2x_1x_2 + b_3x_1x_4 + u_3] \\ &+ \lambda_4[a_6x_2 + a_7x_4 - b_4x_1x_3 + u_4], \end{aligned} \quad (12)$$

where, λ_i, $(i = 1, 2, 3, 4)$ are co-state variables. According to the PMP, we obtain the Hamiltonian equations:

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1},\quad \dot{\lambda}_2 = -\frac{\partial H}{\partial x_2},\quad \dot{\lambda}_3 = -\frac{\partial H}{\partial x_3},\quad \dot{\lambda}_4 = -\frac{\partial H}{\partial x_4}, \tag{13}$$

Substituting (12) into (13), the co-state equations can be derived in the form:

$$\begin{cases} \dot{\lambda}_1 = \alpha_1(x_1 - \bar{x}_1) - a_1\lambda_1 + a_3\lambda_2 - b_1\lambda_2 x_3 \\ \qquad - b_2\lambda_3 x_2 - b_3\lambda_3 x_4 + b_4\lambda_4 x_3, \\ \dot{\lambda}_2 = \alpha_2(x_2 - \bar{x}_2) + \lambda_1 x_3 - a_4\lambda_2 - b_2\lambda_3 x_1 - a_6\lambda_4, \\ \dot{\lambda}_3 = \alpha_3(x_3 - \bar{x}_3) + \lambda_1 x_2 - b_1\lambda_2 x_1 - a_5\lambda_3 + b_4\lambda_4 x_1 \\ \dot{\lambda}_4 = \alpha_4(x_4 - \bar{x}_4) - a_2\lambda_1 - b_3\lambda_3 x_1 - a_7\lambda_4. \end{cases} \tag{14}$$

The optimal control functions that have to be used are determined from the conditions $\frac{\partial H}{\partial u_i} = 0, (i = 1, 2, 3, 4)$. Hence, we get

$$u_i{}^* = \frac{\lambda_i}{\beta_i}, \quad (i = 1, 2, 3, 4). \tag{15}$$

Substituting from (15) into (9) we get the nonlinear controlled state equations:

$$\begin{cases} \dot{x}_1 = a_1x_1 + a_2x_4 - x_2x_3 + \frac{\lambda_1}{\beta_1}, \\ \dot{x}_2 = -a_3x_1 + a_4x_2 + b_1x_1x_3 + \frac{\lambda_2}{\beta_2}, \\ \dot{x}_3 = a_5x_3 + b_2x_1x_2 + b_3x_1x_4 + \frac{\lambda_3}{\beta_3} \\ \dot{x}_4 = a_6x_2 + a_7x_4 - b_4x_1x_3 + \frac{\lambda_4}{\beta_4}. \end{cases} \tag{16}$$

This system of nonlinear differential equations in addition to (14) form a complete system to solve the optimal control of the novel bounded 4D chaotic system. This system has the following boundary conditions

$$\begin{cases} x_1(0) = x_{1,0}, \quad x_1(t_f) = \bar{x}_1, \\ x_2(0) = x_{2,0}, \quad x_2(t_f) = \bar{x}_2, \\ x_3(0) = x_{3,0}, \quad x_3(t_f) = \bar{x}_3, \\ x_4(0) = x_{4,0}, \quad x_4(t_f) = \bar{x}_4, \\ \lambda_i(t_f) = 0, \quad i = 1, 2, 3, 4. \end{cases} \tag{17}$$

Then, by solving the nonlinear systems (14) and (16) with the boundary conditions of (17), we obtain the optimal control law and the optimal state trajectory.

3.2 Analysis and numerical simulation

In this section to demonstrate and verify the effectiveness of the theoretical analysis, we solve the systems (16) and (17). In the following numerical simulations, the MATLAB's `bvp4c` in-built solver is used to solve the systems. The initial values and system parameters are selected as $x_1(0) = -1$, $x_2(0) = 2$, $x_3(0) = -3$, $x_4(0) = 4$, in all simulations so that new bounded 4D chaotic system exhibits a chaotic behavior if no control is applied.

