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ORIGINAL ARTICLE

Exponential synchronization of stochastic chaotic neural networks with mixed time delays and Markovian switching

Cheng-De Zheng · Huaguang Zhang · Zhanshan Wang

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Abstract This paper studies the exponential synchronization problem for a class of stochastic perturbed chaotic neural networks with both Markovian jump parameters and mixed time delays. The mixed delays consist of discrete and distributed time-varying delays. At first, based on a Halanay-type inequality for stochastic differential equations, by virtue of drive-response concept and time-delay feedback control techniques, a delay-dependent sufficient condition is proposed to guarantee the exponential synchronization of two identical Markovian jumping chaotic-delayed neural networks with stochastic perturbation. Then, by utilizing the Jensen integral inequality and a novel Lemma, another delay-dependent criterion is established to achieve the globally stochastic robust synchronization. With some parameters being fixed in advance, these conditions can be solved numerically by employing the Matlab software. Finally, a numerical example with their simulations is provided to illustrate the effectiveness of the presented synchronization scheme.

Keywords Halanay-type inequality · Exponential synchronization · Chaotic-delayed neural networks · Markovian jump

C.-D. Zheng (✉)
School of Science, Dalian Jiaotong University,
Dalian 116028, People's Republic of China
e-mail: zhengchengde@dl.cn

H. Zhang · Z. Wang
School of Information Science and Engineering, Northeastern University, Shenyang 110004, People's Republic of China
e-mail: hgzhang@ieee.org

Z. Wang
e-mail: zhanshan_wang@163.com

1 Introduction

In 1990, Pecora and Carroll [10] addressed the synchronization of chaotic systems using a drive-response concept. The idea is to use the output of the drive system to control the response system so that they oscillate in a synchronized manner. Research on the synchronization of chaotic activity has broadened considerably in the last few decades. Besides the original master–slave mechanism for chaos synchronization, a wide variety of approaches have been presented for the synchronization of chaotic systems which include linear feedback control, nonlinear feedback control, impulsive control method, and adaptive design control, among many others. Synchronization in chaotic systems has been utilized in many applications. It was used to understand self-organization behavior in the brain as well as in ecological systems and has been applied to secure communications, among others [13–15, 17, 23, 27, 28].

Meanwhile, many neural networks may experience abrupt changes in their structure and parameters caused by some phenomena such as component failures or repairs, changing subsystem interconnections, and abrupt environmental disturbances. In this situation, there exist finite modes in the neural networks, and the modes may be switched (or jumped) from one to another at different times. These kinds of systems are known as Markovian jump neural networks. When noise disturbances are considered in Markovian jump neural networks, this class of neural networks is usually called Markovian jump stochastic neural networks or stochastic neural networks with Markovian switching. It is known that a Markovian jump stochastic neural network is more complicated and comprises a general stochastic neural network as its special case. Owing to the practical importance, many papers have recently devoted to study the stability analysis issue for

Markovian jump stochastic neural networks [4, 17, 18, 21, 24, 27]. However, up to now, the synchronization problem for stochastic chaotic neural networks with Markovian switching has received little research attention, despite its practical importance. This situation motivates our present investigation.

This paper studies the exponential synchronization problem for a class of stochastic perturbed chaotic neural networks with both Markovian jump parameters and mixed delays. The mixed delays consist of discrete and distributed time-varying delays. Firstly, by using drive-response concept, a Halanay-type inequality for stochastic differential equations and time-delay feedback control techniques, a delay-dependent sufficient condition is obtained to guarantee the exponential synchronization of two identical Markovian jumping chaotic-delayed neural networks with stochastic perturbation. Next, by means of the Jensen integral inequality and a novel Lemma, another delay-dependent result is established to achieve the globally stochastic robust synchronization. With some parameters being fixed in advance, these conditions are expressed in terms of linear matrix inequalities, which can be solved numerically by employing the Matlab software. Finally, a numerical example is provided to illustrate the effectiveness of the presented synchronization scheme.

Notations Throughout this paper, W^T,W^{-1} denote the transpose and the inverse of a square matrix W, respectively. $W>0(<0)$ denotes a positive (negative) definite symmetric matrix, I denotes the identity matrix with compatible dimension, the symbol "$*$" denotes a block that is readily inferred by symmetry. The shorthand $\mathrm{col}\{M_1, M_2, \ldots, M_k\}$ denotes a column matrix with the matrices $M_1, M_2, \ldots, M_k$. $\mathrm{sym}(A)$ is defined as $A+A^T$, $\mathrm{diag}\{\cdot\}$ stands for a diagonal or block-diagonal matrix. For $\tau>0, \mathcal{C}([-\tau,0];\mathbb{R}^n)$ denotes the family of continuous functions ϕ from $[-\tau, 0]$ to $\mathbb{R}^n$ with the norm $||\phi|| = \sup_{-\tau\leq s\leq 0}|\phi(s)|$. Moreover, let $(\Omega,\mathbb{F},\mathbb{P})$ be a complete probability space with a filtration $\{\mathbb{F}_t\}_{t\geq 0}$ satisfying the usual conditions and $\mathbb{E}\{\cdot\}$ representing the mathematical expectation. Denote by $\mathcal{C}^p_{\mathbb{F}_0}([-\tau,0];\mathbb{R}^n)$ the family of all bounded, $\mathbb{F}_0$-measurable, $\mathcal{C}([-\tau,0];\mathbb{R}^n)$-valued random variables $\xi=\{\xi(s): -\tau\leq s\leq 0\}$ such that $\sup_{-\tau\leq s\leq 0}\mathbb{E}|\xi(s)|^p<\infty$. $||\cdot||$ stands for the Euclidean norm; Matrices, if not explicitly stated, are assumed to have compatible dimensions.

2 Problem description and preliminaries

In this paper, we consider the following stochastic neural networks with both discrete and distributed time-varying delays

$$
\begin{aligned}
\mathrm{d}x(t) = \Bigg(& -\tilde{\beta}(x(t)) + A(t,\eta(t))\tilde{f}(x(t)) \\
& + B(t,\eta(t))\tilde{f}(x(t-\tau(t,\eta(t)))) \\
& + G(t,\eta(t))\int_{t-\varrho(t,\eta(t))}^{t}\tilde{f}(x(s))\mathrm{d}s + \Im\Bigg)\mathrm{d}t \\
& + \tilde{\rho}(t,x(t),x(t-\tau(t,\eta(t))))\mathrm{d}\omega(t), \\
x(t) = \varphi_1(t), & \; t\in[-\hat{\tau},0],
\end{aligned} \tag{1}
$$

where $x(t)=(x_1(t),x_2(t),\ldots,x_n(t))^T\in\mathbb{R}^n$ is the state vector associated with n neurons, $\tilde{\beta}(x(t)) = (\tilde{\beta}_1(x_1(t)), \tilde{\beta}_2(x_2(t)), \ldots, \tilde{\beta}_n(x_n(t)))^T\in\mathbb{R}^n$ is the behaved function. $A(t,\eta(t)) = A(\eta(t)) + \Delta A(t,\eta(t))$, $B(t,\eta(t)) = B(\eta(t)) + \Delta B(t,\eta(t))$, $G(t,\eta(t)) = G(\eta(t)) + \Delta G(t,\eta(t))$ are the interconnection matrices representing the weight coefficients of the neurons. $A(\eta(t))$, $B(\eta(t))$, $G(\eta(t))$ are known real constant matrices. $\Delta A(t,\eta(t)), \Delta B(t,\eta(t))$, $\Delta G(t,\eta(t))$ are the time-varying structured uncertainties. $\tilde{f}(x(t)) = \left(\tilde{f}_1(x_1(t)),\tilde{f}_2(x_2(t)),\ldots,\tilde{f}_n(x_n(t))\right)^T\in\mathbb{R}^n$ denotes the neural activation function. The bounded functions $\tau(t, \eta(t))$, $\varrho(t, \eta(t))$ represent unknown time-varying delays with $0\leq\tau(t,\eta(t))\leq\bar{\tau}(\eta(t))\leq\bar{\tau}, \dot{\tau}(t,\eta(t))\leq\tau_d(\eta(t))$, $0\leq\varrho(t,\eta(t))\leq\bar{\varrho}(\eta(t))\leq\bar{\varrho}$, where $\bar{\tau}(\eta(t)), \bar{\tau}, \bar{\varrho}(\eta(t)), \bar{\varrho}$ are positive scalars, $\hat{\tau}=\max\{\bar{\tau},\bar{\varrho}\}$. $\Im=(I_1,I_2,\ldots,I_n)^T$ is an external input, $\varphi_1(t)$ is a real-valued initial vector function that is continuous on the interval $[-\hat{\tau},0]$. $\{\eta(t), t\geq 0\}$ is a homogeneous, finite-state Markovian process with right continuous trajectories and taking values in finite set $\mathcal{N}=\{1,2,\ldots,N\}$ based on given probability space $(\Omega,\mathbb{F},\mathbb{P})$ and the initial model η_0. $\tilde{\rho}(t,x(t),x(t-\tau(t,\eta(t))))$ is called the noise intensity vector. $\omega(t)$ is a one-dimensional Brownian motion defined on space $(\Omega,\mathbb{F},\mathbb{P})$ with $\mathbb{E}\{\mathrm{d}\omega(t)\}=0$, $\mathbb{E}\{[\mathrm{d}\omega(t)]^2\}=\mathrm{d}t$. Let $\Pi=[\pi_{ij}]_{N\times N}$ denote the transition rate matrix with transition probability:

$$
\mathbb{P}(\eta(t+\delta)=j|\eta(t)=i) = \begin{cases} \pi_{ij}\delta+o(\delta), & i\neq j, \\ 1+\pi_{ii}\delta+o(\delta), & i=j, \end{cases}
$$

where $\delta>0, \lim_{\delta\to 0^+}\frac{o(\delta)}{\delta}=0$ and π_{ij} is the transition rate from mode i to mode j satisfying $\pi_{ij}\geq 0$ for $i\neq j$ with

$$
\pi_{ii} = -\sum_{j=1,j\neq i}^{N}\pi_{ij}, \quad i,j\in\mathcal{N}.
$$

For convenience, each possible value of $\eta(t)$ is denoted by $i(i\in\mathcal{N})$ in the sequel. Then, we have

$$
\begin{aligned}
& A_i = A(\eta(t)), \quad B_i = B(\eta(t)), \quad G_i = G(\eta(t)), \\
& \Delta A_i(t) = \Delta A(t,\eta(t)), \quad \Delta B_i(t) = \Delta B(t,\eta(t)), \\
& \Delta G_i(t) = \Delta G(t,\eta(t)).
\end{aligned}
$$

Throughout this paper, we make the following assumptions:

Assumption 1 The noise intensity vector is assumed to be of the form:

$$\tilde{\rho}(t, x(t), x(t-\tau_i(t))) = C_i(t)x(t) + D_i(t)x(t-\tau_i(t)),$$

where

$$C_i(t) = C_i + \Delta C_i(t), D_i(t) = D_i + \Delta D_i(t),$$

with C_i, D_i being known real constant matrices.

Assumption 2 The admissible parameter uncertainties are assumed to be of the following form:

$$\begin{aligned}&[\,\Delta A_i(t) \quad \Delta B_i(t) \quad \Delta G_i(t) \quad \Delta C_i(t) \quad \Delta D_i(t)\,]\\ &\quad = E_i\Phi_i(t)[\,H_{1i} \quad H_{2i} \quad H_{3i} \quad H_{4i} \quad H_{5i}\,],\end{aligned}$$

where $E_i, H_{ji}(j = 1, \ldots, 5)$ are known real constant matrices with appropriate dimensions, and $\Phi_i(t) \in \mathbb{R}^{n\times n}$ is the time-varying uncertain matrix satisfying $\Phi_i(t)^T\Phi_i(t) \le I$ for any $t > 0$.

