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A semantics for Fuzzy Logic

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Abstract We present a semantics for certain Fuzzy Logics of vagueness by identifying the fuzzy truth value an agent gives to a proposition with the number of independent arguments that the agent can muster in favour of that proposition.

Key words Fuzzy Logic, Vagueness, Truth Functionality

1 Introduction

In the literature the expression 'Fuzzy Logic' is used in two separate ways (at least). One is where 'truth values' are intended to stand for measures of belief (or confidence, or certainty of some sort) and the expression 'Fuzzy Logic' is taken as a synonym for the assumption that belief values are truth functional. That is, if $w(\theta)$ denotes an agent's belief value (on the usual scale $[0, 1]$) for $\theta \in SL$, where SL is the set of sentences from a finite propositional language L built up using the connectives $\neg$, $\wedge$, $\vee$ (we shall consider implication later), then w satisfies

$$w(\neg\theta) = F_{\neg}(w(\theta)),$$

$$w(\theta \wedge \phi) = F_{\wedge}(w(\theta), w(\phi)),$$

$$w(\theta \vee \phi) = F_{\vee}(w(\theta), w(\phi)), \quad (1)$$

for some *fixed* functions $F_{\neg}: [0, 1] \to [0, 1]$ and $F_{\wedge}, F_{\vee}: [0, 1]^2 \to [0, 1]$, where $\theta, \phi \in SL$.

Two popular choices here for $F_{\neg}, F_{\wedge}, F_{\vee}$ are

$$F_{\neg}(x) = 1 - x,$$

$$F_{\wedge}(x, y) = min\{x, y\},$$

$$F_{\vee}(x, y) = max\{x, y\}, \quad (2)$$

$$F_{\neg}(x) = 1 - x,$$

$$F_{\wedge}(x, y) = xy,$$

$$F_{\vee}(x, y) = x + y - xy. \quad (3)$$

Concerning the origins of these schemata, (2) was first introduced by Łukasiewicz in his *infinitely valued logic* in 1923, see [1]. Along with these functions, however, Łukasiewicz also introduced the function $F_{\to}(x, y) = min\{1, 1 - x + y\}$ for implication, which allows the *strong* conjuction, $max\{0, x + y - 1\}$, and strong disjunction, $min\{1, x + y\}$, to be defined, and, in the current terminology, it is this version which is now frequently referred to as 'Łukasiewicz Logic'. At a somewhat later date (1932) Gödel also introduced a logic (see [2]) with conjunction and disjunction as in (2) (but with a different negation). The earliest clear use of (2), without also implication, appears to have been by Chang and Lee in [3] in 1971. For the connectives conjunction and disjunction (but not negation) the second schema, (3), corresponds to *Product Logic* (less its implication), see [4]. [For an excellent survey of Fuzzy Logics we refer the reader to [5].]

The use of 'Fuzzy Logic' in the context of belief values dates back to Mycin (see [6]) and is rather widespread in *theoretical* expert systems (although not, apparently, in genuine working systems, see [7]). Its use in this context is difficult to justify (see, for example, [8] p53), except possibly on the pragmatic grounds of computational simplicity (see [9], [10], [11]).

A second way that the expression 'Fuzzy Logic(s)' is used, and the one which will concern us in this paper, is as the *logic(s) of vagueness*, see [12], [13]. To elaborate, it seems to be the case that we, as examples of so called intelligent agents, can give (subjective) degrees of truth to assertions involving vague predicates. Thus, for example an agent might give *degree of truth* 1/3 to the assertion

A 178cm high woman is tall (4)

involving the vague predicate 'tall', or more precisely 'tall for a woman'. By 'Fuzzy Logic is the logic of vagueness' we mean that for such an agent the values, $w(\theta)$, that the agent assigns to vague assertions $\theta \in SL$ satisfy a schema of the form (1).

One difficulty that relative 'outsiders' (like the author) sometimes have with this intention is how to interpret 'fuzzy truth values', such as the figure 1/3 assigned to (4) in such a way that these values actually do respect the schema (1) (with the function w now, of course, assigning not 'belief values' but 'fuzzy truth values'). In particular how to give a sensible meaning, or semantics, to the values of w so that the schema (2) (the most popular choice of all as we judge it) is satisfied. It is the purpose of this short note to propose one such

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semantics. [We refer the reader to [14], [15][1], [16], [18], [11], [19] for alternate proposals.]

