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Pseudo-t-norms and pseudo-BL algebras

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Abstract BL algebras were introduced by Hájek as algebraic structures for his Basic Logic, starting from continuous t-norms on $[0,1]$. MV algebras, product algebras and Gödel algebras are particular cases of BL algebras. On the other hand, the pseudo-MV algebras extend the MV-algebras in the same way in which the arbitrary l-groups extend the abelian l-groups. We have generalized the BL algebras and pseudo-MV algebras, introducing the pseudo-BL algebras. In this paper we introduce weak-BL algebras and weak-pseudo-BL algebras. We also introduce non-commutative t-norms (we call them pseudo-t-norms) and use them in constructing pseudo-BL algebras and weak-pseudo-BL algebras.

Keywords MV algebra, BL algebra, Pseudo-MV algebra, Pseudo-BL algebra, Weak-BL algebra, Weak-pseudo-BL algebra, t-norm, t-conorm, Pseudo-t-norm, Pseudo-t-conorm, Left continuity, Right continuity

Dedicated to Prof. Beloslav Riečan on the occasion of his 65th birthday

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1 Introduction

Wehave started from the following situation:

(1) We know BL algebras, algebraic structures for which the principal examples come from:
  – the real interval [0, 1] with the structure given by a continuous t-norm and
  – abelian l-groups.
(2) The non-commutative case (pseudo-MV algebras and pseudo-BL algebras) was developed starting from arbitrary l-groups.

The natural problem was: can be defined a concept of pseudo-t-norm (by weaking the axioms of t-norms) on $[0,1]$ or, more general, on bounded chains, bounded lattices, in order to obtain new examples of pseudo-BL algebras?

The present paper tries to answer to this problem in the following way:

(1) We define the notion of pseudo-t-norm, by throwing away the axiom of commutativity;
(2) On arbitrary l-groups, we refind the already known examples;
(3) On [0, 1], the condition of continuity is replaced by the weaker condition of left continuity in both variables. Then, [0, 1] is endowed with a weaker structure, that of weak-pseudo-BL algebra. This structure gives birth to the new concept of weak-BL algebra, in the commutative case.

In conclusion, we define a concept of pseudo-t-norm, that leads to pseudo-BL algebras on arbitrary l-groups; on [0, 1], we can not do this, without exiting from commutative case, the adequate structure being that of weak-pseudo-BL algebra. In this paper we present a general up-today picture of the algebra of non-commutative logic.

2 t-Norms and Φ-operators

First, we shall recall the definitions of t-norms (t-conorms) and of their associated Φ-operators defined on the real interval [0, 1].

Definition 2.1 (Cf. [32])

(a) A binary operation T on the real interval [0, 1] is a t-norm iff:
  (t0) it is commutative,
  (t1) it is associative,
  (t2) it is non-decreasing (isotone) in the first argument (i.e. if $x \leq y$, then $T(x,z) \leq T(y,z)$, for

every $x, y, z \in [0, 1]$), and hence in the second argument too,
(t3) it has 1 as neutral element, i.e. $T(x, 1) = x$ (and consequently, $T(x, 0) = 0$), for every $x \in [0, 1]$;
(b) A binary operation S in the real interval [0, 1] is a t-conorm iff:
(t0) it is commutative,
(t1) it is associative,
(t2) it is non-decreasing (isotone) in the first argument (i.e. if $x \leq y$, then $S(x, z) \leq S(y, z)$, for every $x, y, z \in [0, 1]$), and hence in the second argument too,
(t3′) it has 0 as neutral element, i.e. $S(x, 0) = x$ (and consequently, $S(x, 1) = 1$), for every $x \in [0, 1]$.

Remark 2.2 Since the t-norms on [0, 1] are connected with left continuity and the t-conorms on [0, 1] are connected with right continuity, we could name them "left-t-norm" and "right-t-norm", respectively, and we shall denote them sometimes by "T_L" and "S_R", respectively ("L" from "left" and "R" from "right"). In any case, this is the reason why we shall put the label "left" ("right") to the algebraic structures in which a t-norm (t-conorm, respectively) is involved.

Concerning notation for t-norms, t-conorms and all the binary operations that will appear in this paper, we always feel free to use them either in prefix or in infix notation, without further comment according to which notation seems to be best in the particular context.

There is a 1-1 correspondence between t-norms and t-conorms on [0, 1]. Indeed, for every t-norm T, the function S_T defined by

$$S_T(x, y) \stackrel{\text{def}}{=} 1 - T(1 - x, 1 - y) \tag{1}$$

is a t-conorm and for every t-conorm S, the function T_S defined by

$$T_S(x, y) \stackrel{\text{def}}{=} 1 - S(1 - x, 1 - y) \tag{2}$$

is a t-norm.

Consequently, one can either focus attention on the t-norms or on the t-conorms. Remark that in defining MV algebras there were considered the t-conorms, while in defining BL algebras, there were considered the t-norms. Therefore, in this paper, we shall consider both "left" and "right" structures, to be able to see better the connections existing between them.

Definition 2.3 (See [32])

(a) A binary operation φ^L in the real interval [0, 1] is called Φ^L-operator connected with a given t-norm T iff for every $x, y, z \in [0, 1]$ the following hold true:
(L′) If $y \leq z$, then $\varphi^L(x, y) \leq \varphi^L(x, z)$,
(L″) $T(\varphi^L(x, y), x) \leq y$,
(L‴) $y \leq \varphi^L(x, T(y, x))$.
(b) A binary operation φ^R in the real interval [0, 1] is called Φ^R-operator connected with a given t-conorm S iff for every $x, y, z \in [0, 1]$ the following hold true:
(R1′) If $y \leq z$, then $\varphi^R(x, y) \leq \varphi^R(x, z)$,
(R1″) $S(\varphi^R(x, y), x) \geq y$,
(R1‴) $\varphi^R(x, S(y, x)) \leq y$.

The Φ-operators have been introduced by Pedrycz [47]. One Φ^L-operator (Φ^R-operator) at most is connected to each t-norm (t-conorm). The following proposition holds.

Proposition 2.4 (Cf. [32])
For a t-norm T (t-conorm S), there exists a Φ^L (Φ^R)-operator iff T is left-continuous (S is right-continuous, respectively) and

$$x \rightarrow_L y = \varphi^L(x, y) = \sup\{z | T(z, x) \leq y\}$$
$$(x \rightarrow_R y = \varphi^R(x, y) = \inf\{z | S(z, x) \geq y\}) \ .$$

The t-norms – a common shorthand for "triangular norms" – have been widely used in investigations into probabilistic metric spaces [25, 43, 48, 49]. From those investigations and not only from them, we can say that the t-norms (t-conorms) can be considered as truth functions of generalized conjunction (disjunction) operators in a suitable many-valued logic, while the corresponding Φ^L-operators (Φ^R-operators) can be considered as truth functions of generalized implication operators (also called residua).

We recall the following important result concerning t-norms on [0, 1] (and there is the similar result concerning the t-conorms):

Theorem 2.5 Let T be a t-norm on $[0, 1]$.

(i) If T is left continuous, then we have, for all $x, y \in [0, 1]$,

$$T(x, \varphi^L(x, y)) \leq x \wedge y \ .$$

(ii) [33], cf. [4]

For all $x, y \in [0, 1]$, $T(x, \varphi^L(x, y)) = x \wedge y$ iff T is continuous.

Notice that the commutativity of T is not used in the proof of the theorem.

We can define two negations on $[0, 1]$: "$^{-L}$", connected with t-norms and "$^{-R}$", connected with t-conorms, for any $x \in [0, 1]$, by:

$$x^{-L} \stackrel{\text{def}}{=} x \rightarrow_L 0 \quad \text{and} \quad x^{-R} \stackrel{\text{def}}{=} x \rightarrow_R 1 \ .$$

The notion of t-norm (t-conorm) on the real interval $[0, 1]$ can be straightforward extended to bounded linearly ordered sets (chains), $C_{0,1}$, (see [4]) and even to bounded lattices, $L_{0,1}$, (see [15]), by simply replacing $[0, 1]$ by $C_{0,1}$ or by $L_{0,1}$ in Definition 2.1. The notion of Φ-operator connected with a given t-norm (t-conorm) on $[0, 1]$ can also be straightforward extended to $C_{0,1}$ or to $L_{0,1}$. It remains to solve the problem of the existence of the Φ-operator in these cases.

3 BL algebras

BL algebras were introduced by Petr Hájek [34–37]:

Definition 3.1 [35] A BL algebra is a structure $(A, \vee, \wedge, \odot, \rightarrow, 0, 1)$ such that for all $x, y, z \in A$,

(B1) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice,
(B2) $(A, \odot, 1)$ is an abelian monoid (i.e. $x \odot y = y \odot x$, $x \odot (y \odot z) = (x \odot y) \odot z$, $x \odot 1 = 1 \odot x = x$),

(B3) $x \odot y \leq z$ iff $x \leq y \to z$ (residuation),
(B4) $x \wedge y = x \odot (x \to y)$ (divisibility),
(B5) $(x \to y) \vee (y \to x) = 1$ (preliniarity).

Remark 3.2 Since the notion of BL algebra is connected with the t-norm "$\odot$" on A, we shall name it "left-BL algebra" and we shall put the label "$_{\mathrm{L}}$" in the sequel. The negation on A_{L} is defined by: $x^{-L} \stackrel{\mathrm{def}}{=} x \to_{\mathrm{L}} 0_{\mathrm{L}}$.

We shall define now, dually, the "right-BL algebra", as follows:

Definition 3.3 A right-BL algebra is a structure

$$\mathcal{A}_{\mathrm{R}} = (A_{\mathrm{R}}, \vee_{\mathrm{R}}, \wedge_{\mathrm{R}}, \oplus_{\mathrm{R}}, \to_{\mathrm{R}}, 0_{\mathrm{R}}, 1_{\mathrm{R}}) \ ,$$

of type (2, 2, 2, 2, 0, 0), which satisfies the following axioms, for all $x, y, z \in A_{\mathrm{R}}$:

(R1) $(A_{\mathrm{R}}, \vee_{\mathrm{R}}, \wedge_{\mathrm{R}}, 0_{\mathrm{R}}, 1_{\mathrm{R}})$ is a bounded lattice,
(R2) $(A_{\mathrm{R}}, \oplus_{\mathrm{R}}, 0_{\mathrm{R}})$ is a monoid ($\oplus_{\mathrm{R}}$ is commutative and associative and $x \oplus_{\mathrm{R}} 0_{\mathrm{R}} = 0_{\mathrm{R}} \oplus_{\mathrm{R}} x = x$),
(R3) $z \leq_{\mathrm{R}} x \oplus_{\mathrm{R}} y$ iff $y \to_{\mathrm{R}} z \leq_{\mathrm{R}} x$ (residuation),
(R4) $x \vee_{\mathrm{R}} y = (x \to_{\mathrm{R}} y) \oplus_{\mathrm{R}} x$ (divisibility)
(R5) $(x \to_{\mathrm{R}} y) \wedge_{\mathrm{R}} (y \to_{\mathrm{R}} x) = 0_{\mathrm{R}}$ (prelinearity),

where "$\oplus_{\mathrm{R}}$" is a t-conorm on A_{R}. The negation on A_{R} is defined by: $x^{-R} \stackrel{\mathrm{def}}{=} x \to_{\mathrm{R}} 1_{\mathrm{R}}$.

The class of BL algebras contains the MV algebras [9, 10], the product algebras [37] and the Gödel algebras [35]. These three types of structures constitute algebraic models for the most significant fuzzy logics: Lukasiewicz logic, product logic and Gödel logic. The algebraic study of these algebras is motivated not only by the logical interest, but also by their relation with some remarkable mathematical structures (see [10, 12, 35]).

We recal that:

An MV algebra is a BL algebra with the additional axiom: $x = (x^{-L})^{-L}$, in t-norms case ($x = (x^{-R})^{-R}$, in t-conorms case);

A Π-algebra (product algebra) is a BL algebra with two additional axioms:

(B6) $x \wedge x^{-L} = 0$,
(B7) $(z^{-L})^{-L} \leq ((T(x,z) \to_{\mathrm{L}} T(y,z)) \to_{\mathrm{L}} (x \to_{\mathrm{L}} y))$;

A G-algebra (Gödel algebra) is a BL algebra satysfying the additional axiom: $T(x,x) = x$, in t-norms case ($S_T(x,x) = x$, in t-conorm case), i.e. is a Heyting algebra satisfying the axiom (B5) (the axiom (R5), respectively).

Remark 3.4 One can call the MV algebras as "right-MV algebras" and denote them by: $\mathcal{A}_{\mathrm{R}} = (A_{\mathrm{R}}, \oplus_{\mathrm{R}}, \odot_{\mathrm{L}}, ^{-R}, 0_{\mathrm{R}}, 1_{\mathrm{R}})$ and then define also the "left-MV algebras" and denote them by: $\mathcal{A}_{\mathrm{L}} = (A_{\mathrm{L}}, \odot_{\mathrm{L}}, \oplus_{\mathrm{R}}, ^{-L}, 0_{\mathrm{L}}, 1_{\mathrm{L}})$.

The same for the other two structures.

We recall two types of t-norms (t-conorms) that give BL algebras.

Type 1: continuous t-norms (t-conorms) on [0, 1]

Concerning continuous t-norms T (t-conorms S_T), together with their corresponding implications, that we shall denote by $\to_{\mathrm{L}} = \varphi^L (\to_{\mathrm{R}} = \varphi^R)$, there exist three basic examples: (cf. [50])

(1) Lukasiewicz

$$T(x,y) = \max(0, x + y - 1),$$

$$x \to_{\mathrm{L}} y = \begin{cases} 1, & \text{if } x \leq y \\ 1 - x + y, & \text{if } x > y \end{cases} = \min(1, 1 - x + y);$$

$$S_T(x,y) = \min(1, x + y),$$

$$x \to_{\mathrm{R}} y = \begin{cases} 0, & \text{if } y \leq x \\ y - x, & \text{if } y > x \end{cases}.$$

We have that $([0,1], T, S_T, ^{-L}, 0, 1)$ is a left-MV algebra and $([0,1], S_T, T, ^{-R}, 0, 1)$ is a right-MV algebra (i.e. MV algebra introduced by C. C. Chang in [9]).

As proved in [45], any continuous t-norm on $[0,1]$ with no idempotents (i.e. $x \odot x = x$), except $0, 1$, and at least one nilpotent (i.e. non-zero elements x such that $x^n = 0$ for some $n \geq 1$), is equivalent to T.

