

A new result for projection neural networks to solve linear variational inequalities and related optimization problems

Bonan Huang · Huaguang Zhang · Dawei Gong ·
Zhanshan Wang

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Abstract In recent years, a projection neural network was proposed for solving linear variational inequality (LVI) problems and related optimization problems, which required the monotonicity of LVI to guarantee its convergence to the optimal solution. In this paper, we present a new result on the global exponential convergence of the projection neural network. Unlike existing convergence results for the projection neural network, our main result does not assume the monotonicity of LVI problems. Therefore, the projection neural network can be further guaranteed to solve a class of non-monotone LVI and non-convex optimization problems. Numerical examples illustrate the effectiveness of the obtained result.

Keywords Non-monotone linear variational inequality · Non-convex optimization · Projection neural network · Global exponential convergence

1 Introduction

Consider the following linear variational inequality (LVI) problem: find a vector $x^* \in \Omega$ such that

$$(Mx^* + q)^T(x - x^*) \geq 0, \quad \forall x \in \Omega, \quad (1)$$

where $M \in R^{n \times n}$, $q \in R^n$, the feasible domain $\Omega = \{x \in R^n \mid l \leq x \leq h\}$. If M is positive semidefinite or positive

definite matrix, the problem (1) is called the monotone LVI or strictly monotone LVI.

LVI problems are of crucial importance in mathematical programming and enjoy a wide variety of science and engineering applications. Many constrained optimization problems, such as linear complementary problems and linear-quadratic programming problems, can be equivalently converted into LVI problems [1, 2]. Various numerical algorithms for solving LVI problems, such as the descent method [3] and the projection-type method [4], have been studied for decades.

In recent years, recurrent neural networks have found successful applications in signal processing, pattern recognition, associative memory, and other engineering or scientific fields [5–10, 26–28]. Moreover, due to its inherent nature of parallelization and distributed information process, recurrent neural networks are promising computational models for realtime applications. During the recent decade, some neural networks have been proposed for solving LVI problems and related optimization problems [11–22]. Among them, the following projection neural network was developed and investigated for solving LVI (1) in [11–13]:

$$\frac{dx(t)}{dt} = \lambda \{P_\Omega(x(t) - \alpha(Mx(t) + q)) - x(t)\}, \quad (2)$$

where $\lambda > 0$, $\alpha > 0$ are two scaling factors and $P_\Omega : R^n \rightarrow \Omega$ is a projection operator defined by $P_\Omega(x) = (P_\Omega(x_1), \dots, P_\Omega(x_n))^T$ with

$$P_\Omega(x_i) = \begin{cases} h_i, & x_i > h_i; \\ x_i, & l_i \leq x_i \leq h_i; \\ l_i, & x_i < l_i. \end{cases} \quad (3)$$

where $l = [l_1, \dots, l_n]^T$ and $h = [h_1, \dots, h_n]^T$. Note that x^* is a solution of problem (1) if and only if it is an equilibrium point of the above neural network.

B. Huang · H. Zhang · D. Gong (✉) · Z. Wang
College of Information Science and Engineering,
Northeastern University, Shenyang 110819, China
e-mail: pzhzhx@126.com

H. Zhang
e-mail: hbn629@163.com; hg Zhang@ieee.org

As was shown in [13], the projection neural network (2) features a very low complexity and superior than many other neural networks for solving LVI and related problems. The global convergence of the projection neural network has been well studied in [11–13]; however, it requires very restrictive conditions that the LVI is monotone or strictly monotone, or equivalently, M is positive semidefinite or positive definite. As it is known, there exist many cases that the LVI problems are non-monotone and related optimization problems are non-convex in real-world applications [23–25]. Therefore, reducing the convergence condition of the projection neural networks (2) is very desirable for wide engineering applications, and there are some researchers working on this issue in these years [14, 15, 17, 18]. For example, in [15], Hu et al. proved that the projection neural networks can also solve LVI problems by requiring that the LVI is pseudo-monotone, which is a weaker condition than monotonicity. In recent years, by introducing the time delays into the projection neural network model (2), several delayed neural networks [14, 17, 18] were proposed to solve the LVI (1), by which a class of non-monotone LVI can be solved under some assumptions. However, the introduction of time delays inevitably increases the complexity of the neural networks, which is a critical concern in hardware implementation.