The initial values of co-states for E_i $(i = 1, 2, 3)$ are taken in Table 1. Also, the positive constants in cost function J, are chosen $\alpha_1 = 0.1$, $\alpha_2 = 0.1$, $\alpha_3 = 0.1$, $\alpha_4 = 0.1$, $\beta_1 = 2$, $\beta_2 = 2$, $\beta_3 = 2$, $\beta_4 = 2$. The behaviors of the states (x_1, x_2, x_3, x_4) of the controlled new bounded 4D chaotic system (1) with time are displayed in Figs. 3, 4 and 5.

Note that, the parameters of $a_1 = -0.16$, $a_2 = -0.35$, $a_3 = -0.75$, $a_4 = -0.15$, $a_5 = -0.45$, $a_6 = -0.5$, $a_7 = -0.4$, $b_1 = 1.5$, $b_2 = 1.1$, $b_3 = 1$, $b_4 = 1.15$. are relate to equilibrium zero (E_1), and the system corresponding with $a_1 = -0.3$, $a_2 = -0.5$, $a_3 = -0.6$, $a_4 = -0.1$, $a_5 = -0.1$, $a_6 = -0.6$, $a_7 = -0.15$, $b_1 = 1.2$, $b_2 = 1.5$, $b_3 = 2.5$, $b_4 = 0.4$ has three equilibria (E_1, E_2, E_3).

4 Adaptive control of the chaotic Zhang–Tang system

4.1 Design of the adaptive controller

In this section, we obtain new results for the adaptive control of the chaotic Zhang–Tang system based on the Lyapunov stability theory and from the conditions of the asymptotic stability of this system about its steady states.

Let us assume that we have the controlled system in the following form

$$\begin{aligned} \dot{x}_1 &= a_1x_1 + a_2x_4 - x_2x_3 + v_1, \\ \dot{x}_2 &= -a_3x_1 + a_4x_2 + b_1x_1x_3 + v_2, \\ \dot{x}_3 &= a_5x_3 + b_2x_1x_2 + b_3x_1x_4 + v_3, \\ \dot{x}_4 &= a_6x_2 + a_7x_4 - b_4x_1x_3 + v_4, \end{aligned} \tag{18}$$

Table 1 The initial values of co-states for different equilibrium points

E_i	$\lambda_1(0)$	$\lambda_2(0)$	$\lambda_3(0)$	$\lambda_4(0)$
E_1	1.0242×10^{-5}	-1.9990×10^{-5}	2.9949×10^{-5}	-3.9793×10^{-5}
E_2	2.1704×10^{-4}	-1.7096×10^{-4}	6.1×10^{-3}	-4.2409×10^{-4}
E_3	-2.3440×10^{-4}	3.8054×10^{-5}	6.1×10^{-3}	-3.8593×10^{-4}

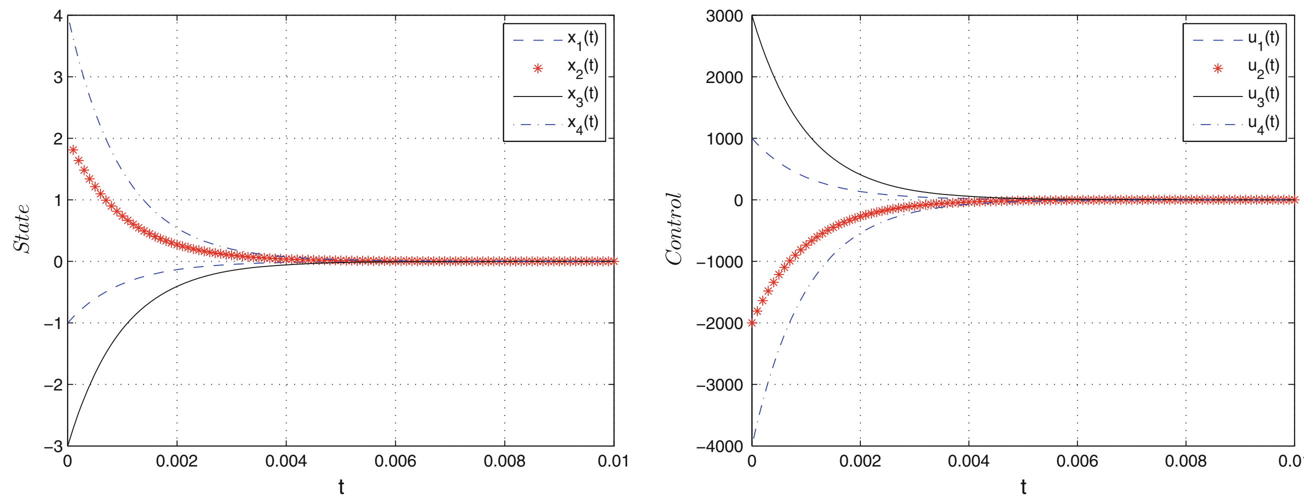

Fig. 3 The stabilized behavior of state and control functions for the equilibrium point E_1