(2) Product (Gaines)

$$T(x,y) = xy,$$

$$x \to_{\mathrm{L}} y = \begin{cases} 1, & \text{if } x \leq y \\ y/x, & \text{if } x > y, \end{cases} \quad \text{(Goguen implication)}$$

$$S_T(x,y) = x + y - xy,$$

$$x \to_{\mathrm{R}} y = \begin{cases} 0, & \text{if } y \leq x \\ \frac{y-x}{1-x}, & \text{if } y > x \end{cases}.$$

We have that $([0,1], \vee = \sup, \wedge = \inf, T, \to_{\mathrm{L}}, 0, 1)$ is a left-BL algebra, called (left) Π (product) algebra and $([0,1], \sup, \inf, S_T, \to_{\mathrm{R}}, 0, 1)$ is a right-BL algebra, called (right) Π (product) algebra. Faucett proved in [23] that any continuous t-norm,"$\odot$", with no idempotents, except $0, 1$, and no nilpotents, is equivalent to T.

(3) Gödel (Brouwer)

$$T(x,y) = \min(x,y),$$

$$x \to_{\mathrm{L}} y = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{if } x > y, \end{cases} \quad \text{(Gödel implication)}$$

$$S_T(x,y) = \max(x,y),$$

$$x \to_{\mathrm{R}} y = \begin{cases} 0, & \text{if } y \leq x \\ y, & \text{if } y > x \end{cases}.$$

It is well-known that $\min(x,y)$ is the greatest t-norm on $[0,1]$.

These three examples are fundamental since any continuous t-norm is either isomorphic to one of them, or it is a combination (ordinal sum) of them [43]. Pictorial representations of these continuous t-norms and of other families of t-norms are presented in [44].

The algebraic structure $([0,1], \sup, \inf, T, \to_{\mathrm{L}}, 0, 1)$, where T is a continuous t-norm on $[0,1]$ and $\to_{\mathrm{L}}$ is the corresponding implication, was the starting point in defining and studying Basic Logic (BL, for short) and BL algebras, structures which correspond to that logical system. Namely, we have:

Proposition 3.5 Let T be a continuous t-norm on the real interval $[0,1]$ and "$\to_{\mathrm{L}}$" be the associated implication. Then $\mathcal{A}_{\mathrm{L}} = ([0,1], \sup, \inf, T, \to_{\mathrm{L}}, 0, 1)$ is a left-BL algebra.

The following proposition gives us examples of right-BL algebras.

Proposition 3.6 Let S_T be a continuous t-conorm in the real interval $[0,1]$, associated to a continuous t-norm T, and let "$\rightarrow_R$" be the associated implication. Then $\mathscr{A}_R = ([0,1], \sup, \inf, S_T, \rightarrow_R, 0, 1)$ is a right-BL algebra, that can be called the dual BL algebra of the left-BL algebra $\mathscr{A}_L$, built with T by the previous Proposition. We also can say that $\mathscr{A}_L$ is the dual BL algebra of $\mathscr{A}_R$.

Type 2: *t*-norms (*t*-conorms) on bounded lattices obtained from abelian l-groups
We present two cases:

Case 2.1
We consider the definition of an MV algebra as given by Chang in [9]. The MV algebras come from abelian l-groups:

Example 3.7 [10, 46], Let us consider an abelian l-group $(G, \vee, \wedge, +, -, 0)$ and let $u \in G, u > 0$. We put by definition:

$$x \oplus y \overset{\text{def}}{=} (x+y) \wedge u, \quad x^- \overset{\text{def}}{=} u - x,$$

$$x \odot y \overset{\text{def}}{=} (x - u + y) \vee 0 \ .$$

Then $(A_R = [0,u], \oplus, \odot, ^-, 0_R = 0, 1_R = u)$ is an MV algebra. The operation "$\oplus$" is a t-conorm and the operation "$\odot$" is a t-norm on the bounded lattice $[0,u]$. If we build the associated implications, we can obtain two BL algebras, a right and a left one, one dual to the other, where $\vee$ and $\wedge$ are those from G.

Following this example, we shall call this MV algebra as "right-MV algebra" and we shall denote it by: $\mathscr{A}_R = (A_R, \oplus_R, \odot_L, ^{-R}, 0_R, 1_R)$.

It follows that we have a similar example of a "left-MV algebra", as follows:

Example 3.8 Let us consider an belian l-group $(G, \vee, \wedge, +, -, 0)$ and let $u' \in G, u' < 0$. We put by definition:

$$x' \odot_L y' \overset{\text{def}}{=} (x' + y') \vee u', \quad x'^{-L} \overset{\text{def}}{=} u' - x',$$

$$x' \oplus_R y' \overset{\text{def}}{=} (x' - u' + y') \wedge 0 \ .$$

Then $(A_L = [u', 0], \odot_L, \oplus_R, ^{-L}, 0_L = u', 1_L = 0)$ is a left-MV algebra. The operation "$\odot_L$" is a t-norm and the operation "$\oplus_R$" is a t-conorm on the bounded lattice $[u', 0]$. If we build the associated implications, we can obtain two BL algebras, a left and a right one, one dual to the other, where $\vee$ and $\wedge$ are those from G.

Case 2.2
The following propositions give us other examples of left-BL algebras (right-BL algebras), examples that come from [12]:

Proposition 3.9 Let $(G, \vee, \wedge, +, -, 0)$ be an abelian l-group and let "$\perp$" be a symbol distinct from the elements of G. If $G^- = \{x' \in G | x' \leq 0\}$, then we define on $G_L = \{\perp\} \cup G^-$ the following structure:

$$x' \odot_L y' = \begin{cases} x' + y', & \text{if } x', y' \in G^- \\ \perp, & \text{otherwise,} \end{cases}$$

$$x' \rightarrow_L y' = \begin{cases} (y' - x') \wedge 0, & \text{if } x', y' \in G^- \\ \perp, & \text{if } x' \in G^-, y' = \perp \\ 0, & \text{if } x' = \perp \ . \end{cases}$$

If we put $\perp \leq x'$ for any $x' \in G_L$, then $(G_L, \leq)$ becomes a lattice with first element, "$\perp$", and last element, "0". The operation "$\odot_L$" is a t-norm on the bounded lattice G_L and "$\rightarrow_L$" is the associated implication.

Then, the structure $(G_L, \vee_L = \vee, \wedge_L = \wedge, \odot_L, \rightarrow_L, 0_L = \perp, 1_L = 0)$ is a left-BL algebra (namely a "left" Π-algebra).

Proposition 3.10 Let $(G, \vee, \wedge, +, -, 0)$ be an abelian l-group and let "$\top$" be a symbol distinct from the elements of G. If $G^+ = \{x \in G | x \geq 0\}$, then we define on $G_R = G^+ \cup \{\top\}$ the following structure:

$$x \oplus_R y = \begin{cases} x + y, & \text{if } x, y \in G^+ \\ \top, & \text{otherwise,} \end{cases}$$

$$x \rightarrow_R y = \begin{cases} (y - x) \vee 0, & \text{if } x, y \in G^+ \\ \top, & \text{if } x \in G^+, y = \top \\ 0, & \text{if } x = \top \ . \end{cases}$$

If we put $x \leq \top$ for any $x \in G_R$, then $(G_R, \leq)$ becomes a lattice with first element, "0", and last element, "$\top$". The operation "$\oplus_R$" is a t-conorm on the bounded lattice G_R and "$\rightarrow_R$" is the associated implication.

Then, the structure $(G_R, \vee_R = \vee, \wedge_R = \wedge, \oplus_R, \rightarrow_R, 0_R = 0, 1_R = \top)$ is a right-BL algebra (namely a "right" Π-algebra).

4 Weak-BL algebras

Concerning left continuous, non-continuous, t-norms (right continuous, non-continuous, t-conorms) on the real interval $[0,1]$, by Theorem 2.5(2), we get that the condition (B4) (the condition (R4)) in the definition of left-(right-) BL algebras is no more verified. Then, we define a more general notion, that of "weak-BL algebra", as follows:

Definition 4.1 A left-weak-BL algebra is a structure $(A, \vee, \wedge, \odot, \rightarrow, 0, 1)$ such that for all $x, y, z \in A$,

(B1) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice,
(B2) $(A, \odot, 1)$ is an abelian monoid (i.e. $x \odot y = y \odot x$, $x \odot (y \odot z) = (x \odot y) \odot z$, $x \odot 1 = 1 \odot x = x$),
(B3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ (residuation),
(B4′) $x \odot (x \rightarrow y) \leq x \wedge y$ (weak-divisibility),
(B5) $(x \rightarrow y) \vee (y \rightarrow x) = 1$ (preliniarity).

The following proposition gives us examples of left-weak-BL algebras:

Proposition 4.2 Let T be a left continuous, non-continuous, t-norm on the real interval $[0,1]$ and "$\rightarrow_L$" be the associated implication. Then $([0,1], \sup, \inf, T, \to_L, 0, 1)$ is a left-weak-BL algebra.

We give the following examples of left continuous (non-continuous) t-norms on $[0,1]$.

Examples 4.3

(1) Let $0 < a_1 < 1$ and let us consider the operation $T_0 : [0,1] \times [0,1] \longrightarrow [0,1]$ defined, for every $x, y \in [0,1]$, by (see Fig. 1):

$$T_0(x,y) = \begin{cases} 0, & \text{if } 0 \le x \le a_1,\ 0 \le y \le a_1 \\ \min(x,y), & \text{otherwise} . \end{cases}$$

Then T_0 is a left continuous t-norm that is not continuous. The associated implication, "$\to_L$", is defined, for every $x, y \in [0,1]$, by:

$$x \to_L y = \begin{cases} a_1, & \text{if } x \le a_1,\ x > y \\ y, & \text{if } x > a_1,\ x > y \\ 1, & \text{if } x \le y . \end{cases}$$

Then $([0,1], \sup, \inf, T_0, \to_L, 0, 1)$ is a left-weak-BL algebra.

The inequality from (B4′) is strict if $x, y \in (0, a_1], x > y$. Indeed, in this case we have:

$T_0(x, x \to_L y) = T_0(x, a_1) = 0$, while $x \wedge y = y > 0$.

Notice that the t-norm T_0 ($a_1 \in [0,1]$) is a non-continuous extension of Gödel continuous t-norm, "min".

(2) Let $0 < a_1 < a_2 < 1$ and let us consider the operation $T_1 : [0,1] \times [0,1] \longrightarrow [0,1]$ defined, for every $x, y \in [0,1]$, by (see Fig. 2):

$$T_1(x,y) = \begin{cases} a_1, & \text{if } a_1 < x \le a_2,\ a_1 < y \le a_2 \\ \min(x,y), & \text{otherwise} . \end{cases}$$

Then T_1 is a left continuous t-norm that is not continuous.

(3) Let $0 < a_1 < a_2 < a_3 < 1$ and let us consider the operation $T_2 : [0,1] \times [0,1] \longrightarrow [0,1]$ defined, for every $x, y \in [0,1]$, by (see Fig. 3):

$$T_2(x,y) = \begin{cases} 0, & \text{if } 0 \le x \le a_1,\ 0 \le y \le a_1 \\ a_2, & \text{if } a_2 < x \le a_3,\ a_2 < y \le a_3 \\ \min(x,y), & \text{otherwise} . \end{cases}$$

Then T_2 is a left continuous t-norm that is not continuous.

(4) Let $0 < a_1 < a_2 < a_3 < a_4 < 1$ and let us consider the operation $T_3 : [0,1] \times [0,1] \longrightarrow [0,1]$ defined, for every $x, y \in [0,1]$, by (see Fig. 4):

$$T_3(x,y) = \begin{cases} a_1, & \text{if } a_1 < x \le a_2,\ a_1 < y \le a_2 \\ a_3, & \text{if } a_3 < x \le a_4,\ a_3 < y \le a_4 \\ \min(x,y), & \text{otherwise} . \end{cases}$$

Then T_3 is a left continuous t-norm that is not continuous.

(5) (Generalization of (1) and (3))

Let $0 < a_1 < a_2 < a_3 < \cdots < a_{2n} < a_{2n+1} < 1 (n \ge 0)$ and let us consider the operation $T_{2n} : [0,1] \times [0,1] \longrightarrow [0,1]$ defined, for every $x, y \in [0,1]$, by (see Fig. 5):

$$T_{2n}(x,y) = \begin{cases} 0, & \text{if } 0 \le x \le a_1, 0 \le y \le a_1 \\ a_{2k}, & \text{if } a_{2k} < x \le a_{2k+1}, a_{2k} < y \le a_{2k+1}, \\ & \quad 1 \le k \le n \\ \min(x,y), & \text{otherwise} . \end{cases}$$

Then T_{2n} is a left continuous t-norm that is not continuous. Notice that

$$T_0 > T_2 > \cdots > T_{2n} .$$

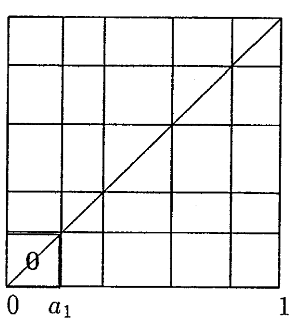

Fig. 1. The t-norm T_0

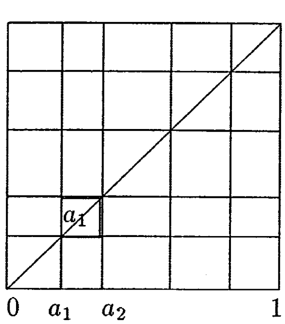

Fig. 2. The t-norm T_1

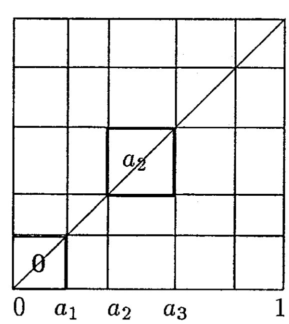

Fig. 3. The t-norm T_2

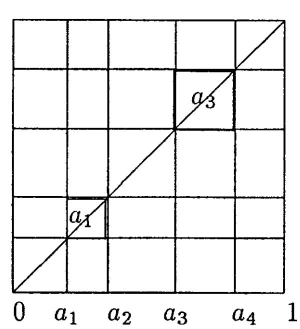

Fig. 4. The t-norm T_3

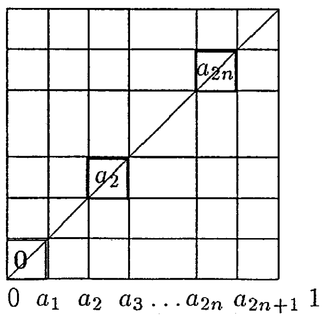

Fig. 5. The t-norm T_{2n}

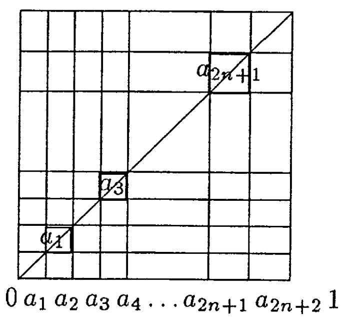

Fig. 6. The t-norm T_{2n+1}

(6) (Generalization of (2) and (4))
Let $0 < a_1 < a_2 < a_3 < a_4 < \cdots < a_{2n+1} < a_{2n+2} < 1 (n \geq 0)$ and let us consider the operation $T_{2n+1} : [0,1] \times [0,1] \longrightarrow [0,1]$ defined, for every $x, y \in [0,1]$, by (see Fig. 6):

$$T_{2n+1}(x,y) = \begin{cases} a_{2k+1}, & \text{if } a_{2k+1} < x \leq a_{2k+2}, a_{2k+1} < y \leq a_{2k+2}, \\ & \quad 0 \leq k \leq n \\ \min(x,y), & \text{otherwise} . \end{cases}$$

Then T_{2n+1} is a left continuous t-norm that is not continuous. Notice that

$$T_1 > T_3 > \cdots > T_{2n+1} .$$

Dually, one can define right-weak-BL algebras, related to right continuous, non-continuous, t-conorms on $[0,1]$.