Motivated by the above discussions, this paper presents a new global convergence condition for the projection neural network (2) to solve the LVI and related problems. By constructing a new Lyapunov–Krasovskii functional, a linear matrix inequality (LMI) approach is developed to establish a sufficient condition for the projection neural network (2) to be globally exponentially convergent to the optimal solution. By the proposed LMI approach, the monotonicity assumption on the LVI is no longer necessary. Therefore, the obtained result guarantees that the projection neural network (2) can also solve effectively a class of non-monotone LVI and non-convex optimization problems.

2 Preliminaries

For the sake of later discussion, a definition and two lemmas and some notations are introduced in this section. In what follows, I denotes the unity matrix with suitable dimension; $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ represent the maximum and minimum eigenvalues of the given matrix, respectively; for a vector $\phi \in R^n$, $\|\phi\|$ denotes the vector l_2 -norm; $\text{diag}(\cdot)$ denotes a diagonal matrix formed from its vector argument.

Definition 1 The projection neural network (2) is said to be globally exponentially stable at the equilibrium point x^*

if every trajectory $x(t)$ of (2) with the initial point $x(t_0) \in R^n$ satisfies that:

$$\|x(t) - x^*\| \leq k_0 \|x(t_0) - x^*\| \exp(-\eta(t - t_0)), \quad \forall t \geq t_0,$$

where k_0 and η are positive constants independent of the initial point.

Lemma 1 [14] *The projection operator P_Ω satisfies the following inequality for any $x, y \in R^n$:*

$$\|P_\Omega(x) - P_\Omega(y)\| \leq \|x - y\|$$

and

$$(P_\Omega(x) - P_\Omega(y))^T (P_\Omega(x) - P_\Omega(y)) \leq (P_\Omega(x) - P_\Omega(y))^T (x - y).$$

Lemma 2 [11] *For each $x(t_0) \in R^n$, there exists a unique continuous solution $x(t)$ for the projection neural network (2) over $t \in [t_0, \infty)$.*

3 Main results

In this section, we analyze the global stability and convergence of the projection neural network (2) and obtain a new condition for its global exponential convergence to the optimal solution of problem (1).

Firstly, we reformulate the projection neural network (2) as follows:

$$\frac{dx(t)}{dt} = P_\Omega(Ax(t) + b) - x(t), \quad (4)$$

where $A = (a_{ij})_{n \times n} = I - \alpha M$ and $b = (b_1, b_2, \dots, b_n)^T = -\alpha q$.

Then, let x^* be the equilibrium point of the neural network (4) and $a_i = [a_{i1}, a_{i2}, \dots, a_{in}]$. By coordinate transformation $u(t) = x(t) - x^*$, we get the following system from (4):

$$\frac{du(t)}{dt} = f(Au(t)) - u(t), \quad (5)$$

where

$$f(Au(t)) = [f_1(a_1u(t)), f_2(a_2u(t)), \dots, f_n(a_nu(t))]^T,$$

and

$$f_i(a_iu(t)) = P_\Omega[a_i(u(t) + x^*) + b_i] - P_\Omega(a_ix^* + b_i), \\ i = 1, 2, \dots, n.$$

Clearly, x^* is globally exponentially stable for system (4) if and only if the zero solution of system (5) is globally exponentially stable.

Theorem 1 *If there exist a positive definite symmetric matrix $P \in R^{n \times n}$ and positive definite diagonal matrices*

$D = \text{diag}\{d_1, d_2, \dots, d_n\}$ and $L = \text{diag}\{l_1, l_2, \dots, l_n\} \in R^{n \times n}$ and a constant $k > 0$ such that the following LMI (6) holds, then the equilibrium point of the neural network (5) is globally exponentially stable.

$$\Xi = \begin{bmatrix} (2k - 2)P & (2k - 1)A^T D + P + A^T L \\ * & DA + A^T D - 2L \end{bmatrix} < 0. \tag{6}$$

Proof Construct the following Lyapunov–Krasovskii functional:

$$V(u(t)) = e^{2k(t-t_0)} u^T(t) P u(t) + 2e^{2k(t-t_0)} \sum_{i=1}^n d_i \int_0^{a_i u} f_i(\eta) d\eta. \tag{7}$$

