MATHEMATICAL METHODS IN DATA SCIENCE

δ-granular reduction in formal fuzzy contexts: Boolean reasoning, graph represent and their algorithms

Zengtai Gong[1] · **Jing Zhang**[1]

Abstract

The fuzzy concept lattice is one of the effective tools for data mining, and granular reduction is one of its significant research contents. However, little research has been done on granular reduction at different granularities in formal fuzzy contexts (FFCs). Furthermore, the complexity of the composition of the fuzzy concept lattice limits the interest in its research. Therefore, how to simplify the concept lattice structure and how to construct granular reduction methods with granularity have become urgent issues that need to be investigated. To this end, firstly, the concept of an object granule with granularity is defined. Secondly, two reduction algorithms, one based on Boolean reasoning and the other on a graph-theoretic heuristic, are formulated while keeping the structure of this object granule unchanged. Further, to simplify the structure of the fuzzy concept lattice, a partial order relation with parameters is proposed. Finally, the feasibility and effectiveness of our proposed reduction approaches are verified by data experiments.

Keywords Formal fuzzy contexts · Formal fuzzy decision contexts · δ-granular reduction · Minimal transversal of hypergraphs

1 Introduction

Formal concept analysis (FCA) was first introduced by Barbut and Monjardet (1970). As an effective data processing tool, FCA is widely used in data mining (Missaoui et al. 1994), semantic web (Maio et al. 2012), machine learning (Xia et al. 2010), information retrieval (Poelmans et al. 2012), medical diagnosis (Zou and Deng 2017), gene expression data analysis (Kaytoue et al. 2011), cooperative game theory (Jiménez-Losada et al. 2023), and other fields. Formal context is an important foundation of FCA, which is a triple set consisting of a set of objects, a set of attributes, and a binary relation between objects and attributes, usually denoted by (U, C, I), where U and C denote the set of objects and the set of attributes, respectively, and I denotes the binary relation between objects and attributes. Formal concept is another important foundation of FCA, which is formed by Galois connections, where all formal concepts form a lattice structure according to a certain partial order, called the concept lattice. A comprehensive theory of FCA was proposed by Wille (1982). Zhang et al. (2005) proposed formal decision context in 2005, which adds decision attributes and binary relations between objects and decision attributes to the formal contexts, and is usually represented by a quintuple (U, C, I, D, J), where U, C, I have the same meaning as that in the formal context, D denotes the set of decision attributes, and J denotes the binary relation between objects and decision attributes. Wei et al. (2008) presented the attribute reduction theory of formal decision contexts based on Zhang et al. (2005), in which two kinds of formal decision contexts are defined, one is called strong consistent formal decision contexts and the other is called weak consistent formal decision contexts. However, the classical formal context is based on the fact that the relationship between objects and attributes is well-defined, i.e., objects have binary relations with attributes with values 0 or 1, which will lead to its limitations in practical applications. For this reason, Burusco and Fuentes-Gonzalez (1994) and Burusco and Fuentes-González (2000) combined formal context with fuzzy sets (Zadeh 1979) and proposed the definition of the fuzzy concept lattice based on fuzzy logic. In the following years, many scholars have proposed various fuzzy concept lattices

---

✉ Zengtai Gong
   zt-gong@163.com; gongzt@nwnu.edu.cn

   Jing Zhang
   jingzhangnwnu@163.com

1   College of Mathematics and Statistics, Northwest Normal University, Anning Dong Lu, Lanzhou 730070, Gansu, China

from different perspectives, combined with different types of fuzzy sets or implication operators, such as T-fuzzy concept lattice, intuitionistic fuzzy concept lattice, variable threshold concept lattice, etc., cf. Addison et al. (2023), Kridlo and Ojeda-Aciego (2017), Bělohlávek (2015), Bělohlávek et al. (2010), Gediga and Düntsch (2002), Georgescu and Popescu (2004), Popescu (2004), Zhang and Wei (2016), Zhang (2018), and Li and Zhang (2010).

It is worth noting that Krajči (2003) and Yahia et al. (2000) independently introduced the concept of "one-side fuzzy concept", which the set of objects is a crisp set and the set of attributes is a fuzzy set. Compared with other fuzzy concepts, it has the advantage that the number of generated formal concepts is significantly reduced. It is also interesting to note that Bělohlávek (2001a, b, c) also proposed a series of fuzzy concept lattice theory as well as definitions of reduction in 2001, and a comprehensive study of fuzzy concept lattices and a comparison of some approaches was made in Bělohlávek et al. (2005). To further investigate the structure and properties of the fuzzy concept lattice, Zhang et al. (2007) generalized the one-sided fuzzy concept based on this, he first constructed a pair of operators that can form Galois connections and then defines three variable threshold concepts, which makes the one-sided fuzzy concept a special case.

As the era of big data approaches, how to accurately eliminate redundant information has become an urgent problem for data analysis and processing. Therefore, attribute reduction of formal contexts naturally becomes a hot problem for data mining and knowledge representation. To address this problem, many scholars have proposed different reduction methods for this purpose, such as rough sets (Yao 2004a, b; Liang et al. 2012; Chen et al. 2007), information entropy (Huang et al. 2016; Singh et al. 2017), and Boolean reasoning (Zhang et al. 2005; Wu et al. 2009; Mi et al. 2010).

The purpose of reduction is to reduce the data dimension, or to simplify the structure of the concept lattice. Some scholars have proposed methods to simplify the structure of the concept lattice by reducing the concepts and thus, some scholars have proposed methods to reduce the attribute dimension without changing some properties of the concept lattice. For the former, numerous scholars have also conducted detailed studies, cf. Elloumi et al. (2004), Jaoua and Elloumi (2002). For the latter, on the one hand, Zhang et al. (2005) discussed the classical concept lattice based on the discernibility matrix and the discernibility function, while ensuring that the structure of the concept lattice remains unchanged. In addition, Konečný (2017) compared the attribute reduction algorithm (DR-method) in Ganter and Wille (1999) with the algorithm in Zhang et al. (2005). He analyzed that the DR-method is more advantageous in time complexity and applied this algorithm to the reduction of various concept lattices. Then, Konečný proved that the DR-method is faster than Zhang et al. (2005) in run-

ning time by the data experiments in Konečný and Krajča (2018). Furthermore, Konečný achieved a new algorithm in Konečný and Krajča (2019) with polynomial time complexity that maintains the structure of the concept lattice unchanged by eliminating unnecessary steps from the algorithm in Zhang et al. (2005). Recently, Shao et al. (2015) extended the approach (Zhang et al. 2005) correspondingly to FFCs. However, one drawback of this reduction approach is that it requires the generation of all formal concepts or formal fuzzy concepts, and this workload is very large, especially for FFCs. On the other hand, in order to be able to ensure that all the reduced attributes can still accurately represent the information of the object, Wu et al. (2009) combined the formal context with the granular computing, and firstly introduced the concept of object granules in the classical formal context and gave the attribute reduction method while keeping the object granules unchanged. It is worth mentioning that, as an extension of the FFC, the formal fuzzy decision context (FFDC) was defined by Shao et al. (2016) by categorizing attributes into conditional and decision attributes. Shao also introduced the definition of fuzzy object granules and fuzzy granular consistent sets and designed the algorithm of granular reduction for FFCs and FFDCs. It is important to know that the reduction methods used here are all Boolean reasoning. In general, the reduction of all formal contexts can be obtained by calculating Boolean function and discernibility matrix.

However, the complexity of the algorithms based on Boolean reasoning is high. Therefore, some heuristic algorithms (Huang et al. 2016; Li et al. 2011) have been proposed by some scholars. In particular, due to the remarkable performance of graph theory in many applications, in recent years, graph theory has been used in the fields of granular computing, FCA, and electrical engineering. For example, Wang and Gong (2018) and Gong and Wang (2017) constructed a hypergraph partitioning model for the granular based on fuzzy equivalence relations. Also, graph theory has been applied to the reduction of coverage decisions by Kulaga et al. (2005). In order to reduce the complexity of constructing fuzzy concepts from FFCs, a reduction method based on directed graph was introduced by Mao (2017). Notably, Chen et al. (2018) combined vertex cover theory in graph theory with Boolean functions to construct a new heuristic reduction algorithm and conducted extensive data experiments, which showed that the algorithm has significant advantages over algorithms based on discernibility matrices and Boolean reasoning both in terms of time complexity and running time.

The motivation for the study in this paper is presented from four perspectives.

(1) Methods for FFCs granular reduction while ensuring all object granules invariance have been investigated in Shao et al. (2016). However, there is a lack of research on granular reduction that keeps object granules of different granulari-

ties invariant. The main difficulty lies in the construction of object granules of different granularity and granular discernibility matrices of discernibility granularity, and in clearly interpreting their actual semantics. Therefore, our intention is to construct object granules and discernibility matrices of different granularities and propose corresponding reduction methods.

(2) The granular reduction method based on Boolean reasoning can find out all the granular reducts, however, the complexity of the reduction algorithm is high. The reduction speed is acceptable when facing small data sets, but it is inefficient when facing large data sets, thus affecting the data processing process. In view of this, we aim to develop a heuristic reduction algorithm to reduce the complexity of the reduction algorithm and improve the efficiency of the reduction.

(3) The traditional partial order relationship makes all fuzzy concepts form a fuzzy concept lattice. The structure of this lattice is complex and all concepts are represented in the lattice. Is it possible to find a partial order relation with parameters that allows concepts satisfying certain properties to be represented in the lattice, while those within a certain range are no longer represented in the lattice?

(4) As far as I know, most of FFCs in the reduction, for multi-valued attributes, is to single out all the multi-valued attributes. However, such a converted data set not only causes the data volume to become larger, but also often fails to represent some features of the original multi-valued attributes and to explain their practical meaning. In addition, a simple regularization of a multi-valued attribute also does not explain its actual meaning very well. Therefore, it is necessary to develop a function for the conversion of multi-valued attributes, especially continuous-type data attributes, to an FFC.

In summary, the contributions and innovations of this thesis are mainly in three areas.

(1) Two maps g_δ and $g_{\bar{\delta}}$, δ-object granule, δ-partial order relation, δ-granular reduction, and δ-discernibility matrix are defined.

(2) δ-granular reduction methods based on Boolean reasoning and underlying graph theory in FFCs and FFDCs, respectively, are given. And the framework for these two approaches is shown in Fig. 1, where an FFC and an FFDC are denoted as K and S, respectively. The δ-distinction matrices of K and S are denoted as D_K^δ and D_S^δ, respectively. The hypergraphs induced by D_K^δ and D_S^δ are denoted as H_K^δ and H_S^δ, respectively. The minimal transversals of H_K^δ and H_S^δ are denoted as $T(H_K^\delta)$ and $T(H_S^\delta)$, respectively.

(3) Finally, a function for conversion from information tables to FFCs is defined.

The rest of the paper is framed as follows. In Sect. 2, some basic concepts in FFCs and graph theory are reviewed and two maps g_δ and $g_{\bar{\delta}}$ with parameters, δ-object granule, and δ-

partial order relations are introduced. In Sect. 3, the definition of δ-granular reduction for FFCs is defined. Furthermore, two reduction algorithms based on Boolean reasoning and graph theory were designed for FFCs without decision attributes. In addition, data experiments were conducted. Similar to Sect. 3, the definition of δ-granular reduction with decision attributes and Boolean reasoning methods and graph theoretic methods for δ-granular reduction are defined and data experiments are performed in Sect. 4. Finally, the summary and outlook of the paper are given in Sect. 5.

2 Preliminaries

In this section, we first review several concepts and results (Yahia et al. 2000; Eiter and Gottlob 1995; Listrovoy and Minukhin 2012) that we are going to use in our work. Then, we introduce some new definitions and properties. Throughout this paper, we always assume that the universe of discourse is a nonempty finite set.

2.1 Formal fuzzy contexts and one-sided fuzzy concepts

The FFC, which forms the basis of the fuzzy formal concept analysis, is generally represented by a triple set $K = (U, C, \tilde{I})$. Where U and C are the sets of objects and attributes, respectively, $\tilde{I}$ is the fuzzy binary relationship of U and C, indicating the degree to which the object has the attribute. For example, $\tilde{I}(x, a) = 0.7$ means the degree to which x has attribute a is 0.7.

Let U be a finite and nonempty set called the universe of discourse. We denote by $P(U)$ and $\mathscr{F}(U)$ the power set and the fuzzy power set of U, respectively. For any $\tilde{U}_1, \tilde{U}_2 \in \mathscr{F}(U)$, then $\tilde{U}_1 \subseteq \tilde{U}_2$ if and only if $\tilde{U}_1(x) \leq \tilde{U}_2(x), \forall x \in U$. Also, the $\cup$ and $\cap$ operation is defined by the following equation

$$\tilde{U}_1 \cup \tilde{U}_2 = \tilde{U}_1(x) \vee \tilde{U}_2(x) = \sup(U_1(x), U_2(x)),$$
$$\tilde{U}_1 \cap \tilde{U}_2 = \tilde{U}_1(x) \wedge \tilde{U}_2(x) = \inf(U_1(x), U_2(x)).$$

Krajči Krajči (2003) and Yahia Yahia et al. (2000) defined two operations between $P(U)$ and $\mathscr{F}(C)$ in FFCs.

Definition 1 Let $K = (U, C, \tilde{I})$ be an FFC. For $X \in P(U)$, $\tilde{B} \in \mathscr{F}(C)$, the operators $f : P(U) \to \mathscr{F}(C)$ and $g : \mathscr{F}(C) \to P(U)$ are defined respectively as follows:

$$f(X)(a) = \bigwedge_{x \in X} \tilde{I}(x, a), a \in C, \tag{1}$$

$$g(\tilde{B}) = \{x \in U | \forall a \in C, \tilde{B}(a) \leq \tilde{I}(x, a)\}. \tag{2}$$

In addition, there are the following properties:

(1) $X_1 \subseteq X_2 \Rightarrow f(X_2) \subseteq f(X_1), \tilde{B}_1 \subseteq \tilde{B}_2 \Rightarrow g(\tilde{B}_2) \subseteq g(\tilde{B}_1)$,
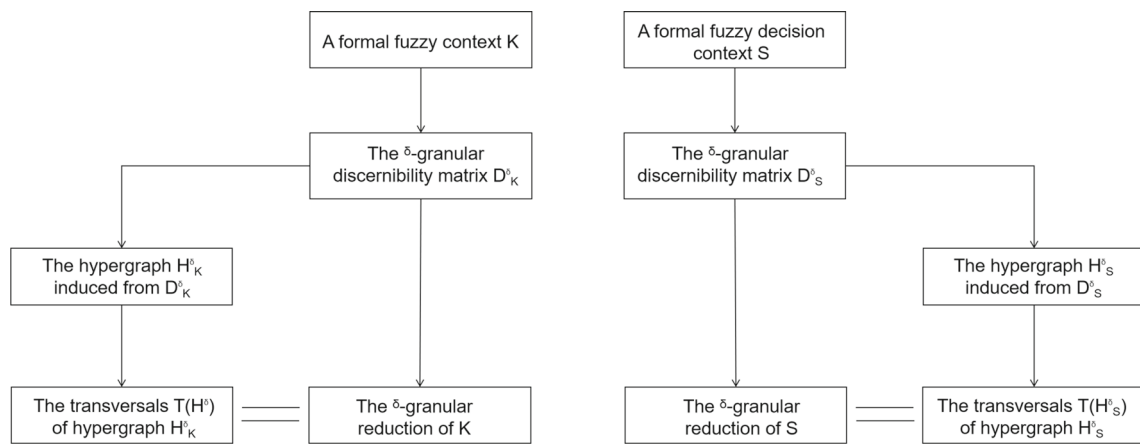
Fig. 1 Structure of the two reduction methods

(2) $X \subseteq g \circ f(X), \tilde{B} \subseteq f \circ g(\tilde{B}),$

(3) $f(X) = f \circ g \circ f(X), g(\tilde{B}) = g \circ f \circ g(\tilde{B}),$

(4) $f(\bigcup_{i \in J} X_i) = \bigcap_{i \in J} f(X_i), \quad g(\bigcup_{i \in J} \tilde{B}_i) = \bigcap_{i \in J} g(\tilde{B}_i).$

where $X, X_1, X_2, X_i \in P(U)$, and $\tilde{B}, \tilde{B}_1, \tilde{B}_2, \tilde{B}_i \in \mathscr{F}(C)$, $i \in J$ (J is an index set). A one-sided fuzzy concept is a binary pairs, usually denoted by $\tilde{F}C = (X, \tilde{B})$, where $X \subseteq U, \tilde{B} \subseteq C$, and satisfy that $f(X) = \tilde{B}, g(\tilde{B}) = X$. X and $\tilde{B}$ are called the extent and intent of $(X, \tilde{B})$, respectively. These properties above point to the fact that the operations f and g are a Galois connection between the power sets $P(U)$ to $\mathscr{F}(C)$. Furthermore, $(g \circ f(X), f(X))$ must be a one-sided fuzzy concept. The set of all concepts we denote by $\mathscr{B}(U, C, \tilde{I})$. Then, all concepts form a concept lattice $\tilde{L}(U, C, \tilde{I})$ through a partial order relation $\preceq$, which means $(X_1, \tilde{B}_1) \preceq (X_2, \tilde{B}_2)$ if and only if $X_1 \subseteq X_2$, or equivalently, $\tilde{B}_2 \subseteq \tilde{B}_1$.

Furthermore, Shao et al. (2016) proposed concept of the fuzzy sub-context. For any $C' \subseteq C$, one can obtain an FFC $K_{C'} = (U, C', \tilde{I}_{C'})$, which is called a sub-context of K, where $\tilde{I}_{C'} = \tilde{I} \cap (U \times C')$. For $X \in P(U)$ and $\tilde{B} \in \mathscr{F}(C')$, the operators $f_{C'} : P(U) \rightarrow \mathscr{F}(C')$ and $g_{C'} : \mathscr{F}(C') \rightarrow P(U)$ are defined respectively as follows:

$$f_{C'}(X)(a) = \bigwedge_{x \in X} \tilde{I}(x, a), \forall a \in C', \quad (3)$$

$$g_{C'}(\tilde{B}) = \{x \in U | \forall a \in C', \tilde{B}(a) \le \tilde{I}(x, a)\}. \quad (4)$$

For any $C' \subseteq C$, its characteristic function $\chi_{C'}$ is defined as $\chi_{C'}(a) = 1$ for $a \in C'$, otherwise is 0. Moreover, there are the following properties.

