Original paper

Soft Computing 3 (1999) 148–153 © Springer-Verlag 1999

A general interpretation of conditioning and its implication on coherence

A. Capotorti, B. Vantaggi

Abstract In this paper we introduce a relation between Boolean events that can be interpreted as a conditioning relation. The goal is to find "elementary" events (atoms) derived from a finite family of conditional events and for this purpose we adopt MV-algebra operations. Two different decompositions (whose elements are called dicotomic and tricotomic conditional atoms, respectively) are proposed and their interpretations are given. Finally, we examine them as a tool for checking the coherence of a partial conditional probability assessment.

Key words Conditional event algebra, MV-algebras, coherent conditional probability assessments, conditional atoms.

1 Introduction

In several fields of uncertainty management the search of "elementary" entities plays a crucial role. In fact, to work with quantities, connected with some relevant events (representing our information), we need to find out the "basic" objects. They practically incorporate the logical constraints (elementary information). These objects (that in the sequel we will call "atoms") could be seen as a sort of little "bricks", to which assign a mass distribution decomposing the initial evaluation.

These atoms are usually "hidden" at the beginning of a decision problem. Therefore, it is crucial to choose the right operations to "build" them. Once we have obtained these elementary entities, we can use them to explore the implicit consequences present in our initial statement (e.g. to detect inconsistencies, to find consistent enlargements of the domain, etc.).

This "factorization" need is present in different approaches, e.g. probabilistic assessments [3, 6], belief functions [15] and fuzzy theory [10].

In this paper we focus on the framework of conditional events (denoted by $a|b$) and on the related problem of checking the coherence of partial conditional probability assessments.

Usually, probabilistic approaches deal with conditional probability $P(a|b)$ but they do not give a meaning to the conditional event $a|b$. This aspect, besides to be a foundational lack, does not allow to evaluate "compound" conditional events, as for example ($a|b$ and $c|d$), or "iterated" conditionals, as $a|(b|c)$. Therefore, the requirement of an algebraic framework for conditional events is essential and relevant for the decisional process. For a wide survey on this subject, the special issue of IEEE Transaction on Systems, Man, and Cybernetics [11] on conditioning, with both theoretical and applied papers, is a valid reference.

In the literature, different definitions of conditional events and of operations among them are proposed. These differences arise from the need to preserve, moving from a Boolean framework to a conditional one, specific probabilistic properties. Unluckily, a unique structure, well adapted to all the different needs, has not been proposed till now. Therefore, there is not a standard algebra for conditional events, while for unconditional events Boolean algebra is commonly adopted.

The main purpose of this paper is to find out suitable operations leading to a decomposition of conditional events into "atoms" and to employ them for the check of coherence of a partial conditional probabilistic assessment.

A common feature of all different proposal is that they are based on a lattice, so the idea is to look for a "reacher" structure, in order to exploit deeper results (as those in [7]). Such a structure is that of an MV-algebra, so that a lattice naturally arises from it.

2 Conditional events and an MV-algebra structure

First of all we give the basic notion of conditional event (as given by de Finetti [4]). Let $(\mathcal{B}, \vee, \wedge, ', \phi, \Omega)$ be a Boolean algebra, where $\vee$ denotes the disjunction, $\wedge$ the conjunction (in the sequel we shall omit it except when the events are singled-out by an index-set), $'$ the contrary and ϕ and Ω the neutral elements of $\vee$ and $\wedge$, respectively. Starting from $a \in \mathcal{B}$ and $b \in \mathcal{B}\setminus\{\phi\}$, a conditional event $a|b$ is an entity (proposition) with three possible truth values, i.e. *true*, *false*, *undetermined*, hence a truth assignment $T(\cdot)$ can be written as

$$T(a|b) = \begin{cases} 1 & \text{if } ab \text{ holds,} \\ 0 & \text{if } a'b \text{ holds,} \\ u & \text{if } b' \text{ holds.} \end{cases} \tag{1}$$

We stress that this approach differs from those where the material implication $b \Rightarrow a$ is taken as "conditional entity". In fact, adopting interpretation (1), the conditional event $a|b$ is

A. Capotorti (✉)
Dipartimento di Matematica,Via Vanvitelli 1, 06123 Perugia, Italy
e-mail: capot@dipmat.unipg.it

B. Vantaggi
Dipartimento di Statistica, Via Pascoli, 06123 Perugia, Italy
e-mail: vant@stat.unipg.it

used to evaluate the occurrence of the event a when assuming the occurrence of b. On the contrary, the use of the material implication as a conditional event is more suitable to assess the validity of the rule *b implies a*, as done for example in the rule-based systems.