5
Pseudo-t norms and Φ-operators

In this section we shall define and study first a generalization of t-norms (t-conorms) from Definition 2.1, that we shall name "pseudo-t-norm" ("pseudo-t-conorm", respectively); for this we follow closely [32].

Definition 5.1 (a) A binary operation T in the real interval $[0,1]$ is a pseudo-t-norm iff:

(pt1) it is associative,
(pt2) it is non-decreasing (isotone) in the first argument and in the second argument, i.e. if $x \leq y$, then $T(x,z) \leq T(y,z)$ and $T(z,x) \leq T(z,y)$, for every $x, y, z \in [0,1]$,
(pt3) it has 1 as neutral element, i.e. $T(x,1) = x = T(1,x)$, for every $x \in [0,1]$;

(b) A binary operation S in the real interval $[0,1]$ is a pseudo-t-conorm iff:

(pt1) it is associative,
(pt2) it is non-decreasing in the first argument and in the second argument, i.e. if $x \leq y$, then $S(x,z) \leq S(y,z)$ and $S(z,x) \leq S(z,y)$, for every $x, y, z \in [0,1]$,
(pt3′) it has 0 as neutral element, i.e. $S(x,0) = x = S(0,x)$, for every $x \in [0,1]$.

(c) A pseudo-t-norm (pseudo-t-conorm) is said to be proper if it is not commutative.

Remarks 5.2

(i) If T is a pseudo-t-norm, then $T(x,0) = 0 = T(0,x)$. Indeed, by (pt3), $T(0,1) = 0 = T(1,0)$; then, since $x \leq 1$, we get, by (pt2), that $T(x,0) \leq T(1,0) = 0$ and $T(0,x) \leq T(0,1) = 0$; hence, $T(x,0) = 0 = T(0,x)$.
(i′) If S is a pseudo-t-conorm, then $S(x,1) = 1 = S(1,x)$.
(ii) A commutative pseudo-t-norm (pseudo-t-conorm) is a t-norm (t-conorm, respectively).
(iii) There is a one-one correspondence between pseudo-t-norms and pseudo-t-conorms given by (1) and (2).

Let now T be a binary operation on $[0,1]$, i.e. $T : [0,1] \times [0,1] \longrightarrow [0,1]$. Then we have the following definitions:

Definition 5.3 (i) The operation T is said to be left continuous in the first argument (right continuous in the first argument) and it is denoted by LC_1 (RC_1 respectively) iff for all $(x_0, y_0) \in [0,1] \times [0,1]$ (i.e. $x_0, y_0 \in [0,1]$) and all convergent sequences $(x_i)_{i \geq 1}$ of points from $[0,1]$ with $\lim_{i \to \infty} x_i = x_0$ and always $x_i < x_0,\ i \geq 1$ ($x_0 < x_i,\ i \geq 1$, respectively), one has

$$\lim_{i \to \infty} T(x_i, y_0) = T(x_0, y_0) = T\left(\lim_{i \to \infty} x_i, y_0 \right) . \tag{3}$$

(ii) The operation T is said to be left continuous in the second argument (right continuous in the second argument) and it is denoted by LC_2 (RC_2 respectively) iff for all $(x_0, y_0) \in [0,1] \times [0,1]$ (i.e. $x_0, y_0 \in [0,1]$) and all convergent sequences $(y_j)_{j \geq 1}$ of points from $[0,1]$ with $\lim_{j \to \infty} y_j = y_0$ and always $y_j < y_0,\ j \geq 1$ ($y_0 < y_j,\ j \geq 1$, respectively), one has

$$\lim_{j \to \infty} T(x_0, y_j) = T(x_0, y_0) = T\left(x_0, \lim_{j \to \infty} y_j \right) . \tag{4}$$

Definition 5.4 (i) The operation T is said to be lower semicontinuous in the first argument (upper semicontinuous in the first argument) and it is denoted by $LSC_1(T)$ ($USC_1(T)$ respectively) iff for all $(x_0, y_0) \in [0,1] \times [0,1]$ (i.e. $x_0, y_0 \in [0,1]$) and each $\varepsilon > 0$, there exists $\delta > 0$, such that

$$T(x, y_0) > T(x_0, y_0) - \varepsilon, \quad \forall x \in (x_0 - \delta, x_0] \tag{5}$$

(respectively

$$T(x, y_0) < T(x_0, y_0) + \varepsilon, \quad \forall x \in [x_0, x_0 + \delta) \quad) . \tag{6}$$

(ii) The operation T is said to be lower semicontinuous in the second argument (upper semicontinuous in the second argument) and it is denoted by $LSC_2(T)$ ($USC_2(T)$ respectively) iff for all $(x_0, y_0) \in [0,1] \times [0,1]$ (i.e. $x_0, y_0 \in [0,1]$) and each $\varepsilon > 0$, there exists $\delta > 0$, such that

$$T(x_0, y) > T(x_0, y_0) - \varepsilon, \quad \forall y \in (y_0 - \delta, y_0] \tag{7}$$

(respectively

$$T(x_0, y) < T(x_0, y_0) + \varepsilon, \quad \forall y \in [y_0, y_0 + \delta) \quad) \ . \tag{8}$$

Corollary 5.5

$$LSC_1(T) \Leftrightarrow T(\sup_{a \in A} x_a, y) = \sup_{a \in A} T(x_a, y), \quad \forall (x_a)_{a \in A},\ y \in [0,1] \ , \tag{9}$$

$$LSC_2(T) \Leftrightarrow T(x, \sup_{b \in B} y_b) = \sup_{b \in B} T(x, y_b), \quad \forall x,\ (y_b)_{b \in B} \in [0,1] \ , \tag{10}$$

$$USC_1(T) \Leftrightarrow T(\inf_{a \in A} x_a, y) = \inf_{a \in A} T(x_a, y), \quad \forall (x_a)_{a \in A},\ y \in [0,1] \ , \tag{11}$$

$$USC_2(T) \Leftrightarrow T(x, \inf_{b \in B} y_b) = \inf_{b \in B} T(x, y_b), \quad \forall x,\ (y_b)_{b \in B} \in [0,1] \ . \tag{12}$$

Proposition 5.6 Let T be a binary operation on $[0,1]$. Then

(1) T is LC_1 iff $LSC_1(T)$,
(2) T is LC_2 iff $LSC_2(T)$,
(3) T is RC_1 iff $USC_1(T)$,
(4) T is RC_2 iff $USC_2(T)$.

Proof: Like in commutative case.

Theorem 5.7 Let T be a pseudo-t-norm on $[0,1]$ and let "φ_1^L" and "φ_2^L" be two operations on $[0,1]$.

(i) The following are equivalent, for all $x, y, z \in [0,1]$:
  (1) (L1′) If $y \le z$, then $\varphi_1^L(x,y) \le \varphi_1^L(x,z)$,
  (L1″) $T(\varphi_1^L(x,y), x) \le y$,
  (L1‴) $y \le \varphi_1^L(x, T(y,x))$;
  (2) $\varphi_1^L(x,y) = \sup\{z \mid T(z,x) \le y\}$;
  (3) $T(z,x) \le y \quad \Leftrightarrow \quad z \le \varphi_1^L(x,y)$.
(ii) The following are equivalent, for all $x, y, z \in [0,1]$:
  (4) (L2′) If $y \le z$, then $\varphi_2^L(x,y) \le \varphi_2^L(x,z)$,
  (L2″) $T(x, \varphi_2^L(x,y)) \le y$,
  (L2‴) $y \le \varphi_2^L(x, T(x,y))$;
  (5) $\varphi_2^L(x,y) = \sup\{z \mid T(x,z) \le y\}$;
  (6) $T(x,z) \le y \quad \Leftrightarrow \quad z \le \varphi_2^L(x,y)$.

Proof: (i) (1) $\Longrightarrow$ (2): By (L1″), we get:

$$\varphi_1^L(x,y) \le \sup\{z \mid T(z,x) \le y\} \ .$$

If, by absurdum hypothesis,

$$\varphi_1^L(x,y) < \sup\{z \mid T(z,x) \le y\} \overset{\text{notation}}{=} z_0 \ ,$$

there would get:

$$\varphi_1^L(x,y) < z_0 \quad \text{and} \quad T(z_0, x) \le y \ .$$

Hence, $z_0 \le \varphi_1^L(x, T(z_0,x)) \le \varphi_1^L(x,y) < z_0$, by (L1‴) and (L1′); contradiction.
(2) $\Longrightarrow$ (3): Obvious.
(3) $\Longrightarrow$ (1): To prove (L1″) first, remark that by (3): $T(\varphi_1^L(x,y), x) \le y$ iff $\varphi_1^L(x,y) \le \varphi_1^L(x,y)$, which is true;
To prove (L1‴), remark that by (3): $y \le \varphi_1^L(x, T(y,x))$ iff $T(y,x) \le T(y,x)$, which is true;
To prove (L1′), remark that, by (L1″) we have that $T(\varphi_1^L(x,y), x) \le y \le z$; hence $T(\varphi_1^L(x,y), x) \le z$, i.e. $\varphi_1^L(x,y) \le \varphi_1^L(x,z)$, by (3) again.
(ii) has a similar proof.

Dually, we have the following

Theorem 5.8 Let S be a pseudo-t-conorm on $[0,1]$ and let "φ_1^R" and "φ_2^R" be two operations on $[0,1]$.
(i′) The following are equivalent, for all $x, y, z \in [0,1]$:
  (1′) (R1′) If $y \le z$, then $\varphi_1^R(x,y) \le \varphi_1^R(x,z)$,
  (R1″) $S(\varphi_1^R(x,y), x) \ge y$,
  (R1‴) $\varphi_1^R(x, S(y,x)) \le y$;
  (2′) $\varphi_1^R(x,y) = \inf\{z \mid S(z,x) \ge y\}$;
  (3′) $S(z,x) \ge y \quad \Leftrightarrow \quad z \ge \varphi_1^R(x,y)$.
(ii′) The following are equivalent, for all $x, y, z \in [0,1]$:
  (4′) (R2′) If $y \le z$, then $\varphi_2^R(x,y) \le \varphi_2^R(x,z)$,
  (R2″) $S(x, \varphi_2^R(x,y)) \ge y$,
  (R2‴) $\varphi_2^R(x, S(x,y)) \le y$;
  (5′) $\varphi_2^R(x,y) = \inf\{z \mid S(x,z) \ge y\}$;
  (6′) $S(x,z) \ge y \quad \Leftrightarrow \quad z \ge \varphi_2^R(x,y)$.

The equivalences from the previous two Theorems show that we can define the "residua" of $T(S)$ in three ways. We choose to extend the definitions from [32]:

Definition 5.9

(i) A binary operation φ_1^L in the real interval $[0,1]$ is called Φ^L-operator (or residuum) connected with a given pseudo-t-norm T in the first argument iff for all $x, y, z \in [0,1]$, the following hold true:
  (L1′) If $y \le z$, then $\varphi_1^L(x,y) \le \varphi_1^L(x,z)$,
  (L1″) $T(\varphi_1^L(x,y), x) \le y$,
  (L1‴) $y \le \varphi_1^L(x, T(y,x))$.
(ii) A binary operation φ_2^L in the real interval $[0,1]$ is called Φ^L-operator (or residuum) connected with a given pseudo-t-norm T in the second argument iff for all $x, y, z \in [0,1]$, the following hold true:
  (L2′) If $y \le z$, then $\varphi_2^L(x,y) \le \varphi_2^L(x,z)$,
  (L2″) $T(x, \varphi_2^L(x,y)) \le y$,
  (L2‴) $y \le \varphi_2^L(x, T(x,y))$.

Dually we have:

Definition 5.10

(i) A binary operation φ_1^R in the real interval $[0,1]$ is called Φ^R-operator (residuum) connected with a given pseudo-t-conorm S in the first argument iff for all $x, y, z \in [0,1]$, the following hold true:
  (R1′) If $y \le z$, then $\varphi_1^R(x,y) \le \varphi_1^R(x,z)$,

(R1″) $S(\varphi_1^R(x,y),x) \geq y$,
(R1‴) $\varphi_1^R(x,S(y,x)) \leq y$.

(ii) A binary operation φ_2^R in the real interval $[0,1]$ is called Φ^R-operator (residuum) connected with a given pseudo-t-conorm S in the second argument iff for all $x,y,z \in [0,1]$, the following hold true:
(R2′) If $y \leq z$, then $\varphi_2^R(x,y) \leq \varphi_2^R(x,z)$,
(R2″) $S(x,\varphi_2^R(x,y)) \geq y$,
(R2‴) $\varphi_2^R(x,S(x,y)) \leq y$.

The existence of the Φ-operators is given by the two following lemmas:

Lemma 5.11 (Left-existence) (See [32])
For a pseudo-t-norm T we have:

(i) there exists a Φ^L operator connected with T in the first argument, φ_1^L, iff $LSC_1(T)$.
(ii) there exists a Φ^L operator connected with T in the second argument, φ_2^L, iff $LSC_2(T)$.

Proof: As in the commutative case.

Lemma 5.12 (Right-existence)
For a pseudo-t-conorm S we have:

(i) there exists a Φ^R operator connected with S in the first argument, φ_1^R, iff $USC_1(S)$.
(ii) there exists a Φ^R operator connected with S in the second argument, φ_2^R, iff $USC_2(S)$.