(1) $f_{C'}(X) = f(X) \cap \chi_{C'}$,

(2) $g \circ f(X) \subseteq g_{C'} \circ f_{C'}(X)$.

For the convenience of description, we will use $f(x)$ instead of $f(\{x\})$.

Table 1 A formal fuzzy context K

$\tilde{I}$	a	b	c	d	e
x_1	0.45	0.67	0.80	0.75	0.80
x_2	0.50	0.79	1.00	0.66	0.99
x_3	0.90	0.80	1.00	0.65	1.00
x_4	0.55	0.80	0.99	0.20	0.99
x_5	0.99	0.60	1.00	0.20	1.00

Table 2 A fuzzy sub-context $(U, C', \tilde{I}_{C'})$

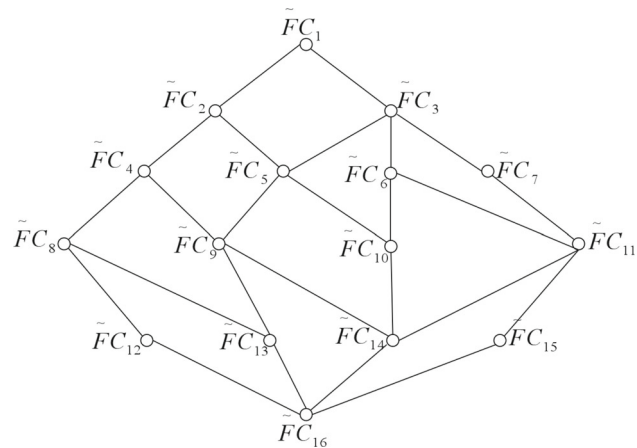| $\tilde{I}$	b	c	e
x_1	0.67	0.80	0.80
x_2	0.79	1.00	0.99
x_3	0.80	1.00	1.00
x_4	0.80	0.99	0.99
x_5	0.60	1.00	1.00

Example 1 Given an FFC $K = (U, C, \tilde{I})$ as depicted in Table 1. Let $C' = \{b, c, e\}$, then $(U, C', \tilde{I}_{C'})$ is a sub-context of K, and is represented by Table 2. For any $X \subseteq U$, it can be easily checked that $f_{C'}(X) = f(X) \cap \chi_{C'}$. All formal concepts induced by K are listed in Table 3. The Hasse diagram corresponding to the fuzzy concept lattice composed of all fuzzy concepts is shown in Fig. 2.

2.2 Formal fuzzy decision contexts

An FFDC is a quintuple set $S = (U, C, \tilde{I}, D, \tilde{J})$, where $(U, C, \tilde{I})$ and $(U, D, \tilde{J})$ are FFCs, $C \cap D = \emptyset$, C and D are conditional attribute set and decision attribute set, respectively. For a given FFDC $S = (U, C, \tilde{I}, D, \tilde{J})$, and $C' \subseteq C$, the operator $f_{C'}$ and $g_{C'}$ in fuzzy context $(U, C, \tilde{I})$ are defined by Eqs. (3) and (4). The corresponding opera-

Table 3 All one-sided fuzzy concepts derived from Table 1

	(objects, attributes)
$\tilde{F}C_1$	$(\{x_1, x_2, x_3, x_4, x_5\}, \{a^{0.45}, b^{0.60}, c^{0.80}, d^{0.20}, e^{0.80}\})$
$\tilde{F}C_2$	$(\{x_1, x_2, x_3, x_4\}, \{a^{0.45}, b^{0.67}, c^{0.80}, d^{0.20}, e^{0.80}\})$
$\tilde{F}C_3$	$(\{x_2, x_3, x_4, x_5\}, \{a^{0.50}, b^{0.60}, c^{0.99}, d^{0.20}, e^{0.99}\})$
$\tilde{F}C_4$	$(\{x_1, x_2, x_3\}, \{a^{0.45}, b^{0.67}, c^{0.80}, d^{0.65}, e^{0.80}\})$
$\tilde{F}C_5$	$(\{x_2, x_3, x_4\}, \{a^{0.50}, b^{0.79}, c^{0.99}, d^{0.20}, e^{0.99}\})$
$\tilde{F}C_6$	$(\{x_3, x_4, x_5\}, \{a^{0.55}, b^{0.60}, c^{0.99}, d^{0.20}, e^{0.99}\})$
$\tilde{F}C_7$	$(\{x_2, x_3, x_5\}, \{a^{0.50}, b^{0.60}, c^{1.00}, d^{0.20}, e^{0.99}\})$
$\tilde{F}C_8$	$(\{x_1, x_2\}, \{a^{0.45}, b^{0.67}, c^{0.80}, d^{0.66}, e^{0.80}\})$
$\tilde{F}C_9$	$(\{x_2, x_3\}, \{a^{0.50}, b^{0.79}, c^{1.00}, d^{0.65}, e^{0.99}\})$
$\tilde{F}C_{10}$	$(\{x_3, x_4\}, \{a^{0.55}, b^{0.80}, c^{0.99}, d^{0.20}, e^{0.99}\})$
$\tilde{F}C_{11}$	$(\{x_3, x_5\}, \{a^{0.90}, b^{0.60}, c^{1.00}, d^{0.20}, e^{1.00}\})$
$\tilde{F}C_{12}$	$(\{x_1\}, \{a^{0.45}, b^{0.67}, c^{0.80}, d^{0.75}, e^{0.80}\})$
$\tilde{F}C_{13}$	$(\{x_2\}, \{a^{0.50}, b^{0.79}, c^{1.00}, d^{0.66}, e^{0.99}\})$
$\tilde{F}C_{14}$	$(\{x_3\}, \{a^{0.90}, b^{0.80}, c^{1.00}, d^{0.65}, e^{1.00}\})$
$\tilde{F}C_{15}$	$(\{x_5\}, \{a^{0.99}, b^{0.60}, c^{1.00}, d^{0.20}, e^{1.00}\})$
$\tilde{F}C_{16}$	$(\emptyset, \{a^{1.0}, b^{1.0}, c^{1.0}, d^{1.0}, e^{1.0}\})$

Table 4 A formal fuzzy decision context S

$\tilde{I}$	a	b	c	d	e	d_1	d_2	d_3
x_1	0.45	0.67	0.80	0.75	0.80	0.67	0.62	0.80
x_2	0.50	0.79	1.00	0.66	0.99	0.70	0.80	0.90
x_3	0.90	0.80	1.00	0.65	1.00	0.91	0.80	0.99
x_4	0.55	0.80	0.99	0.20	0.99	0.75	0.55	0.90
x_5	0.99	0.60	1.00	0.20	1.00	0.75	0.54	1.00

Table 5 All one-sided fuzzy concepts of $\tilde{L}(U, D, \tilde{J})$

	(objects, attributes)
$\tilde{F}C_1$	$(\{x_1, x_2, x_3, x_4, x_5\}, \{d_1^{0.67}, d_2^{0.54}, d_3^{0.80}\}$
$\tilde{F}C_2$	$(\{x_1, x_2, x_3, x_4\}, \{d_1^{0.67}, d_2^{0.55}, d_3^{0.80}\})$
$\tilde{F}C_3$	$(\{x_2, x_3, x_4, x_5\}, \{d_1^{0.70}, d_2^{0.54}, d_3^{0.90}\})$
$\tilde{F}C_4$	$(\{x_1, x_2, x_3\}, \{d_1^{0.67}, d_2^{0.62}, d_3^{0.80}\})$
$\tilde{F}C_5$	$(\{x_2, x_3, x_4\}, \{d_1^{0.70}, d_2^{0.55}, d_3^{0.90}\})$
$\tilde{F}C_6$	$(\{x_3, x_4, x_5\}, \{d_1^{0.75}, d_2^{0.54}, d_3^{0.90}\})$
$\tilde{F}C_7$	$(\{x_2, x_3\}, \{d_1^{0.70}, d_2^{0.80}, d_3^{0.90}\})$
$\tilde{F}C_8$	$(\{x_3, x_4\}, \{d_1^{0.75}, d_2^{0.55}, d_3^{0.90}\})$
$\tilde{F}C_9$	$(\{x_3, x_5\}, \{d_1^{0.75}, d_2^{0.54}, d_3^{0.99}\})$
$\tilde{F}C_{10}$	$(\{x_3\}, \{d_1^{0.91}, d_2^{0.80}, d_3^{0.99}\})$
$\tilde{F}C_{11}$	$(\emptyset, \{d_1^{1.0}, d_2^{1.0}, d_3^{1.0}\})$

Fig. 2 The Hasse diagram of the fuzzy concept lattice $\tilde{L}(U, C, \tilde{I})$

Fig. 3 The Hasse diagram of the concept lattice $\tilde{L}(U, D, \tilde{J})$

tors in $(U, D, \tilde{J})$ are denoted as f_D and g_D so as to avoid confusion.

Example 2 For a given FFDC $S = (U, C, \tilde{I}, D, \tilde{J})$ as shown in Table 4, where, $U = \{x_1, x_2, x_3, x_4, x_5\}$, $C = \{a, b, c, d, e\}$ and $D = \{d_1, d_2, d_3\}$. Table 5 lists all one-sided fuzzy concepts of $\tilde{L}(U, D, \tilde{J})$ and the Hasse diagram of concept lattice $\tilde{L}(U, D, \tilde{J})$ is depicted as Fig. 3.

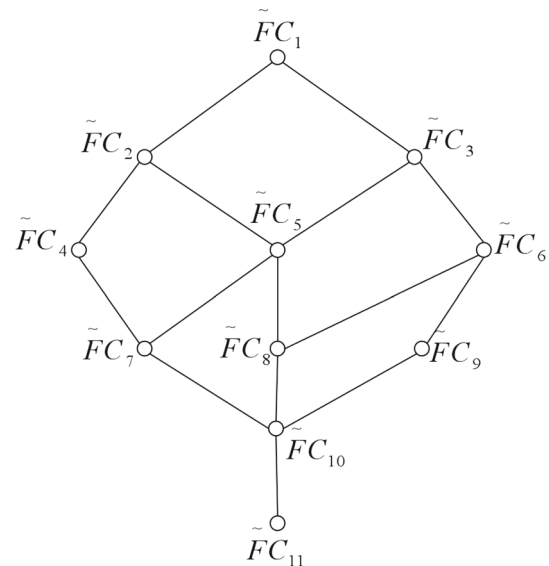## 2.3 Mapping and concepts with parameters in formal fuzzy contexts

Definition 2 Let $K = (U, C, \tilde{I})$ be an FFC. For $X \in P(U)$, $\tilde{B} \in \mathscr{F}(C)$, $\delta \in [0, 1]$, the operators $g_\delta : \mathscr{F}(C) \to P(U)$ and $g_{\bar{\delta}} : \mathscr{F}(C) \to P(U)$ are defined respectively as follows:

$$g_\delta(\tilde{B}) = \{x \in U | \forall a \in C, 0 \leq \tilde{I}(x, a) - \tilde{B}(a) \leq \delta\}\},$$
$$g_{\bar{\delta}}(\tilde{B}) = \{x \in U | \forall a \in C, \tilde{I}(x, a) - \tilde{B}(a) \geq 0 \,\&\, \exists b \in C,$$
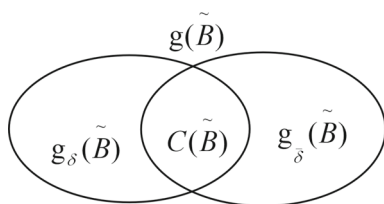$$\tilde{I}(x, b) - \tilde{B}(b) \geq \delta\}.$$

Fig. 4 Diagrammatic representation of the relationship between $g(\tilde{B})$, $g_\delta(\tilde{B})$, $g_{\bar{\delta}}(\tilde{B})$

From the above definition, it is easy to obtain the following properties.

Property 1 *Let $(U, C, \tilde{I})$ be an FFC and $C' \subseteq C$. Then, for all $X \subseteq U$,*

(1) $g_\delta \circ f(X) \subseteq g_{\delta, C'} \circ f_{C'}(X)$,
(2) $\delta_1 < \delta_2 \Rightarrow g_{\delta_1}(\tilde{B}) \subseteq g_{\delta_2}(\tilde{B})$,
(3) $\delta_1 < \delta_2 \Rightarrow g_{\bar{\delta_1}}(\tilde{B}) \supseteq g_{\bar{\delta_2}}(\tilde{B})$,
(4) $\tilde{B}_1 \subseteq \tilde{B}_2 \Rightarrow g_\delta(\tilde{B}_2) \cup g_{\bar{\delta}}(\tilde{B}_2) \subseteq g_\delta(\tilde{B}_1) \cup g_{\bar{\delta}}(\tilde{B}_1)$,
(5) $X \subseteq g_\delta \circ f(X) \cup g_{\bar{\delta}} \circ f(X)$, $\tilde{B} \subseteq f \circ g_\delta(\tilde{B}) \cup f \circ g_{\bar{\delta}}(\tilde{B})$,
(6) $g(\bigcup_{i \in J} \tilde{B}_i) = \bigcap_{i \in J}(g_\delta(\tilde{B}_i) \cup g_{\bar{\delta}}(\tilde{B}_i))$.

Remark 1 From Definitions 1 and 2, we have that $g(\tilde{B}) = g_\delta(\tilde{B}) \cup g_{\bar{\delta}}(\tilde{B})$, where $\delta_1, \delta_2 \in [0, 1]$, $\tilde{B}_1, \tilde{B}_2, \tilde{B}_i \subseteq \mathscr{F}(C)$ and J is the index set. Their relationship is shown in Fig. 4, where the intersecting part $C(\tilde{B}) = \{x \in U| \tilde{B}(a) + \delta = \tilde{I}(x, a), \forall a \in C\}$. Furthermore, g_δ degenerates into Eq. (2) when $\delta = 1$. It is worth noting that $(g_\delta \circ f(X), f(X))$ and $(g_{\bar{\delta}}(\tilde{(B)}), f \circ g_\delta(\tilde{B}))$ are not necessarily one-sided fuzzy concepts. However, $(g_\delta \circ f(x), f(x))$ must be a one-sided fuzzy concept for $x \in U$, and is called an δ-object concept.

Example 3 (Continue from Example 1) When taking $\delta = 0.1$ and $X = \{x_1, x_2\}$ and according to Definitions 1 and 2, we have:

$f(X) = a^{0.45}b^{0.67}c^{0.80}d^{0.66}e^{0.80} = \tilde{B}$,
$g_{0.1}(\tilde{B}) = \{x_1\}$,
$f \circ (g_{0.1} \circ f(X)) = f(x_1) = \{a^{0.45}b^{0.67}c^{0.80}d^{0.75}e^{0.80}\} \neq f(X)$.

Thus, it is known that $(g_\delta \circ f(X), f(X))$ is not necessarily a one-sided fuzzy concept.

2.4 A novel partial order relation

Next, we define a partial order relation between object concepts.

Definition 3 Let $\tilde{F}C_1 = \{X_1, \tilde{B}_1\}$ and $\tilde{F}C_2 = \{X_2, \tilde{B}_2\}$ be two one-sided fuzzy object concepts generated by an FFC $K = (U, C, \tilde{I})$. Then $\tilde{F}C_1 \overset{\delta}{\preceq} \tilde{F}C_2$, if $\tilde{F}C_1$ and $\tilde{F}C_2$ satisfy the following properties:
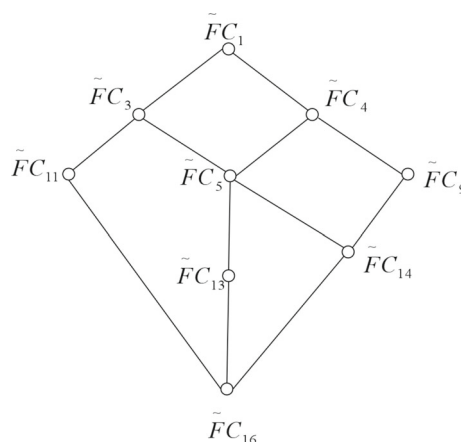

Fig. 5 The Hasse diagram of the concept lattice $L^{0.1}(U, AT, \tilde{I})$

(1) $X_1 \supseteq X_2$,
(2) For all $a \in C$, there are $\tilde{B}_1(a) \leq \tilde{B}_2(a)$,
(3) There exists $b \in C$, such that $\tilde{B}_1(b) + \delta \leq \tilde{B}_2(b)$.

Remark 2 It is easy to known that $\overset{\delta}{\preceq}$ is the hierarchical partial order of fuzzy concept lattices. According to the partial order relationship, all one-sided fuzzy concepts can generate a lattice, noted as $\tilde{L}^\delta(U, C, \tilde{I})$. When $\delta = 0$, L^δ degenerates to $\tilde{L}(U, C, \tilde{I})$.

Example 4 (Continued from Example 1). When taking $\delta = 0.1$, all one-sided fuzzy concepts are shown in Table 3. Let's take the first concept $\tilde{F}C_1 = (\{x_1, x_2, x_3, x_4, x_5\}, \{a^{0.45}, b^{0.60}, c^{0.80}, d^{0.20}, e^{0.80}\})$. Then, the following concept is found based on the partial order relationship defined in Definition 3. Finally, we can obtain a concept lattice $L^{0.1}$ as shown in Fig. 5.