Conditional events can be introduced algebraically, for example by the following relation $\mathscr{R}$ on the Cartesian product $\mathscr{B} \times \mathscr{B}$:

– $(a, a) \in \mathscr{R} \quad \forall a \in \mathscr{B}$;

– $(a, b) \in \mathscr{R} \Rightarrow a \subseteq b$;

– $(a, b) \in \mathscr{R} \Leftrightarrow (b', a') \in \mathscr{R}$;

– $(a, b) \in \mathscr{R}$ and $(c, d) \in \mathscr{R} \Rightarrow (a \vee c \vee bd, b \vee d) \in \mathscr{R}$

This relation is the particular case of the more general relation R_n (with dimension $n=2$) proposed in [8] to introduce the definition of *Abstract Conditional MV-Space*.

By the last two properties of $\mathscr{R}$ it is immediate to define two operations $(\cdot)^{\sim}$ and $\oplus$ (the first unary and the second binary) in the following way:

– $(a, b)^{\sim} = (b', a')$;

– $(a, b) \oplus (c, d) = (a \vee c \vee bd, b \vee d)$;

Combining these two operations it is possible to obtain an other binary operation $\odot$ by

$(a, b) \odot (c, d) = (a^{\sim} \oplus b^{\sim})^{\sim} = (ac, ad \vee bc)$.

The set $\mathscr{R}$ endowed with the operations $\oplus$, $\odot$, $^{\sim}$ is actually an MV-algebra (the proof is immediate) with the pairs (ϕ, ϕ), (Ω, Ω) as neutral elements for $\oplus$ and $\odot$, respectively.

We recall that an algebraic structure $(\mathscr{A}, \oplus, \odot, ^{\sim}, \phi, \Omega)$ is an MV-algebra if $(\mathscr{A}, \oplus, \phi)$ is an Abelian monoid, and the operations have the following properties:

– $a \oplus \Omega = \Omega$;

– $a^{\sim\sim} = a$;

– $\phi^{\sim} = \Omega$;

– $a \odot b = (a^{\sim} \oplus b^{\sim})^{\sim}$ (De Morgan law);

– $(a^{\sim} \oplus b)^{\sim} \oplus b = (b^{\sim} \oplus a)^{\sim} \oplus a$.

Note that a Boolean algebra is an MV-algebra with the additional condition:

$a \oplus a = a$.

We denote by $\mathscr{B}(\mathscr{A}) = \{a \in \mathscr{A} : a \oplus a = a\}$ the maximal Boolean sub-algebra of $\mathscr{A}$, and in our case it reduces to $\mathscr{B}(\mathscr{R}) = \{(a, a) : a \in \mathscr{B}\}$.

Note that, in general, the operations $\oplus$ and $\odot$ do not satisfy the distributive rule (except among elements of $\mathscr{B}(\mathscr{R})$).

We give now a "link" between the two different representations of conditional events.

Denoting with $\mathscr{E}$ the set of all possible conditional events obtained by $\mathscr{B}$, with the map

$$\varphi : \mathscr{R} \backslash \{(\phi, \Omega)\} \to \mathscr{E},$$
$$(a, b) \mapsto a | (a \vee b'), \qquad (2)$$

the MV-algebra $(\mathscr{R}, \oplus, \odot, ^{\sim}, (\phi, \phi), (\Omega, \Omega))$ can be interpreted as a general structure for conditional events.

Note that in (2) b' is disjoint from a, in fact, if $(a, b) \in \mathscr{R}$ then $a \subseteq b$; it means that we actually consider only conditional events $a|b$ with this constraint. This is not a lack of generality, in fact, the truth assignment $T(\cdot)$ induces an equivalence relation

$$e|h \sim f|k \quad \text{iff} \quad eh = fk \quad \text{and} \quad h = k.$$

Since each equivalence class has only one representative $e|h$ with $e \subseteq h$, these equivalence classes are in one-to-one correspondence with the pairs belonging to $\mathscr{R}$.