We should, perhaps, point out at this stage that amongst some of those close to the pulse of vagueness there is a viewpoint that no such semantics are necessary, just as no such semantics are necessary for the two truth values 0,1 in classical two-valued logic. That would seem, however, to still leave open the question of *why* these values should satisfy any particular properties (such as schema (2) or (3)), unless, as is the case with classical two-valued logic, they follow in some way by the commonly agreed *meaning* of the connectives *not*, *and*, *or*.

2
A Semantics for Fuzzy Logic(s)

The idea is to identify the 'fuzzy truth value' that an agent $\mathcal{A}$ would give to $\theta \in SL$ with the ease with which $\mathcal{A}$ can *accept*, or be *convinced* of, θ. This 'ease' is measured simply by the proportion of the set of *independent* arguments that $\mathcal{A}$ has for or against θ which are actually **for** θ.

Precisely, for $\theta \in SL$ the agent's set of arguments for θ, θ^+, and set of arguments against θ, θ^-, are assumed to satisfy the following for some fixed, finite, non-empty, set T of worlds[2]:

(1) For $\theta \in L$ (i.e. θ a propositional variable) $\theta^+ = \{V \in T | V(\theta) = 1 \text{ (\textit{i.e. true})}\}$ and

$$\theta^- = \{V \in T | V(\theta) = 0 \text{ (\textit{i.e. false})}\}.$$

(2) $(\neg\theta)^+ = \theta^-$, $(\neg\theta)^- = \theta^+$.

(3) $(\theta \wedge \phi)^+$ is a maximal set of *independent* pairs $\langle a_1, a_2\rangle$ such that $a_1 \in \theta^+$ and $a_2 \in \phi^+$. (We will come to the question of what 'independent' means later.) $(\theta \wedge \phi)^-$ is a maximal set of independent pairs $\langle a_1, a_2\rangle$ such that $a_1 \in \theta^-$, $a_2 \in Z_\phi$, or , $a_2 \in \phi^-$, $a_1 \in Z_\theta$, (where Z_ϕ, Z_θ will be specified later).

(4) $(\theta \vee \phi)^+$ is a maximal set of independent pairs $\langle a_1, a_2\rangle$ such that $a_1 \in \theta^+$, $a_2 \in Z_\phi$, or, $a_2 \in \phi^+$, $a_1 \in Z_\theta$. $(\theta \vee \phi)^-$ is a maximal set of independent pairs $\langle a_1, a_2\rangle$ such that $a_1 \in \theta^-$ and $a_2 \in \phi^-$.

Thus an argument in favour of a conjuction is a pair of arguments, $\langle a_1, a_2\rangle$, where a_1 is an argument in favour of the first conjunct and a_2 is an argument in favour of the second conjunct. Similarly an argument in favour of a disjunction is a pair of arguments, $\langle b_1, b_2\rangle$, such that either b_1 is an argument in favour of the first disjunct and b_2 belongs to some allowed set of 'place holders' (whose presence tells us that it is the *first* disjunct that b_1 is an argument for), or b_2 is an argument in favour of the second disjunct and b_1 belongs to some allowed set of 'place holders'.

For this agent $\mathcal{A}$ the 'fuzzy truth value' $\mathcal{A}$ gives to θ is defined by

$$w_A(\theta) = \frac{|\theta^+|}{|\theta^+| + |\theta^-|}.$$

The idea here is that the extent to which $\mathcal{A}$ can accept, or be convinced of, θ is measured simply by the number of independent arguments that $\mathcal{A}$ can muster in favour of θ (as opposed to $\neg\theta$). Notice that this definition immediately forces that

$$w_A(\neg\theta) = 1 - w_A(\theta),$$

and that de Morgan's Laws, i.e.

$$w_A(\neg\theta \wedge \neg\phi) = w_A(\neg(\theta \vee \phi)),$$

$$w_A(\neg\theta \vee \neg\phi) = w_A(\neg(\theta \wedge \phi)),$$

hold, and hence is incapable of providing possible semantics for the multitude of fuzzy logics which fall outside this class.