Assumption 3 Each $\tilde{\beta}_j(\zeta)$ is differentiable and satisfies the following condition

$$0 < \lambda_j \le \frac{\tilde{\beta}_j(\xi) - \tilde{\beta}_j(\zeta)}{\xi - \zeta} \le \delta_j, \quad \forall \xi, \quad \zeta \in \mathbb{R}, \xi \ne \zeta,$$

where λ_j, δ_j are known positive real constants.

For notational simplicity, we denote $\Delta = \mathrm{diag}\{\delta_1, \ldots, \delta_n\}$, $\Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$.

Assumption 4 Each neural activation function $\tilde{f}_j(\cdot)(j = 1, 2, \ldots, n)$ is bounded, differentiable and satisfies the following condition

$$\gamma_j \le \frac{\tilde{f}_j(\xi) - \tilde{f}_j(\zeta)}{\xi - \zeta} \le \sigma_j, \quad \forall \xi, \zeta \in \mathbb{R}, \xi \ne \zeta,$$

where γ_j, σ_j are known real constants.

We denote $\Gamma = \mathrm{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$, $\Sigma = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$, $\Theta = \mathrm{diag}\{\theta_1, \theta_2, \ldots, \theta_n\}$, where $\theta_j = \max\{|\gamma_j|, |\sigma_j|\}$.

The system (1) is considered as a drive system. Based on the drive-response concept for synchronization of coupled chaotic systems, which was initially proposed by Pecora and Carroll in [10], the corresponding response system of (1) is given in the following form:

$$\begin{aligned}\mathrm{d}y(t) = &\Bigg(-\tilde{\beta}(y(t)) + A_i(t)\tilde{f}(y(t)) + B_i(t)\tilde{f}(y(t-\tau_i(t))).\\ &+ G_i(t)\int_{t-\varrho_i(t)}^{t} \tilde{f}(y(s))\mathrm{d}s + \Im + u_i(t)\Bigg)\mathrm{d}t\\ &+ \tilde{\rho}(t, y(t), y(t-\tau_i(t)))\mathrm{d}\omega(t),\\ y(t) = &\varphi_2(t), t \in [-\hat{\tau}, 0],\end{aligned} \tag{2}$$

where $y(t) = (y_1(t), y_2(t), \ldots, y_n(t))^T \in \mathbb{R}^n$ is the state vector associated with n neurons, $u_i(t) = (u_{i1}(t), \ldots, u_{in}(t))^T \in \mathbb{R}^n$ is the state feedback controller given to achieve the exponential synchronization between drive-response system, $\varphi_2(t)$ is a real-valued continuous vector function on the interval $[-\hat{\tau}, 0]$.

In order to investigate the problem of exponential synchronization for the chaotic-delayed neural networks with stochastic perturbation, $e_j(t) = y_j(t) - x_j(t)$ is defined as the synchronization error, where $x_j(t)$ and $y_j(t)$ are the ith state variables of drive system (1) and response system (2), respectively. Therefore, the error dynamical system between (1) and (2) is given as follows:

$$\begin{aligned}\mathrm{d}e(t) = &\Bigg(-\beta(e(t)) + A_i(t)f(e(t)) + B_i(t)f(e(t-\tau_i(t))).\\ &+ G_i(t)\int_{t-\varrho_i(t)}^{t} f(e(s))\mathrm{d}s + u_i(t)\Bigg)\mathrm{d}t\\ &+ \rho(t, e(t), e(t-\tau_i(t)))\mathrm{d}\omega(t) \doteq \chi_i(t)\mathrm{d}t + \rho_i(t)\mathrm{d}\omega(t),\\ e(t) = &\varphi(t) \doteq \varphi_2(t) - \varphi_1(t), t \in [-\hat{\tau}, 0],\end{aligned} \tag{3}$$

where $e(t) = (e_1(t), e_2(t), \ldots, e_n(t))^T, \beta(e(t)) = \big(\tilde{\beta}_1(y_1(t)) - \tilde{\beta}_1(x_1(t)), \tilde{\beta}_2(y_2(t)) - \tilde{\beta}_2(x_2(t)), \ldots, \tilde{\beta}_n(y_n(t)) - \tilde{\beta}_n(x_n(t))\big)^T$, $f(e(t)) = \big(\tilde{f}_1(y_1(t)) - \tilde{f}_1(x_1(t)), \tilde{f}_2(y_2(t)) - \tilde{f}_2(x_2(t)), \ldots, \tilde{f}_n(y_n(t)) - \tilde{f}_n(x_n(t))\big)^T$, $f(e(t-\tau_i(t))) = \big(\tilde{f}_1(y_1(t-\tau_i(t))) - \tilde{f}_1(x_1(t-\tau_i(t))), \tilde{f}_2(y_2(t-\tau_i(t))) - \tilde{f}_2(x_2(t-\tau_i(t))), \ldots, \tilde{f}_n(y_n(t-\tau_i(t))) - \tilde{f}_n(x_n(t-\tau_i(t)))\big)^T, \rho_i(t) \doteq \rho(t, e(t), e(t-\tau_i(t))) = \tilde{\rho}(t, y(t), y(t-\tau_i(t))) - \tilde{\rho}(t, x(t), x(t-\tau_i(t))) = C_i(t)e(t) + D_i(t)e(t-\tau_i(t))$.

In this paper, the control input vector with state feedback is designed as follows:

$$u_i(t) = Y_{1i}e(t) + Y_{2i}e(t-\tau_i(t)). \tag{4}$$

From Assumptions 2,3, we obtain that

$$\lambda_j \le \frac{\beta_j(\zeta)}{\zeta} \le \delta_j, f_j(0) = 0, \gamma_j \le \frac{f_j(\zeta)}{\zeta} \le \sigma_j, \quad \forall \zeta \in \mathbb{R}, \zeta \ne 0. \tag{5}$$

Therefore, it follows from [3] that system (3) admits a trivial solution $e(t) = 0$.

To prove our main theorem, we need the following preliminaries.

Definition 1 Let $\psi : \mathbb{R} \to \mathbb{R}$ be a continuous function, the upper right Dini derivative of $\psi(t)$ is defined as

$$D^+\psi(t) = \lim_{\Delta t \to 0^+} \sup \frac{\psi(t+\Delta t) - \psi(t)}{\Delta t}.$$

Definition 2 The drive system (1) and the response system (2) are said to be exponentially synchronized if, for a suitably designed feedback controller, there exist constants $\upsilon \geq 1$ and $\vartheta > 0$ such that

$$\mathbb{E}\left\{||y(t) - x(t)||^2\right\} \leq \upsilon \mathbb{E}\left\{||y(0) - x(0)||^2\right\} e^{-\vartheta t}$$

for any $t \geq 0$, and the constant ϑ is defined as the exponential synchronization rate.

The development of the work in this paper requires the following lemmas.

Lemma 1 (see [8, 12, 16, 25]). *Let A, B and C be real constant matrices with appropriate dimensions, matrix $\Phi(t)$ satisfies $\Phi(t)\Phi(t)^T \leq I$. Then for any matrix $P > 0$ and scalar $\varepsilon > 0$, we have the following inequalities:*

(1) $A^T B + B^T A \leq A^T P^{-1} A + B^T P B$;

(2) $(A + B\Phi(t)C)^T P^{-1} (A + B\Phi(t)C) \leq A^T (P - \varepsilon BB^T)^{-1} A + \varepsilon C^T C$ *if* $P - \varepsilon BB^T > 0$.

Lemma 2 (see [5], Jensen integral inequality). *For any positive symmetric constant matrix $M \in \mathbb{R}^{n \times n}$, scalars $r_1 < r_2$ and vector function $\varpi : [r_1, r_2] \to \mathbb{R}^n$ such that the integrations concerned are well defined, the following matrix inequality holds:*

$$\left(\int_{r_1}^{r_2} \varpi(s)\mathrm{d}s\right)^T M \left(\int_{r_1}^{r_2} \varpi(s)\mathrm{d}s\right) \leq (r_2 - r_1) \int_{r_1}^{r_2} \varpi^T(s) M \varpi(s) \mathrm{d}s.$$

From Lemma 2 of [26], we can easily establish the following Lemma 3, which plays an important role in obtaining our delay-dependent stability result.

Lemma 3 *Set λ_j, μ_j be scalars satisfying $\lambda_j \leq 1, \lambda_j + \mu_j \leq 4 (j = 1, 2), \Sigma_l (l = 1, \ldots, 4)$ be any nonnegative symmetric matrices, $\theta(t)$, $v(t)$ be real functions such that $\theta : \mathbb{R}^+ \to (\underline{\theta}, \bar{\theta}), v : \mathbb{R}^+ \to (\underline{v}, \bar{v})$, then we have*

$$\begin{aligned} &-\frac{\Sigma_1}{\theta(t) - \underline{\theta}} - \frac{\Sigma_2}{\bar{\theta} - \theta(t)} - \frac{\Sigma_3}{v(t) - \underline{v}} - \frac{\Sigma_4}{\bar{v} - v(t)} \\ &\leq \max\left\{ -\frac{\lambda_1\Sigma_1 + \mu_1\Sigma_2}{\bar{\theta} - \underline{\theta}} - \frac{\lambda_2\Sigma_3 + \mu_2\Sigma_4}{\bar{v} - \underline{v}}, \right. \\ &\quad -\frac{\lambda_1\Sigma_1 + \mu_1\Sigma_2}{\bar{\theta} - \underline{\theta}} - \frac{\mu_2\Sigma_3 + \lambda_2\Sigma_4}{\bar{v} - \underline{v}}, -\frac{\mu_1\Sigma_1 + \lambda_1\Sigma_2}{\bar{\theta} - \underline{\theta}} \\ &\quad \left. -\frac{\lambda_2\Sigma_3 + \mu_2\Sigma_4}{\bar{v} - \underline{v}}, -\frac{\mu_1\Sigma_1 + \lambda_1\Sigma_2}{\bar{\theta} - \underline{\theta}} - \frac{\mu_2\Sigma_3 + \lambda_2\Sigma_4}{\bar{v} - \underline{v}} \right\}. \end{aligned} \tag{6}$$

If we set $\lambda_j = \mu_j = 1 (j = 1,2)$ in (6), noticing that

$$\begin{aligned} -\frac{\Sigma_1 + \Sigma_2}{\bar{\theta} - \underline{\theta}} - \frac{\Sigma_3 + \Sigma_4}{\bar{v} - \underline{v}} \leq \max\left\{ -\frac{\Sigma_1}{\bar{\theta} - \underline{\theta}} - \frac{\Sigma_3}{\bar{v} - \underline{v}}, \right. \\ \left. -\frac{\Sigma_1}{\bar{\theta} - \underline{\theta}} - \frac{\Sigma_4}{\bar{v} - \underline{v}}, -\frac{\Sigma_2}{\bar{\theta} - \underline{\theta}} - \frac{\Sigma_3}{\bar{v} - \underline{v}}, -\frac{\Sigma_2}{\bar{\theta} - \underline{\theta}} - \frac{\Sigma_4}{\bar{v} - \underline{v}} \right\}, \end{aligned}$$

then we have the following lemma which is used in [11, 20].