It is interesting now to show how the operations $\oplus$, $\odot$, $^{\sim}$ induce, by the map φ, analogous operations on $\mathscr{E}$. The following diagram makes a picture of the composition of φ^{-1}, $\oplus$ and φ:

$$\begin{array}{c} e|h \xrightarrow{\varphi^{-1}} (e, e \vee h') \qquad (f, f \vee k') \xleftarrow{\varphi^{-1}} f|k \\ \searrow \oplus \swarrow \\ (e \vee f \vee h'k', e \vee f \vee h' \vee k') \\ \downarrow \varphi \\ (e \vee f \vee h'k') | (e \vee f \vee h'k' \vee e'f'hk) \end{array}$$

This composition determines a disjunction operation between conditional events, for which we will use the same symbol $\oplus$.

Analogously, with $\odot$ we obtain the conjunction

$$\begin{array}{c} e|h \xrightarrow{\varphi^{-1}} (e, e \vee h') \qquad (f, f \vee k') \xleftarrow{\varphi^{-1}} f|k \\ \searrow \odot \swarrow \\ (ef, ef \vee ek' \vee fh') \\ \downarrow \varphi \\ ef | (ef \vee e'f' \vee e'h \vee f'k) \end{array}$$

Finally, for the unary operation $^{\sim}$

$$e|h \xrightarrow{\varphi^{-1}} (e, e \vee h') \xrightarrow{(\cdot)^{\sim}} (e'h, e') \xrightarrow{\varphi} e'h|h.$$

Note that φ maps the Boolean elements (a, a) into the unconditional events $a \equiv a|\Omega$ and over them $\oplus$, $\odot$, reduce to the classical Boolean operations $\vee$, $\wedge$, $'$.

Note also that there is an immediate decomposition

$$(e, e \vee h') = (e, e) \oplus (\phi, h')$$

and we will call the first component *Boolean* part (denoted with $(e|h)|_{\beta}$) and the second one *not-Boolean* part $((e|h)|_{\not\beta})$.

As mentioned before, the MV-algebra $(\mathscr{R}, \oplus, \odot, ^{\sim}, (\phi, \phi), (\Omega, \Omega))$ induces naturally a lattice structure $(\mathscr{R}, \varovee, \varowedge, (\phi, \phi), (\Omega, \Omega))$ by

$$(a, b) \varovee (c, d) = ((a, b) \odot (c, d)^{\sim}) \oplus (c, d) = (a \vee c, b \vee d),$$

$$(a, b) \varowedge (c, d) = ((a, b) \oplus (c, d)^{\sim}) \odot (c, d) = (ac, bd).$$

These are the most natural extensions of the corresponding Boolean operators, since they are obtained by applying component-wise the Boolean operations.

Composing with φ, we have analogous operations between conditional events (that we will denote by the same symbols)

$$e|h \xrightarrow{\varphi^{-1}} (e, e \vee h') \qquad (f, f \vee k') \xleftarrow{\varphi^{-1}} f|k$$
$$\searrow \circledvee \swarrow$$
$$(e \vee f, e \vee h' \vee f \vee k')$$
$$\downarrow \varphi$$
$$(e \vee f)|(e \vee f \vee e'f'hk)$$

and

$$e|h \xrightarrow{\varphi^{-1}} (e, e \vee h') \qquad (f, f \vee k') \xleftarrow{\varphi^{-1}} f|k$$
$$\searrow \circledwedge \swarrow$$
$$(ef, ef \vee fh' \vee ek' \vee h'k')$$
$$\downarrow \varphi$$
$$ef|(ef \vee e'h \vee f'k)$$

These lattice operations are the same proposed by different authors (see [1, 4, 12] and they are called "Lukasiewicz" logical operations.

The lattice operations induce a natural order among elements of $\mathscr{R}$

$$(a, b) \leqslant (c, d) \Leftrightarrow \begin{cases} a \subseteq c, \\ b \subseteq d. \end{cases} \tag{3}$$

By means of the map φ the order relation among conditional events is the following:

$$e|h \leqslant f|k \Leftrightarrow \begin{cases} e \subseteq f, \\ e \vee h' \subseteq f \vee k' \ \text{(or equivalently } f'k \subseteq e'h) \end{cases} \tag{4}$$

and it is the same proposed in [12].