Remark 5.13 A pseudo-t-norm T, that is RC_1 (or RC_2) has no operation φ_1^R (or φ_2^R) connected with it. Indeed, if T is RC_1, for instance, then φ_1^R given by Theorem 5.8(2′), do not fulfils (R1″): there exist $x=0$ and $y=1$ such that $T(\varphi_1^R(0,1),0) = 0 < 1$, by Remarks 5.2(i). Dually, a pseudo-t-conorm S, that is LC_1 (or LC_2), has no operation φ_1^L (φ_2^L) connected with it. This proves that the axiom (pt3) (pt3′) from the definition of a pseudo-t-norm (pseudo-t-conorm) is related to the axioms (L1″) and (L2″) ((R1″) and (R2″), respectively).

Remark 5.14 If the pseudo-t-norm T is commutative (i.e. T is a t-norm), then we have:

$$\varphi_1^L = \varphi_2^L \overset{\text{notation}}{=} \varphi^L \text{ and}$$

if the pseudo-t-conorm S is commutative (i.e. S is a t-conorm), then we have:

$$\varphi_1^R = \varphi_2^R \overset{\text{notation}}{=} \varphi^R \ .$$

Theorem 5.15 If a pseudo-t-norm (pseudo-t- conorm) on $[0,1]$ is continuous, then it is commutative.

Proof: (Radko Mesiar) Any continuous pseudo-t-norm (pseudo-t-conorm) is an I-semigroup and hence it is commutative, by [45].

By this theorem we immediately get

Corollary 5.16 The proper pseudo-t-norms (pseudo-t-conorms) on $[0,1]$ cannot be continuous.

Lemma 5.17 Let T be a pseudo-t-norm on $[0,1]$ that is LC_1 and LC_2 and let φ_1^L and φ_2^L be the Φ^L-operators connected with T. Then the following hold, for every $x,y \in [0,1]$:

(1) $T(x,y) \leq x \wedge y$;
(2) $x \leq y$ iff $\varphi_1^L(x,y) = 1$ iff $\varphi_2^L(x,y) = 1$;
(3) $\varphi_1^L(x,y) \vee \varphi_1^L(y,x) = 1 = \varphi_2^L(x,y) \vee \varphi_2^L(y,x)$;
(4) $T(\varphi_1^L(x,y),x) \leq x \wedge y, \quad T(x,\varphi_2^L(x,y)) \leq x \wedge y$,

where $\vee = \sup = \max$ and $\wedge = \inf = \min$, since $[0,1]$ is a dense chain.

Proof:

(1) $y \leq 1$ implies $T(x,y) \leq T(x,1) = x$ and $x \leq 1$ implies $T(x,y) \leq T(1,y) = y$, by (pt2), (pt3). It follows that $T(x,y) \leq x \wedge y$.
(2) $x \leq y \Leftrightarrow T(1,x) \leq y \Leftrightarrow 1 \leq \varphi_1^L(x,y) \Leftrightarrow \varphi_1^L(x,y) = 1$ and $x \leq y \Leftrightarrow T(x,1) \leq y \Leftrightarrow 1 \leq \varphi_2^L(x,y) \Leftrightarrow \varphi_2^L(x,y) = 1$, by Theorem 5.7(3) and (6).
(3) Since $[0,1]$ is a chain, then for $x,y \in [0,1]$ we have either $x \leq y$ or $y \leq x$; hence , by (2), we get that either $\varphi_1^L(x,y) = 1$ or $\varphi_1^L(y,x) = 1$, therefore $\varphi_1^L(x,y) \vee \varphi_1^L(y,x) = 1$
(4) Since $\varphi_1^L(x,y) \leq 1$, we get that $T(\varphi_1^L(x,y),x) \leq T(1,x) = x$, by (pt2), (pt3) and $T\ (\varphi_1^L(x,y),x) \leq y$, by (L1″). It follows that $T(\varphi_1^L(x,y),x) \leq x \wedge y$.

We have a dual Lemma for pseudo-t-conorms on $[0,1]$.
Now we can prove the following

Theorem 5.18 Let T be a pseudo-t-norm on $[0,1]$ that is LC_1 and LC_2 and let φ_1^L and φ_2^L be the Φ^L-operators connected with T. Then the structure

$$([0,1],\vee,\wedge,T,\varphi_1^L,\varphi_2^L,0,1)$$

verify the following properties, for every $x,y \in [0,1]$:

(C1) $([0,1],\vee,\wedge,0,1)$ is a bounded lattice,
(C2) $([0,1],T,1)$ is a monoid,
(C3) $T(x,y) \leq z$ iff $x \leq \varphi_1^L(y,z)$ iff $y \leq \varphi_2^L(x,z)$,
(C4′) $T(\varphi_1^L(x,y),x) \leq x \wedge y$, $T(x,\varphi_2^L(x,y)) \leq x \wedge y$,
(C5) $\varphi_1^L(x,y) \vee \varphi_1^L(y,x) = 1 = \varphi_2^L(x,y) \vee \varphi_2^L(y,x)$.

Proof: (C1) and (C2) are obvious; (C3) follows by Theorem 5.7; (C4) is (4) from Lemma 5.17; (C5) is (3) from Lemma 5.17.

We have a dual theorem for pseudo-t-conorms on [0,1].

Remark 5.19 If T is proper pseudo-t-norm on $[0,1]$ that is LC_1 and LC_2, can we have "=" in (C4′), for all $x,y \in [0,1]$? The answer is no, by Corollary 5.16 and by Theorem 2.5.

We can extend Definitions 5.1, 5.9 and 5.10 to arbitrary bounded chains, $C_{0,1}$, and even to bounded lattices, $L_{0,1}$, by simply replacing $[0,1]$ by $C_{0,1}$ or by $L_{0,1}$. In this cases the existance of the associated implications is an open problem.

In the rest of the paper we shall use the following notations for the implications associated with a pseudo-t-norm and a pseudo-t-conorm:

$x \rightarrow_{\mathrm{L}} y = \varphi_1^L(x,y), \quad x \rightsquigarrow_{\mathrm{L}} y = \varphi_2^L(x,y)$ and

$x \rightarrow_{\mathrm{R}} y = \varphi_1^R(x,y), \quad x \rightsquigarrow_{\mathrm{R}} y = \varphi_2^R(x,y)$,

respectively.

Definition 5.20

(i) If T is a pseudo-t-norm on a bounded lattice $L_{0,1}$ and if the two Φ^L-operators connected with T, "$\rightarrow_{\mathrm{L}}$" and "$\rightsquigarrow_{\mathrm{L}}$", exist, then we define two negations on $L_{0,1}$ by:

$$x^{-L} \stackrel{\text{def}}{=} x \rightarrow_{\mathrm{L}} 0 \quad \text{and} \quad x^{\sim L} \stackrel{\text{def}}{=} x \rightsquigarrow_{\mathrm{L}} 0 \ .$$

(i′) If S is a pseudo-t-conorm on a bounded lattice $L_{0,1}$ and if the two Φ^R-operators connected with S, "$\rightarrow_{\mathrm{R}}$" and "$\rightsquigarrow_{\mathrm{R}}$", exist, then we define two negations on $L_{0,1}$ by:

$$x^{R} \stackrel{\text{def}}{=} x \rightarrow_{\mathrm{R}} 1 \quad \text{and} \quad x^{\sim R} \stackrel{\text{def}}{=} x \rightsquigarrow_{\mathrm{R}} 1 \ .$$

Since the pseudo-t-norms, T, on $[0,1]$ are related to left continuity and the pseudo-t-conorms, S, are related to right continuity, we shall denote in the sequel T by T_{L} and S by S_{R} (L from "left", R from "right") and we shall put the label "left" or "right" to the corresponding structures determined by them.

6 Pseudo-MV algebras

In [26, 27] there were introduced and studied the pseudo-MV algebras as a non-commutative extension of MV algebras:

Definition 6.1 [26, 27] A pseudo-MV algebra is a structure $(A, \oplus, \odot, ^-, ^\sim, 0, 1)$ of type $(2,2,1,1,0,0)$, such that the following axioms are satisfied for all $x, y, z \in A$:

(R1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
(R2) $x \oplus 0 = 0 \oplus x = x$,
(R3) $x \oplus 1 = 1 \oplus x = 1$,
(R4) $1^\sim = 0,\ 1^- = 0$,
(R5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$,
(R6) $x \oplus (x^\sim \odot y) = y \oplus (y^\sim \odot x) = (x \odot y^-) \oplus y = (y \odot x^-) \oplus x$,
(R7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$,
(R8) $(x^-)^\sim = x$,

where $y \odot x \stackrel{\text{def}}{=} (x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$.
Some of these axioms are superfluous (see [40]), but this is not essential for the present paper.

Proposition 6.2 The following properties are true in a pseudo-MV algebra $\mathscr{A}$ [26, 27]:

(0) $(x^-)^\sim = x = (x^\sim)^-$,
(1) $x \vee y \stackrel{\text{def}}{=} x \oplus (x^\sim \odot y) = y \oplus (y^\sim \odot x) = (x \odot y^-) \oplus y = (y \odot x^-) \oplus x$,
(2) $x \wedge y \stackrel{\text{def}}{=} x \odot (x^- \oplus y) = y \odot (y^- \oplus x) = (x \oplus y^\sim) \odot y = (y \oplus x^\sim) \odot x$,
(3) $x \le y$ iff $x^- \oplus y = 1$ iff $y^\sim \odot x = 0$ iff $y = x \oplus (x^\sim \odot y)$ iff $x = x \odot (x^- \oplus y)$ iff $x \odot y^- = 0$ iff $y \oplus x^\sim = 1$,
(4) $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice,
(5) $x \odot 1 = 1 \odot x = x$ and $x \odot 0 = 0 \odot x = 0$,
(6) $x \oplus y = (y^- \odot x^-)^\sim = (y^\sim \odot x^\sim)^-$,
(7) $x \odot (y \odot z) = (x \odot y) \odot z$,
(8) $x \odot y \le z$ iff $y \le x^- \oplus z$ iff $x \le z \oplus y^\sim$ and $z \le x \oplus y$ iff $z \odot y^- \le x$ iff $x^\sim \odot z \le y$,
(9) $(x \odot y^-) \wedge (y \odot x^-) = (x^\sim \odot y) \wedge (y^\sim \odot x) = 0$ and $(y \oplus x^\sim) \vee (x \oplus y^\sim) = (y^- \oplus x) \vee (x^- \oplus y) = 1$,
(10) $x \le y$ implies $x \oplus z \le y \oplus z$ and $z \oplus x \le z \oplus y$,
(11) $x \le y$ implies $x \odot z \le y \odot z$ and $z \odot x \le z \odot y$.

Remarks 6.3
(1) By (R1), (R2) and Proposition 6.2(10),(4), it follows that "$\oplus$" is a pseudo-t-conorm on the bounded lattice $(A, \vee, \wedge, 0, 1)$.
(1′) By Proposition 6.2(7),(5),(11) it follows that "$\odot$" is a pseudo-t-norm on $(A, \vee, \wedge, 0, 1)$.

Following these remarks, we shall call the pseudo-MV algebra from Definition 6.1 as "right-pseudo-MV algebra" and we shall denote it by: $\mathscr{A}_{\mathrm{R}} = (A_{\mathrm{R}}, \oplus_{\mathrm{R}}, \odot_{\mathrm{L}}, ^{-R}, ^{\sim R}, 0_{\mathrm{R}}, 1_{\mathrm{R}})$.
The right-pseudo-MV algebras come from arbitrary l-groups:

Example 6.4 [26, 27] Let us consider an arbitrary l-group $(G, \vee, \wedge, +, -, 0)$ and let $u \in G,\ u > 0$. We put by definition:

$$x \oplus_{\mathrm{R}} y \stackrel{\text{def}}{=} (x + y) \wedge u, \quad x^{-R} \stackrel{\text{def}}{=} u - x,$$

$$x^{\sim R} \stackrel{\text{def}}{=} -x + u, \quad x \odot_{\mathrm{L}} y \stackrel{\text{def}}{=} (x - u + y) \vee 0 \ .$$

Then $(A_{\mathrm{R}} = [0,u], \oplus_{\mathrm{R}}, \odot_{\mathrm{L}}, ^{-R}, ^{\sim R}, 0_{\mathrm{R}} = 0, 1_{\mathrm{R}} = u)$ is a right-pseudo-MV algebra. The operation "$\oplus_{\mathrm{R}}$" is a pseudo-t-conorm and the operation "$\odot_{\mathrm{L}}$" is a pseudo-t-norm in the bounded lattice $[0,u]$. For the associated implications of "$\oplus_{\mathrm{R}}$" see Example 7.9 and for the associated implications of "$\odot_{\mathrm{L}}$" see [6, 7] and Example 7.13.

Remark 6.5 In the pseudo-MV algebra $A_{\mathrm{R}} = [0,u]$ we have [26, 27]:

$$x \odot_{\mathrm{L}} y^{-R} = (x - y) \vee 0, \quad x^{\sim R} \odot_{\mathrm{L}} y = (-x + y) \vee 0,$$

$$x \vee_{\mathrm{R}} y = x \vee y, \quad x \wedge_{\mathrm{R}} y = x \wedge y,$$

$$x \oplus_{\mathrm{R}} 0 = x, \quad x \oplus_{\mathrm{R}} u = u \ .$$

Example of non-commutative l-group
Let $G = (0, \infty) \times \Re$ and define a binary operation "+" on G by:

$$(a,b) + (c,d) \stackrel{\text{def}}{=} (ac, bc + d) \ .$$

The operation "+" is associative, non-commutative, the element $(1,0)$ is the element "0_G" and

$$-(a,b) \stackrel{\text{def}}{=} \left(\frac{1}{a}, -\frac{b}{a}\right) \ .$$

The order relation is the lexicographic order: $(a,b) < (c,d)$ iff $a < c$ or $a = c$ and $b < d$. It makes G a lattice and the structure $(G, \vee, \wedge, +, -, 0_G)$ is a non abelian l-group.

It follows that we have a similar definition for a "left-pseudo-MV algebra", as follows:

Definition 6.6 A left-pseudo-MV algebra is a structure $(A_L, \odot_L, \oplus_R, ^{-L}, ^{\sim L}, 0_L, 1_L)$ of type $(2, 2, 1, 1, 0, 0)$, such that the following axioms are satisfied for all $x, y, z \in A_L$:

(L1) $x \odot_L (y \odot_L z) = (x \odot_L y) \odot_L z$,
(L2) $x \odot_L 1_L = 1_L \odot_L x = x$,
(L3) $x \odot_L 0_L = 0_L \odot_L x = 0_L$,
(L4) $0_L^{-L} = 1_L$, $0_L^{\sim L} = 1_L$,
(L5) $(x^{-L} \odot_L y^{-L})^{\sim L} = (x^{\sim L} \odot_L y^{\sim L})^{-L}$,
(L6) $x \odot_L (x^{\sim L} \oplus_R y) = y \odot_L (y^{\sim L} \oplus_R x) = (x \oplus_R y^{-L}) \odot_L y = (y \oplus_R x^{-L}) \odot_L x$,
(L7) $x \oplus_R (x^{-L} \odot_L y) = (x \odot_L y^{\sim L}) \oplus_R y$,
(L8) $(x^{-L})^{\sim L} = x$,
where $y \oplus_R x \overset{\text{def}}{=} (x^{-L} \odot_L y^{-L})^{\sim L} = (x^{\sim L} \odot_L y^{\sim L})^{-L}$.