Lemma 4 (See [11, 20]). *Set $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$ be any nonnegative symmetric matrices, $\theta(t)$, $v(t)$ be real functions such that $\theta : \mathbb{R}^+ \to (\underline{\theta}, \bar{\theta}), v : \mathbb{R}^+ \to (\underline{v}, \bar{v})$, then we have*

$$\begin{aligned} &-\frac{\Sigma_1}{\theta(t) - \underline{\theta}} - \frac{\Sigma_2}{\bar{\theta} - \theta(t)} - \frac{\Sigma_3}{v(t) - \underline{v}} - \frac{\Sigma_4}{\bar{v} - v(t)} \\ &\leq \max\left\{ -\frac{\Sigma_1}{\bar{\theta} - \underline{\theta}} - \frac{\Sigma_3}{\bar{v} - \underline{v}}, -\frac{\Sigma_1}{\bar{\theta} - \underline{\theta}} - \frac{\Sigma_4}{\bar{v} - \underline{v}}, \right. \\ &\quad \left. -\frac{\Sigma_2}{\bar{\theta} - \underline{\theta}} - \frac{\Sigma_3}{\bar{v} - \underline{v}}, -\frac{\Sigma_2}{\bar{\theta} - \underline{\theta}} - \frac{\Sigma_4}{\bar{v} - \underline{v}} \right\}. \end{aligned}$$

Remark 1 Usually, we choose $\lambda_j = 1$, $\mu_j = 3 (j = 1, 2)$ in Lemma 3. Apparently the result derived from Lemma 3 is less conservative than Lemma 4.

3 Main result

As well known, Itô's formula plays important role in the stability analysis of stochastic Markovian systems and we cite some related results here [1]. Consider a general stochastic Markovian delay system

$$\begin{aligned} \mathrm{d}z(t) = &f(t, z(t), z(t - \kappa), \eta(t))\mathrm{d}t + g(t, z(t), z(t - \vartheta), \\ &\eta(t))\mathrm{d}\omega(t), \end{aligned} \tag{7}$$

on $t \geq t_0$ with initial value $z(t_0) = z_0 \in \mathbb{R}^n$, where $\vartheta > 0$ is time delay, $f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{N} \to \mathbb{R}^n$ and $g : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{N} \to \mathbb{R}^{n+m}$. Let $\mathcal{C}^{2,1}(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{N}, \mathbb{R}^+)$ denote the family of all nonnegative functions $V(t, z, v, \eta(t))$ on $\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{N}$ which are continuously twice differentiable in z, v and once differentiable in t. Let £ be the weak infinitesimal generator of the random process $\{z(t), \eta(t)\}_{t \geq t_0}$ along the system (7) (see [9, 19]), i.e.,

$$\begin{aligned} \pounds V(t, z_t, v_t, i) := \lim_{\delta \to 0^+} \frac{1}{\delta} [\mathbb{E}\{V(t + \delta, z_{t+\delta}, v_{t+\delta}, \eta(t + \delta))| \\ z_t, v_t, \eta(t) = i\} - V(t, z_t, v_t, \eta(t) = i)], \end{aligned} \tag{8}$$

then, by the Dynkin's formula [24, 28], one can get

$$\mathbb{E}V(t, z(t), v(t), i) = \mathbb{E}V(t_0, z(t_0), v(t_0), i) + \mathbb{E}\int_{t_0}^{t} \pounds V(s, z(s), v(s), i)\mathrm{d}s.$$

In order to get our main results, we propose the following Halanay-type inequality for stochastic differential equations:

Lemma 5 [2, 7]. *Let constants $a > b \geq 0$, $d > 0$. Assume that there exists a positive continuous function $V(t, x)$ satisfying the following inequality*

$$D^{+}\mathbb{E}(V(t,x(t))) \leq -a\mathbb{E}(V(t,x(t))) + b\sup_{s\in[t-d,t]}\mathbb{E}(V(s,x(s))), t \geq t_0,$$

then

$$\mathbb{E}(V(t,x(t))) \leq \sup_{s\in[t_0-d,t_0]}\mathbb{E}(V(s,x(s)))e^{-\upsilon(t-t_0)},$$

where $\upsilon \in (0, a-b]$ is the unique positive solution of the equation: $\upsilon = a - be^{\upsilon d}$.

First, we consider model (1) with $G_i(t) = 0$, i.e., the error dynamical system between (1) and (2) is given as follows:

$$\begin{aligned} de(t) &= [-\beta(e(t)) + A_i(t)f(e(t)) + B_i(t)f(e(t-\tau_i(t))) \\ &\quad + u_i(t)]dt + \rho_i(t)d\omega(t), \\ e(t) &= \varphi(t), \quad t \in [-\bar{\tau}, 0]. \end{aligned} \tag{9}$$

Before presenting our first result, for simplicity, we introduce a new vector as

$$\xi_i(t) = \mathrm{col}\{\, e(t), \quad \beta(e(t)), \quad f(e(t)), \quad e(t-\tau_i(t)), \quad f(e(t-\tau_i(t)))\,\}.$$

Let $\varpi_j (j = 1, 2, \ldots, 5)$ be row vectors with block matrix entries, i.e., the j-th block is an identity matrix and the others are zero blocks, such that $e(t) = \varpi_1\xi_i(t), \beta(e(t)) = \varpi_2\xi_i(t)$, and so on.

Now, we begin to state our result for error system (9).

Theorem 1 (See Appendix 1 for a proof). *Assume that Assumptions 1–4 hold and $\bar{\tau}_i, \bar{\tau}, \tau_{di}$ are given scalars. The drive system (1) and the response system (2) with $G_i(t) = 0$ can be exponentially synchronized for any $0 \leq \tau_i(t) \leq \bar{\tau}_i \leq \bar{\tau}, \dot{\tau}_i(t) \leq \tau_{di}$, if there exist symmetric definite positive matrices P_i, F_i, U, K_i, diagonal positive matrices Q_i, R_i, S_i, W, J_i, Z_i, M_i, L_i, and positive scalars $\varepsilon_i, \epsilon_i, \iota, \nu$, such that the following matrix inequalities hold*

$$\nu\bar{P}_i \leq F_i, \tag{10}$$

$$\nu\Theta W\Theta - \iota F_i + \nu U \leq 0, \tag{11}$$

$$(1 + \iota\bar{\tau}_i)(K_i + \Theta L_i\Theta) - \nu P_i \leq 0, \tag{12}$$

$$\Omega_i < 0, \tag{13}$$

where

$$\bar{P}_i = P_i + Q_i(\Delta - \Lambda) + (S_i + R_i)(\Sigma - \Gamma),$$

$$\begin{aligned} \Omega_i &= \varpi_1{}^T F_i\varpi_1 + \sum_{j=1}^{N}\pi_{ij}\varpi_1^T P_j\varpi_1 - \varpi_4^T K_i\varpi_4 - \varpi_5^T L_i\varpi_5 \\ &\quad + \mathrm{sym}\{\varpi_1^T M_i\varpi_2\} - 2\varpi_1^T M_i\Gamma\varpi_1 \\ &\quad + \begin{bmatrix}\varpi_1\\ \varpi_3\end{bmatrix}^T \begin{bmatrix}-\Sigma\Gamma J_i & \frac{1}{2}(\Sigma+\Gamma)J_i\\ * & -J_i\end{bmatrix}\begin{bmatrix}\varpi_1\\ \varpi_3\end{bmatrix} \\ &\quad + \begin{bmatrix}\varpi_4\\ \varpi_5\end{bmatrix}^T \begin{bmatrix}-\Sigma\Gamma Z_i & \frac{1}{2}(\Sigma+\Gamma)Z_i\\ * & -Z_i\end{bmatrix}\begin{bmatrix}\varpi_4\\ \varpi_5\end{bmatrix} \\ &\quad + \sum_{j=1}^{N}\pi'_{ij}\{\varpi_1^T[Q_j(\Delta-\Lambda) + (R_j+S_j)(\Sigma-\Gamma)]\varpi_1 \\ &\quad + \bar{\tau}_j(\varpi_4^T U\varpi_4 + \varpi_5^T W\varpi_5)\} + \varpi_1^T U\varpi_1 \\ &\quad + \varpi_3^T W\varpi_3 - (1-\tau_{di})(\varpi_4^T U\varpi_4 + \varpi_5^T W\varpi_5) \\ &\quad + \epsilon_i(H_{1i}\varpi_3 + H_{2i}\varpi_5)^T(H_{1i}\varpi_3 + H_{2i}\varpi_5) \\ &\quad + \epsilon_i^{-1}\psi_{0i}^T E_i E_i^T\psi_{0i} + \mathrm{sym}\{\psi_{0i}^T(Y_{1i}\varpi_1 - \varpi_2 + A_i\varpi_3 \\ &\quad + Y_{2i}\varpi_4 + B_i\varpi_5)\} + (C_i\varpi_1 + D_i\varpi_4)^T \\ &\quad \times (\bar{P}_i^{-1} - \varepsilon_i E_i E_i^T)^{-1}(C_i\varpi_1 + D_i\varpi_4) \\ &\quad + \varepsilon_i^{-1}(H_{4i}\varpi_1 + H_{5i}\varpi_4)^T(H_{4i}\varpi_1 + H_{5i}\varpi_4), \end{aligned}$$

with

$$\psi_{0i} = \widetilde{P}_i\varpi_1 + Q_i\varpi_2 + (S_i - R_i)\varpi_3,$$
$$\widetilde{P}_i = P_i - \Lambda Q_i + \Sigma R_i - \Gamma S_i, \quad \pi'_{ij} = \max\{0, \pi_{ij}\}.$$

Remark 2 If we set $U = W = 0$ in Theorem 1, then we can obtain a criterion to verify the exponential synchronization of system (1) and system (2) with $\dot{\tau}(t, \eta(t))$ are not known or the time-varying delays $\tau(t, \eta(t))$ are not differentiable.

Remark 3 Based on Schur complements, inequality (13) is equivalent to the following linear matrix inequality which can be solved numerically by employing the Matlab software:

$$\begin{bmatrix} \Psi_i & \mathcal{A}_i E_i & \mathcal{B}_i & \mathcal{C}_i & 0 \\ * & -\epsilon_i I & 0 & 0 & 0 \\ * & * & -\varepsilon_i E_i E_i^T & 0 & I \\ * & * & * & -\varepsilon_i I & 0 \\ * & * & * & * & -\bar{P}_i \end{bmatrix} < 0, \tag{14}$$

where

$$\Psi_i = \begin{bmatrix} F_i + \psi_{1i} & \psi_{2i} & \psi_{3i} + \widetilde{P}_i A_i & \widetilde{P}_i Y_{2i} & \widetilde{P}_i B_i \\ * & -2Q_i & R_i - S_i + Q_i A_i & Q_i Y_{2i} & Q_i B_i \\ * & * & \psi_{4i} + W - J_i & (S_i - R_i) Y_{2i} & (S_i - R_i) B_i + \epsilon_i H_{1i}^T H_{2i} \\ * & * & * & \psi_{5i} & \frac{1}{2} Z_i (\Sigma + \Gamma) \\ * & * & * & * & \psi_{6i} + \epsilon_i H_{2i}^T H_{2i} \end{bmatrix}$$

$$\mathcal{A}_i = \mathrm{col}\{\, \widetilde{P}_i \quad Q_i \quad S_i - R_i \quad 0 \quad 0 \,\}, \quad \mathcal{B}_i = \mathrm{col}\{\, C_i^T \quad 0 \quad 0 \quad D_i^T \quad 0 \,\}, \quad \mathcal{C}_i = \mathrm{col}\{\, H_{4i}^T \quad 00 \quad H_{5i}^T \quad 0 \,\},$$

with

$$\psi_{1i} = \mathrm{sym}\big(\widetilde{P}_i Y_{1i}\big) + U - \Sigma\Gamma J_i - 2M_i\Gamma + \sum_{j=1}^{N} \pi_{ij} P_j + \sum_{j=1}^{N} \pi'_{ij} \big[Q_j(\Delta - \Lambda) + (R_j + S_j)(\Sigma - \Gamma)\big],$$

$$\psi_{2i} = -\widetilde{P}_i + Y_{1i}^T Q_i + M_i, \quad \psi_{3i} = Y_{1i}^T (S_i - R_i) + \frac{1}{2}(\Sigma + \Gamma) J_i,$$

$$\psi_{4i} = \mathrm{sym}((S_i - R_i) A_i) + \epsilon_i H_{1i}^T H_{1i},$$

$$\psi_{5i} = -(1 - \tau_{di}) U - K_i - \Sigma\Gamma Z_i + \sum_{j=1}^{N} \pi'_{ij} \bar{\tau}_j U,$$

$$\psi_{6i} = -(1 - \tau_{di}) W - Z_i - L_i + \sum_{j=1}^{N} \pi'_{ij} \bar{\tau}_j W.$$

Remark 4 For the inequalities (10–12), note that several nonlinear terms, such as, νP_i and ιK_i, coexist in the LMIs (10–12); thus, the conventional LMI solvers (feasp, mincx, and GEVP) are not applicable directly. In view of this fact, we present the following procedure to solve the inequalities (10–13).