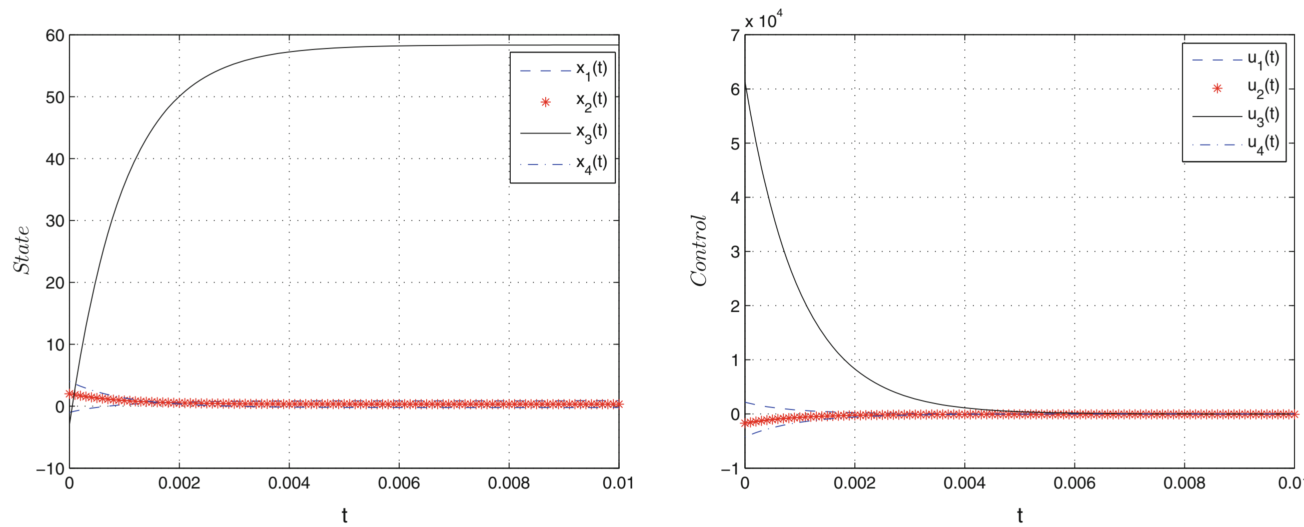

Fig. 4 The stabilized behavior of state and control functions for the equilibrium point E_2