Let $K = (U, C, \tilde{I})$ be an FFC and $C' \subset C$. A binary relation $R_{C'}^\delta$ is defined by

$$R_{C'}^\delta = \{(x, y) \in U \times U | 0 \leq \tilde{I}(y, a) - \tilde{I}(x, a) \leq \delta, \forall a \in C'\}. \tag{5}$$

$R_{C'}^\delta$ is called an ordered relation on the object set, where $(x, y) \in R_{C'}^\delta$ means y is not less than x and y will not be greater than x by more than δ with respect to all attributes in C'. It is evident to obtain that

(1) $R_{C'}^\delta$ is reflexive, transitive, and asymmetric,
(2) $R_{C'}^\delta = \bigcap_{a \in C'} R_{\{a\}}^\delta$,
(3) if $C_1 \subseteq C_2 \subseteq C_3$, then $R_{C_1}^\delta \supseteq R_{C_2}^\delta \supseteq R_{C_3}^\delta$,
(4) if $C_1, C_2 \subseteq C$, then $R_{C_1 \cup C_2}^\delta = R_{C_1}^\delta \cap R_{C_2}^\delta$,
(5) if $C_1 \subseteq C_2 \subseteq C_3$, then $[x]_{R_{C_1}^\delta} \supseteq [x]_{R_{C_2}^\delta} \supseteq [x]_{R_{C_3}^\delta}$,
(6) $[x]_{R_{C'}^\delta} = [y]_{R_{C'}^\delta}$ if and only if $\tilde{I}(x, a) = \tilde{I}(y, a), \forall a \in C'$,

(7) $\tau = \{[x]_{R_{C'}^{\delta}} \mid x \in U\}$ forms a covering of U.

For any $x \in U$, its granule of knowledge induced by the ordered relation $R_{C'}^{\delta}$ is

$$[x]_{R_{C'}^{\delta}} = \{y \in U \mid (x, y) \in R_{C'}^{\delta}\}$$
$$= \{y \in U \mid 0 \le \tilde{I}(y, a) - \tilde{I}(x, a) \le \delta, \forall\, a \in C'\},$$

where $[x]_{R_{C'}^{\delta}}$ is the set of objects that are larger than x, but whose larger value does not exceed δ with respect to all attributes in C'.

Remark 3 When $\delta = 0$, the object granule of x contains only itself, when the granule is the finest. As δ increases, the object granule contains more information. When $\delta = 1$, the object granule degenerates to the object granule in Shao et al. (2016), at which point the granule is the coarsest.

Lemma 1 *Let $K = (U, C, \tilde{I})$ be an FFC, $C' \subset C$ and $x \in U$. Then,*

$$f_{C'}([x]_{R_{C'}^{\delta}})(a) = f_{C'}(x)(a), \forall\, a \in C'.$$

Proof By Eq. (3), we can obtain that

$$f_{C'}([x]_{R_{C'}^{\delta}})(a) = \bigwedge_{x \in [x]_{R_{C'}^{\delta}}} \tilde{I}(x, a)$$
$$= \tilde{I}(x, a)$$
$$= f_{C'}(x)(a).$$

$\square$

Theorem 1 *Let $K = (U, C, \tilde{I})$ be an FFC, $C' \subset C$ and $x \in U$. Then, $([x]_{R_{C'}^{\delta}}, f_{C'}([x]_{R_{C'}^{\delta}}))$ is a one-sided fuzzy concept of $(U, C', \tilde{I}_{C'})$ and $[x]_{R_{C'}^{\delta}} = g_{\delta, C'} \circ f_{C'}(x)$.*

Proof For any $z \in [x]_{R_{C'}^{\delta}}$, there is

$$z \in \{y \mid \forall a \in C', 0 \le \tilde{I}(y, a) - \tilde{I}(x, a) \le \delta\}.$$

By the definition of g_{δ}, it follows that $z \in g_{\delta, C'}(\tilde{I}(x, a))$. It also follows from Lemma 1 that

$$g_{\delta, C'}(\tilde{I}(x, a)) = g_{\delta, C'} \circ f_{C'}(x)(a).$$

Thus,

$$[x]_{R_{C'}^{\delta}} \subseteq g_{\delta, C'} \circ f_{C'}(x).$$

For any $z \in g_{\delta, C'} \circ f_{C'}(x)$, by Lemma 1, we have $g_{\delta, C'} \circ f_{C'}(x) = g_{\delta, C'} \circ (\tilde{I}(x, a)), \forall\, a \in C'$. Again, based on the definition of g_{δ}, it is known that

$$g_{\delta, C'} \circ (\tilde{I}(x, a)) = \{y \mid \forall a \in C', 0 \le \tilde{I}(y, a) - \tilde{I}(x, a) \le \delta\}$$
$$= [x]_{R_{C'}^{\delta}}.$$

Therefore, $[x]_{R_{C'}^{\delta}} = g_{\delta, C'} \circ f_{C'}(x)$, and $([x]_{R_{C'}^{\delta}}, f_{C'}([x]_{R_{C'}^{\delta}}))$ is a one-sided fuzzy concept. $\square$

2.5 Hypergraph and its minimal transversal

A hypergraph is a class of graph structures in which a hyperedge can contain any number of vertices, and is a generalisation of the traditional graph, generally denoted by $H = (V, E)$. Where $V = \{v_1, v_2, \ldots, v_m\}$ denotes the set of vertices and $E = \{e_1, e_2, \ldots, e_n\}$ denotes the set of hyperedges. If there is only one vertex in a hyperedge of a hypergraph, the hyperedge is called a loop, and if there are two or more hyperedges with identical set of vertices in the hypergraph, we say that these hyperedges are multiple hyperedges. In addition, a hypergraph can also be expressed as a incidence matrix $M = (M(ij))\ i \in \{1, 2, \ldots, n\}$ $j \in \{1, 2, \ldots, m\}$, where the i-th row represents the hyperedge e_i and the jth column represents the vertex v_j, and $M(ij) = 1$ if $v_j \in e_i$, otherwise, $M(ij) = 0$.

A transversal Bondy and Murty (2008) T of a hypergraph H refers to the set of vertices that can cover all the hyperedges in H. If there is no subset of the transversal T that is still a transversal, then the T is called the minimal transversal of H. The set of all minimal transversals of H is denoted by $\mathcal{T}(H)$.

We use $N_H(e) = \{v : v \in e\}$ to denote the set of all vertices associated with the hyperedge e. Denote $\mathcal{N}_H = \{N_H(e) : e \in E\}$. It has been verified in Eiter and Gottlob (1995) and Listrovoy and Minukhin (2012) that all minimal transversals of a hypergraph can be obtained by a Boolean formula. Like the approach in the Chen et al. (2020), we define a Boolean function f_H of a hypergraph H as follows, which is a function on vertices $v_1, v_2, \ldots, v_n$ of n Boolean variables $v_1', v_2', \ldots, v_n'$.

$$f_H(v_1', v_2', \ldots, v_n') = \wedge \{\vee N_H(e) : N_H(e) \in \mathcal{N}_H\},$$

where $\vee N_H(e)$ is the disjunction of all Boolean variables v' satisfying $v \in N_H(e)$.

The following lemma describes a method for calculating all minimal transversals of a hypergraph.

Lemma 2 (Eiter and Gottlob 1995; Listrovoy and Minukhin 2012) *Let $H = (V, E)$ be a hypergraph. Then a subset $V' \subseteq V$ is a minimal transversal of H if and only if $\bigwedge_{v_i \in V'} \bar{v}_i$ is a prime implicant of the Boolean function f_H.*

Table 6 The incidence matrix of H

	v_1	v_2	v_3	v_4	v_5	v_6
e_1	1	1	0	0	0	0
e_2	1	1	0	1	0	0
e_3	0	1	1	0	1	0
e_4	0	0	1	1	0	1
e_5	0	0	0	0	0	1

From Lemma 2, it is clear that if

$$f_H(v_1', v_2', \cdots, v_n') = \wedge\{\vee N_H(e) : N_H(e) \in \mathcal{N}_H\}$$
$$= \overset{t}{\underset{i=1}{\vee}} (\overset{s_i}{\underset{j=1}{\wedge}} \bar{v}_j),$$

where $\overset{s_i}{\underset{j=1}{\wedge}} \bar{v}_j, i \leq t$ are all the prime implicants of the Boolean function f_H, then $I_i = \{v_j : j \leq s_i\}, i \leq t$, are all the minimal transversal of H. We will also write v_i instead of $\bar{v}_i$ in the discussion to follow.

Example 5 Let $H = (V, E)$ be a hypergraph with $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E = \{e_1, e_2, e_3, e_4, e_5\} = \{\{v_1, v_2\}, \{v_1, v_2, v_4\}, \{v_2, v_3, v_5\}, \{v_3, v_4, v_6\}, \{v_6\}\}$. The incidence matrix of H shown in Table 6. Furthermore, after simplification, we obtain the Boolean function f_H in prime implicants as: $f_H(v_1, v_2, v_3, v_4, v_5, v_6) = (v_1 \vee v_2) \wedge (v_1 \vee v_2 \vee v_4) \wedge (v_2 \vee v_3 \vee v_5) \wedge (v_3 \vee v_4 \vee v_6) \wedge (v_6) = (v_2 \wedge v_6) \vee (v_1 \wedge v_3 \wedge v_6) \vee (v_1 \wedge v_5 \wedge v_6)$. Therefore, $\mathcal{T}(H) = \{\{v_2, v_6\}, \{v_1, v_3, v_6\}, \{v_1, v_5, v_6\}\}$.

3 δ-granular reduct in formal fuzzy contexts

In this section, we first provide the definition of δ-granular consistent set and δ-granular reduction and introduce two different reduction methods.

3.1 Boolean reasoning for δ-granular reduct in formal fuzzy contexts

Definition 4 Let $K = (U, C, \tilde{I})$ be an FFC. An attribute subset $C' \subseteq C$ is referred to as a δ-granular consistent set of K if $g_{\delta,C'} \circ f_{C'}(x) = g_\delta \circ f(x)$ for all $x \in U$. Furthermore, if there is no proper subset $C \subset C'$ such that C is a δ-granular consistent set, then C' is referred to as a δ-granular reduct of K. The intersection of all δ-granular reducts of K is called the δ-granular core of K.

Definition 4 implies that a δ-granular reduct is a minimal attribute set that retains object δ-granular $[x]_{R_{C'}^\delta}$ in a fuzzy concept lattice.

Theorem 2 *Let $K = (U, C, \tilde{I})$ be an FFC and $C' \subseteq C$. Then C' is a δ-granular consistent set of K if and only if*

$$g_{\delta,C'} \circ f_{C'}(x) \subseteq g^\delta \circ f(x), \forall x \in U. \tag{6}$$

Proof By Property 1 (1), we know that $g_\delta \circ f(x) \subseteq g_{\delta,C'} \circ f_{C'}(x), \forall x \in U$. Hence, we conclude that C' is a δ-granular consistent set if and only if Eq.(6) holds. □

We denote the set of all δ-granular reducts of K as $\mathcal{R}(K^\delta)$. According to the significance of the attributes, based on δ-granular reducts, the attribute set C is divided into three parts:

(1) δ-indispensable attribute (core attribute) set Co_{K^δ} : $Co_{K^\delta} = \bigcap \mathcal{R}(K^\delta)$,

(2) δ-relatively necessary attribute set N_{K^δ} : $N_{K^\delta} = \bigcup \mathcal{R}(K^\delta) - \bigcap \mathcal{R}(K^\delta)$,

(3) δ-unnecessary attribute set $I_{K^\delta} : I_{K^\delta} = C - \bigcup \mathcal{R}(K^\delta)$.

Corollary 1 *Let $K = (U, C, \tilde{I})$ be an FFC and $a \in C$. Then a is an δ-indispensable attribute if and only if there exists $x \in U$ such that $g_{\delta,C-\{a\}} \circ f_{C-\{a\}}(x) \nsubseteq g_\delta \circ f(x)$.*

Proof It can easily be proved from Theorem 1 and the definition of δ-indispensable attribute.

Corollary 1 provides a way to determine whether an attribute is δ-indispensable attribute.

To better represent the degree of difference in the membership of different objects to the same attribute, we introduce the concept of δ-granular discernibility attribute set. □

Definition 5 Let $K = (U, C, \tilde{I})$ be an FFC and $(x, y) \in U \times U$. We define

$$D_K^\delta(x, y) = \begin{cases} \{a \in C| \ \tilde{I}(x, a) > \tilde{I}(y, a)||\tilde{I}(x, a) \\ +\delta < \tilde{I}(y, a), \exists a \in C)\}, \\ \emptyset, \quad \text{others,} \end{cases}$$

where $D_K^\delta(x, y)$ is referred to as the $\delta-$granular discernibility attribute set of x and y, which means that either x is greater than the value of y on some attribute, or the value of some attribute of x plus δ is still less than the value of the attribute of y. And $\mathcal{D}_K^\delta = (D_K^\delta(x, y) : (x, y) \in U \times U)$ is called the $\delta-$granular discernibility matrix of K. We denote $[D]_K^\delta = \{D_K^\delta(x, y) : D_K^\delta(x, y) \neq \emptyset, (x, y) \in U \times U\}$.

Remark 4 It is important to note here that when $\delta = 1$, $D_K^\delta(x, y)$ degenerates to the discernibility attribute set of x and y, $\mathcal{D}_K^\delta$ degenerates to the discernibility matrix, and $[D]_K^\delta$ degenerates to M_0 in Shao et al. (2016).

And, $D_K^\delta(x, y)$ has the following properties.

Theorem 3 *Let $K = (U, C, \tilde{I})$ be an FFC, for any $x, y \in U$, $\delta_1, \delta_2 \in [0, 1]$, if $\delta_1 \geq \delta_2$, then $D_K^{\delta_1}(x, y) \subseteq D_K^{\delta_2}(x, y)$.*

Table 7 The fuzzy discernibility matrix $D_K^{0.05}$

$D_K^{0.05}(x_i, x_j)$	x_1	x_2	x_3	x_4	x_5
x_1	$\emptyset$	$bcde$	$abcde$	$abcde$	$abcde$
x_2	$abcde$	$\emptyset$	ad	cd	abd
x_3	$abcde$	abe	$\emptyset$	$acde$	abd
x_4	$abcde$	abd	ad	$\emptyset$	ab
x_5	$abcde$	$abde$	abd	$abce$	$\emptyset$

Table 8 The fuzzy discernibility matrix $D_K^{0.2}$

$D_K^{0.2}(x_i, x_j)$	x_1	x_2	x_3	x_4	x_5
x_1	$\emptyset$	d	ad	d	abd
x_2	$abce$	$\emptyset$	ad	cd	abd
x_3	$abce$	abe	$\emptyset$	$acde$	bd
x_4	$abcde$	abd	ad	$\emptyset$	ab
x_5	$acde$	ade	ad	ace	$\emptyset$

Table 9 The fuzzy discernibility matrix $D_K^{0.5}$

$D_K^{0.5}(x_i, x_j)$	x_1	x_2	x_3	x_4	x_5
x_1	$\emptyset$	d	d	d	abd
x_2	$abce$	$\emptyset$	d	cd	bd
x_3	$abce$	abe	$\emptyset$	$acde$	bd
x_4	$abcde$	ab	$\emptyset$	$\emptyset$	b
x_5	$acde$	ae	a	ace	$\emptyset$

Proof If $a \in D_K^{\delta_1}(x, y)$ and $\delta_1 \geq \delta_2$, we have $\tilde{I}(x, a) - \tilde{I}(y, a) > \delta_1 \geq \delta_2$ or $\tilde{I}(y, a) - \tilde{I}(x, a) > \delta_1 \geq \delta_2$. By Definition 5, we have $a \in D_K^{\delta_2}(x, y)$, i.e. we have proved that $D_K^{\delta_1}(x, y) \subseteq D_K^{\delta_2}(x, y)$. □

Example 6 (Continued from Example 1) The Tables 7, 8 and 9 present, respectively, the discernibility matrix of K of Example 1 when $\delta = 0.05, 0.2, 0.5$.

It is easy to know that $D_K^{0.05}(x, y) \supseteq D_K^{0.2}(x, y) \supseteq D_K^{0.5}(x, y)$.

To be able to better determine whether $C' \subseteq C$ is a δ-granular consistent set, we introduce the necessary and sufficient condition for δ-granular consistent set.

Theorem 4 Let $K = (U, C, \tilde{I})$ be an FFC and $C' \subseteq C$. Then, C' is a δ-granular consistent set if and only if $C' \cap D_K^{\delta}(x, y) \neq \emptyset$ for all $D_K^{\delta}(x, y) \in [D]_K^{\delta}$.

Proof ($\Rightarrow$) Let C' be a δ-granular consistent set. By Theorem 2, we have $g_{C'}^{\delta} \circ f_{C'}(x) \subseteq g^{\delta} \circ f(x)$ for all $x \in U$. Using Theorem 1 we obtain

$$[x]_{R_{C'}^{\delta}} \subseteq [x]_{R_C^{\delta}}, \forall x \in U. \tag{7}$$

For any $D_K^{\delta}(x, y) \in [D]_K^{\delta}$ with $D_K^{\delta}(x, y) \neq \emptyset$, there exists $a \in C$ such that $\tilde{I}(x, a) > \tilde{I}(y, a)$ or $\tilde{I}(y, a) - \tilde{I}(x, a) > \delta$. Hence, $y \notin [x]_{R_C^{\delta}}$. By Eq. (7), we have $y \notin [x]_{R_{C'}^{\delta}}$. Thus, there exists $c \in C'$ such that $\tilde{I}(x, c) > \tilde{I}(y, c)$ or $\tilde{I}(y, c) - \tilde{I}(x, c) > \delta$. By Definition 5, we conclude that $c \in D_K^{\delta}(x, y)$. Hence, $C' \cap D_K^{\delta}(x, y) \neq \emptyset$.

($\Leftarrow$) Suppose that $C' \cap D_K^{\delta}(x, y) \neq \emptyset$ for all $D_K^{\delta}(x, y) \in [D]_K^{\delta}$ with $D_K^{\delta}(x, y) \neq \emptyset$. For any $x, y \in U$, if $y \notin [x]_{R_C^{\delta}}$, i.e. there exists $a \in C$ such that $\tilde{I}(x, a) > \tilde{I}(y, a)$ or $\tilde{I}(y, a) - \tilde{I}(x, a) > \delta$, we have $D_K^{\delta}(x, y) \neq 0$, hence $C' \cap D_K^{\delta}(x, y) \neq \emptyset$. Thus, there exists $c \in C'$ such that $\tilde{I}(x, c) > \tilde{I}(y, c)$ or $\tilde{I}(y, c) - \tilde{I}(x, c) > \delta$, which means $y \notin [x]_{R_{C'}^{\delta}}$, and we conclude that $[x]_{R_{C'}^{\delta}} \subseteq [x]_{R_C^{\delta}}$. Since $[x]_{R_{C'}} = g_{\delta, C'} \circ f_{C'}(x)$ and $[x]_{R_C^{\delta}} = g_{\delta} \circ f(x)$, it follows that $g_{\delta, C'} \circ f_{C'}(x) \subseteq g_{\delta} \circ f(x)$. By Theorem 2, we obtain that C' is a δ-granular consistent set of K. □

By employing the δ-granular discernibility matrix, we obtain the following judgment theorem of δ-indispensable attribute (δ-core attribute).