1. Initialize the system parameters, including $A_i, B_i, C_i, D_i, E_i, H_{ji} (j = 1, 2, 4, 5), \pi_{il}, \Sigma, \Gamma, \Delta, \Lambda, \Theta, \bar{\tau}_i, \bar{\tau}, \tau_{di} \quad (i, l = 1, 2, \ldots, N)$, and two given coefficients of accuracy $\kappa > 0$ and $\tilde{\kappa} > 0$.
2. To obtain the theoretic maximum value of ν that satisfies the LMIs (10)-(13), we can solve the following generalized eigenvalue minimization problem (GEVP) (see [6, 22]):
   minimize $\nu_1 > 0$
   s.t. inequalities (13) and $\bar{P}_i \le \nu_1 F_i$ hold.

Denote the obtained minimum value by $\underline{\nu}_1$, then we obtain the maximum value $\bar{\nu} = 1/\underline{\nu}_1$ of ν. Set $\nu = \bar{\nu}$ (i.e., choose $\bar{\nu}$ as the first value of ν in the beginning of the new iterative operation).

3. Check whether $\nu < 0$; if yes, go to step 9). Otherwise, go to step 4).
4. Fix ν and use the following GEVP technique to obtain the maximum value of ι:
   minimize $\iota_1 > 0$
   s.t. inequalities (10), (13) and the following inequalities hold
   $\bar{\tau}_i (K_i + \Theta L_i \Theta) \le \iota_1 (\nu P_i - K_i - \Theta L_i \Theta)$.

Denote the obtained minimum value by $\underline{\iota}_1$, then we obtain the maximum value $\bar{\iota} = 1/\underline{\iota}_1$ of ι. By setting $\iota = \bar{\iota}$ (i.e., choose $\bar{\iota}$ as the first value of ι in the beginning of the new iterative operation).

5. Check whether $\iota < 0$; if yes, go to step 7). Otherwise, go to step 6).
6. Check whether the LMIs (10–13) are feasible; if yes, go to step 8). Otherwise, set $\iota = \iota - \kappa$, and then go to step 5).
7. Set $\nu = \nu - \tilde{\kappa}$, go to step 3).
8. Terminate the program and output the maximum values ν, ι.
9. Terminate the program and print "The LMIs (10–13) are infeasible".

Before presenting our result of error system (3), we introduce another new vector as

$$\zeta_i(t) = \mathrm{col}\Bigg\{ \xi_i(t), \int_{t-\varrho_i(t)}^{t} f(e(s))\mathrm{d}s, \int_{t-\tau_i(t)}^{t} \chi_i(s)\mathrm{d}s, \int_{t-\bar{\tau}_i}^{t-\tau_i(t)} \chi_i(s)\mathrm{d}s, \int_{t-\tau_i(t)}^{t} \rho_i(s)\mathrm{d}\omega(s), \int_{t-\bar{\tau}_i}^{t-\tau_i(t)} \rho_i(s)\mathrm{d}\omega(s), \int_{t-\bar{\varrho}_i}^{t-\varrho_i(t)} f(e(s))\mathrm{d}s, \chi_i(t) \Bigg\}.$$

Let ς_j $(j = 1, 2, \ldots, 12)$ be row vectors with block matrix entries, i.e., the j-th block is an identity matrix and the others are zero blocks, such that $e(t) = \varsigma_1 \zeta_i(t)$, $\chi_i(t) = \varsigma_{12} \zeta_i(t)$, and so on.

Now, based on Lemma 3, we can propose the following delay-dependent stability criterion for error system (3):

Theorem 2 (See Appendix 2 for a proof). *Assume that Assumptions 1–4 hold and $\bar{\tau}_i, \bar{\tau}, \tau_{di}$ are given scalars. The drive system (1) and the response system (2) can be exponentially synchronized for any $0 \le \tau_i(t) \le \bar{\tau}_i \le \bar{\tau}$, $\dot{\tau}_i(t) \le \tau_{di}$, if there exist symmetric definite positive matrices P_i, U, K_i, T_j, diagonal positive matrices Q_i, R_i, S_i, W, J_i, Z_i, M_i, L_i, real matrices X_{1i}, X_{2i}, X_{3i} and positive scalars $\varepsilon_i, \epsilon_i, \imath, v$, such that the following matrix inequalities hold*

$$\widetilde{\Omega}_i - 2\varsigma_l^T \bar{T}_1 \varsigma_l - 2\varsigma_k^T \bar{T}_2 \varsigma_k < 0, \quad l = 7, 8; k = 6, 11, \tag{15}$$

where

$$\begin{aligned}
\widetilde{\Omega}_i = & \begin{bmatrix} \varsigma_1 \\ \varsigma_3 \end{bmatrix}^T \begin{bmatrix} -\Sigma\Gamma J_i & \frac{1}{2}(\Sigma+\Gamma)J_i \\ * & -J_i \end{bmatrix} \begin{bmatrix} \varsigma_1 \\ \varsigma_3 \end{bmatrix} \\
& + \begin{bmatrix} \varsigma_4 \\ \varsigma_5 \end{bmatrix}^T \begin{bmatrix} -\Sigma\Gamma Z_i & \frac{1}{2}(\Sigma+\Gamma)Z_i \\ * & -Z_i \end{bmatrix} \begin{bmatrix} \varsigma_4 \\ \varsigma_5 \end{bmatrix} \\
& + \sum_{j=1}^{N} \pi'_{ij} \{ \varsigma_1^T [Q_j(\Delta - \Lambda) + (R_j + S_j)(\Sigma - \Gamma)] \varsigma_1 \\
& + \bar{\tau}_j (\varsigma_4^T U \varsigma_4 + \varsigma_5^T W \varsigma_5) \} + \varsigma_1^T U \varsigma_1 \\
& + \varsigma_3^T W \varsigma_3 - (1 - \tau_{di})(\varsigma_4^T U \varsigma_4 + \varsigma_5^T W \varsigma_5) \\
& + \mathrm{sym}\{\varsigma_1^T M_i \varsigma_2\} - 2\varsigma_1^T M_i \Gamma \varsigma_1 + \varsigma_1^T \sum_{j=1}^{N} \pi_{ij} P_j \varsigma_1 \\
& + \mathrm{sym}\{\varsigma_1^T P_i \varsigma_{12}\} + \bar{\tau}_i^2 \varsigma_{12}^T T_1 \varsigma_{12} + \bar{\varrho}_i^2 \varsigma_3^T T_2 \varsigma_3 \\
& - \left(1 - \sum_{j=1}^{N} \pi_{ij} \bar{\tau}_j \right) (\varsigma_9^T T_3 \varsigma_9 + \varsigma_{10}^T T_3 \varsigma_{10}) \\
& + \mathrm{sym}\{(X_{2i}\varsigma_1 + X_{3i}\varsigma_4)^T (\varsigma_1 - \varsigma_4 - \varsigma_7 - \varsigma_9)\} \\
& + \mathrm{sym}\{\psi_{ai}^T (Y_{1i}\varsigma_1 - \varsigma_{12} - \varsigma_2 + A_i \varsigma_3 \\
& + B_i \varsigma_5 + G_i \varsigma_6 + Y_{2i} \varsigma_4)\} \\
& + \epsilon_i (H_{1i}\varsigma_3 + H_{2i}\varsigma_5 + H_{3i}\varsigma_6)^T (H_{1i}\varsigma_3 + H_{2i}\varsigma_5 + H_{3i}\varsigma_6) \\
& + \epsilon_i^{-1} \psi_{ai}^T E_i E_i^T \psi_{ai} + (C_i \varsigma_1 + D_i \varsigma_4)^T \\
& [(\bar{P}_i + T_3)^{-1} - \varepsilon_i E_i E_i^T]^{-1} (C_i \varsigma_1 + D_i \varsigma_4) \\
& + \varepsilon_i^{-1} (H_{4i}\varsigma_1 + H_{5i}\varsigma_4)^T (H_{4i}\varsigma_1 + H_{5i}\varsigma_4) \\
& - \varsigma_7^T \bar{T}_1 \varsigma_7 - \varsigma_8^T \bar{T}_1 \varsigma_8 - \varsigma_6^T \bar{T}_2 \varsigma_6 - \varsigma_{11}^T \bar{T}_2 \varsigma_{11},
\end{aligned}$$

$$\bar{T}_1 = \left(1 - \sum_{j=1}^{N} \pi_{ij} \bar{\tau}_j \right) T_1, \quad \bar{T}_2 = \left(1 - \sum_{j=1}^{N} \pi_{ij} \bar{\varrho}_j \right) T_2,$$

with

$$\psi_{ai} = \tilde{P}_i \varsigma_1 + Q_i \varsigma_2 + (S_i - R_i)\varsigma_3 + X_{1i}\varsigma_{12}.$$

Remark 5 Similar to Remark 2, if we set $U = W = 0$ in Theorem 1, then we can obtain a criterion to verify the exponential synchronization of system (1) and system (2) with $\dot{\tau}(t, \eta(t))$ are not known or the time-varying delays $\tau(t, \eta(t))$ are not differentiable.

Remark 6 Similar to Remark 3, by Schur complements, inequalities (15) can be equivalently transformed into linear matrix inequalities which could be solved numerically by employing the Matlab software.

4 Illustrative example

In this section, we give a example to demonstrate the effectiveness of our theoretic results.

Example 1 Consider system (1) with $n = 2$ and the following parameters:

$$A_1 = \begin{bmatrix} 1.9 & -0.11 \\ 5.0 & 3.2 \end{bmatrix}, A_2 = \begin{bmatrix} 2.0 & -0.14 \\ 5.0 & 3.1 \end{bmatrix}, B_1 = \begin{bmatrix} -1.8 & -0.1 \\ 0.2 & -2.7 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} -1.7 & -0.1 \\ 0.2 & -2.7 \end{bmatrix}, C_1 = \begin{bmatrix} 2.5 & 0 \\ 0 & 2 \end{bmatrix}, C_2 = \begin{bmatrix} 2.8 & 0 \\ 0 & 2 \end{bmatrix}, D_1 = 2I,$$
$$D_2 = \begin{bmatrix} 1.8 & 0 \\ 0 & 2 \end{bmatrix}, E_2 = \begin{bmatrix} 0.5 & -0.2 \\ -1 & 0.6 \end{bmatrix}, \Phi_i(t) = \begin{bmatrix} \cos(t) & 0 \\ 0 & \sin(t) \end{bmatrix},$$
$$E_1 = 0.1I, H_{ji} = 0.15I, G_i(t) = 0, \Im = 0, j = 1, 2, 4, 5, i = 1, 2.$$

The behaved functions are $\tilde{\beta}_1(x) = \tilde{\beta}_2(x) = 1.5x + 0.5\sin x$, the activation functions are $\tilde{f}_1(x) = \tilde{f}_2(x) = \tanh(x)$, and the time-varying delays are $\tau_1(t) = \tau_2(t) = 0.8 + 0.2\sin t$. Then Assumptions 1–4 are satisfied with $\Delta = 2I, \Lambda = \Sigma = \Theta = I, \Gamma = 0$ and $\bar{\tau}_1 = \bar{\tau}_2 = \bar{\tau} = 1, \tau_{d1} = \tau_{d2} = 0.2$.