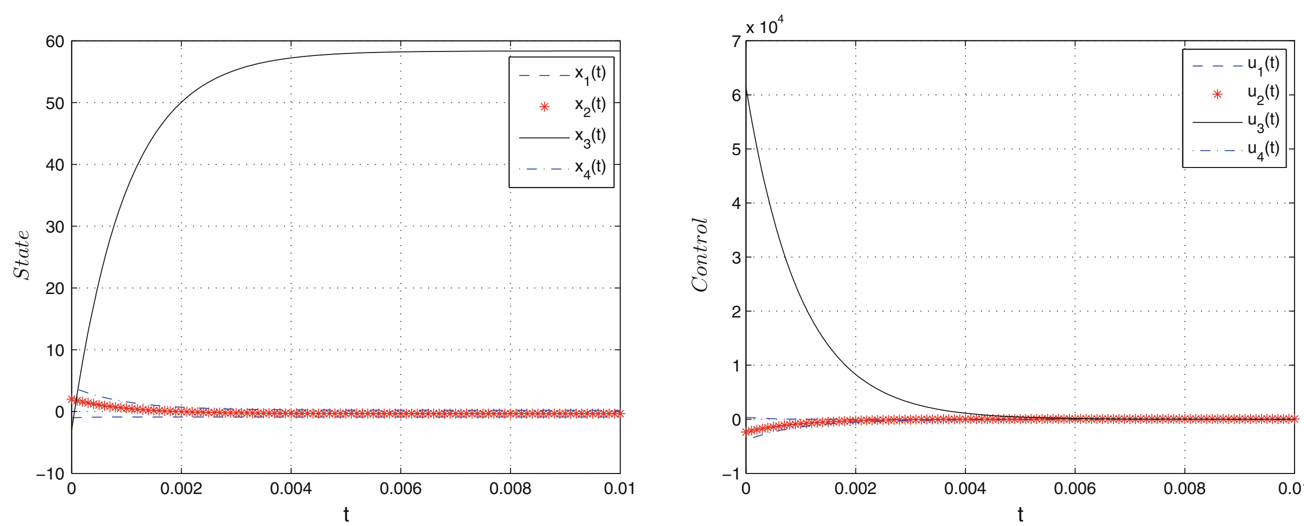

Fig. 5 The stabilized behavior of state and control functions for the equilibrium point E_3

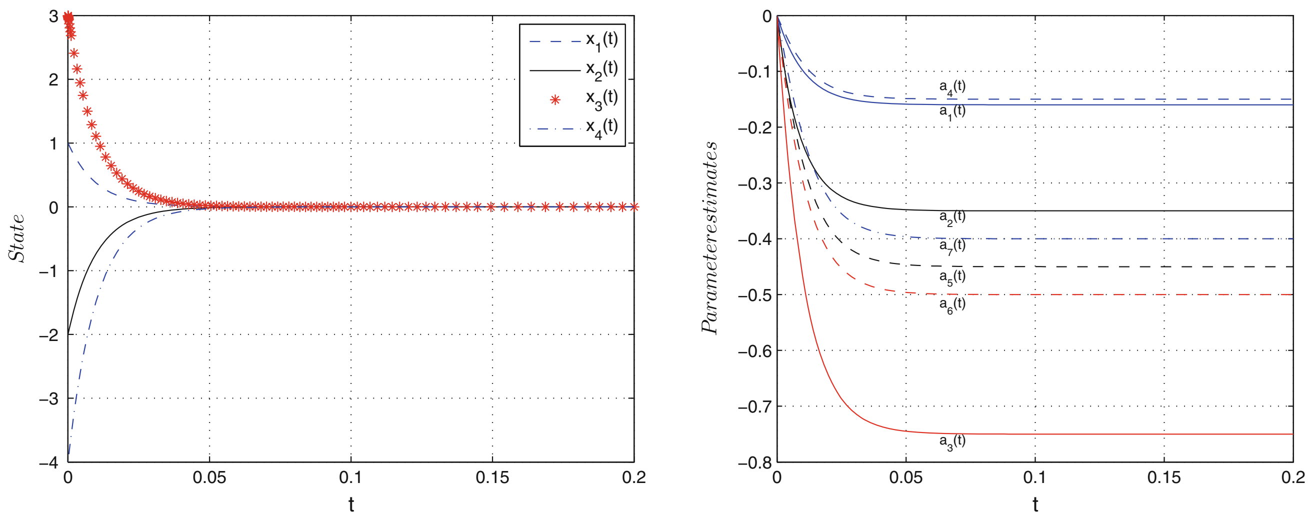

Fig. 6 Time history of state functions and parameter estimates for the equilibrium point E_1

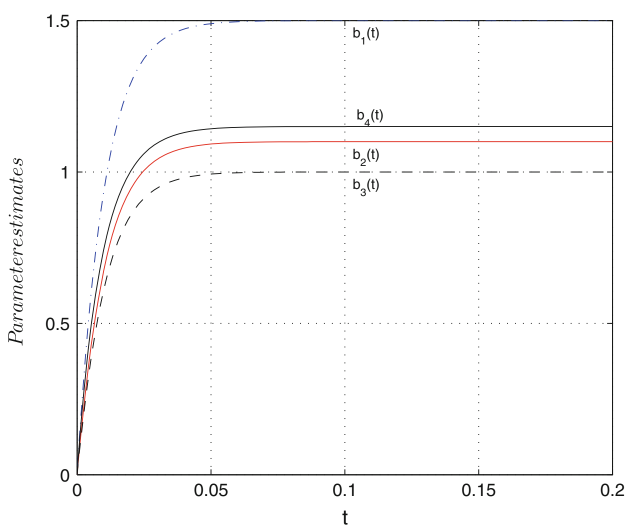

Fig. 7 Time history of parameter estimates for the equilibrium point E_1

where x_1, x_2, x_3 and x_4 are the states of the system, $a_i, (i = 1, 2, \ldots, 7), b_j, (j = 1, 2, 3, 4)$ are unknown parameters of the system, and v_1, v_2, v_3 and v_4 are the adaptive controllers to be designed.