Theorem 5 Let $K = (U, C, \tilde{I})$ be an FFC and $a \in C$. Then, a is δ-indispensable attribute (δ-core attribute) in K if and only if there exists $(x, y) \in U \times U$ such that $D_K^{\delta}(x, y) = \{a\}$.

Proof ($\Rightarrow$) Assume that a is an δ-indispensable attribute in K, then $C - \{a\}$ is not a δ-granular consistent set of K. By Corollary 1 we have $g_{\delta, C-\{a\}} \circ f_{C-\{a\}}(x) \nsubseteq g_{\delta} \circ f(x)$, i.e. $[x]_{R_{C-\{a\}}^{\delta}} \nsubseteq [x]_{R_C^{\delta}}$. Thus, $[x]_{R_C^{\delta}} \subseteq [x]_{R_{C-\{a\}}^{\delta}}$. Hence, there exists $y \in U$ such that $y \in [x]_{R_{C-\{a\}}^{\delta}}$ and $y \notin [x]_{R_C^{\delta}}$, which means $0 \leq \tilde{I}(y, b) - \tilde{I}(x, b) \leq \delta$ ($\forall b \in C - \{a\}$) and $\tilde{I}(x, a) > \tilde{I}(y, a)$ or $\tilde{I}(x, a) + \delta < \tilde{I}(y, a)$. Therefore, by Definition 5, we conclude that $D_K^{\delta}(x, y) = \{a\}$.

($\Leftarrow$) If there exists $(x, y) \in U \times U$ such that $D_K^{\delta}(x, y) = \{a\}$, then, by Definition 5, we obtain $0 \leq \tilde{I}(y, b) - \tilde{I}(x, b) \leq \delta$ ($\forall b \in C - \{a\}$) and $\tilde{I}(x, a) > \tilde{I}(y, a)$ or $\tilde{I}(x, a) + \delta < \tilde{I}(y, a)$. Thus, $[x]_{R_{C-\{a\}}^{\delta}} \nsubseteq [x]_{R_C^{\delta}}$, that is, $g_{\delta, C-\{a\}} \circ f_{C-\{a\}}(x) \nsubseteq g_{\delta} \circ f(x)$. Hence, $a \in \bigcap \mathcal{R}(K^{\delta})$. Therefore, a is an δ−indispensable attribute in K. □

Skowron and Rauszer (1992) and Skowron (1993) has shown that the computation of a reduct in rough set theory can be transformed into the computation of a Boolean function. In particular, Wu et al. (2009) completed the computation of the granular reduction in the classical formal context by computing the prime implicants of a Boolean function. Further, Shao et al. (2016) used a similar approach for the granular reduction of the FFCs. Now, we use the same method to compute the δ-granular reduction in FFCs.

Definition 6 Let $K = (U, C, \tilde{I})$ be an FFC and $(x, y) \in U \times U$. A δ-granular discernibility function f_K^{δ} for K is

a Boolean function of m Boolean variables $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_m$ corresponding to the attributes $a_1, a_2, \ldots, a_m$ respectively, and it is defined as:

$$f_{K^\delta}(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_m) = \wedge\{\vee D^\delta(x, y) : D_K^\delta(x, y) \in \mathcal{D}_K^\delta\},$$

where $\vee D_K^\delta(x, y)$ is the disjunction of all variables a such that $a \in D_K^\delta(x, y)$ and $D_K^\delta(x, y) \in [D]_K^\delta$. Each conjunctor of the reduced disjunctive form is referred to as a prime implicant.

Theorem 6 *Let $K = (U, C, \tilde{I})$ be an FFC and $C' \subseteq C$. Then, C' is a δ-granular reduct of K if and only if $\underset{a \in C'}{\wedge} a$ is a prime implicant of the δ-discernibility function f_{K^δ}.*

Proof $(\Rightarrow)$ Let $C' \subseteq C$ be a $\delta-$granular reduct of K. By Theorem 4, we have $C' \cap D_K^\delta(x, y) \neq \emptyset$, for all $D_K^\delta(x, y) \in [D]_K^\delta$. Then, there exists $D_K^\delta(x, y) \in [D]_K^\delta$ such that $C' \cap D_K^\delta(x, y) = \{a\}$ for $a \in C'$. It follows that $\underset{a \in C'}{\wedge} a$ is a prime implicant of the δ-granular discernibility function.

$(\Leftarrow)$ Suppose $\underset{a \in C'}{\wedge} a$ is a prime implicant of the δ-granular discernibility function f_K^δ. Then $C' \cap D_K^\delta(x, y) \neq \emptyset$, for all $D_K^\delta(x, y) \in [D]_K^\delta$. And, there must exist $D_K^\delta(x, y) \in [D]_K^\delta$ such that $(C' - \{a\}) \cap D_K^\delta(x, y) = \emptyset$. Therefore, we conclude that C' is a δ-granular reduct of K. $\square$

Let

$$f_{K^\delta} = \underset{(x, y) \in U \times U}{\wedge} \vee D_K^\delta(x, y) = \overset{t}{\underset{k=1}{\vee}} \left(\overset{q_k}{\underset{s=1}{\wedge}} a_s \right),$$

where $\wedge_{s=1}^{q_k} a_s, k \leq t$ are all the prime implicants of the δ-granular discernibility function f_{K^δ}. We denote $N_k = \{a_s | s = 1, 2, \ldots, q_k\}$. Then, $\{N_k | k = 1, 2, \ldots, t\}$ is the set of all δ-granular reducts of K.

δ-granular discernibility functions are monotonic Boolean functions and their prime implications uniquely determine all the δ-granular reducts of FFCs.

Example 7 (Continued from Example 6) From Tables 7, 8 and 9, using δ-granular discernibility function we have

$$f_{K^{0.05}} = \underset{(x,y) \in U \times U}{\wedge} \vee D_K^{0.05}(x, y)$$
$$= (a \vee b \vee c \vee d \vee e) \wedge (b \vee c \vee d \vee e)$$
$$\wedge (a \vee b \vee e) \wedge (a \vee b \vee d) \wedge (a \vee b \vee d \vee e)$$
$$\wedge (a \vee d) \wedge (c \vee d \vee e) \wedge (a \vee c \vee d)$$
$$\wedge (a \vee b \vee c \vee e) \wedge (a \vee b)$$
$$= (a \wedge c) \vee (a \wedge d) \vee (a \wedge e) \vee (b \wedge d).$$

$$f_{K^{0.2}} = \underset{(x,y) \in U \times U}{\wedge} \vee D_K^{0.2}(x, y)$$
$$= d \wedge (a \vee b) \wedge (a \vee d) \wedge (b \vee d)$$

$$\wedge (c \vee d) \wedge (a \vee b \vee d) \wedge (a \vee b \vee e)$$
$$\wedge (a \vee c \vee e) \wedge (a \vee d \vee e) \wedge (a \vee b \vee c \vee e)$$
$$\wedge (a \vee c \vee d \vee e) \wedge (a \vee b \vee c \vee d \vee e)$$
$$= (a \wedge d) \vee (b \wedge c \wedge d) \vee (b \wedge d \wedge e).$$

$$f_{K^{0.5}} = \underset{(x,y) \in U \times U}{\wedge} \vee D_K^{0.5}(x, y)$$
$$= a \wedge b \wedge d \wedge (a \vee b) \wedge (a \vee e) \wedge (b \vee d)$$
$$\wedge (c \vee d) \wedge (a \vee b \vee d) \wedge (a \vee b \vee e) \wedge (a \vee c \vee e)$$
$$\wedge (a \vee b \vee c \vee e) \wedge (a \vee c \vee d \vee e)$$
$$\wedge (a \vee b \vee c \vee d \vee e)$$
$$= a \wedge b \wedge d.$$

This implies that the FFC K has four 0.05-granular reductions: $\mathcal{R}_K^{0.05} = \{a, c\}, \{a, d\}, \{a, e\}, \{b, d\}$, and has no 0.05-granular core. Also, $\{a, c\}, \{a, d\}, \{a, e\}, \{b, d\}$ is 0.05-granular consistent sets of K, and any proper subsets of them is not 0.05-granular consistent sets. By the reduction, we have the 0.05-reduced FFCs $(U, \{a, c\}, \tilde{I}_{\{a,c\}}), (U, \{a, d\}, \tilde{I}_{\{a,d\}}), (U, \{a, e\}, \tilde{I}_{\{a,e\}})(U, \{b, d\}, \tilde{I}_{\{b,d\}})$. We obtain the same number of 0.05-object concepts from the reduced FFCs $(U, \{a, c\}, \tilde{I}_{\{a,c\}}), (U, \{a, d\}, \tilde{I}_{\{a,d\}}), (U, \{a, e\}, \tilde{I}_{\{a,e\}}), (U, \{b, d\}, \tilde{I}_{\{b,d\}})$. Similarly, K has three 0.2-granular reducts: $\mathcal{R}_K^{0.2} = \{a, d\}, \{b, c, d\}, \{b, d, e\}$. And K has a unique 0.2-granular core attribute $\{d\}$. K has a unique 0.5-granular reduct: $\{a, b, d\}$, and three 0.5-granular core $\{a\}, \{b\}, \{d\}$.

With Definition 5, we have designed Algorithm 1 to calculate the δ-granular discernibility matrix. As can be seen in Algorithm 1, $D_K^\delta(x_i, x_j)$ corresponds to the $(i - 1)m + j$-th row in $\mathcal{D}_K^\delta$. By Theorem 6, the procedure for computing $\delta-$granular reducts is given in Algorithm 2. Steps 4–5 of the Algorithm 2 use the idea of an incremental algorithm. The time complexity of Algorithm 1 is $O(|U|^2)$. The time complexity of steps 2–6 of Algorithm 2 is $O(|U|^2|C|^3)$. Therefore, the time complexity of the total algorithm is $O(|U|^2|C|^3)$.

Algorithm 1 Calculate $\mathcal{D}_K^\delta$

Require: An FFC $K = (U, C, \tilde{I})$ where $U = \{x_1, x_2, \ldots, x_{|U|}\}$ and $C = \{a_1, a_2, \ldots, a_{|C|}\}$
Ensure: $\mathcal{D}_K^\delta$// the δ-granular discernibility matrix of K
1: Let $\mathcal{D}_K^\delta = \emptyset$
2: **for** $i = 1$ to $|U|$ **do**
3: Generate a matrix $B_{|U||C|}$ obtained by 1-by-$|U|$ tiling of copies of the i-th row $K(i)$ of K
4: Let $E = B - K$
5: Numbers greater than 0 and less than $-\delta$ in E are replaced by 1 and all other numbers are replaced by 0
6: $\mathcal{D}_K^\delta \leftarrow [\mathcal{D}_K^\delta; E]$
7: **end for**
8: Return $\mathcal{D}_K^\delta$

Algorithm 2 Calculate all δ-granular reducts of K (BRK))

Require: An FFC $K = (U, C, \tilde{I})$ where $U = \{x_1, x_2, \ldots, x_{|U|}\}$ and $C = \{a_1, a_2, \ldots, a_{|C|}\}$
Ensure: $\mathcal{R}(K^\delta)$ // the set of δ−granular reducts of K
1: Calculate the fuzzy granular discernibility matrix $\mathcal{D}_K^\delta$ using Algorithm 1
2: Delete all full 0 rows and duplicate rows in $\mathcal{D}_K^\delta$
3: Consider each row in $\mathcal{D}_K^\delta$ as a $D_K^\delta(x, y)$ and store it in a new empty cell array $\mathcal{R}(K^\delta)$
4: Calculate $f_{K^\delta} = \underset{(x,y) \in U \times U}{\wedge} \vee D_K^\delta(x, y)$
5: Calculate $f_{K^\delta} = \overset{t}{\underset{i=1}{\vee}} (\overset{s_i}{\underset{j=1}{\wedge}} a_j)$
6: Let $N_k = \{a_j | j \le s_i\}$ and $\mathcal{R}^\delta(K) = \{N_k | \ k \le t\}$
7: Return $\mathcal{R}(K^\delta)$

3.2 Graph represent for δ-granular reduct in formal fuzzy contexts

Definition 7 Let $K = (U, C, \tilde{I})$ be an FFC, D_K^δ be the δ-granular discernibility matrix of K. Denote $V = C$ and $E = \{e \in D_K^\delta | e \ne \emptyset\}$. We call the pair (V, E) an δ-induced graph from K, denoted by $H_K^\delta = (V, E)$.

In fact, a δ-granular discernibility attribute set as a crisp hyperedge. Also, the granular δ-discernibility matrix can view as a crisp hypergraph.

Example 8 (Continued from Example 6) When taking $δ = 0.5$, we can obtain a hypergraph $H_K^{0.5}$, which has the matrix expression in Table 10.

Theorem 7 Let $H_K^\delta = (V, E)$ be an induced graph of a given FFC $K = (U, C, \tilde{I})$. Then $\mathcal{T}(H_K^\delta) = \mathcal{R}(K^\delta)$.

Proof By Definition 7, it is clear that every hyperedge of the induced graph H_K^δ is a granular δ-discernibility attribute set of x and y in K. In fact, the set of hyperedges in H_K^δ is the same as the granular δ-discernibility matrix of K. Thus, by Lemmas 2 and Theorem 6, one can see that $f_{H_K^\delta}(v_1, v_2, \ldots, v_m) = f_{K^\delta}(a_1, a_2, \ldots, a_m)$. This implies that $\mathcal{T}(H_K^\delta) = \mathcal{R}(K^\delta)$. □

The above result shows that finding the δ-granular reducts of a given FFC can be translated into computing the minimal transversals of a hypergraph. Furthermore, an attribute is a δ-granular core of a given FFC $K = (U, C, \tilde{I})$ if and only if it is a vertex with loops in H_K^δ.

Theorem 8 Let $H_K^\delta = (V, E)$ be an induced hypergraph of a given FFC $K = (U, C, \tilde{I})$. If $a \in C$ is a δ-granular core of K, then a is a vertex with self-loop for the hypergraph H_K^δ.

Proof In fact, if $a \in C$ is a δ-granular core of K, then according to Theorem 5, there are $x, y \in U$ such that

Table 10 Incidence matrix representation of hypergraph induced from $[D]_K^{0.5}$

	a	b	c	d	e
e_1	0	0	0	1	0
e_2	0	0	0	1	0
e_3	0	0	0	1	0
e_4	0	0	0	1	0
e_5	1	1	0	1	0
e_6	1	1	1	0	1
e_7	1	1	1	0	1
e_8	1	1	1	1	1
e_9	1	0	1	1	1
e_{10}	1	0	1	1	1
e_{11}	1	1	0	0	1
e_{12}	1	1	0	0	0
e_{13}	1	0	0	0	1
e_{14}	1	0	0	0	0
e_{15}	0	0	1	1	0
e_{16}	1	0	1	0	1
e_{17}	0	1	0	1	0
e_{18}	0	1	0	1	0
e_{19}	0	1	0	0	0

$D_K^\delta(x, y) = \{a\}$. This means that there is a hyperedge $e \in E$ such that $e = (a, a)$ by Definition 5, that is, the hyperedge e is a loop since it has only the unique vertex a. □

Example 9 (Continued from Example 6). From Tables 7, 8 and 9, we can obtain the corresponding Boolean function of H_K as follows.

$$f_{H_K^{0.05}} = (a \vee b \vee c \vee d \vee e) \wedge (b \vee c \vee d \vee e)$$
$$\wedge (a \vee b \vee e) \wedge (a \vee b \vee d) \wedge (a \vee b \vee d \vee e)$$
$$\wedge (a \vee d) \wedge (c \vee d \vee e) \wedge (a \vee c \vee d \vee e)$$
$$\wedge (a \vee b \vee c \vee e) \wedge (a \vee b)$$
$$= (a \wedge c) \vee (a \wedge d) \vee (a \wedge e) \vee (b \wedge d).$$

$$f_{H_K^{0.2}} = d \wedge (a \vee b) \wedge (a \vee d) \wedge (b \vee d)$$
$$\wedge (c \vee d) \wedge (a \vee b \vee d)$$
$$\wedge (a \vee b \vee e) \wedge (a \vee c \vee e) \wedge (a \vee d \vee e)$$
$$\wedge (a \vee b \vee c \vee e) \wedge (a \vee c \vee d \vee e)$$
$$\wedge (a \vee b \vee c \vee d \vee e)$$
$$= (a \wedge d) \vee (b \wedge c \wedge d) \vee (b \wedge d \wedge e).$$

$$f_{H_K^{0.5}} = a \wedge b \wedge d \wedge (a \vee b) \wedge (a \vee e)$$
$$\wedge (b \vee d) \wedge (c \vee d)$$
$$\wedge (a \vee b \vee d) \wedge (a \vee b \vee e) \wedge (a \vee c \vee e)$$
$$\wedge (a \vee b \vee c \vee e) \wedge (a \vee c \vee d \vee e)$$

$$\wedge (a \vee b \vee c \vee d \vee e)$$
$$= a \wedge b \wedge d.$$

Combined with Example 7, it is easy to verify that $\mathcal{T}(H_K^\delta) = \mathcal{R}(K^\delta)$.

Having transformed a problem of finding the δ-granular reduction of an FFC into finding the minimal transversals of a hypergraph, we can then implement it by using some heuristic algorithms from graph theory. Next, we use the idea of greedy algorithms to construct Algorithm 3 for finding a minimal transversal of a hypergraph. It is easy to see the time complexity of Algorithm 3 is $O(|U|^2|C|)$. However, the complexity of granular reduction Algorithm 2 is $O(|U|^2|C|^3)$. This means that Algorithm 3 is more efficient than the Algorithm 2.