3
Two different kinds of conditional atoms

Starting from a generic (i.e. without any structure) finite family $\mathscr{F}$ of conditional events $e_i|h_i$ (with $i = 1, \ldots, n$), we look for a proper decomposition of each one into "atoms" (disjoint respect to $\odot$). This idea arises from the procedure used for unconditional events.

The unconditional atoms are used as a tool to detect "all the possible situations" arising from the combination of the given events.

Following this idea for the conditional events we have two possibilities for the "case detection": the first procedure is to consider all the combinations among the events in "affirmative" or "negative" form; the second one extends the combinations also to the "undetermined" form.

We will describe the two approaches separately, in the former the atoms will be called *dicotomic*, in the other one *tricotomic* (but we will not use this distinction when from the context it will be clear).

For the first approach, even if we are in a conditional framework, the logical procedures involved are the same as the unconditional situation (see [6]). We recall the unconditional procedure: starting from $\mathscr{A} = \{a_1, \ldots, a_n\}$ with a_i belonging to a Boolean algebra $\mathscr{B}$, we define an atom as

$$c_I = \left(\bigwedge_I a_i\right) \wedge \left(\bigwedge_{N \setminus I} a_i'\right) \tag{5}$$

with $I \subseteq N = \{1, \ldots, n\}$.

It is easy to check that these atoms are a partition of the sure event Ω. The atoms are elementary objects used to factorize a probabilistic assessment on the family $\mathscr{A}$ for checking its coherence [6].

Following this path, we build the conditional atoms generated by the given family $\mathscr{F}$ using the MV-algebra operations $\odot$, $\tilde{}$ introduced above.

Note that in the following, we will use for conditional events both notation $e_i|h_i$ and $(e_i, e_i \vee h_i')$ (i.e. the map φ will be understood).

We will define dicotomic conditional atoms as

$$d_I \stackrel{\text{def}}{=} \left(\bigodot_I (e_i|h_i)\right) \odot \left(\bigodot_{N \setminus I} (e_i|h_i)^{\sim}\right)$$
$$= \left(\bigodot_I (e_i, e_i \vee h_i')\right) \odot \left(\bigodot_{N \setminus I} (e_i' h_i, e_i')\right), \tag{6}$$

where $I \subseteq N$.

Using the definition of operation $\odot$, it is possible, by induction, to obtain an explicit expression for these atoms (we present the two parts separately because the entire expression is not readable)

$$d_{I|\beta} = \left(\bigwedge_I e_i \bigwedge_{N \setminus I} e_k' h_k, \quad \bigwedge_I e_i \bigwedge_{N \setminus I} e_k' h_k\right),$$
$$d_{I|\not\beta} = \left(\phi, \bigvee_{j \in I}\left(\bigwedge_{i \in I \setminus \{j\}} e_i h_j' \bigwedge_{k \in N \setminus I} e_k' h_k\right)\right.$$
$$\left.\vee \bigvee_{s \in N \setminus I}\left(\bigwedge_{k \in N \setminus (I \cup \{s\})} e_k' h_k h_s' \bigwedge_{i \in I} e_i\right)\right). \tag{7}$$

Note that we are considering all the index subsets $I \subseteq N$, but some of the conditional atoms can be equal to (ϕ, ϕ) (this is possible when there are logical constraints among events), however these are irrelevant for the results shown in Sect. 4.

We want to stress that a conditional atom is exactly identified by the index set I, whose elements point to the conditional events with an "affirmative" (i.e. without $(.)^{\sim}$) form in expression (6) of d_I.

This is practically useful for a simpler notation (i.e. we will use always the notation d_I instead of expression (7)) and also it could have a central role in a possible implementation.

Applying the operations $\oplus$ and $\odot$ it is easy to verify the following properties:

$$d_I \odot d_{I'} = (\phi, \phi) \quad \forall I \neq I',$$
$$d_I \odot \left(\phi, \bigwedge_N h_i'\right) = (\phi, \phi) \quad \forall I,$$
$$\bigoplus_{I \subseteq N} d_I \oplus \left(\phi, \bigwedge_N h_i'\right) = (\Omega, \Omega). \tag{8}$$

In the sequel, we denote with d_0 the couple $(\phi, \bigwedge_N h_i')$.