Theorem 4.1 The novel chaotic system (18) with unknown system parameters is globally and asymptotically stabilized for all initial states $(x_1(0), x_2(0), x_3(0), x_4(0)) \in \mathbb{R}^4$ by the adaptive control law:

$$
\begin{aligned}
v_1 &= -\hat{a}_1 x_1 - \hat{a}_2 x_4 + x_2 x_3 - k_1(x_1 - \bar{x}_1),\\
v_2 &= \hat{a}_3 x_1 - \hat{a}_4 x_2 - \hat{b}_1 x_1 x_3 - k_2(x_2 - \bar{x}_2),\\
v_3 &= -\hat{a}_5 x_3 - \hat{b}_2 x_1 x_2 - \hat{b}_3 x_1 x_4 - k_3(x_3 - \bar{x}_3),\\
v_4 &= -\hat{a}_6 x_2 - \hat{a}_7 x_4 + \hat{b}_4 x_1 x_3 - k_4(x_4 - \bar{x}_4),
\end{aligned}
\tag{19}
$$

and the following parameter estimation update law

$$
\begin{aligned}
\dot{\hat{a}}_1 &= (x_1 - \bar{x}_1)x_1 + k_5(a_1 - \hat{a}_1),\\
\dot{\hat{a}}_2 &= (x_1 - \bar{x}_1)x_4 + k_6(a_2 - \hat{a}_2),\\
\dot{\hat{a}}_3 &= -(x_2 - \bar{x}_2)x_1 + k_7(a_3 - \hat{a}_3),\\
\dot{\hat{a}}_4 &= (x_2 - \bar{x}_2)x_2 + k_8(a_4 - \hat{a}_4),\\
\dot{\hat{a}}_5 &= (x_3 - \bar{x}_3)x_3 + k_9(a_5 - \hat{a}_5),\\
\dot{\hat{a}}_6 &= (x_4 - \bar{x}_4)x_2 + k_{10}(a_6 - \hat{a}_6),\\
\dot{\hat{a}}_7 &= (x_4 - \bar{x}_4)x_4 + k_{11}(a_7 - \hat{a}_7),\\
\dot{\hat{b}}_1 &= (x_2 - \bar{x}_2)x_1 x_3 + k_{12}(b_1 - \hat{b}_1),\\
\dot{\hat{b}}_2 &= (x_3 - \bar{x}_3)x_1 x_4 + k_{13}(b_2 - \hat{b}_2),\\
\dot{\hat{b}}_3 &= (x_3 - \bar{x}_3)x_1 x_4 + k_{14}(b_3 - \hat{b}_3),\\
\dot{\hat{b}}_4 &= -(x_4 - \bar{x}_4)x_1 x_3 + k_{15}(b_4 - \hat{b}_4),
\end{aligned}
\tag{20}
$$

where $\hat{a}_i, (i = 1, 2, \ldots, 7), \hat{b}_j, (j = 1, 2, 3, 4)$ are estimate values of uncertain parameters a_i, b_j and $k_r(r = 1, 2, \ldots, 15)$ are positive constants, respectively.

Proof Substituting (19) into (18), we get the closed-loop system as

$$
\begin{cases}
\dot{x}_1 = (a_1 - \hat{a}_1)x_1 + (a_2 - \hat{a}_2)x_4 - k_1(x_1 - \bar{x}_1),\\
\dot{x}_2 = -(a_3 - \hat{a}_3)x_1 + (a_4 - \hat{a}_4)x_2 + (b_1 - \hat{b}_1)x_1 x_3 - k_2(x_2 - \bar{x}_2),\\
\dot{x}_3 = (a_5 - \hat{a}_5)x_3 + (b_2 - \hat{b}_2)x_1 x_2 + (b_3 - \hat{b}_3)x_1 x_4 - k_3(x_3 - \bar{x}_3),\\
\dot{x}_4 = (a_6 - \hat{a}_6)x_2 + (a_7 - \hat{a}_7)x_4 - (b_4 - \hat{b}_4)x_1 x_3 - k_4(x_4 - \bar{x}_4),
\end{cases}
\tag{21}
$$