In this paper, the transition rate matrix is given as follows

$$\Pi = \begin{bmatrix} -0.7 & 0.7 \\ 0.3 & -0.3 \end{bmatrix},$$

and the control input vector with state feedback is designed as (4) with

$$Y_{11} = -14.5I,\ Y_{12} = -10.7I,\ Y_{21} = 2.1I,\ Y_{22} = 2.3I.$$

Set $v = 0.2, \imath = 0.1$, solving the LMIs (10–13) in Theorem 1 by resorting to the Matlab LMI Control Toolbox, we have one feasible solution as follows

$$P_1 = \begin{bmatrix} 0.0071 & -0.0002 \\ -0.0002 & 0.0007 \end{bmatrix}, P_2 = \begin{bmatrix} 0.0016 & -0.0001 \\ -0.0001 & 0.0002 \end{bmatrix},$$
$$U = \begin{bmatrix} 0.9308 & -0.0465 \\ -0.0465 & 0.3153 \end{bmatrix} \times 10^{-3},$$
$$K_1 = \begin{bmatrix} 0.0013 & -0.0000 \\ -0.0000 & 0.0001 \end{bmatrix}, K_2 = \begin{bmatrix} 0.2756 & -0.0099 \\ -0.0099 & 0.0290 \end{bmatrix}$$
$$\times 10^{-3}, F_1 = \begin{bmatrix} 0.0019 & -0.0001 \\ -0.0001 & 0.0007 \end{bmatrix},$$

$$F_2 = \begin{bmatrix} 0.0023 & -0.0003 \\ -0.0003 & 0.0008 \end{bmatrix}, Q_1 = \begin{bmatrix} 0.4592 & 0 \\ 0 & 0.0495 \end{bmatrix}$$
$$\times 10^{-3}, Q_2 = \begin{bmatrix} 0.2088 & 0 \\ 0 & 0.0243 \end{bmatrix} \times 10^{-3},$$
$$R_1 = \begin{bmatrix} 0.1832 & 0 \\ 0 & 0.0569 \end{bmatrix} \times 10^{-3}, R_2 = \begin{bmatrix} 0.2956 & 0 \\ 0 & 0.0817 \end{bmatrix}$$
$$\times 10^{-3}, J_1 = \begin{bmatrix} 0.0483 & 0 \\ 0 & 0.0038 \end{bmatrix},$$
$$J_2 = \begin{bmatrix} 0.0070 & 0 \\ 0 & 0.0012 \end{bmatrix}, S_1 = \begin{bmatrix} 0.5399 & 0 \\ 0 & 0.0490 \end{bmatrix}$$
$$\times 10^{-4}, S_2 = \begin{bmatrix} 0.1862 & 0 \\ 0 & 0.1634 \end{bmatrix} \times 10^{-4},$$
$$Z_1 = \begin{bmatrix} 0.0020 & 0 \\ 0 & 0.0004 \end{bmatrix}, Z_2 = \begin{bmatrix} 0.8249 & 0 \\ 0 & 0.3355 \end{bmatrix}$$
$$\times 10^{-3}, L_1 = \begin{bmatrix} 0.9356 & 0 \\ 0 & 0.0969 \end{bmatrix} \times 10^{-5},$$
$$L_2 = \begin{bmatrix} 0.3488 & 0 \\ 0 & 0.3402 \end{bmatrix} \times 10^{-5}, M_1 = \begin{bmatrix} 0.0015 & 0 \\ 0 & 0.0002 \end{bmatrix},$$
$$M_2 = \begin{bmatrix} 0.6877 & 0 \\ 0 & 0.1575 \end{bmatrix} \times 10^{-3},$$
$$W = \begin{bmatrix} 0.6496 & 0 \\ 0 & 0.4187 \end{bmatrix} \times 10^{-5}, \epsilon_1 = 0.0029,$$
$$\epsilon_2 = 0.0033, \varepsilon_1 = 13.9798, \varepsilon_2 = 16.3399.$$

Figure 1 shows the neural network model has a chaotic attractor with initial values $x_1(t) = 0.2, x_2(t) = 0.5$, $t \in [-1, 0]$. The initial values of the response system are taken as $y_1(t) = -1.3, y_2(t) = 2.1, t \in [-1, 0]$. Figures 2 and 3 depict the phase trajectories of the drive system and response system, respectively. Figure 4 shows the error states. By numerical simulation, we can see that the dynamical behaviors of response system (2) synchronize with master system (1) as shown in Figs. 2, 3.

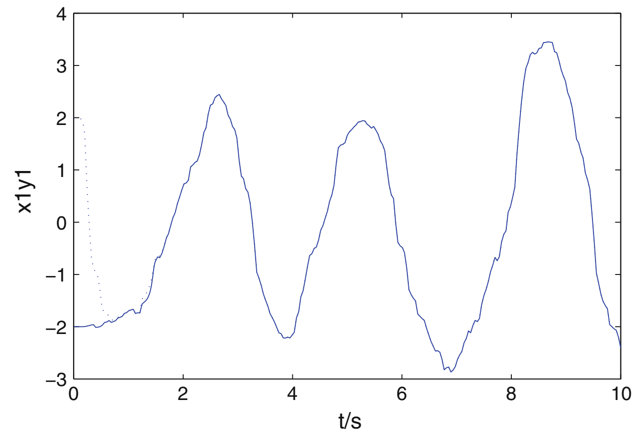

Fig. 2 The phase trajectories of $t - x_1(t) - y_1(t)$

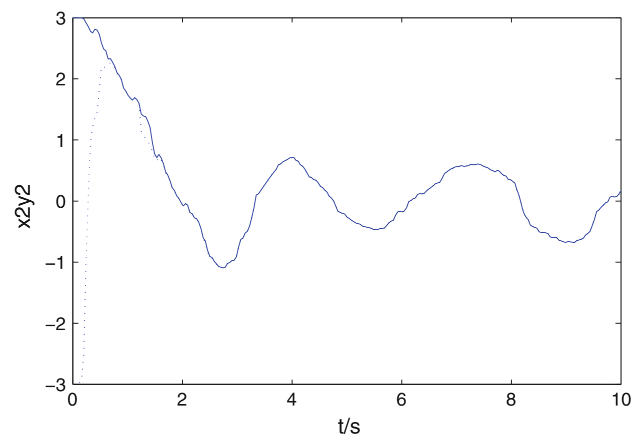

Fig. 3 The phase trajectories of $t - x_2(t) - y_2(t)$

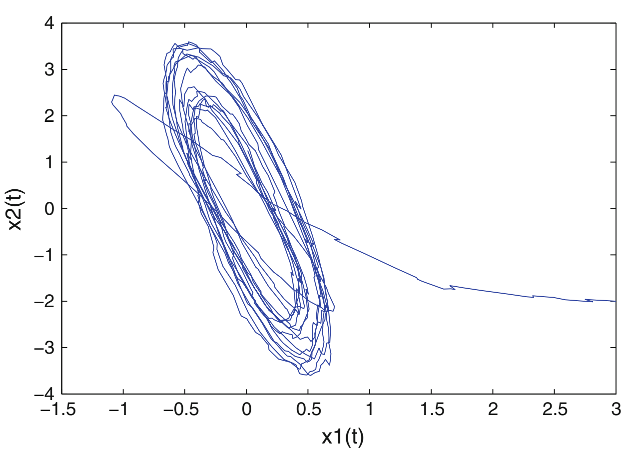

Fig. 1 Chaotic attractor of Example 1

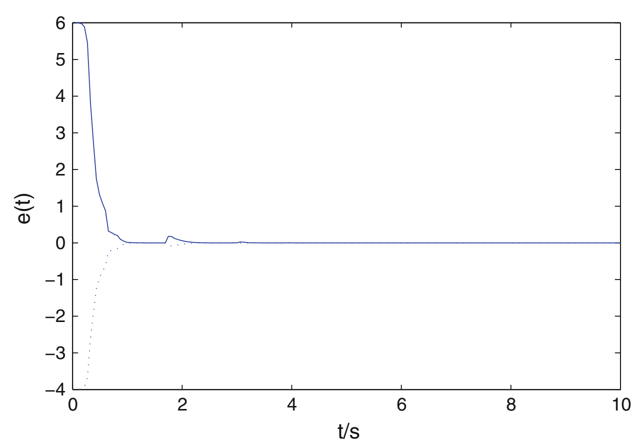

Fig. 4 The error state of $t - e_1(t) - e_2(t)$

5 Conclusion

In this paper, the exponential synchronization problem has been investigated for a class of stochastic perturbed chaotic neural networks with discrete and distributed time-varying delays as well as Markovian jump parameters. Based on a Halanay-type inequality for stochastic differential equations, the Jensen integral inequality and a novel Lemma, two delay-dependent sufficient condition are proposed to guarantee the exponential synchronization of two identical Markovian jumping chaotic-delayed neural networks with stochastic perturbation. With some parameters being fixed in advance, these conditions are expressed in terms of linear matrix inequalities, which can be solved numerically by employing the Matlab software. Finally, a numerical example with simulations is provided to illustrate the effectiveness and usefulness of the presented synchronization scheme.

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Appendix 1

Proof of Theorem 1

Consider the following Lyapunov-Krasovskii functional:

$$\begin{aligned} V_1(t,e_t,i) &= e(t)^T P_i e(t) + 2\sum_{j=1}^{N} q_{ji} \int_0^{e_j(t)} \left[\beta_j(s) - \lambda_j s\right] \mathrm{d}s \\ &\quad + 2\sum_{j=1}^{n} r_{ji} \int_0^{e_j(t)} \left[\sigma_j s - f_j(s)\right] \mathrm{d}s \\ &\quad + 2\sum_{j=1}^{n} s_{ji} \int_0^{e_j(t)} \left[f_j(s) - \gamma_j s\right] \mathrm{d}s \\ &\quad + \int_{t-\tau_i(t)}^{t} \left[e(s)^T U e(s) + f(e(s))^T W f(e(s))\right] \mathrm{d}s, \end{aligned} \tag{16}$$

where $Q_i = \mathrm{diag}\{q_{1i}, q_{2i}, \ldots, q_{ni}\}, R_i = \mathrm{diag}\{r_{1i}, r_{2i}, \ldots, r_{ni}\}$, $S_i = \mathrm{diag}\{s_{1i}, s_{2i}, \ldots, s_{ni}\}$.