From the definition of P_Ω , it is obvious that $P_\Omega[a_i(u(t) + x^*) + b_i]$ is a non-decreasing function, so $f_i(a_i u(t))$ is also non-decreasing. In addition, it easy to see that $f_i(0) = 0$, for $i = 1, 2, \dots, n$. Therefore, if $a_i u \geq 0$, then for $0 \leq \eta \leq a_i u$, it can be obtained that

$$0 \leq \int_0^{a_i u} f_i(\eta) d\eta \leq a_i u f_i(a_i u),$$

and if $a_i u < 0$, then for $0 \geq \eta \geq a_i u$, we obtain

$$0 \geq \int_{a_i u}^0 f_i(\eta) d\eta \geq -a_i u f_i(a_i u).$$

Consequently, for any $a_i u$, we can obtain that

$$0 \leq \int_0^{a_i u} f_i(\eta) d\eta \leq a_i u f_i(a_i u). \tag{8}$$

Calculating the time derivative of $V[u(t)]$ along the trajectory of (5) yields:

$$\begin{aligned} \frac{dV(u(t))}{dt} &= 2ke^{2k(t-t_0)} u^T(t) P u(t) + 2e^{2k(t-t_0)} u^T(t) P \frac{du(t)}{dt} \\ &\quad + 4ke^{2k(t-t_0)} \sum_{i=1}^n d_i \int_0^{a_i u} f_i(\eta) d\eta \\ &\quad + 2e^{2k(t-t_0)} f^T(Au(t)) DA \frac{du(t)}{dt} \\ &\leq (2k - 2)e^{2k(t-t_0)} u^T(t) P u(t) \\ &\quad + 2e^{2k(t-t_0)} u^T(t) [P + (2k - 1)A^T D] f(Au(t)) \\ &\quad + 2e^{2k(t-t_0)} f^T(Au(t)) DA f(Au(t)) \end{aligned} \tag{9}$$

From Lemma 1, we can obtain that:

$$\begin{aligned} &[P_\Omega(Ax + b) - P_\Omega(Ax^* + b)]^T [P_\Omega(Ax + b) - P_\Omega(Ax^* + b)] \\ &\leq [P_\Omega(Ax + b) - P_\Omega(Ax^* + b)]^T A(x - x^*), \end{aligned}$$

that is

$$f^T(Au(t)) f(Au(t)) \leq f^T(Au(t)) Au(t).$$

For any positive definite diagonal matrix $L = \text{diag}\{l_1, l_2, \dots\}$, the following inequality holds:

$$f^T(Au(t)) L f(Au(t)) \leq f^T(Au(t)) L Au(t). \tag{10}$$

Substituting (10) into (9), it yields:

$$\begin{aligned} \frac{dV(u(t))}{dt} &\leq (2k - 2)e^{2k(t-t_0)} u^T(t) P u(t) \\ &\quad + 2e^{2k(t-t_0)} u^T(t) [P + (2k - 1)A^T D + A^T L] f(Au(t)) \\ &\quad + 2e^{2k(t-t_0)} f^T(Au(t)) (DA - L) f(Au(t)) \end{aligned}$$

Let $v^T(t) = [u^T f^T(Au(t))]$, then

$$\frac{dV(u(t))}{dt} \leq e^{2k(t-t_0)} v^T \Xi v,$$

where

$$\Xi = \begin{bmatrix} (2k - 2)P & (2k - 1)A^T D + P + A^T L \\ * & DA + A^T D - 2L \end{bmatrix}. \tag{11}$$

Therefore, $V(u(t)) \leq V(u(t_0))$ if (6) holds. Furthermore,

$$\begin{aligned} V(u(t_0)) &= u^T(t_0) P u(t_0) + 2 \sum_{i=1}^n d_i \int_0^{a_i u(t_0)} f_i(\eta) d\eta \\ &\leq \lambda_{\max}(P) \|u(t_0)\|^2 + 2 \max_i(d_i) [Au(t_0)]^T f[Au(t_0)] \\ &\leq \lambda_{\max}(P) \|u(t_0)\|^2 + 2 \max_i(d_i) u^T(t_0) (A^T A) u(t_0) \\ &\leq \left[\lambda_{\max}(P) + 2 \max_i(d_i) \lambda_{\max}(A^T A) \right] \|u(t_0)\|^2. \end{aligned} \tag{12}$$

Meanwhile, it is easy to see that:

$$\begin{aligned} V(u(t)) &\geq e^{2k(t-t_0)} u^T(t) P u(t) \\ &\geq e^{2k(t-t_0)} \lambda_{\min}(P) \|u(t)\|^2 \end{aligned} \tag{13}$$

Therefore, from (12) and (13) we obtain,

$$\|u(t) - u^*\| \leq \phi \|u(t_0) - u^*\| e^{-k(t-t_0)},$$

where

$$\phi = \sqrt{\frac{\lambda_{\max}(P) + 2 \max_i(d_i) \lambda_{\max}(A^T A)}{\lambda_{\min}(P)}}.$$

By Lyapunov stability theory and Definition 1, the equilibrium point of neural network (5) is globally exponentially stable if (6) holds. This completes the proof.