Algorithm 3 A heuristic algorithm based on graph theory to compute the δ-granular reducts of K (GTK)

Require: An FFC $K = (U, C, \tilde{I})$
Ensure: A δ-granular reduct $\mathcal{T}(H_K^\delta)$ of K based graph theory
1: Calculate the δ-granular discernibility matrix D_K^δ using Algorithm 1
2: Delete all full 0 rows and duplicate rows in D_K^δ
3: Generate the incidence matrix of the induced graph from $\mathcal{D}_K^\delta$
4: Let $\mathcal{T}(H_K^\delta) = \emptyset$
5: Find all the vertices with self-loops and denote it by $\mathcal{T}(H_K^\delta)$
6: For each $v \in \mathcal{T}(H_K^\delta)$, delete all the hyperedges incident to the vertex v
7: **while** $\mathcal{D}_K^\delta \neq \emptyset$ **do**
8: Find the vertex v_0 with the maximum degree, and let $\mathcal{T}(H_K^\delta) = \mathcal{T}(H_K^\delta) \cup \{v_0\}$
9: Delete all the hyperedges incident to the vertex v_0
10: **end while**
11: For any $v \in \mathcal{T}(H_K^\delta)$, if the hyperedges incident to v can be covered by the set of vertices $\mathcal{T}(H_K^\delta) - \{v\}$, delete the vertex v
12: Return D_S^δ

Proposition 9 *Let $K = (U, C, \tilde{I})$ be an FFC, then the result $\mathcal{T}(H_K^\delta)$ output by Algorithm 3 is a δ-granular reduct of K.*

Proof It can be seen from Theorem 7 that a minimal transversal $\mathcal{T}(H_K^\delta)$ of induced hypergraph H corresponds to a δ-granular reduct of K. In order to prove this result, it is only necessary to explain that the result $\mathcal{T}(H_K^\delta)$ output by Algorithm 3 is a minimal transversal of induced graph H_K^δ. According to steps 7–10 of Algorithm 3, it is clear that $\mathcal{T}(H_K^\delta)$ is a transversal of induced graph H_K^δ, because $\mathcal{T}(H_K^\delta)$ covers all the hyperedges of H_K^δ. Let's use the counter proof method to assume that $\mathcal{T}(H_K^\delta)$ is not a minimal transversal, that is, there is $v \in \mathcal{T}(H_K^\delta)$, so that the vertex set $\mathcal{T}(H_K^\delta) - \{v\}$ can still cover all hyperedges of H_K^δ. However, it can be seen from steps 11 of Algorithm 3 that there is no such $v \in \mathcal{T}(H_K^\delta)$, so that the vertex set $\mathcal{T}(H_K^\delta) - \{v\}$ can

still cover all hyperedges of H_K^δ. Therefore, the proposition is proved. □

3.3 Experimental setup and data sets

From these datasets we can see that the values under each attribute do not all belong to [0, 1], so our first priority is to fuzzify this data sets. The idea of fuzzification is to first grade each attribute and then use a Gaussian fuzzy function to calculate the affiliation of the data in each grade. Because most data set conforms to the normal distribution, that is, Gaussian distribution, we use Gaussian fuzzy function to fuzzify the data.

Definition 8 (Poelmans et al. 2013) An information system is denoted by $I = (U, C, V, F)$, where $U = \{x_1, x_2, \ldots, x_n\}$ (called universe set) is a nonempty set of objects, $C = \{a_1, a_2, \ldots, a_m\}$ (called attribute set) is a nonempty set of attributes, $F = \{f_j | j = 1, 2, \ldots, m\}$ is a family of mapping set between universe U and attribute set C, where $f_j : U \times C \to V$ and $V = \bigcup_{a_j \in C} V_j (j = 1, 2, \ldots, m)$.

Definition 9 Let $I = \{U, C, V, F\}$ be an information system, where $U = \{x_1, x_2, \ldots, x_n\}$, $C = \{a_1, a_2, \ldots, a_m\}$, $F = \{f_l | l = 1, 2, \ldots, m\}$. Let $\mu_C = \{\mu_{a_l}^{I_{a_l}(i)}(x_j) : 1 \leq j \leq n, 1 \leq l \leq m, 1 \leq i \leq k_l\}$, where $\mu_C = \{\mu_{a_l}^{I_{a_l}(i)}(x_j)\}$ represents the membership degree of the $l-$th attribute value of object x_j belong to interval $I_{a_l}(i)$, $\bigcup_{i=1}^{k_l} I_{a_l}(i) = V_{a_l}$, V_{a_l} is the value domain of f_j. For simplicity, we write V_l instead of V_{a_l}, I_l instead of I_{a_l}. And $I_l(i) = [I_l^-(i), I_r^+(i)]$, $|I_l(i)| = I_r^+(i) - I_l^-(i)$, $\bar{I}_l(i) = \frac{I_l^-(i)+I_l^+(i)}{2}$ and $I_l^-(1) = \min(V_l)$, $I_l^+(k_l) = \max(V_l)$, $I_l^+(i-1) \geq I_l^-(i)$, $\mu_l^{I_l(i)}(x_j)$ is defined as follows.

$$\mu_l^{I_l(1)}(x_j) = \begin{cases} 1, & I_l^-(1) = V_l(x_j), \\ e^{-\frac{(V_l(x_j)-I_l^-(1))^2}{2(|I_l(1)|)^2}}, & I_l^-(1) < V_l(x_j) \leq I_l^+(1), \end{cases}$$

$$\mu_l^{I_l(i)}(x_j) = \begin{cases} e^{-\frac{(\bar{I}_l(i)-V_l(x_j))^2}{2(\frac{1}{2}|I_l(i)|)^2}}, & I_l^-(i) < V_l(x_j) \leq \bar{I}_l(i), \\ e^{-\frac{(V_l(x_j)-\bar{I}_l(i))^2}{2(\frac{1}{2}|I_l(i)|)^2}}, & \bar{I}_l(i) < V_l(x_j) \leq I_l^+(i), \end{cases}$$

$$\mu_l^{I_l(k_l)}(x_j) = \begin{cases} 0, & V_l(x_j) = I_l^-(k_l) \\ e^{-\frac{(I_l^+(k_l)-V_l(x_j))^2}{2(|I_l(k_l)|)^2}}, & I_l^-(k_l) < V_l(x_j) \leq I_l^+(k_l). \end{cases}$$

For each subinterval $I_l(i)$, it also needs to satisfy a condition: it cannot be included in the union of one or more other subintervals. It should be emphasized that $I_l(i) \cap I_l(i+1)$ ($i = 1, 2, \ldots, k_l - 1$) is allowed nonempty whereas it must be an empty set in the classical interval partition, as shown in Fig. 6.
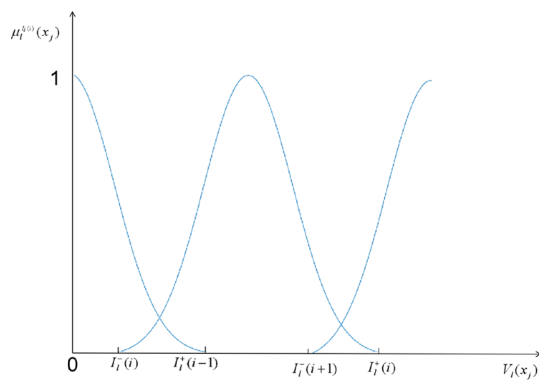
Fig. 6 The Gaussian fuzzy number

Table 11 Data description of the 4 data sets

No	Data sets	Objects	Features	Type
1	Glass	214	9	Continuous
2	Heart	270	13	Continuous
3	seeds	210	7	Continuous
4	Wine	178	13	Continuous

The larger value of k_l, the finer the information granule. The smaller value of k_l, the coarser the information granule. By Definition 9, an information table with continuous data can be transformed into an FFC.

The fuzzified data is better interpreted. For example, in a certain dataset, there are 2000 objects, one of the attributes is "height", and the value of this attribute is continuous. Then we can classify the height values into "high, medium and low" three grades, each two grades can be intersected. Then according to the formula in Definition 9 to calculate the corresponding degree of affiliation, the calculated value indicates that the object belongs to the "high, medium, low" each grade of the degree of affiliation.

Table 11 summarizes the set of data used in the experiment. They are real-world data sets which were collected from UCI Machine Learning Repository.[1]

Each data set in Table 11 can be regarded as a large information system. For discrete attributes with no more than 5 attribute values, we use the method in Ganter and Wille (1999) to convert discrete attributes into Boolean ones. For continuous attributes or a discrete attribute with more than 5 values, we apply Definition 9 to convert continuous attributes into fuzzy attributes with a value of [0-1]. In this paper, we divide the entire range of each continuous attribute into three subintervals. The first and third intervals are half of the second interval, and the repetition of each two adjacent intervals is 1/5 of the length of the second interval. Therefore, we get $K1, K2, K3$, and $K4$, which is shown in Table 12, where $|U|$

Table 12 Formal fuzzy contexts used in tests

Data sets	$|U|$	$|C|$	Density (%)
$K1$	214	27	40.00
$K2$	271	38	38.90
$K3$	210	21	43.40
$K4$	178	39	42.48

and $|C|$ represent the cardinalities of object sets and condition attribute sets respectively. Density represents the proportion of non-zero items in the entire data set.

In our experiments, we have obtained reduct sets of attributes while keeping object granules of a certain granularity unchanged, using the BRK and GTK algorithms, respectively. Each algorithm can be seen as a two-step process, where the first step is generated by Algorithm 1. The difference between the two algorithms lies in the second step of each algorithm.

Table 13 lists the running times for generating the difference matrices at different granularities for each data set. Tables 14, 15, 16 and 17 present the experimental results, including the cardinality of the reduced set of attributes, whether it is a reduct or not and the running time of the second step in each algorithm. As can be seen from these tables, there are 40 cases, of which only 12 cases are different, but ninety-nine percent of all datasets with different results differ by no more than ten percent. This means that the difference in cardinality after reduction is very small. This is mainly because the reduction algorithm is an approximation algorithm and there are errors. To reduce this error, we can adjust the order of the attributes in the dataset. In order to see more directly how the two algorithms compare in terms of running time on the second step, we have drawn Figs. 7, 8, 9 and 10. The X-axis in each figure represents the granularity value and the Y-axis represents the running time in seconds. Note that the Y-axis is of logarithmic. It is straightforward to see from the figures that GTK performs better than BRK in terms of running time. All experiments were run on an i5-10210U (1.60GHz), 8.0GB RAM and a personal computer in Matlab.

Algorithm 2 (BRK) and Algorithm 3 (GTK) both aim at finding the granular reducts of FFCs, but the difference is that BRK calculates all the granular reducts of an FFC, but GTK only calculates one granular reduct of an FFC. By analyzing the algorithms and the results of data experiments, we can know that the BRK algorithm has high complexity and long running time, while the GTK algorithm has low complexity and high reduction efficiency.

---

[1] [Online]. Available: https://archive.ics.uci.edu/ml/index.php.

Table 13 The running time of the first step of BRK and the GTK

δ	$K1$(Glass)	$K2$(Heart)	$K3$(Seeds)	$K4$(Wine)
$\delta = 0.1$	0.218755	0.70626	0.162852	0.179857
$\delta = 0.2$	0.208647	0.568689	0.153116	0.182404
$\delta = 0.3$	0.219642	0.618025	0.155187	0.193528
$\delta = 0.4$	0.227668	0.63504	0.151183	0.180815
$\delta = 0.5$	0.209841	0.589859	0.155161	0.177767
$\delta = 0.6$	0.204755	0.526602	0.168484	0.180411
$\delta = 0.7$	0.208848	0.539101	0.156221	0.181718
$\delta = 0.8$	0.212355	0.531659	0.156312	0.179879
$\delta = 0.9$	0.220561	0.553766	0.16174	0.192384
$\delta = 1$	0.216132	0.53885	0.157429	0.188992

Table 14 Comparison of the output of the two algorithms ($K1$ (Glass))

$K1$ (Glass)	The cardinality of the reduct		Whether the output is a reduct?		The whole running time (s)	
	BRK	GTK	BRK	GTK	BRK	GTK
$\delta = 0.1$	10	9	Yes	Yes	11.13556	0.6972
$\delta = 0.2$	14	10	Yes	Yes	21.257265	1.881056
$\delta = 0.3$	11	12	Yes	Yes	32.22656	3.056033
$\delta = 0.4$	11	12	Yes	Yes	30.307366	3.131181
$\delta = 0.5$	11	12	Yes	Yes	31.163007	3.169799
$\delta = 0.6$	11	12	Yes	Yes	32.220366	3.084411
$\delta = 0.7$	12	11	Yes	Yes	27.608736	2.504474
$\delta = 0.8$	11	11	Yes	Yes	51.537105	2.322138
$\delta = 0.9$	11	11	Yes	Yes	23.00938	1.655343
$\delta = 1$	12	12	Yes	Yes	12.169715	1.399923

Table 15 Comparison of the output of the two algorithms ($K2$ (Heart))

$K2$ (Heart)	The cardinality of the reduct		Whether the output is a reduct?		The whole running time (s)	
	BRK	GTK	BRK	GTK	BRK	GTK
$\delta = 0.1$	11	14	Yes	Yes	969.301679	46.622603
$\delta = 0.2$	15	12	Yes	Yes	2191.288822	24.324593
$\delta = 0.3$	17	16	Yes	Yes	11473.38567	51.610193
$\delta = 0.4$	17	16	Yes	Yes	5839.711817	48.243427
$\delta = 0.5$	17	16	Yes	Yes	5569.253667	36.990687
$\delta = 0.6$	17	16	Yes	Yes	8829.95628	36.431264
$\delta = 0.7$	15	16	Yes	Yes	2508.310875	46.118352
$\delta = 0.8$	15	15	Yes	Yes	2366.940844	20.77905
$\delta = 0.9$	19	16	Yes	Yes	952.443765	43.901989
$\delta = 1$	20	19	Yes	Yes	102.757187	24.277777

4 δ-granular reduct in formal fuzzy decision contexts

In this section, we introduce the definition of δ-consistent and δ-granular reduction in FFDCs. Then, we provide two different reduction approaches.

4.1 Boolean reasoning for δ-granular reduction in formal fuzzy decision contexts

Definition 10 Let $S = (U, C, \tilde{I}, D, \tilde{J})$ be an FFDC. S is said to be δ-consistent if $g_\delta \circ f(x) \subseteq g_{\delta, D} \circ f_D(x)$ for all $x \in U$. Otherwise, it is said to be δ-inconsistent.

Example 10 (Continue from Example 2) According to the properties of f and g_δ, S is δ-consistent when we take

Table 16 Comparison of the output of the two algorithms (K3 (Seeds))

| K3 (seeds)	The cardinality of the reduct		Whether the output is a reduct?		The whole running time (s)	
	BRK	GTK	BRK	GTK	BRK	GTK
$\delta = 0.1$	11	11	Yes	Yes	30.272436	2.095256
$\delta = 0.2$	12	12	Yes	Yes	72.759872	4.621699
$\delta = 0.3$	12	12	Yes	Yes	76.265352	2.16971
$\delta = 0.4$	13	13	Yes	Yes	101.483164	3.166776
$\delta = 0.5$	13	13	Yes	Yes	101.345067	3.173729
$\delta = 0.6$	13	13	Yes	Yes	101.784218	3.2003
$\delta = 0.7$	13	13	Yes	Yes	84.172087	4.475752
$\delta = 0.8$	13	13	Yes	Yes	48.722745	1.981423
$\delta = 0.9$	14	14	Yes	Yes	58.280346	2.313625
$\delta = 1$	14	14	Yes	Yes	9.618673	0.847385

Table 17 Comparison of the output of the two algorithms (K4 (Wine))

K4 (Wine)	The cardinality of the reduct		Whether the output is a reduct?		The whole running time (s)	
	BRK	GTK	BRK	GTK	BRK	GTK
$\delta = 0.1$	11	11	Yes	Yes	495.418016	22.295554
$\delta = 0.2$	12	12	Yes	Yes	163.842852	23.076288
$\delta = 0.3$	12	12	Yes	Yes	1494.513452	4.061028
$\delta = 0.4$	13	13	Yes	Yes	668.294656	13.399942
$\delta = 0.5$	13	13	Yes	Yes	666.869961	13.702262
$\delta = 0.6$	13	13	Yes	Yes	667.553298	13.537314
$\delta = 0.7$	13	13	Yes	Yes	284.716192	9.826793
$\delta = 0.8$	13	13	Yes	Yes	1965.858836	3.47196
$\delta = 0.9$	14	14	Yes	Yes	246.0072	4.803754
$\delta = 1$	14	14	Yes	Yes	146.430889	3.328255
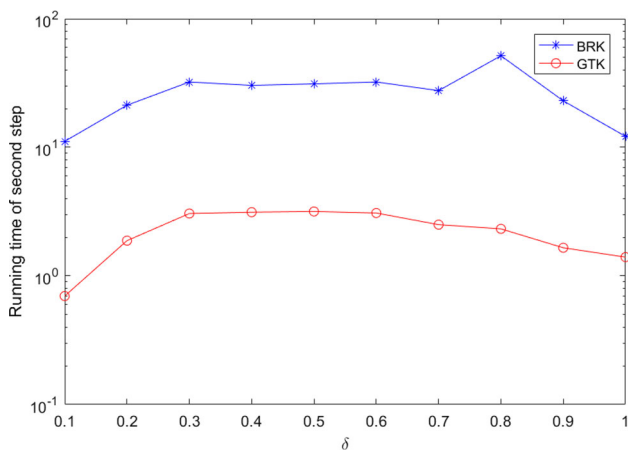
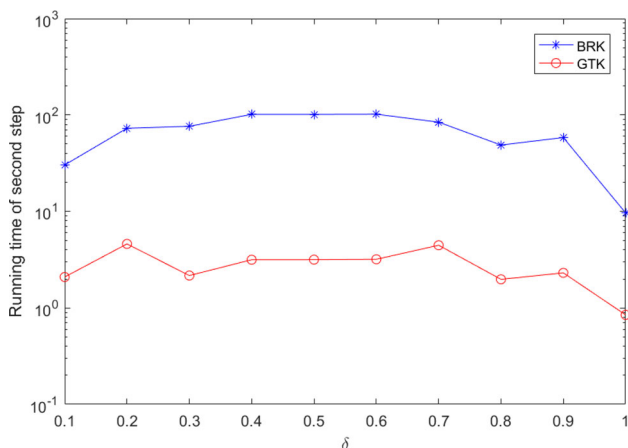**Fig. 7** Comparison of the running times of the second step of the BRK and the GTK (Glass)
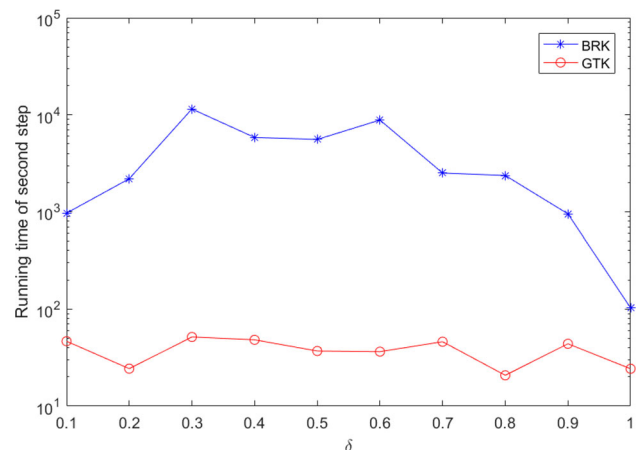
Fig. 8 Comparison of the running times of the second step of the BRK and the GTK (Heart)