These properties are the generalization of that requested for a Boolean partition. But for an MV-algebra, in literature, there is an other definition of partition. In particular, we give the definition proposed in [14] and "tailored" for our purpose.

Definition 1 Let $(\mathscr{A}, \oplus, \odot, \tilde{}, \phi, \Omega)$ be an MV-algebra, a MV-partition of $\mathscr{A}$ is a finite subset $\{a_1, \ldots, a_k\}$ of $\mathscr{A}$ and a set of integer $m_i \geqslant 1$ $(i=1, \ldots, k)$, satisfying the following conditions:

(i) $\bigoplus_{i=1}^{k} m_i a_i = \Omega$,
(ii) if $\bigoplus_{i=1}^{k} n_i a_i = \phi$ with integer coefficients n_i, then all n_i are zero (where $m_i a_i = \bigoplus_{j=1}^{m_i} a_i$ and $n_i a_i = \bigoplus_{j=1}^{|n_i|} (a_i)\tilde{}$ if $n_i < 0$).

We are now in a position to assert the following result.

Proposition 1 The set $\{d_I\}_{I \subseteq N} \cup \{d_0\}$, together with the integers $m_0 = m_I = 1$ (where m_0 and m_I are the coefficients associated to d_0 and d_I, respectively, in the Definition 1), is a MV-partition of $(\mathscr{R}, \oplus, \odot, \tilde{}, (\phi, \phi), (\Omega, \Omega))$.

Proof: Property (i) of Definition 1 is exactly the third property of (8). Suppose, by absurd, that there exist n_I's and n_0 not all zeroes such that

$$\bigoplus_{I \subseteq N} n_I d_I \oplus n_0 d_0 = (\phi, \phi).$$

By the definition of $\oplus$ and $\tilde{}$ it derives that d_I must be equal to (ϕ, ϕ) for all I such that $n_I \neq 0$ and analogously for d_0 if $n_0 \neq 0$, but this is an absurd. $\square$

We show now that, like in the unconditional case, the conditional atoms d_I, contained in one event $(e_z, e_z \vee h'_z)$ of the family $\mathscr{F}$, are characterized by the belonging of z to the index set I.

Lemma 1

$$d_I \leqslant (e_z, e_z \vee h'_z) \Leftrightarrow z \in I. \tag{9}$$

Proof: The implication $\Rightarrow$ comes directly from (7) and (3).
For the implication $\Leftarrow$, since $z \in I$ we can rewrite (6) as

$$d_I = (e_z, e_z \vee h'_z) \odot \left(\bigodot_{I \setminus \{z\}} (e_i, e_i \vee h'_i) \right) \odot \left(\bigodot_{N \setminus I} (e'_i h_i, e'_i) \right), \tag{10}$$

so the result follows from the property $(a, b) \odot (c, d) = (ac, ad \vee bc) \leqslant (a, b)$. $\square$

The following proposition allows us to rewrite the events of the family $\mathscr{F}$ by means of d_I and d_0.

Proposition 2 For all $k \in N$ we have

$$\bigoplus_{I \ni k} d_I \oplus d_0 = (e_k, e_k \vee h'_k) \tag{11}$$

(where $I \ni k$ is used to extend the operation over all the index set I containing k).

Note that, in general, it is not possible to have a similar decomposition only for the Boolean part (e_k, e_k) of an event $(e_k, e_k \vee h'_k)$, i.e.

$$\bigoplus_{I \ni k} d_I|_{\beta} \neq (e_k, e_k).$$

This does not allow to operate separately with the Boolean $d_I|_{\beta}$ and the not-Boolean $d_I|_{\not\beta}$ parts of dicotomic atoms: this happens because $\oplus$ and $\odot$ are not distributive .

We describe now the second possible approach where all the three truth values are considered. We define tricotomic conditional atoms as

$$t_{I,J} = \bigodot_{I} (e_i, e_i \vee h'_i) \odot \bigodot_{J} (e'_i h_i, e'_i) \odot \bigodot_{N \setminus \{I \cup J\}} (h'_i, h'_i) \tag{12}$$

with I and J disjoint subsets of N. From expression (12) it is clear that I identifies the events with an "affirmative" form, J that ones with a "negative" form and the rest are in an "undetermined " form (i.e. $T(h'_i) = 1 \Leftrightarrow T(e_i|h_i) = u$).