It can be easily verified that $V_1(t, e_t, i)$ is a nonnegative function over $[-\bar{\tau}, +\infty)$. Evaluating the time derivative of $V_1(t, e_t, i)$ along the trajectory of system (9), we have that

$$\mathrm{d}V_1(t,e_t,i) = \pounds V_1(t,e_t,i)\mathrm{d}t + \frac{\partial}{\partial e} V_1(t,e_t,i)\rho_i(t)\mathrm{d}\omega(t), \tag{17}$$

where

$$\begin{aligned} \pounds V_1(t,e_t,i) &= 2\big\{e(t)^T P_i + [\beta(t) - \Lambda e(t)]^T Q_i \\ &\quad + [\Sigma e(t) - f(e(t))]^T R_i + [f(e(t)) - \Gamma e(t)]^T S_i\big\} \\ &\quad \times \Big[- \beta(t) + A_i(t) f(e(t)) + B_i(t) f(e(t-\tau_i(t))) \\ &\quad + Y_{1i} e(t) + Y_{2i} e(t-\tau_i(t))\Big] \\ &\quad + 2\sum_{k=1}^{N} \pi_{ik} \sum_{j=1}^{n} \int_0^{e_j(t)} \Big\{ q_{jk}[\beta_j(s) - \lambda_j(s)] + r_{jk}[\sigma_j s - f_j(s)] \\ &\quad + s_{jk}[f_j(s) - \gamma_j(s)]\Big\}\mathrm{d}s + e(t)^T \left(\sum_{j=1}^{N} \pi_{ij} P_j + U\right) e(t) \\ &\quad - (1-\dot{\tau}_i(t)) e(t-\tau_i(t))^T U e(t-\tau_i(t)) \\ &\quad + f(e(t))^T W f(e(t)) \\ &\quad + \sum_{j=1}^{N} \pi_{ij}\tau_j(t)\big[e(t-\tau_i(t))^T U e(t-\tau_i(t)) \\ &\quad + f(e(t-\tau_i(t)))^T W f(e(t-\tau_i(t)))\big] \\ &\quad -(1-\dot{\tau}_i(t)) f(e(t-\tau_i(t)))^T W f(e(t-\tau_i(t))) \\ &\quad + \frac{1}{2}\mathrm{trace}\left[\rho_i(t)^T \frac{\partial^2}{\partial e^2} V_1(t,e_t,i)\rho_i(t)\right]. \end{aligned} \tag{18}$$

From Assumptions 3 and 4, we get that

$$\begin{aligned} &2\sum_{k=1}^{N} \pi_{ik} \sum_{j=1}^{n} \int_0^{e_j(t)} \{q_{jk}[\beta_j(s) - \lambda_j(s)] + r_{jk}[\sigma_j s - f_j(s)] + s_{jk}[f_j(s) - \gamma_j(s)]\}\mathrm{d}s \\ &\le 2\sum_{k=1}^{N} \pi'_{ik} \sum_{j=1}^{n} \int_0^{e_j(t)} \{q_{jk}(\delta_j - \lambda_j)s + r_{jk}(\sigma_j - \gamma_j)s + s_{jk}(\sigma_j - \gamma_j)s\}\mathrm{d}s \\ &= e(t)^T \sum_{k=1}^{N} \pi'_{jk}[Q_k(\Delta - \Lambda) + (R_k + S_k)(\Sigma - \Gamma)] e(t). \end{aligned} \tag{19}$$

In addition, we derive that

$$\begin{aligned} &\sum_{j=1}^{N} \pi_{ij}\tau_j(t)\big[e(t-\tau_i(t))^T U e(t-\tau_i(t)) \\ &\quad + f(e(t-\tau_i(t)))^T W f(e(t-\tau_i(t)))\big] \\ &\quad \le \zeta_i(t)^T \sum_{j=1}^{N} \pi'_{ij}\bar{\tau}_j\left(\varsigma_4^T U \varsigma_4 + \varsigma_5^T U \varsigma_5\right)\zeta_i(t). \end{aligned} \tag{20}$$

For any $j=1,2,\ldots,n$, it follows from (5) that

$$0\le \frac{\mathrm{d}(\beta_j(e_j)-\lambda_j e_j)}{\mathrm{d}e_j}\le \delta_j-\lambda_j,$$
$$0\le \frac{\mathrm{d}(f_j(e_j)-\gamma_j e_j)}{\mathrm{d}e_j}\le \sigma_j-\gamma_j,$$
$$0\le \frac{\mathrm{d}(\sigma_j e_j-f_j(e_j))}{\mathrm{d}e_j}\le \sigma_j-\gamma_j.$$

Thus, we have that

$$\begin{aligned}
&\frac{1}{2}\frac{\partial^2}{\partial e^2}V_1(t,e_t,i)=P_i+Q_i\\
&\times \mathrm{diag}\left\{\frac{\mathrm{d}(\beta_1(\mathrm{e}_1)-\lambda_1\mathrm{e}_1)}{\mathrm{d}e_1},\ldots,\frac{\mathrm{d}(\beta_n(e_n)-\lambda_n e_n)}{\mathrm{d}e_n}\right\}\\
&\quad+R_i\times \mathrm{diag}\left\{\frac{\mathrm{d}(\sigma_1\mathrm{e}_1-\mathrm{f}_1(\mathrm{e}_1))}{\mathrm{d}e_1},\ldots,\frac{\mathrm{d}(\sigma_\mathrm{n}\mathrm{e}_\mathrm{n}-\mathrm{f}_\mathrm{n}(\mathrm{e}_\mathrm{n}))}{\mathrm{d}e_n}\right\}\\
&\quad+S_i\times \mathrm{diag}\left\{\frac{\mathrm{d}(\mathrm{f}_1(\mathrm{e}_1)-\gamma_1\mathrm{e}_1)}{\mathrm{d}e_1},\ldots,\frac{\mathrm{d}(\mathrm{f}_\mathrm{n}(\mathrm{e}_\mathrm{n})-\gamma_\mathrm{n}\mathrm{e}_\mathrm{n})}{\mathrm{d}e_n}\right\}\\
&\le \bar{P}_i.
\end{aligned}\tag{21}$$

$$\begin{aligned}
&\frac{1}{2}\mathrm{trace}\left[\rho_i(t)^T\frac{\partial^2}{\partial e^2}V_1(t,e_t,i)\rho_i(t)\right]\\
&=[C_ie(t)+D_ie(t-\tau_i(t))+E_i\Phi_i(t)(H_{3i}e(t)+H_{4i}e(t-\tau_i(t)))]^T\\
&\quad\times\bar{P}_i[C_ie(t)+D_ie(t-\tau_i(t))+E_i\Phi_i(t)(H_{3i}e(t)+H_{4i}e(t-\tau_i(t)))]\\
&\le \varepsilon_i^{-1}(H_{3i}e(t)+H_{4i}e(t-\tau_i(t)))^T(H_{3i}e(t)+H_{4i}e(t-\tau_i(t)))\\
&\quad+(C_ie(t)+D_ie(t-\tau_i(t)))^T\left(\bar{P}_i^{-1}-\varepsilon_iE_iE_i^T\right)^{-1}(C_ie(t)\\
&\quad+D_ie(t-\tau_i(t))).
\end{aligned}\tag{24}$$

From (5), the following inequalities hold for any positive diagonal matrix M_i with compatible dimension

$$0\le 2\{e(t)^TM_i\beta(e(t))-e(t)^TM_i\Gamma e(t)\}.\tag{25}$$

From (18–25), we obtain that

$$\begin{aligned}
£V_1(t,e_t,i)&\le \xi_i(t)^T\bar{\Omega}_i(t)\xi_i(t)+e(t-\tau_i(t))^TK_ie(t-\tau_i(t))\\
&\quad+f(e(t-\tau_i(t)))^TL_if(e(t-\tau_i(t))).
\end{aligned}\tag{26}$$

where

$$\bar{\Omega}_i(t)=\begin{bmatrix}
\psi_{1i}+\psi_{7i} & \psi_{2i} & \psi_{3i}+\widetilde{P}_iA_i(t) & \widetilde{P}_iY_{2i}+\psi_{8i} & \widetilde{P}_iB_i(t)\\
* & -2Q_i & R_i-S_i+Q_iA_i(t) & Q_iY_{2i} & Q_iB_i(t)\\
* & * & W-J_i+\mathrm{sym}((S_i-R_i)A_i(t)) & (S_i-R_i)Y_{2i} & (S_i-R_i)B_i(t)\\
* & * & * & \psi_{5i}+\psi_{9i} & \frac{1}{2}Z_i(\Sigma+\Gamma)\\
* & * & * & * & \psi_{6i}
\end{bmatrix},$$

with

$$\psi_{7i}=C_i^T\left(\bar{P}_i^{-1}-\varepsilon_iE_iE_i^T\right)^{-1}C_i+\varepsilon_i^{-1}H_{3i}^TH_{3i},$$
$$\psi_{8i}=C_i^T\left(\bar{P}_i^{-1}-\varepsilon_iE_iE_i^T\right)^{-1}D_i+\varepsilon_i^{-1}H_{3i}^TH_{4i},$$
$$\psi_{9i}=D_i^T\left(\bar{P}_i^{-1}-\varepsilon_iE_iE_i^T\right)^{-1}D_i+\varepsilon_i^{-1}H_{4i}^TH_{4i}.$$

For any $j=1,2,\ldots,n$, from (5) we obtain that

$$\left(f_j(e_j(t))-\sigma_je_j(t)\right)\left(f_j(e_j(t))-\gamma_je_j(t)\right)\le 0,$$
$$\left(f_j(e_j(t-\tau(t)))-\sigma_je_j(t-\tau(t))\right)$$
$$\left(f_j(e_j(t-\tau(t)))-\gamma_je_j(t-\tau(t))\right)\le 0.$$

Therefore, the following matrix inequalities hold for any positive diagonal matrices J_i, Z_i with compatible dimensions

$$\begin{aligned}
0\le& -e(t)^T\Sigma\Gamma J_ie(t)+e(t)^TJ_i(\Sigma+\Gamma)f(e(t))\\
&-f(e(t))^TJ_if(e(t)),
\end{aligned}\tag{22}$$

$$\begin{aligned}
0\le& -e(t-\tau(t))^T\Sigma\Gamma Z_ie(t-\tau_i(t))\\
&+e(t-\tau_i(t))^TZ_i(\Sigma+\Gamma)f(e(t-\tau_i(t)))\\
&-f(e(t-\tau_i(t)))^TZ_if(e(t-\tau_i(t))).
\end{aligned}\tag{23}$$

According to Assumption 1 and Lemma 1, for any positive scalar ε_i we have that

Now, by (14), it is easy to see that there exists a scalar $\alpha>1$ such that

$$\begin{bmatrix}
\tilde{\Psi}_i & \mathcal{A}_iE_i & \mathcal{B}_i & \mathcal{C}_i & 0\\
* & -\epsilon_iI & 0 & 0 & 0\\
* & * & -\varepsilon_iF_iF_i^T & 0 & I\\
* & * & * & -\varepsilon_iI & 0\\
* & * & * & * & -\bar{P}_i
\end{bmatrix}<0,\tag{27}$$

where

$$\tilde{\Psi}_i = \begin{bmatrix} \alpha F_i + \psi_{1i} & \psi_{2i} & \psi_{3i} + \widetilde{P}_i A_i & \widetilde{P}_i Y_{2i} + \psi_{8i} & \widetilde{P}_i B_i \\ * & -2Q_i & R_i - S_i + Q_i A_i & Q_i Y_{2i} & Q_i B_i \\ * & * & W - J_i + \mathrm{sym}((S_i - R_i)A_i) & (S_i - R_i)Y_{2i} & (S_i - R_i)B_i \\ * & * & * & \psi_{5i} + \psi_{9i} & \frac{1}{2}Z_i(\Sigma + \Gamma) \\ * & * & * & * & \psi_{6i} \end{bmatrix}.$$

Applying Schur complements to (27) results in

$$\begin{bmatrix} \alpha F_i + \psi_{1i} + \psi_{7i} & \psi_{2i} & \psi_{3i} + \widetilde{P}_i A_i & \widetilde{P}_i Y_{2i} + \psi_{8i} & \widetilde{P}_i B_i \\ * & -2Q_i & R_i - S_i + Q_i A_i & Q_i Y_{2i} & Q_i B_i \\ * & * & W - J_i + \mathrm{sym}((S_i - R_i)A_i) & (S_i - R_i)Y_{2i} & (S_i - R_i)B_i \\ * & * & * & \psi_{5i} + \psi_{9i} & \frac{1}{2}Z_i(\Sigma + \Gamma) \\ * & * & * & * & \psi_{6i} \end{bmatrix}$$
$$+\epsilon_i^{-1}\begin{bmatrix} \widetilde{P}_i E_i \\ Q_i E_i \\ (S_i - R_i)E_i \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} \widetilde{P}_i E_i \\ Q_i E_i \\ (S_i - R_i)E_i \\ 0 \\ 0 \end{bmatrix}^T + \epsilon_i \begin{bmatrix} 0 \\ 0 \\ H_{1i}^T \\ 0 \\ H_{2i}^T \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ H_{1i}^T \\ 0 \\ H_{2i}^T \end{bmatrix}^T < 0. \tag{28}$$