$\delta = 0.01, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$, respectively. For simplicity, we only present the determination of the consistent of S for $\delta = 0.01$ and $\delta = 0.45$, other cases can be obtained in a similar way.

$$g_{0.01} \circ f(x_1) = \{x_1\} = g_{0.01,D} \circ f_D(x_1),$$

$$g_{0.01} \circ f(x_2) = \{x_2\} = g_{0.01,D} \circ f_D(x_2),$$
$$g_{0.01} \circ f(x_3) = \{x_3\} = g_{0.01,D} \circ f_D(x_3),$$
$$g_{0.01} \circ f(x_4) = \{x_4\} = g_{0.01,D} \circ f_D(x_4),$$
$$g_{0.01} \circ f(x_5) = \{x_5\} = g_{0.01,D} \circ f_D(x_5),$$
$$g_{0.45} \circ f(x_1) = \{x_1\} \subseteq \{x_1, x_2, x_3\} = g_{0.45,D} \circ f_D(x_1),$$

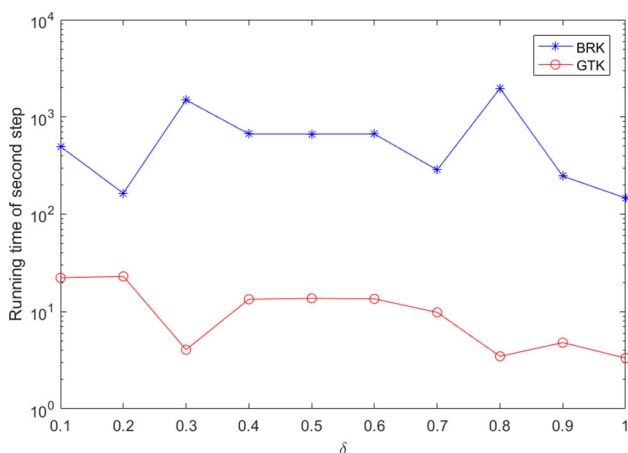**Fig. 9** Comparison of the running times of the second step of the BRK and the GTK (Seeds)

Fig. 10 Comparison of the running times of the second step of the BRK and the GTK (Wine)

$$g_{0.45} \circ f(x_2) = \{x_2\} \subseteq \{x_2, x_3\} = g_{0.45, D} \circ f_D(x_2),$$
$$g_{0.45} \circ f(x_3) = \{x_3\} = g_{0.45, D} \circ f_D(x_3),$$
$$g_{0.45} \circ f(x_4) = \{x_3, x_4\} = g_{0.45, D} \circ f_D(x_4),$$
$$g_{0.45} \circ f(x_5) = \{x_5\} = g_{0.45, D} \circ f_D(x_5).$$

Definition 11 Let $S = (U, C, \tilde{I}, D, \tilde{J})$ be a δ-consistent FFDC and $C' \subseteq C$. If $g_{\delta, C'} \circ f_{C'}(x) \subseteq g_{\delta, D} \circ f_D(x)$ for all $x \in U$, then C' is referred to as a δ-granular consistent set of S. If C' is a δ-granular consistent set of S and no proper subset of C' is a δ-granular consistent set, then C' is referred to as a δ-granular reduct of S.

We denote the set of all δ-granular reducts of S as $\mathcal{R}(S^\delta)$. According to the significance of the attributes, based on δ-granular reducts, the attribute set C is divided into three parts:

(1) δ-indispensable attribute (core attribute) set Co_{S^δ} : $Co_{S^\delta} = \bigcap \mathcal{R}(S^\delta)$,

(2) δ-relatively necessary attribute set K_{S^δ} : $K_{S^\delta} = \bigcup \mathcal{R}(S^\delta) - \bigcap \mathcal{R}(S^\delta)$,

(3) δ-unnecessary attribute set I_{S^δ} : $I_{S^\delta} = C - \bigcup \mathcal{R}(S^\delta)$.

Similar to the definition of ordered relation $R_{C'}^\delta$ defined by Eq. (5), the ordered relation R_D^δ with respect to the decision attribute set D in $(U, C, \tilde{I}, D, \tilde{J})$ is defined by $R_D^\delta = \{(x, y) \in U \times U | 0 \leq \tilde{J}(y, d) - \tilde{J}(x, d) \leq \delta, \forall d \in D\}$.

Definition 12 Let $S = (U, C, \tilde{I}, D, \tilde{J})$ be a δ-consistent FFDC and $(x, y) \in U \times U$. We define

$$D_S^\delta(x, y) = \begin{cases} \{a \in C | \tilde{I}(x, a) > \tilde{I}(y, a) || \tilde{I}(x, a) + \delta \\ < \tilde{I}(y, a)(\exists a \in C) \, \& \, \tilde{J}(x, c) \\ > \tilde{J}(y, c) || \tilde{J}(x, c) + \delta \\ < \tilde{J}(y, c), (\exists c \in D)\}, \\ \emptyset, \quad \text{others,} \end{cases}$$

where $D_S^\delta(x, y)$ is referred to as the granular δ-granular discernibility attribute set of x and y in S, and $\mathcal{D}_S^\delta = (D_S^\delta(x, y) | (x, y) \in U \times U)$ is called the δ-granular discernibility matrix of S. We denote $[D]_S^\delta = \{D_S^\delta(x, y) | D_S^\delta(x, y) \neq \emptyset, (x, y \in U)\}$.

Remark 5 When $\delta = 1$, $D_S^\delta(x, y)$ degenerates to the discernibility attribute set of x and y in S, $\mathcal{D}_S^\delta$ degenerates to the discernibility matrix of S, and $[D]_S^\delta$ degenerates to M_0^S in Shao et al. (2016).

Theorem 10 Let $S = (U, C, \tilde{I}, D, \tilde{J})$ be a consistent FFDC, if $\delta_1 \geq \delta_2$, then $D_S^{\delta_1}(x, y) \subseteq D_S^{\delta_2}(x, y)$.

Proof If $a \in D_S^{\delta_1}(x, y)$, that is $\tilde{I}(x, a) > \tilde{I}(y, a)$ or $\tilde{I}(y, a) - \tilde{I}(x, a) > \delta_1 \geq \delta_2$ and there exists $d \in D$ such that $\tilde{J}(x, d) > \tilde{J}(y, d)$ or $\tilde{J}(y, d) - \tilde{J}(x, d) > \delta_1 \geq \delta_2$. That is, $a \in D_S^{\delta_2}(x, y)$. Then $D_S^{\delta_1}(x, y) \subseteq D_S^{\delta_2}(x, y)$. $\square$

Theorem 11 Let $S = (U, C, \tilde{I}, D, \tilde{J})$ be an FFDC and $C' \subseteq C$. Then, C' is a δ-granular consistent set if and only if $C' \cap D_S^\delta(x, y) \neq \emptyset$ for all $D_S^\delta(x, y) \in [D]_S^\delta$.

Proof ($\Rightarrow$) For any $D_S^\delta \in [D]_S^\delta$, according to Definition 12, it is known that $y \notin [x]_{R_D^\delta}$. And since C' is a δ-granular consistent set, we know that $[x]_{R_{C'}^\delta} = g_{C'}^\delta \circ f_{C'}(x) \subseteq g_D^\delta \circ f_D = [x]_{R_D^\delta}$. Thus, $y \notin [x]_{R_{C'}^\delta}$, that is, there exists an attribute $c \in C'$, such that $\tilde{I}(x, c) > \tilde{I}(y, c)$ or $\tilde{I}(x, c) + \delta < \tilde{I}(y, c)$, which implies that $c \in D_S^\delta(x, y)$. Then, we have $c \in C' \cap D_S^\delta(x, y)$. Therefore, $C \cap D_S^\delta(x, y) \neq \emptyset$.

($\Leftarrow$) Suppose that $C \cap D_S^\delta(x, y) \neq \emptyset$. For any $D_S^\delta(x, y) \in [D]_S^\delta$, since S is δ-consistent, then for any $x, y \in U$, if $y \notin [x]_{R_D^\delta}$, there must be $y \notin [x]_{R_C^\delta}$, i.e. exists $a \in C$, such that $\tilde{I}(x, a) > \tilde{I}(y, a)$ or $\tilde{I}(x, a) + \delta < \tilde{I}(y, a)$. Thus, we obtain that $D_S^\delta(x, y) \neq \emptyset$. And since it is assumed that $C \cap D_S^\delta(x, y) \neq \emptyset$, there exists $c \in C'$ such that $\tilde{I}(x, c) > \tilde{I}(y, c)$ or $\tilde{I}(x, c) + \delta < \tilde{I}(y, c)$, which implies that $y \notin [x]_{R_{C'}^\delta}$. It follows that $[x]_{R_{C'}^\delta} \subseteq [x]_{R_D^\delta}$. Therefore, we have $[x]_{R_{C'}^\delta} = g_{\delta, C'} \circ f_{C'}(x) \subseteq g_{\delta, D} \circ f_D(x) = [x]_{R_D^\delta}$, i.e. C' is a δ-granular consistent set of S. $\square$

Theorem 12 provides a method to determine whether the attribute set is a δ-granular consistent set of S.

Theorem 12 *Let* $S = (U, C, \tilde{I}, D, \tilde{J})$ *be a δ-consistent FFDC and* $a \in C$. *Then,* a *is a δ-indispensable (core) attribute of* S *if and only if there exists* $(x, y) \in U \times U$ *such that* $D_S^\delta(x, y) = \{a\}$.

Proof ($\Rightarrow$) Assume that a is an δ-indispensable attribute in S, then $C - \{a\}$ is not a δ$-$granular consistent set of S. By Definition 11, we have $g_{\delta, C-\{a\}} \circ f_{C-\{a\}}(x) \not\subseteq g_{\delta, D} \circ f_D(x)$, that is, $[x]_{R^\delta_{C-\{a\}}} \not\subseteq [x]_{R^\delta_D}$. Thus, there exists $y \in U$ such that $y \in [x]_{R^\delta_{C-a}}$ and $y \notin [x]_{R^\delta_D}$, which means $\tilde{I}(x, a) - \tilde{I}(y, a) > 0$ or $\tilde{I}(x, a) + \delta < \tilde{I}(y, a)$, and $0 \le \tilde{I}(y, b) - \tilde{I}(x, b) \le \delta$ for all $b \in C - \{a\}$, and, there exists $d \in D$ $\tilde{J}(x, d) > \tilde{J}(y, d)$ or $\tilde{J}(x, d) + \delta < \tilde{J}(y, d)$. Therefore, by the definition of $D_S^\delta(x, y)$, we conclude that $D_S^\delta(x, y) = \{a\}$.

($\Leftarrow$) If there exists $(x, y) \in U \times U$ such that $D_S^\delta(x, y) = \{a\}$, then, by Definition 12, we obtain $0 \le \tilde{I}(y, b) - \tilde{I}(x, b) \le \delta$, ($\forall b \in C - \{a\}$ and $\tilde{I}(x, a) > \tilde{I}(y, a)$ or $\tilde{I}(x, a) + \delta < \tilde{I}(y, a)$. And, there exists $d \in D$, such that $\tilde{J}(x, d) > \tilde{J}(y, d)$ or $\tilde{J}(x, d) + \delta < \tilde{J}(y, d)$, i.e. $y \in [x]_{R^\delta_{C-\{a\}}}$ and $y \notin [x]_{R^\delta_D}$. Thus, we have, $[x]_{R^\delta_{C-\{a\}}} \not\subseteq [x]_{R^\delta_D}$, that is, $g_{\delta, C-\{a\}} \circ f_{C-\{a\}} \not\subseteq g_{\delta, D} \circ f_D(x)$. Hence, $a \in \cap \mathcal{R}_{S^\delta}$. Therefore, a is an δ-indispensable attribute (core attribute) of S. □

Let

$$f_{S^\delta} = \underset{(x,y) \in U \times U}{\wedge} \vee\ D_S^\delta(x, y) = \overset{t}{\underset{k=1}{\vee}} \left(\overset{q_k}{\underset{s=1}{\wedge}} a_s \right),$$

where $\wedge_{s=1}^{q_k}$, $k \le t$ are all the prime implicants of the δ-granular discernibility function f_S^δ. We denote $N_k = \{a_s | s = 1, 2, \ldots, q_k\}$. Then, $\{N_k | k = 1, 2, \ldots, t\}$ is the set of all δ-granular reducts of S.

Theorem 13 *Let* $S = (U, C, \tilde{I}, D, \tilde{J})$ *be an FFDC and* $C' \subseteq C$. *Then,* C' *is a δ-granular reduct of* S *if and only if* $\underset{a \in C'}{\wedge} a$ *is a prime implicant of the δ-discernibility function* f_{S^δ}.

Proof Its proof is similar to that of Theorem 6. □

Example 11 Continued from Example 2. When δ is taken as 0.05, 0.2 and 0.5, the δ-granular discernibility matrices are shown in Tables 18, 19, and 20 respectively.

$f_{S^{0.05}}(a, b, c, d, e)$
$= (b \vee c \vee d \vee e) \wedge (a \vee b \vee c \vee d \vee e)$
$\quad \wedge (a \vee d) \wedge (c \vee d) \wedge (a \vee b \vee d)$
$\quad \wedge (a \vee b \vee e) \wedge (a \vee c \vee d \vee e) \wedge (a \vee b)$
$\quad \wedge (a \vee b \vee d \vee e) \wedge (a \vee b \vee c \vee e)$

Table 18 0.05-granular discernibility matrix of S

$D_S^{0.05}(x, y)$	x_1	x_2	x_3	x_4	x_5
x_1	$\emptyset$	$bcde$	$abcde$	$abcde$	$abcde$
x_2	$abcde$	$\emptyset$	ad	cd	abd
x_3	$abcde$	abe	$\emptyset$	$acde$	abd
x_4	$abcde$	abd	ad	$\emptyset$	ab
x_5	$abcde$	$abde$	abd	$abce$	$\emptyset$

Table 19 0.2-granular discernibility matrix of S

$D_S^{0.2}(x, y)$	x_1	x_2	x_3	x_4	x_5
x_1	$\emptyset$	$\emptyset$	ad	d	abd
x_2	$abce$	$\emptyset$	ad	cd	abd
x_3	$abce$	abe	$\emptyset$	$acde$	bd
x_4	$abcde$	abd	ad	$\emptyset$	ab
x_5	$acde$	ade	ad	ace	$\emptyset$

Table 20 0.5-granular discernibility matrix of S

$D_S^{0.5}(x, y)$	x_1	x_2	x_3	x_4	x_5
x_1	$\emptyset$	$\emptyset$	$\emptyset$	d	abd
x_2	$abce$	$\emptyset$	$\emptyset$	cd	bd
x_3	$abce$	abe	$\emptyset$	$acde$	bd
x_4	$abce$	ab	$\emptyset$	$\emptyset$	b
x_5	$acde$	ae	a	ace	$\emptyset$

$= (a \wedge c) \vee (a \wedge d) \vee (b \wedge d).$

$f_{S^{0.2}}(a, b, c, d, e)$
$= (a \vee d) \wedge d \wedge (a \vee b \vee d) \wedge (a \vee b \vee c \vee e)$
$\quad \wedge (c \vee d) \wedge (a \vee b \vee e)$
$\quad \wedge (a \vee c \vee d \vee e) \wedge (b \vee d) \wedge (a \vee b \vee c \vee d \vee e)$
$\quad \wedge (a \vee b) \wedge (a \vee d \vee e) \wedge (a \vee c \vee e)$
$= (a \wedge d) \vee (b \wedge c \wedge d) \vee (b \wedge d \wedge e).$

$f_{S^{0.5}}(a, b, c, d, e)$
$= d \wedge (a \vee b \vee d) \wedge (a \vee b \vee c \vee e) \wedge (c \vee d) \wedge (b \vee d)$
$\quad \wedge (a \vee b \vee e) \wedge (a \vee c \vee d \vee e)$
$\quad \wedge (a \vee b) \wedge b \wedge (a \vee e) \wedge a \wedge (a \vee c \vee e)$
$= a \wedge b \wedge d.$

Next, we discuss the calculation of D_S^δ from a different perspective. In fact, a δ-consistent FFDC $S = (U, C, \tilde{I}, D, \tilde{J})$ can be seen an FFC without decision attributes. It can be expressed as $B = (U, A, \tilde{R})$ satisfies $A = C \cup D$ and $\tilde{R} = \tilde{I} \cup \tilde{J}$.

By Definition 5 and 12, we can observe that for a given FFDC S, the δ-granular discernibility matrix of S and that of the union FFC are different. However, we can obtain the following result.