The fact that we use worlds as arguments for, or against, propositional variables should not (necessarily) be interpreted as saying that one of these worlds is the 'correct one' and defines the absolutely true state of the 'real' world. Rather they might be thought of as convenient idealisations of arguments in the form of examples, instances, etc.. For example, some of my arguments in favour of a 178cm high woman being tall might be that I know of women of around that height who chose the size 'long' in clothes, who play basketball, who stand in the back row in group photos, etc.. and these might in turn be used to furnish examples of worlds where 178cms was 'tall'.

We now consider some definitions of 'independent' and the sets Z_θ.

First definition $Z_\theta = \{u_0\}$, where u_0 is some new dummy argument (standing for 'unspecified'), and two pairs of arguments $\langle a_1, a_2\rangle$, $\langle b_1, b_2\rangle$ are *independent* if $a_1 \neq b_1$ and $a_2 = b_2 = u_0$ or $a_2 \neq b_2$ and $a_1 = b_1 = u_0$ or $a_1, a_2, b_1, b_2 \neq u_0$ and $a_1 \neq b_1$, $a_2 \neq b_2$.

With this definition we find that $|\theta^+| + |\theta^-| = |T|$ for all $\theta \in SL$ and

$$w_A(\neg\theta) = 1 - w_A(\theta),$$

$$w_A(\theta \wedge \phi) = min\{w_A(\theta), w_A(\phi)\},$$

$$w_A(\theta \vee \phi) = max\{w_A(\theta), w_A(\phi)\}.$$

In other words this agrees with schema (2). Notice that this is about the strictest version of 'independence' that one could expect. According to this version even arguments $\langle a_1, u_0\rangle$, $\langle u_0, b_2\rangle$ for a disjunction are considered to be 'dependent'. One explanation for agent $\mathcal{A}$ arriving at such an interpretation of 'independent' is that $\mathcal{A}$*'s* arguments may be represented (internally[3]) in such a way that $\mathcal{A}$ is only able to check if two arguments are, or are not, *really different*, if they are arguments for (or against) the same sentence θ. In such a case a particularly paranoid $\mathcal{A}$ might not wish to treat arguments $\langle a_1, u_0\rangle$, $\langle u_0, b_2\rangle$ as independent on the grounds that a_1 and b_2 *may* not actually be *really* different. [Notice that $\mathcal{A}$ need have no concerns the s/he will choose independent arguments

[1] In connection with these papers see also [17]

[2] A *world* is just a valuation, V, on L, except that we do not demand that if $V_1(\theta) = V_2(\theta)$ for all $\theta \in SL$ then $V_1 = V_2$. The use of worlds here could be replaced (generalised even) by the use instead of *weighted* valuations, but for the sake of simplicity we shall stick with worlds.

[3] For notational simplicity and mathematical clarity we have suppressed mention of additional features, such as the sentence actually being argued for or against, which one might reasonably expect would form part of the agent's internal representation of an argument. Clearly this could, if necessary, be built in to facilitate *explanation*.

$\langle a_1, a_2\rangle$, $\langle b_1, b_2\rangle$ for $\theta \wedge \phi$ with a_1 being *really* equal to b_2 and a_2 being *really* equal to b_1 since in such a case s/he could replace $\langle a_1, a_2\rangle$, $\langle b_1, b_2\rangle$ in his/her maximal independent set by $\langle a_1, b_2\rangle$, $\langle b_1, a_2\rangle$, respectively, without altering the figures. Similar considerations show that nothing is lost by using the same figures in the special case of $\theta = \phi$, where $\mathscr{A}$ *can* compare the arguments a_1 and b_2 etc..]

Second Definition $Z_\theta = \theta^+ \cup \theta^-$ and two pairs of arguments $\langle a_1, a_2\rangle$, $\langle b_1, b_2\rangle$ are *independent* if $\langle a_1, a_2\rangle \neq \langle b_1, b_2\rangle$.

With this definition,

$$w_A(\neg\theta) = 1 - w_A(\theta),$$

$$w_A(\theta \wedge \phi) = w_A(\theta) w_A(\phi),$$

$$w_A(\theta \vee \phi) = w_A(\theta) + w_A(\phi) - w_A(\theta) w_A(\phi).$$

In other words this agrees with schema (3). In this case then, unlike the first case, we do require that in an argument in favour of a disjunction the 'dummy coordinate' is a definite argument[4], but we do not care whether it is for or against the corresponding sentence.