Using Assumption 1 and Lemma 1, for any positive scalar ϵ_i we have that

$$\begin{bmatrix} 0 & 0 & \widetilde{P}_i \Delta A_i(t) & 0 & \widetilde{P}_i \Delta B_i(t) \\ * & 0 & Q_i \Delta A_i(t) & 0 & Q_i \Delta B_i(t) \\ * & * & \mathrm{sym}((S_i - R_i)\Delta A_i(t)) & 0 & (S_i - R_i)\Delta B_i(t) \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix}$$
$$= \begin{bmatrix} \widetilde{P}_i E_i \\ Q_i E_i \\ (S_i - R_i)E_i \\ 0 \\ 0 \end{bmatrix} \Phi_i(t) \begin{bmatrix} 0 \\ 0 \\ H_{1i}^T \\ 0 \\ H_{2i}^T \end{bmatrix}^T + \begin{bmatrix} 0 \\ 0 \\ H_{1i}^T \\ 0 \\ H_{2i}^T \end{bmatrix} \Phi_i(t)^T \begin{bmatrix} \widetilde{P}_i E_i \\ Q_i E_i \\ (S_i - R_i)E_i \\ 0 \\ 0 \end{bmatrix}^T$$
$$\le \epsilon_i^{-1}\begin{bmatrix} \widetilde{P}_i E_i \\ Q_i E_i \\ (S_i - R_i)E_i \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} \widetilde{P}_i E_i \\ Q_i E_i \\ (S_i - R_i)E_i \\ 0 \\ 0 \end{bmatrix}^T + \epsilon_i \begin{bmatrix} 0 \\ 0 \\ H_{1i}^T \\ 0 \\ H_{2i}^T \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ H_{1i}^T \\ 0 \\ H_{2i}^T \end{bmatrix}^T.$$

This together with (28) provides that

$$\bar{\Omega}_i(t) + \mathrm{diag}\{\alpha F_i \quad 0 \quad 0 \quad 0 \quad 0\} < 0.$$

By this inequality and (26), it is easy to see that

$$\pounds V_1(t, e_t, i) < -\alpha e(t)^T F_i e(t) + e(t - \tau_i(t))^T K_i e(t - \tau_i(t)) + f(e(t - \tau_i(t)))^T L_i f(e(t - \tau_i(t))).$$

Taking the mathematical expectations on both sides of (17), from above inequality we have that

$$\mathrm{d}\mathbb{E}\{V_1(t, e_t, i)\} = \mathbb{E}\pounds V_1(t, e_t, i)\mathrm{d}t + \mathbb{E}\left\{\frac{\partial}{\partial e} V_1(t, e_t, i)\rho_i(t)\mathrm{d}\omega(t)\right\}$$
$$< [-\alpha e(t)^T F_i e(t)\mathrm{d}t + e(t - \tau_i(t))^T K_i e(t - \tau_i(t))\mathrm{d}t + f(e(t - \tau_i(t)))^T L_i f(e(t - \tau_i(t)))]\mathrm{d}t.$$

By integrating above inequality from $t - \tau(t)$ to t, we obtain that

$$\mathbb{E}\{V_1(t, e_t, i)\} - \mathbb{E}\{V_1(t, e_{t-\tau_i(t)}, i)\} = \int_{t-\tau_i(t)}^{t} \mathbb{E}\{V_1(s, e_s, i)\}\mathrm{d}s$$
$$< -\alpha \int_{t-\tau_i(t)}^{t} e(s)^T F_i e(s)\mathrm{d}s + \int_{t-\tau_i(t)}^{t} [e(s - \tau_i(s))^T K_i e(s - \tau_i(s)) + f(e(s - \tau_i(s)))^T L_i f(e(s - \tau_i(s)))]\mathrm{d}s.$$

It follows that

$$\mathbb{E}\left[\frac{\mathrm{d}V_1(t, e_t, i)}{\mathrm{d}t}\right] + \iota\mathbb{E}[V_1(t, e_t, i) - V_1(t, e_{t-\tau_i(t)}, i)]$$
$$< -\alpha e(t)^T F_i e(t) + e(t - \tau_i(t))^T K_i e(t - \tau_i(t)) + f(e(t - \tau_i(t)))^T L_i f(e(t - \tau_i(t))) - \iota\alpha \int_{t-\tau_i(t)}^{t} e(s)^T F_i e(s)\mathrm{d}s$$
$$+ \iota \int_{t-\tau(t)}^{t} [e(s - \tau_i(s))^T K e(s - \tau_i(s)) + f(e(s - \tau_i(s)))^T L_i f(e(s - \tau_i(s)))]\mathrm{d}s. \tag{29}$$

In view of (10) and (11), we have that

$$-e(t)^T F_i e(t) \le -v e(t)^T \bar{P}_i e(t), \tag{30}$$

$$-\iota \int_{t-\tau_i(t)}^{t} e(s)^T F_i e(s)\mathrm{d}s \le -v \int_{t-\tau_i(t)}^{t} [f(e(s))^T W f(e(s)) + e(s)^T U e(s)]\mathrm{d}s. \tag{31}$$

Noticing that

$$2\sum_{j=1}^{n} q_{ji} \int_0^{e_j(t)} [\beta_j(s) - \lambda_j s]\mathrm{d}s \le 2\sum_{j=1}^{n} q_{ji} \int_0^{e_j(t)} (\delta_j - \gamma_j)s\mathrm{d}s = e(t)^T Q_i(\Delta - \Lambda)e(t),$$

$$2\sum_{j=1}^{n} r_{ji} \int_0^{e_j(t)} [\sigma_j s - f_j(s)]\mathrm{d}s \le 2\sum_{j=1}^{n} r_{ji} \int_0^{e_j(t)} (\sigma_j - \gamma_j)s\mathrm{d}s = e(t)^T R_i(\Sigma - \Gamma)e(t),$$

$$2\sum_{j=1}^{n} s_{ji} \int_0^{e_j(t)} [f_j(s) - \gamma_j s]\mathrm{d}s \le 2\sum_{j=1}^{n} s_{ji} \int_0^{e_j(t)} (\sigma_j - \gamma_j)s\mathrm{d}s = e(t)^T S_i(\Sigma - \Gamma)e(t).$$

Therefore, from (16) we have that

$$\mathbb{E}\{V_1(t,e_t,i)\} \le e(t)^T\bar{P}_i e(t) + \int_{t-\tau_i(t)}^{t} \left[e(s)^T U e(s) + f(e(s))^T W f(e(s))\right]\mathrm{d}s.$$

This together with (30–31) yields that

$$-e(t)^T F_i e(t) - \imath \int_{t-\tau_i(t)}^{t} \left[e(s)^T F_i e(s)\right]\mathrm{d}s \le - v\mathbb{E}\{V_1(t,e_t,i)\}. \quad (32)$$

Moreover, $\mathbb{E}\{V_1(t,e_t,i)\} \ge e(t)^T P_i e(t)$, therefore, it follows from (5) and (12) that

$$\begin{aligned} &e(t-\tau_i(t))^T K_i e(t-\tau_i(t)) + f(e(t-\tau_i(t)))^T L_i f(e(t-\tau_i(t)))\\ &\le e(t-\tau_i(t))^T (K_i + \Theta L_i \Theta) e(t-\tau_i(t))\\ &\le \frac{v}{1+\imath\bar{\tau}_i} e(t-\tau_i(t))^T P_i e(t-\tau_i(t))\\ &\le \frac{v}{1+\imath\bar{\tau}_i}\mathbb{E}\{V_1(t,e_{t-\tau_i(t)},i)\}. \end{aligned} \quad (33)$$

Thus, we obtain that

$$\begin{aligned} &\int_{t-\tau_i(t)}^{t} \left[e(s-\tau_i(s))^T K_i e(s-\tau_i(s)) + f(e(s-\tau_i(s)))^T L_i f(e(s-\tau_i(s)))\right]\mathrm{d}s \quad (34)\\ &\le \frac{v}{1+\imath\bar{\tau}_i} \int_{t-\tau_i(t)}^{t} \mathbb{E}\{V_1(s,e_{s-\tau_i(s)},i)\}\mathrm{d}s. \end{aligned}$$

Substituting (32–34) into (29) derives that

$$\begin{aligned} \frac{\mathrm{d}\mathbb{E}\{V_1(t,e_t,i)\}}{\mathrm{d}t} &< -(\imath+\alpha v)\mathbb{E}\{V_1(t,e_t,i)\} + \imath\mathbb{E}\{V_1(t,e_{t-\tau_i(t)},i)\}\\ &\quad + \frac{v}{1+\imath\bar{\tau}_i}\left[\mathbb{E}\{V_1(t,e_{t-\tau_i(t)},i)\} + \imath\int_{t-\tau_i(t)}^{t}\mathbb{E}\{V_1(s,e_{s-\tau_i(s)})\}\mathrm{d}s\right]\\ &\le -(\imath+\alpha v)\mathbb{E}\{V_1(t,e_t,i)\} + \imath\mathbb{E}\{V_1(t,e_{t-\tau_i(t)},i)\}\\ &\quad + \frac{v}{1+\imath\bar{\tau}_i}\left[\mathbb{E}\{V_1(t,e_{t-\tau_i(t)},i)\} + \imath\tau_i(t)\sup_{[t-2\bar{\tau},t]}\mathbb{E}\{V_1(s,e_{s-\tau_i(s)})\}\right]\\ &\le -(\imath+\alpha v)\mathbb{E}\{V_1(t,e_t,i)\} + (\imath+v)\sup_{[t-2\bar{\tau},t]}\mathbb{E}\{V_1(s,e_s,i)\}. \end{aligned}$$

Noting that $\alpha > 1$, applying Lemma 5 to above inequality results in

$$\mathbb{E}\{V_1(t,e_t,i)\} \le \sup_{[-2\bar{\tau},0]}\mathbb{E}\{V_1(s,e_s,i)\}e^{-\kappa t},$$

where κ is the unique positive solution of the following equation:

$$\kappa = \imath + \alpha v - (v+\imath)e^{2\kappa\bar{\tau}}.$$

Therefore, we arrive at the conclusion that

$$\mathbb{E}\{||e(t)||^2\} \le e^{-\kappa t}\mathbb{E}\{||\varphi(t)||^2\}.$$

The proof is completed.