Theorem 14 *Let $S = (U, C, \tilde{I}, D, \tilde{J})$ be a δ-consistent FFDC, $B = (U, A, \tilde{R})$ be the union FFC induced from S. For any $x, y \in U$, if $D_B^\delta(x, y)$ and $D_S^\delta(x, y)$ are the δ-granular discernibility attribute sets of x and y in B and S, respectively, then $D_S^\delta(x, y) \subseteq D_B^\delta(x, y)$.*

Proof For each $x, y \in U$, if $D_S^\delta(x, y) = \emptyset$, then $D_S^\delta(x, y) \subseteq D_B^\delta(x, y)$. Thus we suppose that $D_S^\delta(x, y) \neq \emptyset$. If $D_S^\delta(x, y) \neq \emptyset$, then by Definition 12, there must exist $a \in A$ such that $\tilde{I}(x, a) > \tilde{I}(y, a)$ or $\tilde{I}(x, a) + \delta < \tilde{I}(y, a)\}$ and $d \in D$ such that $\tilde{J}(x, d) > \tilde{J}(y, d)$ or $\tilde{J}(x, d) + \delta < \tilde{J}(y, d)$. Therefore, one can conclude that $\{d \in D | \tilde{J}(x, d) > \tilde{J}(y, d) || \tilde{J}(x, d) + \delta < \tilde{J}(y, d)\} \neq \emptyset$. Note that $D_B^\delta(x, y)$ is the δ-granular discernibility attribute set of x and y in the union FFC B, then by the definition of $D_B^\delta(x, y)$, we have $D_B^\delta(x, y) = D_S^\delta(x, y) \cup \{d \in D | \tilde{J}(x, d) > \tilde{J}(y, d) || \tilde{J}(x, d) + \delta < \tilde{J}(y, d)\}$. Thus $D_S^\delta(x, y) \subseteq D_B^\delta(x, y)$ holds. □

By the proof of Theorem 14, it is also obvious to see that we can obtain the D_S^δ via $D_B^\delta(x, y)$. Actually, we have the following result.

Theorem 15 *Let $B = (U, C, \tilde{R})$ be the union FFC induced from a given δ-consistent FFDC $S = (U, C, \tilde{I}, D, \tilde{J})$. For any $x, y \in U$, then the following formula holds:*

$$D_S^\delta(x, y) = \begin{cases} D_B^\delta(x, y) - D, & D_B^\delta(x, y) \cap D \neq \emptyset, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Proof For any $x, y \in U$, if there is $d \in D$ such that $\tilde{J}(x, d) > \tilde{J}(y, d) || \tilde{J}(x, d) + \delta < \tilde{J}(y, d)$, then $\{d \in D | \tilde{J}(x, d) > \tilde{J}(y, d) || \tilde{J}(x, d) + \delta < \tilde{J}(y, d)\} \neq \emptyset$. From Theorem 14, we know that $D_B^\delta(x, y) = D_S^\delta(x, y) \cup \{d \in D : \tilde{J}(x, d) > \tilde{J}(y, d) || \tilde{J}(x, d) + \delta < \tilde{J}(y, d)\}$. Since $C \cap D = \emptyset$, and $D_S^\delta(x, y) \subseteq C$. Thus, we have $D_B^\delta(x, y) \cap D = \{d \in D : \tilde{J}(x, d) > \tilde{J}(y, d) || \tilde{J}(x, d) + \delta < \tilde{J}(y, d)\} \neq \emptyset$. Moreover, we have $D_S^\delta(x, y) = D_B^\delta(x, y) - \{d \in D : \tilde{J}(x, d) > \tilde{J}(y, d) || \tilde{J}(x, d) + \delta < \tilde{J}(y, d)\} = D_B^\delta(x, y) - D$. In addition, if $0 \leq \tilde{J}(y, d) - \tilde{J}(x, d) \leq \delta$ for all $d \in D$, then, we have $D_S^\delta(x, y) = \emptyset$. □

Example 12 (Continue from Example 2) Table 21 shows the 0.2-granular discernibility matrix of the union FFC B induced from the FFDC S of Example 2. It is easy to obtain the 0.2-granular discernibility matrix of S according to Theorem 15. It is the same as shown in Table 19.

Following the idea of Theorem 15, we give the algorithms for computing the δ-granular discernibility matrix D_S^δ and

δ-granular reduction of a given S, i.e., Algorithm 4 and 5. The time complexity of the Algorithm 4 is $O(|U|^2(|D| + |C|))$. The time complexity of steps 2–4 of Algorithm 5 is $O(|U|^2|C|^3)$. Therefore, the maximum time complexity of Algorithm 5 is $O(|U|^2(|D| + |C| + |C|^3))$.

Algorithm 4 Calculate D_S^δ

Require: An $\mathscr{S} = (U, C, \tilde{I}, D, \tilde{J})$ with $|U| = m, |C| = n, |D| = n_1$
Ensure: D_S^δ.// The $delta$-granular discernibility matrix of S
1: $D_S^\delta = \emptyset$
2: **for** $i = 1$ to m **do**
3: Generate a matrix $B_{|U||A|}$ obtained by 1-by-$|U|$ tiling of copies of the i-th row $S(i)$ of S
4: Let $E = B - S$, $E_1 = E(:, 1 : n)$, $E_2 = E(:, n + 1 : n + n_1)$
5: **end for**
6: Numbers greater than 0 and less than $-\delta$ in E_2 are replaced by 1 and all other numbers are replaced by 0
7: **for** $i = 1$ to m **do**
8: If the elements in i-row of E_2 are not all 0, replace the elements in i-row of E_1 that are greater than 0 or less than $-\delta$ with 1, and replace the other elements with 0
9: if the elements in i-row of E_2 are all 0, replace the elements in i-row of E_1 with 0
10: **end for**
11: $D_S^\delta = E_1$
12: Return D_S^δ

Algorithm 5 Calculate all δ-granular reducts of S (BRS)

Require: A δ-consistent FFDC $S = (U, C, \tilde{I}, D, \tilde{J})$ with $|U| = m, |C| = n, |D| = n_1$
Ensure: $\mathcal{R}(S^\delta)$ // the set of $\delta-$granular reducts of S
1: Calculate the δ-granular discernibility matrix D_S^δ by Algorithm 4
2: Calculate $f_{S^\delta} = \bigwedge\limits_{(x, y) \in U \times U} \vee D_S^\delta(x, y)$
3: Calculate $f_{S^\delta} = \bigvee\limits_{i=1}^{t} (\bigwedge\limits_{j=1}^{s_i} a_j)$
4: Let $N_k = \{a_j | j \leq s_i\}$ and $\mathcal{R}(S^\delta) = \{N_k | k \leq t\}$
5: Return D_S^δ

4.2 Graph representation for δ-granular reduction in formal fuzzy decision contexts

In this section, we will discuss the δ-granular reduction based on graph theory in FFDCs.

Definition 13 Let $S = (U, C, \tilde{I}, D, \tilde{J})$ be a δ-consistent FFDC, $\mathcal{D}_S^\delta$ be the δ-granular discernibility matrix of S. Denote $V = C$ and $E = \{e \in \mathcal{D}_S^\delta | e \neq \emptyset\}$. We call $H_S^\delta = (V, E)$ an δ-induced hypergraph from S.

Definition 13 provides a method for generating induced hypergraph from $\mathcal{D}_S^\delta$. According to it, we can generate a hypergraph from $\mathcal{D}_S^\delta$ in which a hyperedge is generated from

Table 21 The 0.2-granular discernibility matrix of B

$D_B^{0.2}$	x_1	x_2	x_3	x_4	x_5
x_1	$\emptyset$	$\emptyset$	add_1	dd_2	$abdd_2$
x_2	$abced_1d_2d_3$	$\emptyset$	add_1	cdd_2	$abdd_2$
x_3	$abced_1d_2d_3$	$abed_1d_3$	$\emptyset$	$acded_1d_2d_3$	bdd_1d_2
x_4	$abced_1d_3$	$abdd_1d_2$	add_2	$\emptyset$	abd_2
x_5	$acded_1d_3$	$aded_1d_2d_3$	add_2d_3	$aced_3$	$\emptyset$

a nonempty set of δ-granular discernibility attribute set in $\mathcal{D}_S^{\delta}$.

Let $\mathcal{R}(S^{\delta})$ and $\mathcal{T}(H_S^{\delta})$ denote the set of all δ-granular reducts of the δ-consistent FFDC and the minimal traversals of induced hypergraph of S. Then by Theorem 7, we can also have the following result.

Theorem 16 *For a giver δ-consistent FFDC $S = (U, C, \tilde{I}, D, \tilde{J})$, if $H_S^{\delta} = (V, E)$ be the induced graph from D_S^{δ}, then $\mathcal{R}(S^{\delta}) = \mathcal{T}(H_S^{\delta})$.*

Example 13 (Continued from Example 11) We first take $\delta = 0.2$ and then, according to Definition 13, are able to generate a hypergraph $H = (V, E)$. Where $V = \{a, b, c, d, e\}$, $E = \{e_1, e_2, \ldots, e_{12}\}$ and $e_1 = \{a, d\}$, $e_2 = \{d\}$, $e_3 = \{a, b, d\}$, $e_4 = \{a, b, c, e\}$, $e_5 = \{c, d\}$, $e_6 = \{a, b, e\}$, $e_7 = \{a, c, d, e\}$, $e_8 = \{b, d\}$, $e_9 = \{a, b, c, d, e\}$, $e_{10} = \{a, b\}$, $e_{11} = \{a, d, e\}$, $e_{12} = \{a, c, e\}$.

$$f_{H_S^{0.2}}(a, b, c, d, e)$$
$$= (a \vee d) \wedge d \wedge (a \vee b \vee d)$$
$$\quad \wedge (a \vee b \vee c \vee e) \wedge (c \vee d) \wedge (a \vee b \vee e)$$
$$\quad \wedge (a \vee c \vee d \vee e) \wedge (b \vee d) \wedge (a \vee b$$
$$\quad \vee c \vee d \vee e) \wedge (a \vee b) \wedge (a \vee d \vee e) \wedge (a \vee c \vee e)$$
$$= (a \wedge d) \vee (b \wedge c \wedge d) \vee (b \wedge d \wedge e).$$

It is easy to see that S has three 0.2-granular reducts $\{a, d\}$, $\{b, c, d\}$ and $\{b, d, e\}$, and $\mathcal{R}(S^{0.2}) = \mathcal{T}(H_S^{0.2})$.

Theorem 17 *Let $S = (U, C, \tilde{I}, D, \tilde{J})$ be a δ-consistent FFDC, $H_S^{\delta} = (V, E)$ be an induced hypergraph of S. If $a \in C$ is a δ-granular core of S, then a is a vertex with loops for the hypergraph H_S^{δ}.*

Proof The proof of the above theorem is similar to the proof of Theorem 8. □

Next, we construct an algorithm for searching a δ-granular reduct of a δ-consistent FFDC.

Very obviously, may see, the time complexity of Algorithm 6 is $O(|U|^2(|C| + |D|))$. However, the complexity of granular reduction Algorithm 5 is $O(|U|^2(|D| + |C| + |C|^3))$. This means that Algorithm 6 is more efficient than the Algorithm 5.

Algorithm 6 Calculate a δ-granular reduct of a δ-consistent FFDC based graph theory (GTS)

Require: A δ-consistent FFDC $S = (U, C, \tilde{I}, D, \tilde{J})$
Ensure: A δ-granular reduct $T(H_S^{\delta})$ based on graph theory of S
1: Let $T(H_S^{\delta}) = \emptyset$
2: Generate the incidence matrix $\mathcal{D}_S^{\delta}$ of induced hypergraph of S by Algorithm 4
3: Find all the vertices with loops and denote it by $T(H_S^{\delta})$
4: For each $v \in T(H_S^{\delta})$, delete all the hyperedges incident to the vertex v
5: **while** $\mathcal{D}_S^{\delta} \neq \emptyset$ **do**
6: Find the vertex v_0 with the maximum degree, and let $T(H_S^{\delta}) = T(H_S^{\delta}) \cup \{v_0\}$
7: Delete all the hyperedges incident to the vertex v_0
8: **end while**
9: For each $v \in T(H_S^{\delta})$, if the hyperedges incident to v can be covered by the set of vertices $T(H_S^{\delta}) - \{v\}$, delete the vertex v
10: Return $T(H_S^{\delta})$

Table 22 Consistent formal fuzzy decision contexts used in tests

Data sets	$|U|$	$|C|$	$|D|$	$|B|$
$S1$	300	5	3	8
$S2$	1000	5	3	8
$S3$	450	15	9	24
$S4$	1800	30	18	48
$S5$	1600	40	24	64

4.3 Experimental setup and data sets

The datasets with decision attributes are listed in Table 22. Where $S1$ and $S2$ are the datasets generated by concatenating 60 and 200 times in Table 4 in Example 2, respectively. $S3$, $S4$, and $S5$ are the data sets generated in Table 4 by first merging 3, 6, and 8 times, respectively, and then concatenating 30, 60, and 40 times, respectively, in series. It is easy to verify that when δ takes 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 and 1, this data set is all δ-consistent. Depending on how the datasets are merged and concatenated, we can know that the consistent of the datasets after mergence or concatenation remains the same as the original dataset. That is, it is ensured that the resulting dataset is δ-consistent.

In our experiments, we obtained approximate sets of attributes while keeping constant the object granules with

Table 23 The running time of the first step of BRS and the GTS

δ	S1	S2	S3	S4	S5
$\delta = 0.1$	0.471342	4.809352	1.531034	131.263964	122.563986
$\delta = 0.2$	0.198509	4.485885	1.401534	129.889472	122.447427
$\delta = 0.3$	0.192808	4.346476	1.57229	126.926357	124.958064
$\delta = 0.4$	0.197799	4.306023	1.449022	126.952863	123.970145
$\delta = 0.5$	0.199039	4.279833	1.451244	131.337362	130.013984
$\delta = 0.6$	0.198132	4.328953	1.444457	130.425004	127.132066
$\delta = 0.7$	0.196264	4.508816	1.423177	129.904496	126.055775
$\delta = 0.8$	0.193214	4.305551	1.380441	127.267965	124.22763
$\delta = 0.9$	0.203292	4.325166	1.417604	126.990282	124.101829
$\delta = 1$	0.207324	4.661825	1.373026	123.557458	119.842988

Table 24 Comparison of the output of the two algorithms (S1)

S1	The cardinality of the reduct		Whether the output is a reduct?		The whole running time (s)	
	BRS	GTS	BRS	GTS	BRS	GTS
$\delta = 0.1$	2	2	Yes	Yes	0.011116	0.042651
$\delta = 0.2$	2	2	Yes	Yes	0.008991	0.014408
$\delta = 0.3$	2	2	Yes	Yes	0.009246	0.011113
$\delta = 0.4$	2	2	Yes	Yes	0.008816	0.011535
$\delta = 0.5$	3	3	Yes	Yes	0.010884	0.011543
$\delta = 0.6$	3	3	Yes	Yes	0.007316	0.012161
$\delta = 0.7$	3	3	Yes	Yes	0.008306	0.011858
$\delta = 0.8$	3	3	Yes	Yes	0.006984	0.011645
$\delta = 0.9$	3	3	Yes	Yes	0.00762	0.012751
$\delta = 1$	3	3	Yes	Yes	0.011381	0.019376

Table 25 Comparison of the output of the two algorithms (S2)

S2	The cardinality of the reduct		Whether the output is a reduct?		The whole running time (s)	
	BRS	GTS	BRS	GTS	BRS	GTS
$\delta = 0.1$	2	2	Yes	Yes	0.009758	0.014093
$\delta = 0.2$	2	2	Yes	Yes	0.008963	0.010775
$\delta = 0.3$	2	2	Yes	Yes	0.008986	0.010907
$\delta = 0.4$	2	2	Yes	Yes	0.009211	0.011193
$\delta = 0.5$	3	3	Yes	Yes	0.007709	0.011102
$\delta = 0.6$	3	3	Yes	Yes	0.00722	0.012759
$\delta = 0.7$	3	3	Yes	Yes	0.007364	0.011834
$\delta = 0.8$	3	3	Yes	Yes	0.007187	0.012101
$\delta = 0.9$	3	3	Yes	Yes	0.007082	0.011646
$\delta = 1$	3	3	Yes	Yes	0.00729	0.012944

decision attributes at different granularities, using both BRS and GTS algorithms. The two algorithms, BRS and GTS, each consist of two steps. The first step both uses Algorithm 4 to generate the δ-granular discernibility matrix for the FFDCs. Table 23 shows the running times for the first step for each data set. The difference between the algorithms BRS and GTS is in the second step. Tables 24, 25, 26, 27 and 28 present a comparison of the output of all datasets under the two algorithms, including the cardinality of the reduced set of attributes, whether it is a reduct or not, and the running time of step 2 of the algorithm. As can be seen from these tables, there are 50 cases, of which 47 output the same result, accounting for ninety-four percent of the total. The reason for the error is mainly because the two algorithms are approx-

Table 26 Comparison of the output of the two algorithms (S3)

S3	The cardinality of the reduct		Whether the output is a reduct?		The whole running time (s)	
	BRS	GTS	BRS	GTS	BRS	GTS
δ = 0.1	8	6	Yes	Yes	0.038171	0.082717
δ = 0.2	6	6	Yes	Yes	0.133986	0.055043
δ = 0.3	6	6	Yes	Yes	0.129259	0.022255
δ = 0.4	6	6	Yes	Yes	0.147201	0.022641
δ = 0.5	9	9	Yes	Yes	0.017294	0.023796
δ = 0.6	9	9	Yes	Yes	0.017683	0.027165
δ = 0.7	9	9	Yes	Yes	0.01757	0.026335
δ = 0.8	9	9	Yes	Yes	0.014167	0.022443
δ = 0.9	9	9	Yes	Yes	0.014041	0.023162
δ = 1	9	9	Yes	Yes	0.010617	0.011558

Table 27 Comparison of the output of the two algorithms (S4)

S4	The cardinality of the reduct		Whether the output is a reduct?		The whole running time (s)	
	BRS	GTS	BRS	GTS	BRS	GTS
δ = 0.1	17	14	Yes	Yes	0.121469	0.257477
δ = 0.2	12	12	Yes	Yes	33.845295	0.099595
δ = 0.3	12	12	Yes	Yes	33.832783	0.094227
δ = 0.4	12	12	Yes	Yes	33.787547	0.101769
δ = 0.5	18	18	Yes	Yes	0.117948	0.11743
δ = 0.6	18	18	Yes	Yes	0.055587	0.145412
δ = 0.7	18	18	Yes	Yes	0.058899	0.123401
δ = 0.8	18	18	Yes	Yes	0.037742	0.096154
δ = 0.9	18	18	Yes	Yes	0.039432	0.11152
δ = 1	18	18	Yes	Yes	0.019146	0.081042

Table 28 Comparison of the output of the two algorithms (S5)

S5	The cardinality of the reduct		Whether the output is a reduct?		The whole running time (s)	
	BRS	GTS	BRS	GTS	BRS	GTS
δ = 0.1	23	20	Yes	Yes	0.143238	0.248335
δ = 0.2	16	16	Yes	Yes	1310.1552	0.148324
δ = 0.3	16	16	Yes	Yes	1315.00261	0.089236
δ = 0.4	16	16	Yes	Yes	1408.81629	0.088159
δ = 0.5	24	24	Yes	Yes	0.199436	0.149124
δ = 0.6	24	24	Yes	Yes	0.112051	0.151874
δ = 0.7	24	24	Yes	Yes	0.116821	0.162886
δ = 0.8	24	24	Yes	Yes	0.075599	0.128042
δ = 0.9	24	24	Yes	Yes	0.072693	0.125623
δ = 1	24	24	Yes	Yes	0.032215	0.084164

imate. To reduce this error, the order of the input attributes can be adjusted.