For the derivation of the update law for adjusting the parameter estimates, the Lyapunov approach is used. We consider the quadratic Lyapunov function

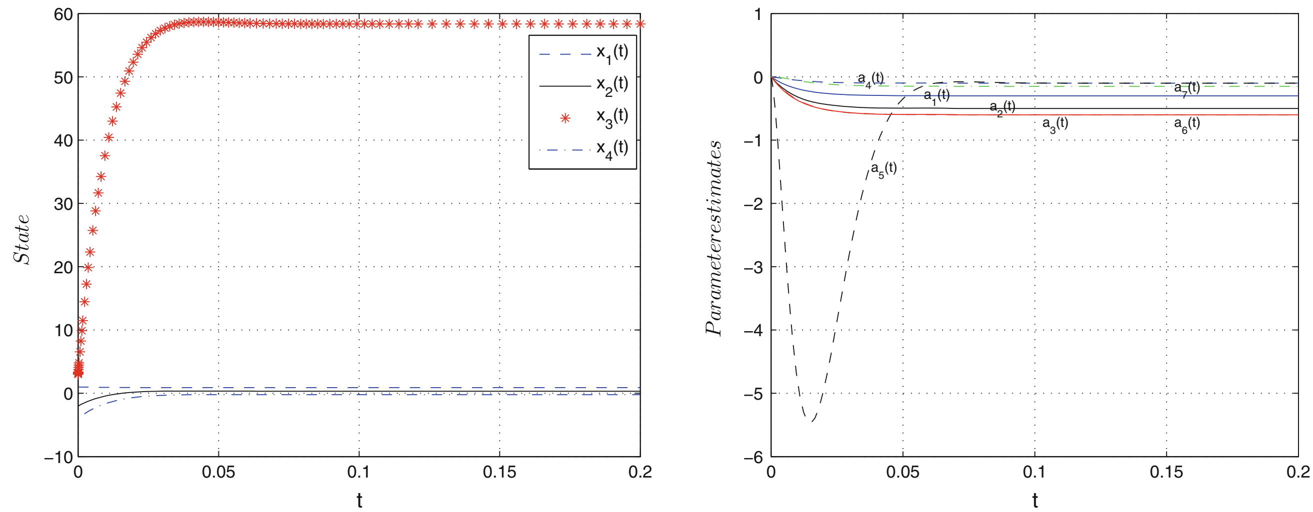

Fig. 8 Time history of state functions and parameter estimates for the equilibrium point E_2

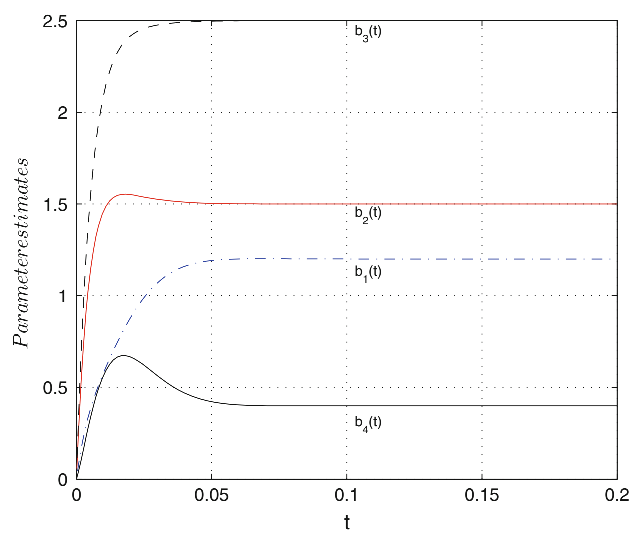

Fig. 9 Time history of parameter estimates for the equilibrium point E_2

$$V(x_1,x_2,x_3,\tilde{a}_i,\tilde{b}_j)=\frac{1}{2}((x_1-\overline{x}_1)^2+(x_2-\overline{x}_2)^2+(x_3-\overline{x}_3)^2 + (x_4-\overline{x}_4)^2+\tilde{a}_1^2+\tilde{a}_2^2+\tilde{a}_3^2+\tilde{a}_4^2+\tilde{a}_5^2 + \tilde{a}_6^2+\tilde{a}_7^2+\tilde{b}_1^2+\tilde{b}_2^2+\tilde{b}_3^2+\tilde{b}_4^2), \tag{22}$$

where the variables $\tilde{a}_i = a_i - \hat{a}_i, \tilde{b}_j = b_j - \hat{b}_j$, $(i = 1, 2, \ldots, 7), (j = 1, 2, 3, 4)$.