Appendix 2

Proof of Theorem 2

Define the following Lyapunov-Krasovskii functional:

$$V(t,e_t,i) = \sum_{j=1}^{2} V_j(t,e_t,i),$$

where $V_1(t, e_t, i)$ ie defined in (16) and

$$\begin{aligned} V_2(t,e_t,i) &= \bar{\tau}_i \int_{t-\bar{\tau}_i}^{t}\int_{v}^{t} \chi_i(s)^T T_1 \chi_i(s)\mathrm{d}s\mathrm{d}v\\ &\quad + \bar{\varrho}_i \int_{t-\bar{\varrho}_i}^{t}\int_{v}^{t} f(e(s))^T T_2 f(e(s))\mathrm{d}s\mathrm{d}v\\ &\quad + \int_{t-\bar{\tau}_i}^{t}\int_{v}^{t} \rho(s)^T T_3 \rho(s)\mathrm{d}s\mathrm{d}v. \end{aligned}$$

It can be easily verified that $V(t, e_t, i)$ is a nonnegative function over $[-\hat{\tau}, +\infty)$. Evaluating the time derivative of $V(t, e_t, i)$ along the trajectory of system (3), we have that

$$\mathrm{d}V(t,e_t,i) = \pounds V(t,e_t,i)\mathrm{d}t + \frac{\partial}{\partial e}V(t,e_t,i)\rho_i(t)\mathrm{d}\omega(t), \quad (35)$$

where

$$\begin{aligned} \pounds V_1(t,e_t,i) &\le 2\{e(t)^T P_i + [\beta(t) - \Lambda e(t)]^T Q_i\\ &\quad + [\Sigma e(t) - f(e(t))]^T R_i + [f(e(t)) - \Gamma e(t)]^T S_i\}\\ &\quad \times \left[-\beta(t) + A_i(t) f(e(t)) + B_i(t) f(e(t-\tau_i(t))) + G_i(t) \int_{t-\varrho_i(t)}^{t} f(e(s))\mathrm{d}s + Y_{1i}e(t) + Y_{2i}e(t-\tau_i(t))\right]\\ &\quad + e(t)^T \sum_{k=1}^{N} \pi'_{jk}[Q_k(\Delta-\Lambda) + (R_k+S_k)(\Sigma-\Gamma)]e(t)\\ &\quad + e(t)^T\left(\sum_{j=1}^{N}\pi_{ij}P_j + U\right)e(t) - (1-\dot{\tau}_i(t))e(t-\tau_i(t))^T U e(t-\tau_i(t)) + \sum_{j=1}^{N}\pi'_{ij}\bar{\tau}_j\big[e(t-\tau_i(t))^T U e(t-\tau_i(t))\\ &\quad + f(e(t-\tau_i(t)))^T W f(e(t-\tau_i(t)))\big] + f(e(t))^T W f(e(t))\\ &\quad - (1-\dot{\tau}_i(t)) f(e(t-\tau_i(t)))^T W f(e(t-\tau_i(t))) + \rho_i(t)^T\widetilde{P}_i\rho_i(t), \end{aligned} \quad (36)$$

$$\pounds V_2(t,e_t,i) = \bar{\tau}_i^2\chi_i(t)^T T_1\chi_i(t) - \bar{\tau}_i\left(1-\sum_{j=1}^{N}\pi_{ij}\bar{\tau}_j\right)\int_{t-\bar{\tau}_i}^{t}\chi_i(s)^T T_1\chi_i(s)\mathrm{d}s + \bar{\varrho}_i^2 f(e(t))^T T_2 f(e(t)) - \bar{\varrho}_i\left(1-\sum_{j=1}^{N}\pi_{ij}\bar{\varrho}_j\right)\int_{t-\bar{\varrho}_i}^{t} f(e(s))^T T_2 f(e(s))\mathrm{d}s + \bar{\tau}_i\rho_i(t)^T T_3\rho_i(t) - \left(1-\sum_{j=1}^{N}\pi_{ij}\bar{\tau}_j\right)\int_{t-\bar{\tau}_i}^{t}\rho_i(s)^T T_3\rho_i(s)\mathrm{d}s. \tag{37}$$

For any t with $0<\tau_i(t)<\bar{\tau}_i$ and $0<\varrho_i(t)<\bar{\varrho}_i$, from Lemma 2 we have the following inequalities

$$\begin{aligned} &-\bar{\tau}_i\left(1-\sum_{j=1}^{N}\pi_{ij}\bar{\tau}_j\right)\int_{t-\bar{\tau}_i}^{t}\chi_i(s)^T T_1\chi_i(s)\mathrm{d}s \\ &= -\bar{\tau}_i\int_{t-\tau_i(t)}^{t}\chi_i(s)^T\bar{T}_1\chi_i(s)\mathrm{d}s - \bar{\tau}_i\int_{t-\bar{\tau}_i}^{t-\tau_i(t)}\chi_i(s)^T\bar{T}_1\chi_i(s)\mathrm{d}s \\ &\leq -\frac{\bar{\tau}_i}{\tau_i(t)}\left(\int_{t-\tau_i(t)}^{t}\chi_i(s)\mathrm{d}s\right)^T\bar{T}_1\left(\int_{t-\tau_i(t)}^{t}\chi_i(s)\mathrm{d}s\right) \\ &\quad -\frac{\bar{\tau}_i}{\bar{\tau}_i-\tau_i(t)}\left(\int_{t-\bar{\tau}_i}^{t-\tau_i(t)}\chi_i(s)\mathrm{d}s\right)^T\bar{T}_1\left(\int_{t-\bar{\tau}_i}^{t-\tau_i(t)}\chi_i(s)\mathrm{d}s\right), \end{aligned} \tag{38}$$

$$\begin{aligned} &-\bar{\varrho}_i\left(1-\sum_{j=1}^{N}\pi_{ij}\bar{\varrho}_j\right)\int_{t-\bar{\varrho}_i}^{t} f(e(s))^T T_2 f(e(s))\mathrm{d}s \\ &= -\bar{\varrho}_i\int_{t-\varrho_i(t)}^{t} f(e(s))^T\bar{T}_2 f(e(s))\mathrm{d}s - \bar{\varrho}_i\int_{t-\bar{\varrho}_i}^{t-\varrho_i(t)} f(e(s))^T\bar{T}_2 f(e(s))\mathrm{d}s \\ &\leq -\frac{\bar{\varrho}_i}{\varrho_i(t)}\left(\int_{t-\varrho_i(t)}^{t} f(e(s))\mathrm{d}s\right)^T\bar{T}_2\left(\int_{t-\varrho_i(t)}^{t} f(e(s))\mathrm{d}s\right) \\ &\quad -\frac{\bar{\varrho}_i}{\bar{\varrho}_i-\varrho_i(t)}\left(\int_{t-\bar{\varrho}_i}^{t-\varrho_i(t)} f(e(s))\mathrm{d}s\right)^T\bar{T}_2\left(\int_{t-\bar{\varrho}_i}^{t-\varrho_i(t)} f(e(s))\mathrm{d}s\right). \end{aligned} \tag{39}$$

Set $\lambda_j = 1$, $\mu_j = 3$, based on Lemma 2 we get from (38–39) that

$$\begin{aligned} &-\bar{\tau}_i\int_{t-\bar{\tau}_i}^{t}\chi_i(s)^T\bar{T}_1\chi_i(s)\mathrm{d}s - \bar{\varrho}_i\int_{t-\bar{\varrho}_i}^{t} f(e(s))^T\bar{T}_2 f(e(s))\mathrm{d}s \\ &\leq \max\{-\varsigma_7^T\bar{T}_1\varsigma_7 - 3\varsigma_8^T\bar{T}_1\varsigma_8 - \varsigma_6^T\bar{T}_2\varsigma_6 - 3\varsigma_{11}^T\bar{T}_2\varsigma_{11}, \\ &-\varsigma_7^T\bar{T}_1\varsigma_7 - 3\varsigma_8^T\bar{T}_1\varsigma_8 - 3\varsigma_6^T\bar{T}_2\varsigma_6 - \varsigma_{11}^T\bar{T}_2\varsigma_{11},. \\ &-3\varsigma_7^T\bar{T}_1\varsigma_7 - \varsigma_8^T\bar{T}_1\varsigma_8 - 3\varsigma_6^T\bar{T}_2\varsigma_6 - \varsigma_{11}^T\bar{T}_2\varsigma_{11}, \\ &-3\varsigma_7^T\bar{T}_1\varsigma_7 - \varsigma_8^T\bar{T}_1\varsigma_8 - \varsigma_6^T\bar{T}_2\varsigma_6 - 3\varsigma_{11}^T\bar{T}_2\varsigma_{11}\}. \end{aligned} \tag{40}$$

It is easy to verify that Eq. (40) holds for any t with $0\leq\tau_i(t)\leq\bar{\tau}_i$ and $0\leq\varrho_i(t)\leq\bar{\varrho}_i$.

From [4, 17], we have that

$$\mathbb{E}\left(\int_{t-\tau_i(t)}^{t}\rho_i(s)^T T_3\rho_i(s)\mathrm{d}s\right) = \mathbb{E}\left\{\left(\int_{t-\tau_i(t)}^{t}\rho_i(s)\mathrm{d}\omega(s)\right)^T T_3\left(\int_{t-\tau_i(t)}^{t}\rho_i(s)\mathrm{d}\omega(s)\right)\right\}, \tag{41}$$

$$\mathbb{E}\left(\int_{t-\bar{\tau}_i}^{t-\tau_i(t)}\rho_i(s)^T T_3\rho_i(s)\mathrm{d}s\right) = \mathbb{E}\left\{\left(\int_{t-\bar{\tau}_i}^{t-\tau_i(t)}\rho_i(s)\mathrm{d}\omega(s)\right)^T T_3\left(\int_{t-\bar{\tau}_i}^{t-\tau_i(t)}\rho_i(s)\mathrm{d}\omega(s)\right)\right\}. \tag{42}$$

On the other hand, by the Leibniz-Newton formula, we get that

$$\int_{t-\tau_i(t)}^{t}\chi_i(s)\mathrm{d}s = e(t) - e(t-\tau_i(t)) - \int_{t-\tau_i(t)}^{t}\rho_i(s)\mathrm{d}\omega(s).$$

Therefore, the following equalities hold for any real matrices $X_{ji}(j = 1, 2, 3)$ with compatible dimensions

$$2\chi_i(t)^T X_{1i}^T\left\{-\chi_i(t) - \beta(e(t)) + A_i(t)f(e(t)) + B_i(t)f(e(t-\tau_i(t))) + G_i(t)\int_{t-\varrho_i(t)}^{t} f(e(s))\mathrm{d}s + Y_{1i}e(t) + Y_{2i}e(t-\tau_i(t))\right\} = 0, \tag{43}$$

$$2(X_{2i}e(t) + X_{3i}e(t-\tau_i(t)))^T\left\{e(t) - e(t-\tau_i(t)) - \int_{t-\tau_i(t)}^{t}\rho_i(s)\mathrm{d}\omega(s) - \int_{t-\tau_i(t)}^{t}\chi_i(s)\mathrm{d}s\right\} = 0. \tag{44}$$

From Lemma 1, the following matrix inequalities hold for any positive scalar ϵ_i

$$2\zeta_i(t)^T\psi_{ai}^T(\Delta A_i(t)\varsigma_3+\Delta B_i(t)\varsigma_5+\Delta G_i(t)\varsigma_6)\zeta_i(t)$$
$$=2\zeta_i(t)^T\psi_{ai}^T E_i\Phi_i(t)(H_{1i}\varsigma_3+H_{2i}\varsigma_5+H_{3i}\varsigma_6)\zeta_i(t)$$
$$\leq\zeta_i(t)^T\{\epsilon_i^{-1}\psi_{ai}^T E_i E_i^T\psi_{ai}^T+\epsilon_i(H_{1i}\varsigma_3+H_{2i}\varsigma_5+H_{3i}\varsigma_6)^T (H_{1i}\varsigma_3+H_{2i}\varsigma_5+H_{3i}\varsigma_6)\}\zeta_i(t). \tag{45}$$

By (22–25), (36–37) and (40–45), and taking the mathematical expectations on both sides of (35), we obtain that

$$\mathrm{d}\mathbb{E}\{V(t,e_t,i)\}=\mathbb{E}\pounds V(t,e_t,i)\mathrm{d}t+\mathbb{E}\left\{\frac{\partial}{\partial e}V(t,e_t,i)\rho_i(t)\mathrm{d}\omega(t)\right\}$$
$$\leq\zeta_i(t)^T\Big(\widetilde{\Omega}_i+2\max\{-\varsigma_8^T\bar{T}_1\varsigma_8-\varsigma_{11}^T\bar{T}_2\varsigma_{11},-\varsigma_8^T\bar{T}_1\varsigma_8-\varsigma_6^T\bar{T}_2\varsigma_6, -\varsigma_7^T\bar{T}_1\varsigma_7-\varsigma_6^T\bar{T}_2\varsigma_6,-\varsigma_7^T\bar{T}_1\varsigma_7-\varsigma_{11}^T\bar{T}_2\varsigma_{11}\}\Big)\zeta_i(t).$$

From (15), there exists a positive scalar α_0 such that

$$\mathrm{d}\mathbb{E}\{V(t,e_t,i)\}<-\alpha_0\mathbb{E}||e(t)||^2.$$

Similar to the proof of Theorem 1 in [27], it implies that the error system (3) is globally exponentially stable. This completes the proof.

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