Remark 1 Compared with the existing exponential convergence results in [11–13], Theorem 1 does not require that M is positive semidefinite or positive definite. It implies that the projection neural network (2) can solve a class of non-monotone LVI. Therefore, Theorem 1 extends

the results obtained in [11–13]. Moreover, the condition in Theorem 1 is described in the LMI form, which is easy to check by many efficient LMI solvers.

Remark 2 Based on the neural network model (2), delayed projection neural networks were developed in [14, 17, 18], which can solve a class of non-monotone LVI (1). However, Theorem 1 shows that the projection neural network (2) can solve a class of non-monotone LVI problems without the need of introducing time delays. Moreover, the results obtained in [14, 17, 18] were under the assumption that $I - \alpha M$ is non-singular. In contrast, Theorem 1 has no such restriction.

4 Numerical simulations

In order to demonstrate the effectiveness of the projection neural network (2) for solving non-monotone LVI and non-convex optimization problems, two illustrative examples are presented in this section.

Example 1 [18] Consider the following third-order LVI problem, where

$$M = \begin{bmatrix} 0.5 & 1 & -0.1 \\ -0.1 & 0.5 & 0.1 \\ 1 & -0.1 & 0.5 \end{bmatrix}, q = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \tag{14}$$

$$\Omega = \{x \in R^3 \mid -5 \leq x_i \leq 5, i = 1, 2, 3\}.$$

It is obvious that M is an asymmetric and indefinite matrix, which means that it is a non-monotone LVI problem, and this problem has a unique solution $x = [-0.3343, -0.5775, 2.5532]^T$. Thus, the results in [11–13] and [15] are not applicable for this example.

Let $k = 0.01, \alpha = 1$, after solving the LMI (6), feasible solution can be obtained as follows:

$$P = \begin{bmatrix} 84.2878 & 12.5512 & 18.4989 \\ 12.5512 & 95.1699 & -1.5830 \\ 18.4989 & -1.5830 & 67.6605 \end{bmatrix},$$

$$D = \begin{bmatrix} 135.5941 & 0 & 0 \\ 0 & 382.2949 & 0 \\ 0 & 0 & 72.6084 \end{bmatrix},$$

$$L = \begin{bmatrix} 178.0005 & 0 & 0 \\ 0 & 298.7208 & 0 \\ 0 & 0 & 94.9591 \end{bmatrix}.$$

Therefore, Theorem 1 guarantee that the neural network (2) can globally exponentially converge to the solution of LVI (14). Figure 1 displays the transient behavior of (2) with ten random initial points. The simulation results show

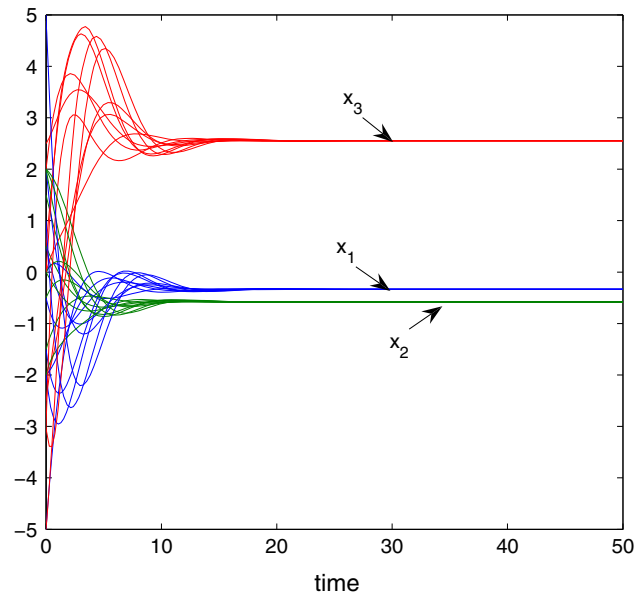


Fig. 1 Transient behavior of the neural network (2) in Example 1

that all the trajectories are convergent to the optimal solution $x^* = [-0.3343, -0.5775, 2.5532]^T$.

Next, we will show that the network (2) can be applied to solve a class of non-convex optimization problems.