Proposition 6.7 The following properties are true in a left-pseudo-MV algebra $\mathscr{A}_L$:

(0′) $(x^{-L})^{\sim L} = x = (x^{\sim L})^{-L}$,
(1′) $x \wedge_L y \overset{\text{def}}{=} x \odot_L (x^{\sim L} \oplus_R y) = y \odot_L (y^{\sim L} \oplus_R x) = (x \oplus_R y^{-L}) \odot_L y = (y \oplus_R x^{-L}) \odot_L x$,
(2′) $x \vee_L y \overset{\text{def}}{=} x \oplus_R (x^{-L} \odot_L y) = y \oplus_R (y^{-L} \odot_L x) = (x \odot_L y^{\sim L}) \oplus_R y = (y \odot_L x^{\sim L}) \oplus_R x$,
(3′) $x \leq y$ iff $y^{-L} \odot_L x = 0_L$ iff $x^{\sim L} \oplus_R y = 1_L$ iff $x = y \odot_L (y^{\sim L} \oplus_R x)$ iff $y = y \oplus_R (y^{-L} \odot_L x)$ iff $y \odot_L x^{-L} = 1_L$ iff $x \oplus_R y^{\sim} = 0_L$,
(4′) $(A_L, \vee_L, \wedge_L, 0_L, 1_L)$ is a bounded distributive lattice,
(5′) $x \oplus_R 0_L = 0_L \oplus_R x = x$ and $x \oplus_R 1_L = 1_L \oplus_R x = 1_L$,
(6′) $x \odot_L y = (y^{-L} \oplus_R x^{-L})^{\sim L} = (y^{\sim L} \oplus_R x^{\sim L})^{-L}$,
(7′) $x \oplus_R (y \oplus_R z) = (x \oplus_R y) \oplus_R z$,
(8′) $z \leq x \oplus_R y$ iff $x^{-} \odot_L z \leq y$ iff $z \odot_L y^{\sim} \leq x$ and $x \odot_L y \leq z$ iff $x \leq z \oplus_R y^{-}$ iff $y \leq x^{\sim} \oplus_R z$,
(9′) $(x \oplus_R y^{-}) \vee (y \oplus_R x^{-}) = (x^{\sim} \oplus_R y) \vee (y^{\sim} \oplus_R x) = 1_L$ and $(y \odot_L x^{\sim}) \wedge (x \odot_L y^{\sim}) = (y^{-} \odot_L x) \wedge (x^{-} \odot_L y) = 0_L$;
(10′) $x \leq y$ implies $x \odot_L z \leq y \odot_L z$ and $z \odot_L x \leq z \odot_L y$;
(11′) $x \leq y$ implies $x \oplus_R z \leq y \oplus_R z$ and $z \oplus_R x \leq z \oplus_R y$.

Remark 6.8 "$\odot_L$" is a pseudo-t-norm and "$\oplus_R$" is a pseudo-t-conorm on the bounded lattice $(A_L, \vee_L, \wedge_L, 0_L, 1_L)$.

The left-pseudo-MV algebras come from arbitrary l-groups as follows:

Example 6.9 Let us consider an arbitrary l-group $(G, \vee, \wedge, +, -, 0)$ and let $u' \in G$, $u' < 0$. We put by definition:

$$x' \odot_L y' \overset{\text{def}}{=} (x' + y') \vee u', \quad x'^{-L} \overset{\text{def}}{=} u' - x',$$

$$x'^{\sim L} \overset{\text{def}}{=} -x' + u', \quad x' \oplus_R y' \overset{\text{def}}{=} (x' - u' + y') \wedge 0 \ .$$

Then $(A_L = [u', 0], \odot_L, \oplus_R, ^{-L}, ^{\sim L}, 0_L = u', 1_L = 0)$ is a left-pseudo-MV algebra. The operation "$\odot_L$" is a pseudo-t-norm and the operation "$\oplus_R$" is a pseudo-t-conorm on the bounded lattice $[u', 0]$. For the associated implications of "$\odot_L$" see Example 7.10 and for the associated implications of "$\oplus_R$" see Example 7.14.

Remark 6.10 In the pseudo-MV algebra $A_L = [u', 0]$ we have:

$$x' \oplus_R y'^{-L} = (x' - y') \wedge 0, \quad x'^{\sim L} \oplus_R y' = (-x' + y') \wedge 0,$$
$$x' \vee_L y' = x' \vee y', \quad x' \wedge_L y' = x' \wedge y',$$
$$x' \odot_L 1_L = x' \odot_L 0 = x', \quad x' \odot_L 0_L = x' \odot_L u' = u' \ .$$

Remark 6.11 [16] If we consider the two pseudo-MV algebras: $A_R = [0, u]$ from Example 6.4 and $A_L = [u', 0]$ from Example 6.9, we remark that if $u' = -u$, then there exists a function $f : [0, u] \longrightarrow [u', 0]$ defined by: $f(x) = -x = x'$ and we have that: (1) f is a bijection, $f(0_R) = 1_L$, $f(1_R) = 0_L$, (2) $f(x \oplus_R y) = f(y) \odot_L f(x)$, (3) $f(x^{-R}) = (f(x))^{\sim L}$ and $f(x^{\sim R}) = (f(x))^{-L}$, (4) $f(x \odot_L y) = f(y) \oplus_R f(x)$, (5) $f(x \vee_R y) = f(x) \wedge_L f(y)$, $f(x \wedge_R y) = f(x) \vee_L f(y)$, since $-[(x + y) \wedge u] = [-(x + y) \vee (-u)] = (-y - x) \vee (-u)$ and $-[(x - u + y) \vee 0] = [-(x - u + y)] \wedge 0 = (-y + u - x) \wedge 0$. This function shows the connection between the right- and left-pseudo-MV algebras $\mathscr{A}_R$ and $\mathscr{A}_L$.

Remark 6.12 [27] If $\mathscr{A}_R = (A_R, \oplus_R, \odot_L, ^{-R}, ^{\sim R}, 0_R, 1_R)$ is a right-pseudo-MV algebra, then $\mathscr{A}_R^a = (A_R, \odot_L, \oplus_R, ^{\sim R}, ^{-R}, 0_R, 1_R)$ is a left-pseudo-MV algebra, called the "associated left-pseudo-MV algebra" of $\mathscr{A}_R$.

Remark 6.13 If $\mathscr{A}_L = (A_L, \odot_L, \oplus_R, ^{-L}, ^{\sim L}, 0_L, 1_L)$ is a left-pseudo-MV algebra, then $\mathscr{A}_L^a = (A_L, \oplus_R, \odot_L, ^{\sim L}, ^{-L}, 0_L, 1_L)$ is a right-pseudo-MV algebra, called the "associated right-pseudo-MV algebra" of $\mathscr{A}_L$.

Remark 6.14 $(\mathscr{A}_R^a)^a = \mathscr{A}_R$ and $(\mathscr{A}_L^a)^a = \mathscr{A}_L$.

In [21], Dvurecenskij proved that any (right) pseudo-MV algebra can be obtained as in Example 6.4. In fact, Dvurecenskij's result asserts that the category of pseudo-MV algebras is equivalent to the category of l-groups with strong unit. This result extends the fundamental theorem of Mundici [46](cf. also [10]) (which asserts the equivalence between the category of MV-algebras and the category of commutative l-groups with strong unit) to non-commutative case. An important open problem is the characterization of the varieties of pseudo-MV algebras.

Proposition 6.15 [26, 27] Every commutative pseudo-MV algebra is an MV algebra.

7 Pseudo-BL algebras

In [28] there were introduced the pseudo-BL algebras and there have presented some basic properties. The theory was developed in [16, 17]. Pseudo-BL algebras are structures which extend BL algebras and pseudo-MV algebras, being a non-commutative version of BL algebras.

We shall define first "right-pseudo-BL algebras" and then we shall define the pseudo-BL algebras from [28] as "left-pseudo-BL algebras".

Definition 7.1 [16] A right-pseudo-BL algebra is a structure

$$\mathscr{A}_R = (A_R, \vee_R, \wedge_R, \oplus_R, \rightarrow_R, \rightsquigarrow_R, 0_R, 1_R) \ ,$$

of type $(2,2,2,2,2,0,0)$, which satisfies the following axioms, for all $x,y,z \in A_R$:

(RC1) $(A_R, \vee_R, \wedge_R, 0_R, 1_R)$ is a bounded lattice,
(RC2) $(A_R, \oplus_R, 0_R)$ is a monoid ($\oplus_R$ is associative and $x \oplus_R 0_R = 0_R \oplus_R x = x$),
(RC3) $z \leq_R x \oplus_R y$ iff $y \rightarrow_R z \leq_R x$ iff $x \rightsquigarrow_R z \leq_R y$,
(RC4) $x \vee_R y = (x \rightarrow_R y) \oplus_R x = x \oplus_R (x \rightsquigarrow_R y)$,
(RC5) $(x \rightarrow_R y) \wedge_R (y \rightarrow_R x) = (x \rightsquigarrow_R y) \wedge_R (y \rightsquigarrow_R x) = 0_R$.

We shall agree that the operations $\vee_R$, $\wedge_R$, $\oplus_R$ have priority towards the operations $\rightarrow_R$, $\rightsquigarrow_R$.

Proposition 7.2 The following properties hold in a right-pseudo-BL algebra $\mathscr{A}_R$, for all $x,y,z \in A_R (\leq = \leq_R)$:

(1) $x \rightarrow_R (y \oplus_R x) \leq y \leq (x \rightarrow_R y) \oplus_R x$,
(1′) $x \rightarrow_R (y \oplus_R x) \leq x \leq (y \rightarrow_R x) \oplus_R y$,
(2) $x \rightsquigarrow_R (x \oplus_R y) \leq y \leq x \oplus_R (x \rightsquigarrow_R y)$,
(2′) $x \rightsquigarrow_R (x \oplus_R y) \leq x \leq y \oplus_R (y \rightsquigarrow_R x)$,
(3) if $y \leq z$, then $x \rightarrow_R y \leq x \rightarrow_R z$ and $x \rightsquigarrow_R y \leq x \rightsquigarrow_R z$,
(4) if $x \leq y$, then $x \oplus_R z \leq y \oplus_R z$ and $z \oplus_R x \leq z \oplus_R y$.

By Definition 7.1 and Proposition 7.2, we get that "$\oplus_R$" is a pseudo-*t*-conorm on A_R and that "$\rightarrow_R$" and "$\rightsquigarrow_R$" are the Φ^R-operators connected with it.
Analogously, we have the following

Definition 7.3 [16, 28] A left-pseudo-BL algebra is a structure

$$\mathscr{A}_L = (A_L, \vee_L, \wedge_L, \odot_L, \rightarrow_L, \rightsquigarrow_L, 0_L, 1_L) \ ,$$

of type $(2,2,2,2,2,0,0)$, which satisfies the following axioms , for all $x,y,z \in A_L$:

(C1) $(A_L, \vee_L, \wedge_L, 0_L, 1_L)$ is a bounded lattice,
(C2) $(A_L, \odot_L, 1_L)$ is a monoid, i.e. $\odot_L$ is associative and $x \odot_L 1_L = 1_L \odot_L x = x$,
(C3) $x \odot_L y \leq z$ iff $x \leq y \rightarrow_L z$ iff $y \leq x \rightsquigarrow_L z$,
(C4) $x \wedge_L y = (x \rightarrow_L y) \odot_L x = x \odot_L (x \rightsquigarrow_L y)$,
(C5) $(x \rightarrow_L y) \vee_L (y \rightarrow_L x) = (x \rightsquigarrow_L y) \vee_L (y \rightsquigarrow_L x) = 1_L$.

We shall agree that the operations $\vee_L$, $\wedge_L$, $\odot_L$ have priority towards the operations $\rightarrow_L$, $\rightsquigarrow_L$. We shall put paranthesis even it is not necessary, for the sick of clearness.

Proposition 7.4 [16, 17, 28] In a left-pseudo-BL algebra $\mathscr{A}_L$, the following properties hold:

(1) $(x \rightarrow_L y) \odot_L x \leq y \leq x \rightarrow_L (y \odot_L x)$,
(1′) $(x \rightarrow_L y) \odot_L x \leq x \leq y \rightarrow_L (x \odot_L y)$,
(2) $x \odot_L (x \rightsquigarrow_L y) \leq y \leq x \rightsquigarrow_L (x \odot_L y)$,
(2′) $x \odot_L (x \rightsquigarrow_L y) \leq x \leq y \rightsquigarrow_L (y \odot_L x)$,
(3) if $y \leq z$, then $x \rightarrow_L y \leq x \rightarrow_L z$ and $x \rightsquigarrow_L y \leq x \rightsquigarrow_L z$,
(4) if $x \leq y$, then $x \odot_L z \leq y \odot_L z$ and $z \odot_L x \leq z \odot_L y$.

By Definition 7.3 and Proposition 7.4, we get that "$\odot_L$" is a pseudo-*t*-norm on A_L and that "$\rightarrow_L$" and "$\rightsquigarrow_L$" are the Φ^L-operators connected with it.

Definition 7.5

(i) If $\mathscr{A}_L$ is a left-pseudo-BL algebra, then we define two negations on A_L by:

$$x^{-L} \stackrel{\text{def}}{=} x \rightarrow_L 0_L \quad \text{and} \quad x^{\sim L} \stackrel{\text{def}}{=} x \rightsquigarrow_L 0_L \ .$$

(i′) If $\mathscr{A}_R$ is a right-pseudo-BL algebra, then we define two negations on A_R by:

$$x^{-R} \stackrel{\text{def}}{=} x \rightarrow_R 1_R \quad \text{and} \quad x^{\sim R} \stackrel{\text{def}}{=} x \rightsquigarrow_R 1_R \ .$$

Proposition 7.6 [16, 28] Every commutative pseudo-BL algebra is a BL algebra.

We shall give now examples of pseudo-BL algebras. First, notice that, by Remark 5.19, it follows that there are not pseudo-BL algebras on $[0,1]$.
We shall present two types of pseudo-*t*-norms (pseudo-*t*-conorms) on bounded lattices obtained from arbitrary l-groups that give pseudo-BL algebras and that are generalizations of the corresponding commutative cases 2.1 and 2.2 from Sect. 2.