It is interesting to consider for these two cases how it is possible for a 'contradiction' such as $\theta \wedge \neg\theta$ to acquire a non-zero truth value. This can happen because our agent $\mathscr{A}$ can have an argument a_1 **for** θ and an argument a_2 **against**. In that case $\mathscr{A}$ will have an argument $\langle a_1, a_2\rangle$ for $\theta \wedge \neg\theta$. Of course we might feel that, classically, a and b should not be able to coexist and that our agent $\mathscr{A}$ should examine his arguments in a bit more depth before combining them. In reply, however, agent $\mathscr{A}$ might point out that s/he is not concerned with truth and belief, but with acceptability, and that as far as s/he is concerned both a_1 and a_2 are accredited arguments which together provide an argument for $\theta \wedge \neg\theta$. This point clearly illustrates the difference between degrees of belief (where believing a contradiction is 'inconsistent', and hence untenable) and the notion(s) of acceptability that we have introduced.

Notice that if we were to take to heart this requirement that only mutually consistent arguments could be combined and, in consequence, required in the second example above that we could only form pairs of arguments $\langle a_1, a_2\rangle$ if $a_1 = a_2$ (and identified $\langle a_1, a_1\rangle$ with a_1) then our resulting truth values would have have the properties of probabilities (see, for example [8] page 7).

In the next section we consider expanding our set of connectives to include also implication.

3 Adding implication

If we add implication to our set of connectives then the corresponding addition to schema (1) is simply

$$w(\theta \to \phi) = F_\to(w(\theta), w(\phi)),$$

where $F_\to : [0, 1]^2 \to [0, 1]$. As indicated earlier, Fuzzy Logics are frequently given with such a function $F_\to$ specified, for example, in Gödel's Logic, by,

$$F_\to(x, y) = \begin{cases} y & \text{if } y < x, \\ 1 & \text{otherwise}, \end{cases}$$

and, in Product Logic, by,

$$F_\to(x, y) = \begin{cases} y/x & \text{if } y < x, \\ 1 & \text{otherwise}. \end{cases}$$

In attempting to enlarge our semantics to cover implication we are faced with a choice of interpretation between (at least) implication as material implication and implication as denoting a conditional (analogous to conditional probability). In the former case it seems our semantics force us to treat $\theta \to \phi$ as if it was $\neg\theta \vee \phi$ in which case (2),(3) would give $F_\to(x, y)$ equal to $max\{1-x, y\}$ and $1 - x + xy$ respectively. The more interesting case is where we treat implication as denoting a conditional. Notice that with this interpretation of implication we would not anticipate having to handle nested implications.

In this case, building on the first semantics given above (which yielded *min*, *max*, *etc*) we could argue that an argument in favour of *if* θ *then* ϕ is a pair $\langle a_1, a_2\rangle$ where $a_1 \in \theta^+$ and $a_2 \in \phi^+$, and an argument against is a pair $\langle a_1, a_2\rangle$ with $a_1 \in \theta^+$ and $a_2 \in \phi^-$. Using the 'independent arguments' requirement again, for $\theta, \phi \in SL$ (where SL is defined as previously using only the connectives $\neg$, $\wedge$, $\vee$, so not mentioning $\to$), this gives a value to $w_A(\theta \to \phi)$ of

$$\frac{min\{s, t\}}{(min\{s, t\} + min\{s, 1-t\})}$$

where $s = w_A(\theta)$, $t = w_A(\phi)$. [If $s = 0$ then this is not defined, although clearly in this context, as with conditional probability, such a gap is not so objectionable.] Whilst this is not the standard Gödel $F_\to$ as given above it seems not so unreasonable. At least, unlike its standard counterpart, it is not always greater or equal to the truth degree of ϕ. We should emphasis here that in obtaining this expression we required that neither θ nor ϕ already mention the connective $\to$. If they did then we may no longer have that

$$|\theta^+| + |\theta^-| = |\phi^+| + |\phi^-| = |T|$$

so that the above derivations of $F_\wedge$, $F_\vee$, $F_\to$ may no longer hold.