Example 2 Consider the following quadratic programming problem:

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}x^T Qx + p^T x \\ \text{s.t.} \quad & Wx \leq c, x \geq 0. \end{aligned} \tag{15}$$

where

$$W = \begin{bmatrix} \frac{12}{5} & -1 \\ \frac{5}{2} & 1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}, Q = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, c = \begin{bmatrix} \frac{35}{12} \\ \frac{35}{2} \\ 5 \\ 5 \end{bmatrix},$$

$$p = \begin{bmatrix} -30 \\ -30 \end{bmatrix}.$$

This is a non-convex quadratic programming problem, and its unique optimal solution is $x^* = [5, 5]^T$. Following a similar line of [14], problem (15) can be solved by the projection neural network (2), by letting

$$M = \begin{bmatrix} Q & W^T \\ -W & 0 \end{bmatrix}, q = \begin{bmatrix} p \\ c \end{bmatrix},$$

$$\Omega = \{z = (x, y) \in R^6 \mid z \geq 0\}.$$

Let $k = 0.1, \alpha = 0.1$, after solving the LMI (6), we can get the following feasible solutions:

$$P = \begin{bmatrix} 12.8048 & 0.0321 & 0.0010 & 0.0010 & -0.0026 & 0 \\ 0.0321 & 13.1617 & 0.0296 & -0.0317 & 0 & -0.0295 \\ 0.0010 & 0.0296 & 12.8549 & 0 & 0 & 0 \\ 0.0010 & -0.0317 & 0 & 12.8128 & 0 & 0 \\ -0.0026 & 0 & 0 & 0 & 12.8564 & 0 \\ 0 & -0.0295 & 0 & 0 & 0 & 12.8561 \end{bmatrix}$$

$$D = \begin{bmatrix} 28.8961 & 0 & 0 & 0 & 0 & 0 \\ 0 & 30.1821 & 0 & 0 & 0 & 0 \\ 0 & 0 & 28.8962 & 0 & 0 & 0 \\ 0 & 0 & 0 & 28.8962 & 0 & 0 \\ 0 & 0 & 0 & 0 & 28.8821 & 0 \\ 0 & 0 & 0 & 0 & 0 & 28.8962 \end{bmatrix}$$

$$L = \begin{bmatrix} 16.2489 & 0 & 0 & 0 & 0 & 0 \\ 0 & 15.2771 & 0 & 0 & 0 & 0 \\ 0 & 0 & 16.0927 & 0 & 0 & 0 \\ 0 & 0 & 0 & 16.2240 & 0 & 0 \\ 0 & 0 & 0 & 0 & 16.0766 & 0 \\ 0 & 0 & 0 & 0 & 0 & 16.0889 \end{bmatrix}$$

According to Theorem 1, the neural network (2) is globally exponentially stable at its equilibrium point. Figure 2 displays the transient behavior of (2) with ten random initial points. The simulation results shows that all the trajectories are convergent to the optimal solution $x^* = [x_1, x_2]^T = [5, 5]^T$.

5 Conclusion

In this paper, we proposed a new result on the exponential convergence of a projection neural network, which was originally proposed for solving monotone LVI and convex optimization problems. Without the monotonicity assumption on the LVI, we proved that the projection neural network is still globally exponentially convergent to

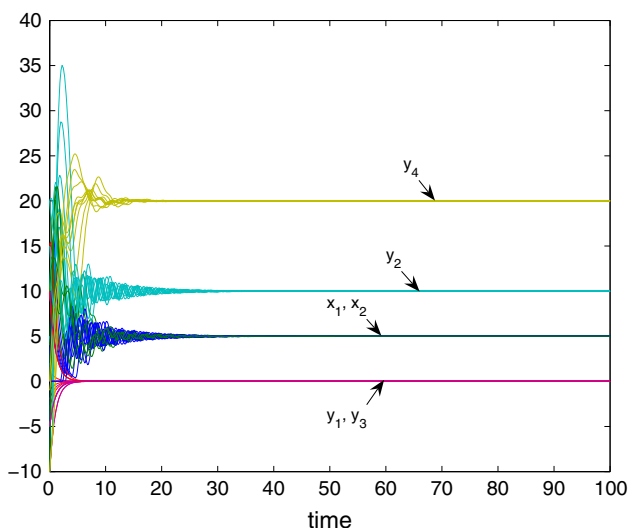


Fig. 2 Transient behavior of the neural network (2) in Example 2

the optimal solution under mild conditions. Therefore, the application scope of the projection neural network was extended to a class of non-monotone LVI and non-convex optimization problems. Simulation results demonstrated the effectiveness of the obtained result.

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