Case 2.1′

Proposition 7.7 [16] Let $\mathscr{A}_R = (A_R, \oplus_R, \odot_L, {}^{-R}, {}^{\sim R}, 0_R, 1_R)$ be a right-pseudo-MV algebra and let $\rightarrow_R$, $\rightsquigarrow_R$ be two implications defined by:

$$x \rightarrow_R y \stackrel{\text{def}}{=} y \odot_L x^{-R} \quad \text{and} \quad x \rightsquigarrow_R y \stackrel{\text{def}}{=} x^{\sim R} \odot_L y \ . \tag{13}$$

Then $\mathscr{A}_R = (A_R, \vee_R, \wedge_R, \oplus_R, \rightarrow_R, \rightsquigarrow_R, 0_R, 1_R)$ is a right-pseudo-BL algebra.

Proposition 7.8 [16] Let $\mathscr{A}_L = (A_L, \odot_L, \oplus_R, {}^{-L}, {}^{\sim L}, 0_L, 1_L)$ be a left-pseudo-MV algebra and let $\rightarrow_L$, $\rightsquigarrow_L$ be two implications defined by:

$$x \rightarrow_L y \stackrel{\text{def}}{=} y \oplus_R x^{-L} \quad \text{and} \quad x \rightsquigarrow_L y \stackrel{\text{def}}{=} x^{\sim L} \oplus_R y \ . \tag{14}$$

Then $\mathscr{A}_L = (A_L, \vee_L, \wedge_L, \odot_L, \rightarrow_L, \rightsquigarrow_L, 0_L, 1_L)$ is a left-pseudo-BL algebra.

Example 7.9 [16] In the right-pseudo-MV algebra $\mathscr{A}_R = ([0,u], \oplus_R, \odot_L, {}^{-R}, {}^{\sim R}, 0_R, 1_R)$ from Example 6.4, we define, by (13):

$$x \rightarrow_R y \stackrel{\text{def}}{=} (y - x) \vee 0 \quad \text{and} \quad x \rightsquigarrow_R y \stackrel{\text{def}}{=} (-x + y) \vee 0 \ .$$

Then $\mathscr{A}_R = ([0,u], \vee_R = \vee, \wedge_R = \wedge, \oplus_R, \rightarrow_R, \rightsquigarrow_R, 0_R = 0, 1_R = u)$ is a right-pseudo-BL algebra, by Proposition 7.7. "$\oplus_R$" is a pseudo-*t*-conorm in the bounded lattice $[0,u]$ and "$\rightarrow_R$" and "$\rightsquigarrow_R$" are the associated implications.

Example 7.10 [16] In the left-pseudo-MV algebra $\mathscr{A}_L = ([u',0], \odot_L, \oplus_R, {}^{-L}, {}^{\sim L}, 0_L, 1_L)$ from Example 6.9, we define, by (14):

$$x' \rightarrow_L y' \stackrel{\text{def}}{=} (y' - x') \wedge 0 \text{ and}$$

$$x' \rightsquigarrow_L y' \stackrel{\text{def}}{=} (-x' + y') \wedge 0 \ .$$

Then $\mathscr{A}_L = ([u',0], \vee_L = \vee, \wedge_L = \wedge, \odot_L, \rightarrow_L, \rightsquigarrow_L, 0_L = u', 1_L = 0)$ is a left-pseudo-BL algebra, by

Proposition 7.8. "$\odot_L$" is a pseudo-t-norm in the bounded lattice $[u', 0]$ and "$\rightarrow_L$" and "$\rightsquigarrow_L$" are the associated implications.

Proposition 7.11 Let $\mathscr{A}^a_R = (A_R, \odot_L, \oplus_R, {}^{\sim R}, {}^{-R}, 0_R, 1_R)$ be the associated left-pseudo-MV algebra of $\mathscr{A}_R = (A_R, \oplus_R, \odot_L, {}^{-R}, {}^{\sim R}, 0_R, 1_R)$ ($\mathscr{A}^a_R$ is a left-pseudo-MV algebra, by Remark 6.12) and let $\rightarrow_L$, $\rightsquigarrow_L$ be two implications defined by (14):

$$x \rightarrow_L y \overset{\text{def}}{=} y \oplus_R x^{\sim R} \quad \text{and} \quad x \rightsquigarrow_L y \overset{\text{def}}{=} x^{-R} \oplus_R y \ . \tag{15}$$

Then $\mathscr{A}^a_R = (A_R, \vee_R, \wedge_R, \odot_L, \rightarrow_L, \rightsquigarrow_L, 0_R = 0, 1_R = u)$ is a left-pseudo-BL algebra, called the associated left-pseudo-BL algebra of the right-pseudo-BL algebra $\mathscr{A}_R = (A_R, \vee_R, \wedge_R, \oplus_R, \rightarrow_R, \rightsquigarrow_R, 0_R, 1_R)$, built by Proposition 7.7.

Proof: By Proposition 7.8.

Proposition 7.12 Let $\mathscr{A}^a_L = (A_L, \oplus_R, \odot_L, {}^{\sim L}, {}^{-L}, 0_L, 1_L)$ be the associated right-pseudo-MV algebra of $\mathscr{A}_L = (A_L, \odot_L, \oplus_R, {}^{-L}, {}^{\sim L}, 0_L, 1_L)$ ($\mathscr{A}^a_L$ is a right-pseudo-MV algebra, by Remark 6.13) and let $\rightarrow_R$, $\rightsquigarrow_R$ be two implications defined by (13):

$$x \rightarrow_R y \overset{\text{def}}{=} y \odot_L x^{\sim L} \quad \text{and} \quad x \rightsquigarrow_R y \overset{\text{def}}{=} x^{-L} \odot_L y \ . \tag{16}$$

Then $\mathscr{A}^a_L = (A_L, \vee_R, \wedge_R, \oplus_R, \rightarrow_R, \rightsquigarrow_R, 0_L, 1_L)$ is a right-pseudo-BL algebra, called the associated right-pseudo-BL algebra of the left-pseudo-BL algebra $\mathscr{A}_L = (A_L, \vee_L, \wedge_L, \odot_L, \rightarrow_L, \rightsquigarrow_L, 0_L, 1_L)$, built by Proposition 7.8.

Proof: By Proposition 7.7.

Example 7.13 Let $\mathscr{A}^a_R = ([0, u], \odot_L, \oplus_R, {}^{\sim R}, {}^{-R}, 0_R, 1_R)$ be the associated pseudo-MV algebra of $\mathscr{A}_R$ from Example 6.4 ($\mathscr{A}^a_R$ is a left-pseudo-MV algebra); we define, by (15):

$$x \rightarrow_L y \overset{\text{def}}{=} y \oplus_R x^{\sim R} = (y - x + u) \wedge u \quad \text{and}$$

$$x \rightsquigarrow_L y \overset{\text{def}}{=} x^{-R} \oplus_R y = (u - x + y) \wedge u \ .$$

Then $\mathscr{A}^a_R = ([0, u], \vee, \wedge, \odot_L, \rightarrow_L, \rightsquigarrow_L, 0_R = 0, 1_R = u)$ is a left-pseudo-BL algebra, called the associated left-pseudo-BL algebra of the right-pseudo-BL algebra $([0, u], \vee, \wedge, \oplus_R, \rightarrow_R, \rightsquigarrow_R, 0, u)$, built in Example 7.9.

"$\odot_L$" is a pseudo-t-norm in the bounded lattice $[0, u]$ and "$\rightarrow_L$" and "$\rightsquigarrow_L$" are the associated implications.

Example 7.14 Let $\mathscr{A}^a_L = ([u', 0], \oplus_R, \odot_L, {}^{\sim L}, {}^{-L}, 0_L = u', 1_L = 0)$ be the associated pseudo-MV algebra of $\mathscr{A}_L$ from Example 6.9 ($\mathscr{A}^a_L$ is a right-pseudo-MV algebra); we define, by (16):

$$x' \rightarrow_R y' \overset{\text{def}}{=} y' \odot_L x'^{\sim L} = (y' - x' + u') \vee u' \text{ and}$$

$$x' \rightsquigarrow_R y' \overset{\text{def}}{=} x'^{-L} \odot_L y' = (u' - x' + y') \vee u' \ .$$

Then $\mathscr{A}^a_L = ([u', 0], \vee, \wedge, \oplus_R, \rightarrow_R, \rightsquigarrow_R, 0_L = u', 1_L = 0)$ is a right-pseudo-BL algebra, called the associated right-pseudo-BL algebra of the left pseudo-BL algebra $([u', 0], \vee, \wedge, \odot_L, \rightarrow_L, \rightsquigarrow_L, u', 0)$, built in Example 7.10.

"$\oplus_R$" is a pseudo-t-conorm in the bounded lattice $[u', 0]$ and "$\rightarrow_R$" and "$\rightsquigarrow_R$" are the associated implications.

Remark 7.15 On the bounded lattice $[0, u]$, the pseudo-t-conorm "$\oplus_R$" has a corresponding pseudo-t-norm, "$\odot_L$". On the bounded lattice $[u', 0]$, the pseudo-t-norm "$\odot_L$" has a corresponding pseudo-t-conorm, "$\oplus_R$".

Proposition 7.16 [16, 28] A left-pseudo-BL algebra $\mathscr{A}_L$ is a left-pseudo-MV algebra iff the property

$$(x^{-L})^{\sim L} = x = (x^{\sim L})^{-L}$$

holds for any $x \in A_L$, where "$\oplus_R$" is defined by

$$y \oplus_R x = (x^{-L} \odot_L y^{-L})^{\sim L} = (x^{\sim L} \odot_L y^{\sim L})^{-L} = x^{\sim L} \rightarrow_L y = y^{-L} \rightsquigarrow_L x \ .$$

We have a dual Proposition for right-pseudo-BL algebras.

Case 2.2′

Example 7.17 [16] Let $(G, \vee, \wedge, +, -, 0)$ be an arbitrary l-group and let "$\top$" be a symbol distinct from the elements of G. If $G^+ = \{x \in G \mid x \geq 0\}$, then we define on $G_R = G^+ \cup \{\top\}$ the following structure:

$$x \oplus_R y = \begin{cases} x + y, & \text{if } x, y \in G^+ \\ \top, & \text{otherwise,} \end{cases}$$

$$x \rightarrow_R y = \begin{cases} (y - x) \vee 0, & \text{if } x, y \in G^+ \\ \top, & \text{if } x \in G^+,\ y = \top \\ 0, & \text{if } x = \top, \end{cases}$$

$$x \rightsquigarrow_R y = \begin{cases} (-x + y) \vee 0, & \text{if } x, y \in G^+ \\ \top, & \text{if } x \in G^+,\ y = \top \\ 0, & \text{if } x = \top\ . \end{cases}$$

If we put $x \leq \top$ for any $x \in G_R$, then $(G_R, \leq)$ becomes a lattice with first element, "0", and last element, "$\top$". "$\oplus_R$" is the only pseudo-t-conorm on the bounded lattice G_R and "$\rightarrow_R$" and "$\rightsquigarrow_R$" are the associated implications.

Then, the structure $(G_R, \vee_R = \vee, \wedge_R = \wedge, \oplus_R, \rightarrow_R, \rightsquigarrow_R, 0_R = 0, 1_R = \top)$ is a right-pseudo-BL algebra (that has no an associated pseudo-BL algebra).

The following example generalizes an example from [12]:

Example 7.18 [16] Let $(G, \vee, \wedge, +, -, 0)$ be an arbitrary l-group and let "$\perp$" be a symbol distinct from the elements of G. If $G^- = \{x' \in G \mid x' \leq 0\}$, then we define on $G_L = \{\perp\} \cup G^-$ the following structure:

$$x' \odot_L y' = \begin{cases} x' + y', & \text{if } x', y' \in G^- \\ \perp, & \text{otherwise,} \end{cases}$$

$$x' \rightarrow_L y' = \begin{cases} (y' - x') \wedge 0, & \text{if } x', y' \in G^- \\ \perp, & \text{if } x' \in G^-,\ y' = \perp \\ 0, & \text{if } x' = \perp, \end{cases}$$

$$x' \rightsquigarrow_L y' = \begin{cases} (-x'+y') \wedge 0, & \text{if } x', y' \in G^- \\ \perp, & \text{if } x' \in G^-,\ y' = \perp \\ 0, & \text{if } x' = \perp \end{cases} .$$

If we put $\perp \leq x'$ for any $x' \in G_L$, then $(G_L, \leq)$ becomes a lattice with first element, "$\perp$", and last element, "0". "$\odot_L$" is the only pseudo-t-norm on the bounded lattice G_L and "$\rightarrow_L$" and "$\rightsquigarrow_L$" are the associated implications.

Then, the structure $(G_L, \vee_L = \vee, \wedge_L = \wedge, \odot_L, \rightarrow_L, \rightsquigarrow_L, 0_L = \perp, 1_L = 0)$ is a left-pseudo-BL algebra (that has no an associated pseudo-BL algebra).

Remark 7.19 [16] By Examples 7.17 and 7.18, if we define a map $f : G_R \longrightarrow G_L$ by:

$$f(x) = \begin{cases} -x = x', & \text{if } x \in G^+ \\ \perp, & \text{if } x = \top \end{cases} ,$$

then we have:

(1) f is a bijection, $f(0_R) = 1_L$, $f(1_R) = 0_L$,
(2) $f(x \vee_R y) = f(x) \wedge_L f(y)$, $f(x \wedge_R y) = f(x) \vee_L f(y)$,
(3) $f(x \oplus_R y) = f(y) \odot_L f(x)$,
(4) $f(x \rightarrow_R y) = f(x) \rightsquigarrow_L f(y)$, $f(x \rightsquigarrow_R y) = f(x) \rightarrow_L f(y)$,

since $-(x+y) = -y - x$ and $-[(y-x) \vee 0] = (x-y) \wedge 0$. This function shows the connection between right- and left-pseudo-BL algebras G_R and G_L.

Remark 7.20 On the bounded lattice G_R, the pseudo-t-conorm "$\oplus_R$" has no a corresponding pseudo-t-norm and on the bounded lattice G_L, the pseudo-t-norm "$\odot_L$" has no a corresponding pseudo-t-conorm.

It is proved in [17] that the pseudo-BL algebras from Examples 7.17 and 7.18 generalize the product algebras (Π-algebras) from [37].