It is easy to check that, analogously to dicotomic atoms,

$$\bigoplus_{I,J} t_{I,J} = (\Omega, \Omega). \tag{13}$$

On the contrary, for the tricotomic atoms we have that, in general,

$$t_{I,J} \odot t_{I',J'} = (\phi, a)$$

with $a \in \mathscr{B}$.

However, also in this case the following result, whose proof is exactly the same as Proposition 2, holds.

Proposition 3 The set $\{t_{I,J}\}_{I,J \subseteq N}$, together with the integers $m_{I,J} = 1$ ($m_{I,J}$'s are the coefficients associated to $t_{I,J}$ in the Definition 1), is a MV-partition of $(\mathscr{R}, \oplus, \odot, \tilde{}, (\phi, \phi), (\Omega, \Omega))$.

The main properties of tricotomic atoms, that we use in the sequel, are

$$(e_i|h_i)_{\beta} = \bigoplus_{I \ni i} t_{I,J}|_{\beta}, \quad (h_i|\Omega) = \bigoplus_{I \cup J \not\ni i} t_{I,J}|_{\beta}, \quad (e_i|h_i)_{\not\beta} = \bigoplus_{I \ni i} t_{I,J}|_{\not\beta}. \tag{14}$$

These properties allow to work separately with the Boolean and not-Boolean parts, so that they are useful to "rebuild" the events belonging to the given family $\mathscr{F}$. This is the peculiar characteristic of tricotomic atoms that differs them from the others.

4
Checking coherence by conditional atoms

We can show how dicotomic and tricotomic atoms can be involved in checking the coherence of a probability assessment on family $\mathscr{F}$.

We recall that the problem of checking coherence consists on verifying if a partial conditional assessment P: $\mathscr{F} \to [0, 1]$ is a restriction of a conditional probability distribution P': $\mathscr{B} \times \mathscr{B} \setminus \{\phi\} \to [0, 1]$ that satisfies all the axiomatic properties given in [5, 9].

Before we show the connection between conditional atoms and coherence of a conditional probability assessment, we introduce further notations. Let $\mathscr{F} = \{e_i|h_i\}_{i=1}^{n}$ be a general finite family of conditional events, we denote by $\mathscr{T}_F$ the set of tricotomic atoms (contained in $\bigoplus_{i=1}^{n} (h_i, h_i)$) associated to $\mathscr{F}$ and with $\mathscr{A}_F$ the MV-algebra generated by $\mathscr{T}_F$.

For the coherent conditional probability there is a characterization theorem given in [3] and we can rewrite it involving the tricotomic atoms.

Theorem 1 Let $P\colon \mathscr{F} \to [0, 1]$ be a numerical assessment. The following propositions are equivalent:
– P is a coherent conditional probability,
– there exists at least one finite class of probabilities $\{P_0, P_1, \ldots P_k\}$ such that

1. P_0 is defined over $\mathscr{A}_{\mathscr{F}}$, while for $\alpha = 1, \ldots, k$ P_α is defined over $\mathscr{A}_{\mathscr{F}_\alpha}$ with $\mathscr{F}_\alpha = \{e_i|h_i\colon\ P_{\alpha-1}((h_i, h_i)) = 0\}$,
2. for all $e_i|h_i$ there exists an unique α such that $P_\alpha((h_i, h_i)) > 0$ and

$$P(e_i|h_i) = \frac{\sum_{I \ni i} P_\alpha(t_{I,J})}{\sum_{I \cup J \ni i} P_\alpha(t_{I,J})} \quad \text{with } t_{I,J} \in \mathscr{T}_{\mathscr{F}_\alpha}.$$