In order to visualise the difference in running time between the two algorithms, we have plotted Figs. 11, 12, 13, 14 and 15. where the X-axis represents the granularity value and the Y-axis is the running time of the second step of the algorithm in seconds. Note that the Y-axis is logarithmic. As can be seen

from the figures, in most cases GTS performs worse than BRS in terms of running time, but GRS is more stable than BRS and does not fluctuate much. As can be seen in Fig. 16, BRS has much more total running time for the second step than GTS in all datasets.

Algorithm 5 (BRS) and Algorithm 6 (GTS) both aim at finding the granular reducts of FFDCs, but the difference is
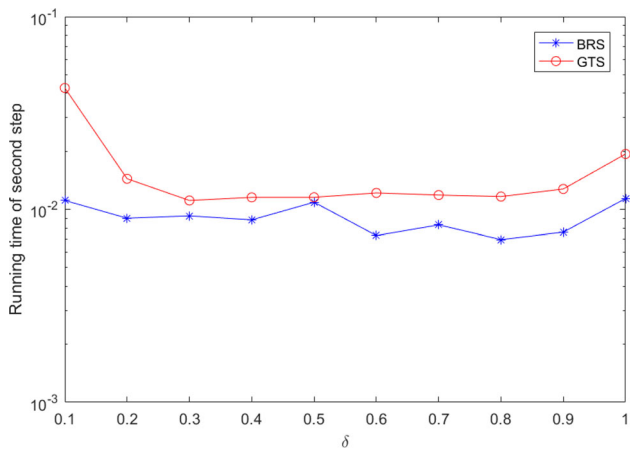
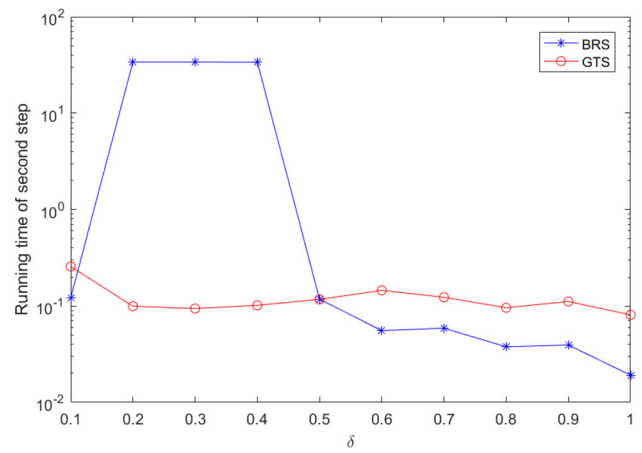**Fig. 11** Comparison of the running times of the second step of the BRS and the GTS (S1)

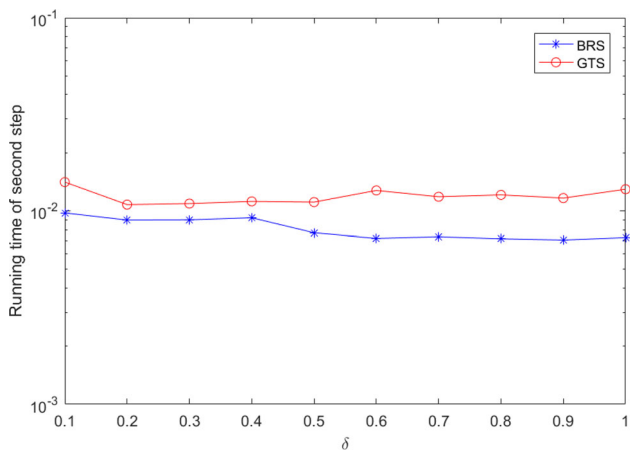**Fig. 14** Comparison of the running times of the second step of the BRS and the GTS (S4)

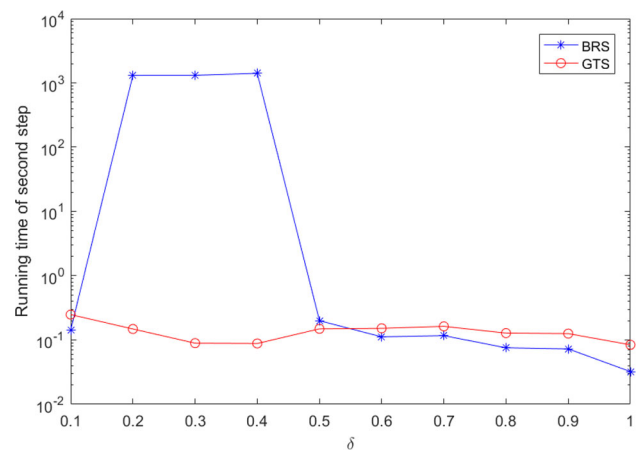**Fig. 12** Comparison of the running times of the second step of the BRS and the GTS (S2)

Fig. 15 Comparison of the running times of the second step of the BRS and the GTS (S5)
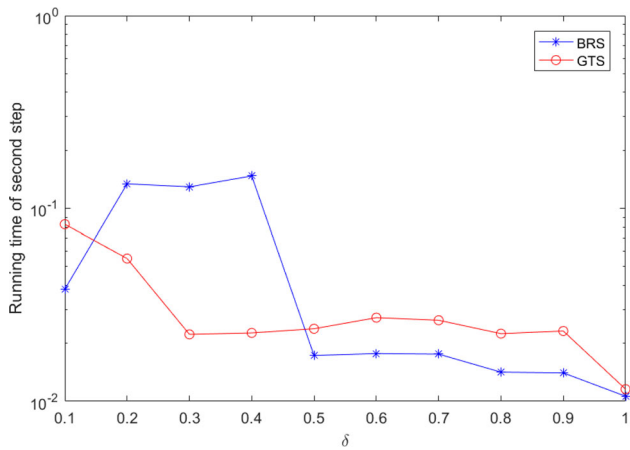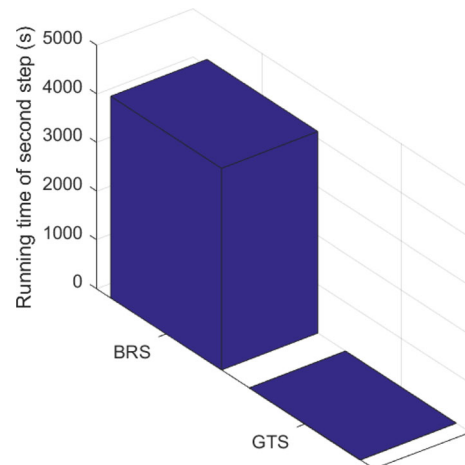

Fig. 13 Comparison of the running times of the second step of the BRS and the GTS (S3)

Fig. 16 Total run time in all datasets for the second step of BRS and GTS

that BRS calculates all the granular reducts of an FFDC, but GTS only calculates one granular reduct of an FFDC. By analyzing the algorithms and the results of data experiments, we can know that the BRS algorithm has high complexity and long running time, while the GTS algorithm has low complexity and high reduction efficiency.

5 Conclusions

In this paper, we introduce two mappings g_δ and $g_{\bar{\delta}}$, presenting the concept of an object granule of different granularity and two methods of attribute reduction with the guarantee that this object granule remains unchanged, one algorithm based on Boolean reasoning and the other on graph theory. Also, in order to simplify the conceptual lattice structure, we introduce a partial order relation with parameters. These works have certain significance for the theoretical framework of attribute reduction in FFCs. Then, there are still some limitations. For example, (1) the proposed mapping g_δ and f cannot form a pair of Galois connections. (2) The proposed partial order relation with parameters simplifies the concept lattice but does not guarantee that important concepts are not missed. (3) The δ-granular reduction method proposed in this paper is only applicable to consistent FFDCs, but real datasets are usually inconsistent, and this method is limited in practical data processing and knowledge mining.

Therefore, there are still many issues that we need to discuss in the future. For example, (1) How improving the partial order relations with parameters allows simplifying the structure of the original concept lattice without losing important concepts. (2) For the practicality of δ-granular reduction, we will investigate δ-granular reduction in the inconsistent FFDCs in the future.

Declarations

References

Addison G, Izadpanahi A, Saha S, Winter M (2023) L-fuzzy concept analysis using fuzzy categories. Fuzzy Sets Syst. https://doi.org/10.1016/j.fss.2022.11.008

Barbut M, Monjardet B (1970) Ordre et classification: Algèbre et combinatoire. Hachette, Paris

Bělohlávek R (2001a) Fuzzy closure operators. J Math Anal Appl 262:473–489

Bělohlávek R (2001b) Reduction and a simple proof of characterization of fuzzy concept lattices. Fundam Inform 46(4):277–285

Bělohlávek R (2001c) Lattices of fixed points of fuzzy Galois connections. Math Log Q 47(1):111–116

Bělohlávek R (2015) Fuzzy Galois connections. Math Logic Q 45(4):497–504

Bělohlávek R, Baets BD, Outrata J, Vychodil V (2010) Computing the lattice of all fixpoints of a fuzzy closure operator. IEEE Trans Fuzzy Syst 18(3):546–557

Bělohlávek R, Sklenar V, Zacpal J (2005) Crisply generated fuzzy concepts. In: Proceedings of ICFCA 2005, Lecture Notes in Artificial Intelligence, vol. 3403, pp. 269–284

Bondy JA, Murty USR (2008) Graph theory. Springer, Berlin

Burusco A, Fuentes-Gonzalez R (1994) The study of the L-fuzzy concept lattices. Mathware Soft Comput 3:209–218

Burusco A, Fuentes-González R (2000) Concept lattices defined from implication operators. Fuzzy Sets Syst 114(3):431–436

Chen D, Wang C, Hu Q (2007) A new approach to attribute reduction of consistent and inconsistent covering decision systems with covering rough sets. Inf Sci 177(17):3500–3518

Chen JK, Mi JS, Lin YJ (2018) A graph approach for knowledge reduction in formal contexts. Knowl Based Syst 148(15):177–188

Chen JK, Mi JS, Lin YJ (2020) A graph approach for fuzzy-rough feature selection. Fuzzy Sets Syst 391:96–116

Eiter T, Gottlob G (1995) Identifying the minimal transversals of a hypergraph and related problems. SIAM J Comput 24(6):1278–1304

Elloumi S, Jaam J, Hasnah A, Jaoua A, Nafkha I (2004) A multilevel conceptual data reduction approach based on the Lukasiewicz implication. Inf Sci 163(4):253–262

Ganter B, Wille R (1999) Formal concept analysis. Mathematic foundations. Springer, Berlin

Gediga, G, Düntsch, I (2002) Modal-style operators in qualitative data analysis. In: Proc. IEEE International Conference on Data Mining, pp. 155–162

Georgescu G, Popescu A (2004) Non-dual fuzzy connections. Arch Math Logic 43(8):1009–1039

Gong ZT, Wang Q (2017) On the connection of fuzzy hypergraph with fuzzy information system. J Intell Fuzzy Syst 33(3):1665–1676

Huang CC, Li JH, Dias SM (2016) Attribute significance, consistency measure and attribute reduction in formal concept analysis. Neural Netw World 26(6):607–623

Jaoua A, Elloumi S (2002) Galois connection, formal concepts and Galois lattice in real relations: application in a real classifier. J Syst Softw 60(2):149–163

Jiménez-Losada A, Ordóez M (2023) Sharing profiits in formal fuzzy contexts, Fuzzy Sets Syst. http://creativecommons.org/licenses/by-nc-nd/4.0/

Kaytoue M, Kuznetsov SO, Napoli A, Duplessis S (2011) Mining gene expression data with pattern structures in formal concept analysis. Inf Sci 181(10):1989–2001

Konečný J (2017) On attribute reduction in concept lattices: methods based on discernibility matrix are outperformed by basic clarification and reduction. Inf Sci 415:199–212

Konečný J, Krajča P (2018) On attribute reduction in concept lattices: experimental evaluation shows discernibility matrix based methods inefficient. Inf Sci 467:431–445

Konečný J, Krajča P (2019) On attribute reduction in concept lattices: the polynomial time discernibility matrix-based method becomes the CR-method. Inf Sci 491:48–62

Krajči S (2003) Cluster based efficient generation of fuzzy concepts. Neural Netw. World 5:521–530

Kridlo O, Ojeda-Aciego M (2017) Extending formal concept analysis using intuitionistic L-fuzzy sets. In: IEEE International Conference on Fuzzy Systems, IEEE

Kulaga P, Sapiecha P, Krzysztof S (2005) Approximation algorithm for the argument reduction problem. Computer recognition systems. Springer, Berlin, pp 243–248

Li L, Zhang J (2010) Attribute reduction in fuzzy concept lattices based on the T implication. Knowl Based Syst 23:497–503

Li JH, Mei CL, Lv YJ (2011) A heuristic knowledge-reduction method for decision formal contexts. Comput Math 4(61):1096–1106

Liang JY, Wang F, Dang C, Qian YH (2012) An efficient rough feature selection algorithm with a multigranulation view. Int J Approx Reason 53(6):912–926

Listrovoy S, Minukhin S (2012) The solution algorithms for problems on the minimal vertex cover in networks and the minimal cover in Boolean matrixes. Int J Comput Sci Issues 9(5):8–15

Maio CD, Fenza G, Loia V, Senatore S (2012) Hierarchical web resources retrieval by exploiting fuzzy formal concept analysis. Inf Process Manage 48(3):399–418

Mao H (2017) Representing attribute reduction and concepts in concept lattice using graphs. Soft Comput 21(24):7293–7311

Mi JS, Leung Y, Wu WZ (2010) Approaches to attribute reduction in concept lattices induced by axialities. Knowl Based Syst 23(6):504–511

Missaoui R, Godin R, Boujenoui A(1994) Extracting exact and approximate rules from databases. In: Workshop on Incompleteness & Uncertainty in Information Systems, DBLP

Poelmans J, Ignatov DI, Kuznetsov SO, Dedene G (2013) Formal concept analysis in knowledge processing: a survey on applications. Expert Syst Appl 40:6538–6560

Poelmans J, Ignatov DI, Viaene S, Dedene G, Kuznetsov SO (2012) Text mining scientific papers: a survey on FCA-based information retrieval research. In: Industrial Conference on Advances in Data Mining: Applications and Theoretical Aspects, Springer, pp. 273–287

Popescu A (2004) A general approach to fuzzy concepts. Math Log Q 50(3):265–280

Shao MW, Yang HZ, Wu WZ (2015) Knowledge reduction in formal fuzzy contexts. Knowl Based Syst 73:265–275

Shao MW, Leung Y, Wang XZ, Wu WZ (2016) Granular reducts of formal fuzzy contexts. Knowl Based Syst 114:156–166

Singh PK, Cherukuri AK, Li J (2017) Concepts reduction in formal concept analysis with fuzzy setting using Shannon entropy. Int J Mach Learn Cybern 8(1):179–189

Skowron A (1993) Boolean reasoning for implication rules generation. Methodologies for intelligent systems. Springer, Berlin, pp 295–305

Skowron A, Rauszer C (1992) The discernibility matrices and functions in information systems. Intelligent decision support: handbook of applications and advances of the rough sets theory. Kluwer Academic Publishers, Dordrecht, pp 331–362

Wang Q, Gong ZT (2018) An application of fuzzy hypergraphs and hypergraphs in granular computing. Inf Sci 429:296–314

Wei L, Qi JJ, Zhang WX (2008) Concept lattice attribute reduction for formal decision contexts. Sci China: Ser E-Inf Sci 38(2):195–208

Wille R (1982) Restructuring lattice theory: an approach based on hierarchies of concepts. In: Rival I (ed) Ordered sets. Reidel, Dordrecht, pp 445–470

Wu WZ, Leung Y, Mi JS (2009) Granular computing and knowledge reduction in formal contexts. IEEE Trans Knowl Data Eng 21(10):1461–1474

Xia H, Chen Y, Gao H, Li Z, Chen Y (2010) Concept lattice-based semantic web service matchmaking. In: Communication Software and Networks, International Conference, pp. 439–443

Yahia SB, Arour K, Slimani A, Jaoua A (2000) Discovery of compact rules in relational databases. Inf Sci J 3(4):497–511

Yao YY (2004a) A comparative study of formal concept analysis and rough set theory in data analysis. Rough sets and current trends in computing. Lecture notes in artificial intelligence. Springer, Berlin, pp 59–68

Yao YY (2004b) Concept lattices in rough set theory. In: Proceedings of the North American Fuzzy Information Processing Society, pp. 796–801

Zadeh LA (1979) Fuzzy sets and information granularity. North-Holland, Amsterdam

Zhang Z (2018) Constructing L-fuzzy concept lattices without fuzzy Galois closure operation. Fuzzy Sets Syst 333:71–86

Zhang J, Wei L (2016) Rules acquisition and attribute reduction of ordered formal decision contexts. Pattern Recognit Artif Intell 29:976–984

Zhang WX, Wei L, Qi JJ (2005) Attribute reduction theory and approach of concept lattices. Sci China Ser E-Inf Sci 35(6):628–639

Zhang WX, Ma JM, Fan SQ (2007) Variable threshold concept lattices. Inf Sci 177(22):4883–4892

Zou CF, Deng HF (2017) Using fuzzy concept lattice for intelligent disease diagnosis. IEEE Access 5:236–242