8 Weak-pseudo-BL algebras

We have seen that proper pseudo-t-norms (pseudo-t-conorms) on $[0,1]$ can not be continuous; remark also that they can not be LC_1 and RC_2 or RC_1 and LC_2 and have two Φ-operators, Φ^L and Φ^R or Φ^R and Φ^L, by Remark 5.13. We have seen that there are not pseudo-BL algebras on $[0,1]$. We shall "weak" the definition of pseudo-BL algebras in order to obtain a structure on $[0,1]$ that generalizes weak-BL algebras on $[0,1]$.

Definition 8.1 A left-weak-pseudo-BL algebra is a structure

$$\mathscr{A}_L = (A_L, \vee_L, \wedge_L, \odot_L, \rightarrow_L, \rightsquigarrow_L, 0_L, 1_L) ,$$

of type $(2,2,2,2,2,0,0)$, which satisfies the following axioms, for all $x, y, z \in A_L$:

(C1) $(A_L, \vee_L, \wedge_L, 0_L, 1_L)$ is a bounded lattice,
(C2) $(A_L, \odot_L, 1_L)$ is a monoid, i.e. $\odot_L$ is associative and $x \odot_L 1_L = 1_L \odot_L x = x$,
(C3) $x \odot_L y \leq z$ iff $x \leq y \rightarrow_L z$ iff $y \leq x \rightsquigarrow_L z$,
(C4′) $(x \rightarrow_L y) \odot_L x \leq x \wedge y$, $\quad x \odot_L (x \rightsquigarrow_L y) \leq x \wedge y$,
(C5) $(x \rightarrow_L y) \vee_L (y \rightarrow_L x) = (x \rightsquigarrow_L y) \vee_L (y \rightsquigarrow_L x) = 1_L$.

The following Proposition gives us exemples of left-weak-pseudo-BL algebras.

Proposition 8.2 Let T be a proper pseudo-t-norm that is LC_1 and LC_2 and let "$\rightarrow_L$" and "$\rightsquigarrow_L$" be the corresponding implications.

Then $([0,1], \vee = \max, \wedge = \min, T, \rightarrow_L, \rightsquigarrow_L, 0, 1)$ is a left-weak-pseudo-BL algebra.

Proof: By Theorem 5.18.

We shall give now examples of proper pseudo-t-norms that are LC_1 and LC_2 (proper pseudo-t-conorms that are RC_1 and RC_2), by extending Gödel t-norm (t-conorm) to non-commutative case.

Examples 8.3 (1) The following example was suggested to authors by Radko Mesiar at FSTA 2000, where pseudo-t-norms and pseudo-BL algebras were presented for the first time.

Let us consider the real interval $[0,1]$ and let $0 < a_1 < b_1 < 1$.

Let us consider the function $T_{0,1} : [0,1] \times [0,1] \longrightarrow [0,1]$ by (see Fig. 7):

$$T_{0,1}(x,y) = \begin{cases} 0, & \text{if } 0 \leq x \leq a_1,\ 0 \leq y \leq b_1 \\ \min(x,y), & \text{otherwise} \end{cases} .$$

Then, $T_{0,1}$ is a proper pseudo-t-norm on $[0,1]$ that is LC_1 and LC_2. Indeed, to prove (pt1), (pt2) and (pt3) is routine. $T_{0,1}$ is not commutative, since $0 = T_{0,1}(a_1, b_1) \neq T_{0,1}(b_1, a_1) = \min(b_1, a_1) = a_1$. Hence, $T_{0,1}$ is a proper pseudo-t-norm on $[0,1]$.
To prove that $T_{0,1}$ is LC_1, take a fixed $y = \beta \in [0,1]$. Then,

$$T_{0,1}(x,\beta) = \begin{cases} 0, & \text{if } x \leq a_1 \\ \min(x,y), & x > a_1 \end{cases} \quad \beta \leq b_1 \text{ and}$$

$T_{0,1}(x,\beta) = \min(x,\beta)$, if $\beta > b_1$. It follows that $T_{0,1}$ is LC_1. Similarly, $T_{0,1}$ is LC_2.

By Lemma 5.11, Proposition 5.6 and Theorem 5.7(2), (5), the two implications associated with $T_{0,1}$ are:

$$x \rightarrow_L y = \begin{cases} \max(a_1, y), & \text{if } x \leq b_1 \quad (x > y) \\ y, & \text{if } x > b_1 \quad (x > y) \\ 1, & \text{if } x \leq y \end{cases}$$

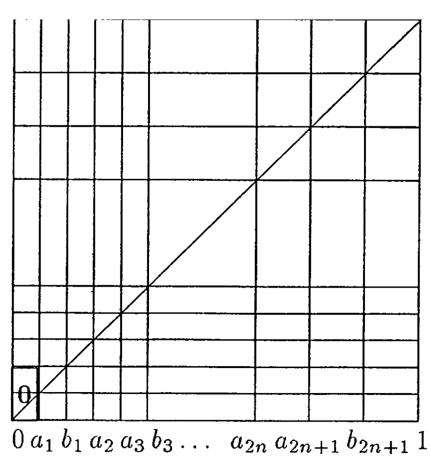

Fig. 7. The pseudo-t-norm $T_{0,1}$

and

$$x \rightsquigarrow_{\mathrm{L}} y = \begin{cases} b_1, & \text{if } x \leq a_1 \quad (x > y) \\ y, & \text{if } x > a_1 \quad (x > y) \\ 1, & \text{if } x \leq y \ . \end{cases}$$

Then $\mathscr{A}_{\mathrm{L}} = ([0,1], \sup, \inf, T_{0,1}, \rightarrow_{\mathrm{L}}, \rightsquigarrow_{\mathrm{L}}, 0, 1)$ is a left-weak-pseudo-BL algebra. The inequalities from (C4′) are strict in the following two cases:

(i) $0 < y < x \leq a_1 < b_1 < 1$:
$T_{0,1}((x \rightarrow_{\mathrm{L}} y), x) = T_{0,1}(a_1, x) = 0$ and
$T_{0,1}(x, (x \rightsquigarrow_{\mathrm{L}} y)) = T_{0,1}(x, b_1) = 0$, while
$x \wedge y = y > 0$;
(ii) $0 < a_1 = y < b_1 = x < 1$:
$T_{0,1}((x \rightarrow_{\mathrm{L}} y), x) = T_{0,1}(a_1, x) = 0$ and
$T_{0,1}(x, (x \rightsquigarrow_{\mathrm{L}} y)) = T_{0,1}(b_1, a_1) = a_1$, while
$x \wedge y = y = a_1$.

Remark that if $a_1 = b_1$, then $T_{0,1} = T_0$ from Examples 4.3(1).

A possible interpretation of $T_{0,1}$ can be: there are two threshholds, a_1 and b_1, such that if the grade of truth of x is less or equal to a_1 and the grade of truth of y is less or equal to b_1, then their conjunction, $T_{0,1}(x,y)$, is drastic.

By duality, one can define right-weak-pseudo-BL algebra and take the pseudo-t-conorm $S_{0,1}$, associated with $T_{0,1}$:

$$S_{0,1}(x,y) = \begin{cases} 1, & \text{if } x > a_1, \ y > b_1 \\ \max(x,y), & \text{otherwise} \ , \end{cases}$$

that is RC_1 and RC_2.

Then one can build, by Theorem 5.8(2′), (5′) its associated implications, "$\rightarrow_{\mathrm{R}}$", "$\rightsquigarrow_{\mathrm{R}}$", to get that: $\mathscr{A}_{\mathrm{R}} = ([0,1], \sup, \inf, S_{0,1}, \rightarrow_{\mathrm{R}}, \rightsquigarrow_{\mathrm{R}}, 0, 1)$ is a right-weak-pseudo-BL algebra, called the dual weak-pseudo-BL algebra of above $\mathscr{A}_{\mathrm{L}}$.

(1′) The following example is the "pair" ("symmetric") of the previous example (1).
Let us consider the real interval $[0,1]$ and let $0 < a_1 < b_1 < 1$.

Let us consider the function $T^s_{0,1} : [0,1] \times [0,1] \longrightarrow [0,1]$ by:

$$T^s_{0,1}(x,y) = \begin{cases} 0, & \text{if } 0 \leq x \leq b_1, \ 0 \leq y \leq a_1 \\ \min(x,y), & \text{otherwise} \ . \end{cases}$$

Then, $T^s_{0,1}$ is a proper pseudo-t-norm on $[0,1]$ that is LC_1 and LC_2. Remark that if $a_1 = b_1$, then $T^s_{0,1} = T_0$ from Examples 4.3(1).

(2) Let us consider the real interval $[0,1]$ and let $0 < a_1 < a_2 < b_2 < 1$.

Let us consider the function $T_{1,2} : [0,1] \times [0,1] \longrightarrow [0,1]$ by (see Fig. 8):

$$T_{1,2}(x,y) = \begin{cases} a_1, & \text{if } a_1 < x \leq a_2, \ a_1 < y \leq b_2 \\ \min(x,y), & \text{otherwise} \ . \end{cases}$$

Then, $T_{1,2}$ is a proper pseudo-t-norm on $[0,1]$ that is LC_1 and LC_2. Remark that if $a_2 = b_2$, then $T_{1,2} = T_1$ from Examples 4.3(2).

(2′) The following example is the "pair" ("symmetric") of the previous example (2). Let us consider the real interval $[0,1]$ and let $0 < a_1 < a_2 < b_2 < 1$. Let us consider the function $T^s_{1,2} : [0,1] \times [0,1] \longrightarrow [0,1]$ by:

$$T^s_{1,2}(x,y) = \begin{cases} a_1, & \text{if } a_1 < x \leq b_2, \ a_1 < y \leq a_2 \\ \min(x,y), & \text{otherwise} \ . \end{cases}$$

Then, $T^s_{1,2}$ is a proper pseudo-t-norm on $[0,1]$ that is LC_1 and LC_2. Remark that if $a_2 = b_2$, then $T^s_{1,2} = T_1$ from Examples 4.3(2).

(3) Let us consider the real interval $[0,1]$ and let $0 < a_1 < b_1 < a_2 < a_3 < b_3 < 1$. Let us consider the function $T_{2,3} : [0,1] \times [0,1] \longrightarrow [0,1]$ by (see Fig. 9):

$$T_{2,3}(x,y) = \begin{cases} 0, & \text{if } 0 \leq x \leq a_1, \ 0 \leq y \leq b_1 \\ a_2, & \text{if } a_2 < x \leq a_3, \ a_2 < y \leq b_3 \\ \min(x,y), & \text{otherwise} \ . \end{cases}$$

Then, $T_{2,3}$ is a proper pseudo-t-norm on $[0,1]$ that is LC_1 and LC_2. Remark that if $a_1 = b_1$ and $a_3 = b_3$, then $T_{2,3} = T_2$ from Examples 4.3(3).

(3′) Example (3) has a "pair" ("symmetric"), the LC_1 and LC_2 pseudo-t-norm $T^s_{2,3}$:

$$T^s_{2,3}(x,y) = \begin{cases} 0, & \text{if } 0 \leq x \leq b_1, \ 0 \leq y \leq a_1 \\ a_2, & \text{if } a_2 < x \leq b_3, \ a_2 < y \leq a_3 \\ \min(x,y), & \text{otherwise} \ . \end{cases}$$

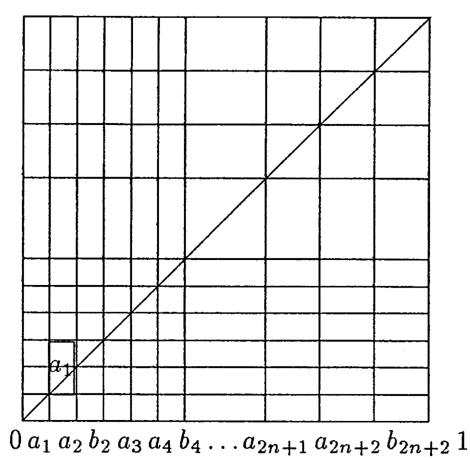

Fig. 8. The pseudo-t-norm $T_{1,2}$

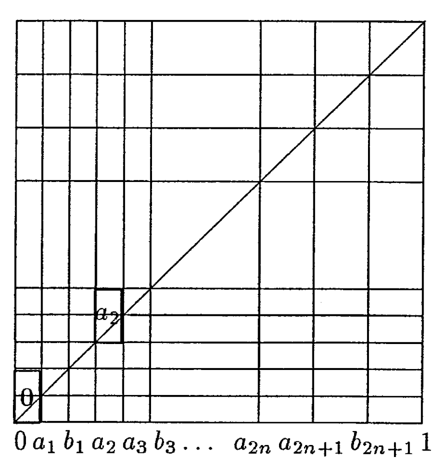

Fig. 9. The pseudo-t-norm $T_{2,3}$

Remark that if $a_1 = b_1$ and $a_3 = b_3$, then $T^s_{2,3} = T_2$ from Examples 4.3(3).

(3″) We can make two combinations of (3) and (3′) (and even more with Examples 4.3(3)), one being the "pair" of the other:

$$T^1_{2,3}(x,y) = \begin{cases} 0, & \text{if } 0 \le x \le a_1,\ 0 \le y \le b_1 \\ a_2, & \text{if } a_2 < x \le b_3,\ a_2 < y \le a_3 \\ \min(x,y), & \text{otherwise;} \end{cases}$$

$$T^{1s}_{2,3}(x,y) = \begin{cases} 0, & \text{if } 0 \le x \le b_1,\ 0 \le y \le a_1 \\ a_2, & \text{if } a_2 < x \le a_3,\ a_2 < y \le b_3 \\ \min(x,y), & \text{otherwise;} \end{cases}$$

$$T^2_{2,3}(x,y) = \begin{cases} 0, & \text{if } 0 \le x \le a_1,\ 0 \le y \le a_1 \\ a_2, & \text{if } a_2 < x \le a_3,\ a_2 < y \le b_3 \\ \min(x,y), & \text{otherwise;} \end{cases}$$

$$T^{2s}_{2,3}(x,y) = \begin{cases} 0, & \text{if } 0 \le x \le a_1,\ 0 \le y \le a_1 \\ a_2, & \text{if } a_2 < x \le b_3,\ a_2 < y \le a_3 \\ \min(x,y), & \text{otherwise;} \end{cases}$$

$$T^3_{2,3}(x,y) = \begin{cases} 0, & \text{if } 0 \le x \le a_1,\ 0 \le y \le b_1 \\ a_2, & \text{if } a_2 < x \le a_3,\ a_2 < y \le a_3 \\ \min(x,y), & \text{otherwise;} \end{cases}$$

$$T^{3s}_{2,3}(x,y) = \begin{cases} 0, & \text{if } 0 \le x \le b_1,\ 0 \le y \le a_1 \\ a_2, & \text{if } a_2 < x \le a_3,\ a_2 < y \le a_3 \\ \min(x,y), & \text{otherwise .} \end{cases}$$

Remark that if $a_1 = b_1$ and $a_3 = b_3$, then all the six previous LC_1 and LC_2 pseudo-t-norms are equal to T_2 from Examples 4.3(3).