In [3] it is also shown that, from a computational point of view, checking coherence is equivalent to the compatibility of a sequence of linear systems $\mathscr{S}_0, \mathscr{S}_1, \ldots, \mathscr{S}_k$ and we can rewrite them by tricotomic atoms properties (14)

$$\mathscr{S}_\alpha = \begin{cases} \sum\limits_{I \cup J \ni i} P_\alpha(t_{I,J})\ P(e_i|h_i) &= \sum\limits_{I \ni i} P_\alpha(t_{I,J}) \quad \text{if } P_{\alpha-1}(h_i, h_i) = 0, \\ \sum\limits_{I,J} P_\alpha(t_{I,J}) &= 1, \\ P_\alpha(t_{I,J}) &\geqslant 0, \\ P_\alpha(t_{\phi,\phi}) &= 0, \\ P_\alpha((\phi, a)) &= 0, \quad \text{if } t_{I,J} = (\phi, a). \end{cases} \tag{15}$$

A particular feature of this probabilistic approach is that the assessment P is given directly on the conditional events of $\mathscr{F}$ (contrary to the classical approach, where the conditional probability is just a numerical ratio) and moreover it is not required the positivity for the probability of the conditioning events h_i.

An alternative way of introducing coherence is based on the betting scheme (actually this was the original way de Finetti introduced the coherence principle, see [6]). This formulation requires that, given an assessment

$$p : \mathscr{F} \to [0, 1]$$

$$e_i|h_i \mapsto p_i,$$

the "possible gain"

$$G = \sum_{i=1}^{n} \lambda_i T(h_i)\,(T(e_i) - p_i), \quad \lambda_i \in \mathbb{R} \tag{16}$$

(where $T(e_i) = T(e_i|\Omega)$ and $T(h_i) = T(h_i|\Omega)$) cannot represent a sure win [or sure loss], i.e. the probability assessment is said to be coherent if $\sup G \geqslant 0$ [or $\inf G \leqslant 0$], where the supremum (infimum) is taken over all possible combinations of values of $T(e_i)$ and $T(h_i)$, except the case $T(h_i) = 0$ for all $i = 1, \ldots, n$ (see [13]).

Gain (16) represents the convention that, betting on the $e_i|h_i$'s, if one pays an amount p_i, when h_i occurs he wins an amount 1 if e_i is true and 0 if e_i is false, but, if h_i does not occur, the bet is called off and the gambler receives the amount p_i back. So the assessment $P(e_i|h_i) = p_i$ is not coherent if it is possible to find out a composition of bets that ensure a win [or a loss].

In expression (16) are involved only the truth values of the Boolean events h_i and e_i. In the previous section we have stressed that the tricotomic atoms well fit the need of working on the components of conditional events separately. It is for this reason that we have used them to rewrite the Characterization Theorem 1 and the systems $\mathscr{S}_\alpha$'s.

If we try to involve dicotomic atoms in the coherence problem we obtain different conclusions. The procedure to obtain the dicotomic atoms is a direct generalization of the one used for Boolean atoms and the suitable way to connect them with the coherence problem is to consider as relevant only the truth or falsity of the conditional events, "neglecting" the undetermined cases. This situation refers to the case that the truth value of an event e_i has no relevance if the conditioning event h_i does not occur. With this assumption, the expression of the gain can be written as

$$G_\beta = \sum_{1}^{n} \lambda_i (I(e_i|h_i) - p_i), \quad \lambda_i \in \mathbb{R}, \tag{17}$$

where $I(\cdot)$ is the restriction of $T(\cdot)$ to the values 0 and 1 (we use the notation G_β to emphasize the analogy with the gain with Boolean events). Obviously, G_β and G are not comparable, so the equivalence between the absence of sure win [or sure loss] with a characterization theorem like 1 does not hold any more.

In fact, the absence of sure win [or sure loss] with the gain G_β is equivalent, by Proposition 3, to the existence of a solution of the linear system

$$\begin{cases} \sum\limits_{I \ni i} p'(d_I) = p_i, \\ \sum\limits_{I} p'(d_I) = 1, \\ p'(d_I) \geqslant 0. \end{cases} \tag{18}$$

A solution of the previous system represents a possible "mass" distribution on the dicotomic atoms such that it is additive respect to the given assessment. Nevertheless, such a potential solution is not sufficient to ensure the coherence of the conditional probabilistic assessments $P(e_i|h_i) = p_i$ because it is not guaranteed that constraints like $P(e_i) = P(e_i|h_i)\ P(h_i)$ are satisfied. These constraints are the characteristic features (even if, by the compound probability theorem, they are only necessary conditions) of conditional probabilities.