Taking time derivative of the Lyapunov function V, we obtain

$$\dot{V} =(x_1-\overline{x}_1)\dot{x}_1+(x_2-\overline{x}_2)\dot{x}_2+(x_3-\overline{x}_3)\dot{x}_3+(x_4-\overline{x}_4)\dot{x}_4 + \tilde{a}_1\dot{\hat{a}}_1+\tilde{a}_2\dot{\hat{a}}_2+\tilde{a}_3\dot{\hat{a}}_3+\tilde{a}_4\dot{\hat{a}}_4+\tilde{a}_5\dot{\hat{a}}_5+\tilde{a}_6\dot{\hat{a}}_6+\tilde{a}_7\dot{\hat{a}}_7 + \tilde{b}_1\dot{\hat{b}}_1+\tilde{b}_2\dot{\hat{b}}_2+\tilde{b}_3\dot{\hat{b}}_3+\tilde{b}_4\dot{\hat{b}}_4. \tag{23}$$

Substituting (21) and (20) into (23), the time derivative of the Lyapunov function becomes

$$\dot{V} = -k_1(x_1-\overline{x}_1)^2-k_2(x_2-\overline{x}_2)^2-k_3(x_3-\overline{x}_3)^2 - k_4(x_4-\overline{x}_4)^2-k_5(a_1-\hat{a}_1)^2-k_6(a_2-\hat{a}_2)^2 - k_7(a_3-\hat{a}_3)^2-k_8(a_4-\hat{a}_4)^2-k_9(a_5-\hat{a}_5)^2 - k_{10}(a_6-\hat{a}_6)^2-k_{11}(a_7-\hat{a}_7)^2-k_{12}(b_1-\hat{b}_1)^2 - k_{13}(b_2-\hat{b}_2)^2-k_{14}(b_3-\hat{b}_3)^2-k_{15}(b_4-\hat{b}_4)^2. \tag{24}$$

The Lyapunov function V is positive definite on $\mathbb{R}^{15}$ and its derivative $\dot{V}$ is negative definite on $\mathbb{R}^{15}$, according to the Lyapunov stability theory [34], the equilibrium solution of the controlled system (18) is asymptotically stable, namely, the controlled system (18) can asymptotically converge to the equilibrium $E(\overline{x}_1,\overline{x}_2,\overline{x}_3,\overline{x}_4,\hat{a}_1,\hat{a}_2,\hat{a}_3,\hat{a}_4, \hat{a}_5,\hat{a}_6,\hat{a}_7, \hat{b}_1,\hat{b}_2,\hat{b}_3,\hat{b}_4)$ with the adaptive control law (19) and the parameter estimation update law (20). This completes the proof. □

4.2 Numerical results

For the numerical simulations, we solve the controlled novel chaotic system (18) with the adaptive control law (19) and the parameter update law (20). In the following numerical simulations, the MATLAB's `ode45` in-built solver is used to solve the systems. The initial values and system parameters are selected as $x_1(0) = -1$, $x_2(0) = 2$, $x_3(0) = -3$, $x_4(0) = 4$. For the adaptive and update laws, we take $k_i = 50$ for $(i = 1, 2, \ldots, 15)$.

Suppose that the initial values of the parameter estimates are chosen as $a_i = 0, b_j = 0, i = 1, 2\ldots 7, j = 1, 2, 3, 4$. Figures 6, 7, 8, 9, 10 and 11 show that the controlled