(4) Let us consider the real interval $[0,1]$ and let $0 < a_1 < a_2 < b_2 < a_3 < a_4 < b_4 < 1$. Let us consider the function $T_{3,4} : [0,1] \times [0,1] \longrightarrow [0,1]$ by (see Fig. 10):

$$T_{3,4}(x,y) = \begin{cases} a_1, & \text{if } a_1 < x \le a_2,\ a_1 < y \le b_2 \\ a_3, & \text{if } a_3 < x \le a_4,\ a_3 < y \le b_4 \\ \min(x,y), & \text{otherwise .} \end{cases}$$

Then, $T_{3,4}$ is a proper pseudo-t-norm on $[0,1]$ that is LC_1 and LC_2. Remark that if $a_2 = b_2$ and $a_4 = b_4$, then $T_{3,4} = T_3$ from Examples 4.3(4).

(4′) Example (4) has a "pair" ("symmetric"), the LC_1 and LC_2 pseudo-t-norm $T^s_{3,4}$. Remark that if $a_2 = b_2$ and $a_4 = b_4$, then $T^s_{3,4} = T_3$ from Examples 4.3(4).

(4″) We can make two combinations of (4) and (4′) (and even more with Examples 4.3(4)), one being the "pair" of the other. Remark that if $a_2 = b_2$ and $a_4 = b_4$, then all these combinations are LC_1 and LC_2 pseudo-t-norms equal to T_3 from Examples 4.3(4).

(5) (Generalization of (1) and (3))
Let $0 = a_0 < a_1 < b_1 < a_2 < a_3 < b_3 < \cdots < a_{2n} < a_{2n+1} < b_{2n+1} < 1$ $(n \ge 0)$.

Let us consider the function $T_{2n,2n+1} : [0,1] \times [0,1] \longrightarrow [0,1]$ by (see Fig. 11):

$$T_{2n,2n+1}(x,y) = \begin{cases} 0, & \text{if } 0 \le x \le a_1,\ 0 \le y \le b_1 \\ a_{2k}, & \text{if } a_{2k} < x \le a_{2k+1}, \\ & \quad a_{2k} < y \le b_{2k+1}, \\ & \quad 1 \le k \le n \\ \min(x,y), & \text{otherwise .} \end{cases}$$

Then, $T_{2n,2n+1}$ is a proper pseudo-t-norm on $[0,1]$ that is LC_1 and LC_2. Remark that if $a_{2k+1} = b_{2k+1}$, $0 \le k \le n$, then $T_{2n,2n+1} = T_{2n}$ from Examples 4.3(5). Remark that

$$T_{0,1} > T_{2,3} > \cdots > T_{2n,2n+1} \ .$$

(5′) Example (5) has a "pair" ("symmetric"), the LC_1 and LC_2 pseudo-t-norm $T^s_{2n,2n+1}$. Remark that if $a_{2k+1} = b_{2k+1}$, $0 \le k \le n$, then $T^s_{2n,2n+1} = T_{2n}$ from Examples 4.3(5).

(5″) We can make many combinations of (5) and (5′) (and even more with Examples 4.3(5)), one being the "pair" of the other. Remark that if $a_{2k+1} = b_{2k+1}$, $0 \le k \le n$, then all these combinations are LC_1 and LC_2 pseudo-t-norms equal to T_{2n} from Examples 4.3(5).

(6) (Generalization of (2) and (4))
Let $0 < a_1 < a_2 < b_2 < a_3 < a_4 < b_4 < \cdots < a_{2n+1} < a_{2n+2} < b_{2n+2} < 1$ $(n \ge 0)$.

Let us consider the function $T_{2n+1,2n+2} : [0,1] \times [0,1] \longrightarrow [0,1]$ by (see Fig. 12):

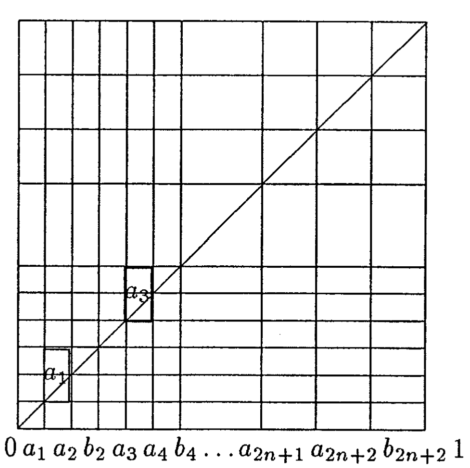

Fig. 10. The pseudo-t-norm $T_{3,4}$

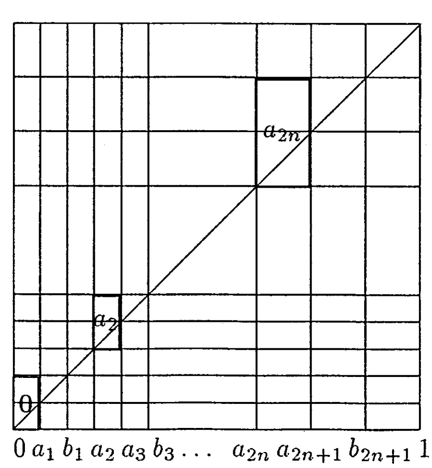

Fig. 11. The pseudo-t-norm $T_{2n,2n+1}$

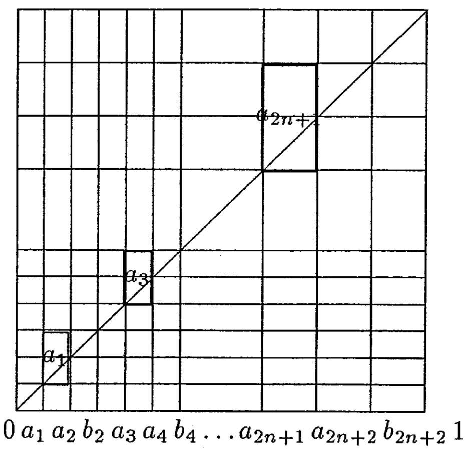

Fig. 12. The pseudo-t-norm $T_{2n+1,2n+2}$

$$T_{2n+1,2n+2}(x,y) = \begin{cases} a_{2k+1}, & \text{if } a_{2k+1} < x \le a_{2k+2}, \\ & \quad a_{2k+1} < y \le b_{2k+2}, \\ & \quad 0 \le k \le n \\ \min(x,y), & \text{otherwise} \end{cases} .$$

Then, $T_{2n+1,2n}$ is a proper pseudo-t-norm on $[0,1]$ that is LC_1 and LC_2. Remark that if $a_{2k} = b_{2k}$, $1 \le k \le n+1$, then $T_{2n+1,2n+2} = T_{2n+1}$ from Examples 4.3(6). Remark that

$$T_{1,2} > T_{3,4} > \cdots > T_{2n+1,2n+2} .$$

(6′) Example (6) has a "pair" ("symmetric"), the LC_1 and LC_2 pseudo-t-norm $T^s_{2n+1,2n+2}$. Remark that if $a_{2k} = b_{2k}$, $1 \le k \le n+1$, then $T^s_{2n+1,2n+2} = T_{2n+1}$ from Examples 4.3(6).

(6″) We can make many combinations of (6) and (6′) (and even more with Examples 4.3(6)), one being the "pair" of the other. Remark that if $a_{2k} = b_{2k}$, $1 \le k \le n+1$, then all these combinations are LC_1 and LC_2 pseudo-t-norms equal to T_{2n+1} from Examples 4.3(6).

Remark 8.4 On $[0,1]$, for each pseudo-t-norm T we can define a corresponding pseudo-t-conorm, S_T.

By using proper pseudo-t-norms (pseudo-t-conorms) on $[0,1]$, one can develop non-commutative fuzzy logic and also non-commutative linear logic.

We shall end the paper with some basic properties true in a left-weak-pseudo-BL algebra, denoted, for simplicity of writing, by:

$$\mathscr{A} = (A, \vee, \wedge, \odot, \to, \rightsquigarrow, 0, 1) .$$

Proposition 8.5 The following properties hold:

(1) $x \odot (x \rightsquigarrow y) \le y \le x \rightsquigarrow (x \odot y)$ and $x \odot (x \rightsquigarrow y) \le x \le y \rightsquigarrow (y \odot x)$,
(2) $(x \to y) \odot x \le x \le y \to (x \odot y)$ and $(x \to y) \odot x \le y \le x \to (y \odot x)$,
(3) if $x \le y$ then $z \rightsquigarrow x \le z \rightsquigarrow y$ and $z \to x \le z \to y$,
(4) if $x \le y$ then $x \odot z \le y \odot z$ and $z \odot x \le z \odot y$,
(5) if $x \le y$ then $y \rightsquigarrow z \le x \rightsquigarrow z$ and $y \to z \le x \to z$,
(6) $x \le y$ iff $x \rightsquigarrow y = 1$ iff $x \to y = 1$,
(7) $x \rightsquigarrow x = x \to x = 1$,
(8) $1 \rightsquigarrow x = 1 \to x = x$,
(9) $y \le x \rightsquigarrow y$ and $y \le x \to y$.

Proof:

(1) $x \odot (x \rightsquigarrow y) \le x \wedge y \le y, x$; the second inequalities follow by (C3): $y \le x \rightsquigarrow (x \odot y) \Leftrightarrow x \odot y \le x \odot y$ and $x \le y \rightsquigarrow (y \odot x) \Leftrightarrow y \odot x \le y \odot x$.
(2) Similarly.
(3) By (1), $z \odot (z \rightsquigarrow x) \le x \le y$, hence $z \rightsquigarrow x \le z \rightsquigarrow y$, by (C3); by (2), $(z \to x) \odot z \le x \le y$; hence $z \to x \le z \to y$, by (C3).
(4) By (2), $x \le y \le z \to (y \odot z)$, hence $x \odot z \le y \odot z$, by (C3); by (1), $x \le y \le z \rightsquigarrow (z \odot y)$, hence $z \odot x \le z \odot y$, by (C3).
(5) If $x \le y$, then $x \odot (y \rightsquigarrow z) \le y \odot (y \rightsquigarrow z) \le z$, by (4) and (1); hence $y \rightsquigarrow z \le x \rightsquigarrow z$, by (C3). If $x \le y$, then $(y \to z) \odot x \le (y \to z) \odot y \le y \wedge z \le z$, by (4) and (C4′); hence $y \to z \le x \to z$, by (C3).
(6) $x \le y$ iff $x \odot 1 \le y$ iff $1 \le x \rightsquigarrow y$ iff $x \rightsquigarrow y = 1$, by (C2), (C3) and (C1).
(7) Obviously, by (6), since $x \le x$.
(8) $x = 1 \wedge x \ge 1 \odot (1 \rightsquigarrow x) = 1 \rightsquigarrow x$, by (C4′) and (C2) and $x \le 1 \rightsquigarrow x \Leftrightarrow 1 \odot x = x \le x$. Hence $1 \rightsquigarrow x = x$.
(9) Since $x \le 1$, then $1 \rightsquigarrow y \le x \rightsquigarrow y$ and $1 \to y \le x \to y$, by (5); hence $y \le x \rightsquigarrow y$ and $y \le x \to y$, by (8).

Proposition 8.6
Let I be an arbitrary set.
$a \odot (\vee_{i \in I} b_i) = \vee_{i \in I}(a \odot b_i)$, $\quad (\vee_{i \in I} b_i) \odot a = \vee_{i \in I}(b_i \odot a)$,
whenever the arbitrary unions exist.

Proof: Obviously, $a \odot b_i \le a \odot (\vee_{i \in I} b_i)$, for each $i \in I$, by (C1) and Proposition 8.5(4). Let $a \odot b_i \le z$, $i \in I$; then $b_i \le a \rightsquigarrow z$, $i \in I$, by (C3) and hence $\vee_{i \in I} b_i \le a \rightsquigarrow z$; it follows that $a \odot (\vee_{i \in I} b_i) \le z$, by (C3) again. Therefore we get that $a \odot (\vee_{i \in I} b_i) = \vee_{i \in I}(a \odot b_i)$.

To prove the second equality, remark first that $b_i \odot a \le (\vee_{i \in I} b_i) \odot a$, $i \in I$, by (C1) and Proposition 8.5(4). Let $b_i \odot a \le u$, $i \in I$; then $b_i \le a \to u$, $i \in I$, by (C3) and hence $\vee_{i \in I} b_i \le a \to u$; it follows that $(\vee_{i \in I} b_i) \odot a \le u$, by (C3) again. Thus $(\vee_{i \in I} b_i) \odot a = \vee_{i \in I}(b_i \odot a)$.

Proposition 8.7
(a) $x \vee y = ((x \rightsquigarrow y) \to y) \wedge ((y \rightsquigarrow x) \to x)$,
(b) $x \vee y = ((x \to y) \rightsquigarrow y) \wedge ((y \to x) \rightsquigarrow x)$.

Proof: (a) Denote by α the right term of the equality from (a).

Since $x \odot (x \rightsquigarrow y) \le x \wedge y \le y$, by (C4′), then $x \le (x \rightsquigarrow y) \to y$, by (C3); but we also have that $y \le (x \rightsquigarrow y) \to y$, by Proposition 8.5(9); it follows that $x \vee y \le (x \rightsquigarrow y) \to y$, by (C1). Similarly, $x \vee y \le (y \rightsquigarrow x) \to x$. Hence $x \vee y \le ((x \rightsquigarrow y) \to y) \wedge ((y \rightsquigarrow x) \to x)$, i.e. $x \vee y \le \alpha$.

On the other hand, $\alpha = \alpha \odot 1 = \alpha \odot ((x \rightsquigarrow y) \vee (y \rightsquigarrow x)) = [\alpha \odot (x \rightsquigarrow y)] \vee [\alpha \odot (y \rightsquigarrow x)]$, by (C2), (C5) and Proposition 8.6. But, $\alpha \odot (x \rightsquigarrow y) = [((x \rightsquigarrow y) \to y) \wedge ((y \rightsquigarrow x) \to x)] \odot (x \rightsquigarrow y) \le ((x \rightsquigarrow y) \to y) \odot (x \rightsquigarrow y) \le (x \rightsquigarrow y) \wedge y \le y$, by (C1), Proposition 8.5(4) and by (C4′), (C1). Similarly, $\alpha \odot (y \rightsquigarrow x) \le x$. Hence, $\alpha = \alpha \odot (x \rightsquigarrow y) \vee \alpha \odot (y \rightsquigarrow x) \le y \vee x$, i.e. $\alpha \le x \vee y$. It follows that $x \vee y = \alpha$. (b) has a similar proof.

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