Such negative result is clearly due to the strong assumption of considering only two truth-values.

5 Conclusions

In the literature there are no logical and algebraic structure for conditional events reaching a wide agreement: in fact, there are many definitions of conditional events and of operations among them.

In this paper we adopt the Lukasiewicz three values logic and, by the algebraic point of view, we endow the conditional events with an MV-algebra structure. This approach leads to detect two different kinds of elementary entities (dicotomic and tricotomic atoms) useful to get two different MV-partitions. Our approach differs from the most common techniques, where lattice operations are adopted, because they do not allow to build a partition. Our proposal is more general since a lattice structure is anyhow derived from the MV-algebra, in particular, this lattice is the same proposed by the other authors. These atoms are an useful tool to explore all the possible configurations and to decompose the conditional events of a given finite family.

Another reason to develop a conditional event's algebra is the foundational lack of the usual probabilistic approach where the conditional probability is defined like a pure numerical ratio of the unconditional probabilities. Hence the conditional probability theory is usually adopted without caring about the meaning of the conditional events and the way how to combine them.

De Finetti's approach of coherent conditional probability assessments differs from the classical theory because conditional events are expressed by truth-values and it is possible to define directly conditional probabilities.

The decomposition properties of conditional atoms suggest the use of both kinds to the problem of checking the coherence of a partial conditional probability assessment. But, while tricotomic atoms allow a rewriting of the characterization theorem and of the betting scheme, the same is not possible with dicotomic atoms.

The interpretation of coherence requires a careful handling of Boolean and not-Boolean parts and this is the feature that distinguish the two kinds of atoms.

References

1. **Capotorti A** (1995) Generalized concept of atoms for conditional events, In Coletti G, Dubois D, Scozzafava R (Eds) Mathematical Models for Handling Partial Knowledge in Artificial Intelligence, New York and London: Plenum Press; pp 183–190
2. **Coletti G** (1993) Numerical and qualitative judgments in probabilistic expert systems, in R. Scozzafava (Ed) Proc of the Int Workshop on Probabilistic Methods in Expert Systems, Roma, SIS, pp 37–55
3. **Coletti G, Scozzafava R** (1996) Characterization of coherent conditional probabilities as a tool for their assessment and extension, Int J Uncertainty Fuzziness Knowledge-Based Systems, **4**(2), 103–127
4. **de Finetti B** (1936) La logique de la probabilité, In Herman et Cie. (Eds) Actes du Congrès Inter. de Philosophie Scientifique, Paris, pp 565–573
5. **de Finetti B** (1949) Sulla impostazione assiomatica del calcolo delle probabilitá, Annali Univ. Trieste, **19**, 3–55 (English trasl. in: (1972) Probability, Induction, Statistics, Ch. 5. London, Wiley)
6. **de Finetti B** (1974) Theory of Probability, vol 1–2. New York: Wiley
7. **Di Nola A** (1991) Representation and reticulation by quotients of MV-algebras, Ricerche di Matematica, **40**(2), 291–297
8. **Di Nola A, Lettieri A** (1998) One chain generated varietes of MV-algebras, preprint
9. **Dubins LE** (1975) Finitely additive conditional probabilities, conglomerability and disintegration, Ann Probab, **3**, 89–99
10. **Dubois D, Prade H** (1980) Fuzzy Sets and Systems, London: Academic Press
11. **Dubois D, Goodman IR, Calabrese PG** (1994) (Guest Eds) IEEE Transaction on Systems, Man, and Cybernetics, Special Issue on Conditional Event Algebra, vol 24(12).
12. **Goodman IR, Nguyen HT, Walker EA** (1991) Conditional Inference and Logic for Intelligent Systems: A Theory of Measure-Free Conditioning, North-Holland: Amsterdam
13. **Holzer S** (1985) On coherence and conditional prevision, Bull. Unione Matematica Italiana, **6**(4), 441–460.
14. **Mundici D** (1997) Non Boolean partitions and many-valued logic, In Proc IFSA 97, vol 1, Prague, pp 25–29
15. **Smets P** (1995) The canonical decomposition of a weighted Belief, In Proc of IJCAI-'95, vol 2 Montreal, Canada, pp 1896–1901