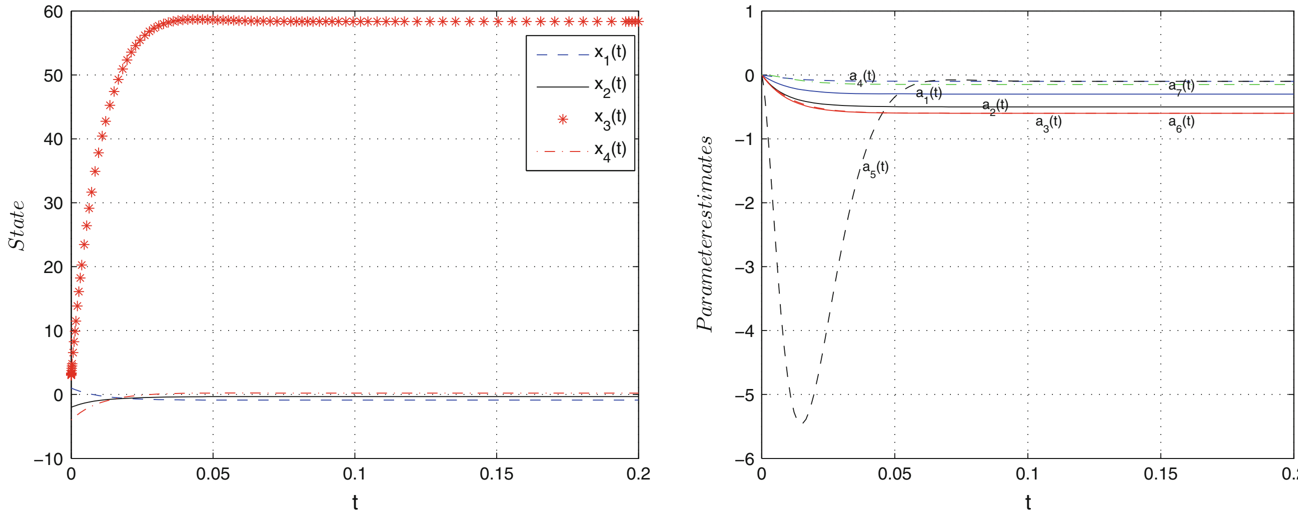

Fig. 10 Time history of state functions and parameter estimates for the equilibrium point E_3

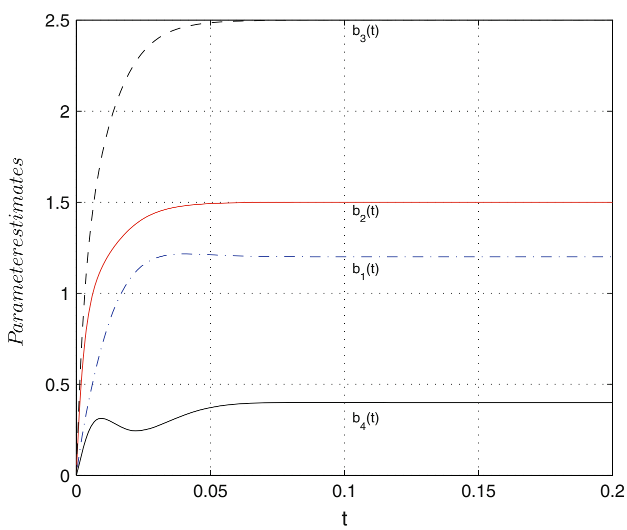

Fig. 11 Time history of parameter estimates for the equilibrium point E_3

chaotic system (18) converges to E_i $(i = 1, \ldots, 3)$ asymptotically with time. Also, these figures show that the parameter estimates $a_i, (i = 1, 2, \ldots, 7), b_j, (j = 1, 2, 3, 4)$ converge to the system parameter values $a_1 = -0.16$, $a_2 = -0.35$, $a_3 = -0.75$, $a_4 = -0.15$, $a_5 = -0.45$, $a_6 = -0.5$, $a_7 = -0.4$, $b_1 = 1.5$, $b_2 = 1.1$, $b_3 = 1$, $b_4 = 1.15$.

5 Conclusion

In this paper, we have studied the problem of optimal control and adaptive control of the chaotic Zhang–Tang system. As the system can be chaotic, it makes sense to determine whether an adequate control method can be applied to control this chaos. To this end, we considered the problems of optimal control of chaos and of parameter estimation for the Zhang–Tang system. Based on the PMP, this system is stabilized to its equilibrium points. The stability and instability of the steady-states of this system are studied using the linear stability approach. In addition, we proposed Lyapunov stability to control the new autonomous chaotic system by a feedback control approach. In fact, we used the feedback control approach for estimating the system of unknown parameters. Numerical simulations demonstrate the effectiveness of the analytical results.

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