Applying the same method (under the same conditions) in the second case we get the answer

$$w_A(\theta \to \phi) = F_\to(w_A(\theta), w_A(\phi)) = w_A(\phi),$$

provided $w_A(\theta) \neq 0$, a not unexpected, if not very satisfactory, answer, given the way $F_\wedge$, $F_\vee$ treat their arguments analogously to statistically independent probabilities in this case.

We finally remark that if we proceeded as at the end of the previous section by allowing arguments to be combined just if they were mutually consistent, then, hardly surprisingly, we obtain the corresponding conditional probability for $w_A(\theta \to \phi)$.

[4] Possibly because $\mathscr{A}$'s arguments carry some additional information, for example confirmation that the 'sentence' in question is actually in SL.

4
Vagueness

The value, or otherwise, of the approach we have taken to providing a semantics for the logic(s) of vagueness now rests, firstly, on one's willingness to identify 'vagueness', or, more precisely, the truth degree of vague statements, with 'acceptability', or 'willingness to be convinced', and secondly on one's willingness to quantify 'acceptability' in terms of the proportion of independent arguements for, as opposed to against.

On this first point, of course, we all have our own understanding of what 'vagueness' is. However if we consider again the example

A 178cm high woman is tall to degree 1/3,

it seems to us reasonable to suppose that the degree to which such a woman is 'tall' (for an agent $\mathscr{A}$) is directly related, identifiable even, with the ease with which $\mathscr{A}$ is willing to accept, or 'agree for the sake of argument', that she is tall. In particular then, one imagines that our agent, unless s/he is a pygmy, would find it very difficult to accept that a 150cm high woman is 'tall', and hence would give it a very low truth value. Similarly one imagines that our agent would have no problem in accepting that a 186cm high woman was 'tall'.

A similar example, often quoted in connection with vagueness, is the Sorites Paradox, in one version of which a hirsute gentleman loses one hair at a time until nothing at all remains to cover his shiny pate. The question is, 'At what stage is he bald?', and the paradox is that if we only allow truth values 0 and 1 here then one is led to the risible conclusion that there must be a stage at which the removal of one hair transformed him from not bald to bald. On the other hand by allowing truth values between 0 and 1 for the vague notion of this poor fellow's baldness the so called paradox is avoided. Again we would argue that it would seem very natural here in this depapillatory process to identify the truth value at any stage of the statement 'he is bald' with one's willingness at that time[5] to agree to, or accept, (as opposed to disagree with) that statement.

Turning now to the second point, the reasonableness of measuring 'willingness to accept' in terms of the proportion of independent arguments for, once the notion of what constitutes independent arguments is resolve it seems hard to argue with this simple proportion (or perhaps a scaled version of it) as providing a suitable measure. More questionable are the notions of independence between arguments that we have used, and certainly there may be other, more satisfactory notions still awaiting discovery.

One should note here that our aim in this paper was, as far as possible, to provide semantics for existing, popular, Fuzzy Logics. It is certainly possible to dream up different notions of independence from the ones we have use here, but our general experience is that the corresponding truth values are not truth functional, i.e. fail to satisfy schema (1). In particular we have so far been unable to find any reasonable notion of independence (etc.) which would yield the negation, and (strong) conjunction and disjunction,

$$F_{\neg}(x) = 1 - x,$$

$$F_{\wedge}(x, y) = max\{0, x + y - 1\},$$

$$F_{\vee}(x, y) = min\{1, x + y\},$$

of Łukasiewicz Logic. An alternative, indirect, route by which this Logic might possibly be furnished with some such semantics is by using the embedding of Łukasiewicz Logic into Product Logic 'with too small truth degrees' as given in [20], although our endeavours to date along these lines have again been unconvincing. [In [18] Mundici does exhibit an intriguing semantics for $(k+2)$-valued Łukasiewicz Logic, although it appears unrelated to the semantics suggested in this paper.]

[5] We might expect that, just as for ourselves, the arguments that an agent might, in practice, muster would also be varying with time.

5
Conclusion

In this short note we have shown that a possible semantics for the logic(s) of vagueness might be obtained by identifying the extent to which a sentence $\theta \in SL$ is true for an agent $\mathscr{A}$ with the extent to which $\mathscr{A}$ can muster independent arguments for accepting, as opposed to rejecting, θ. We have shown that for suitable formulations of 'independent' this can yield the schemas (2) and (3).

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