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FUZZY SYSTEMS AND THEIR MATHEMATICS

L-fuzzy filters on complete residuated lattices

Ahmed A. Ramadan[1]

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Abstract

This paper is toward the establishment of relationships between L-fuzzy filters, L-fuzzy topological spaces and L-fuzzy pre-proximity spaces in complete residuated lattices. We have demonstrated the existence of functors between the categories of L-fuzzy filter spaces, L-fuzzy topological spaces and L-fuzzy pre-proximity spaces.

Keywords Complete residuated lattice · L-fuzzy filter · L-fuzzy topology · L-fuzzy pre-proximity · Functors

1 Introduction

Ward and Dilworth (1939) introduced the notion of complete residuated lattice as a primitive concept which is highly useful for structure of truth value in many valued logic. Bělohlávek (2002) proved that fuzzy relations with truth values in complete residuated lattice are capable of modeling intelligent systems with insufficient and incomplete information. Höhle and Šostak (1999) used different algebraic structures (cqm, quantales, MV-algebra) of truth value to introduce concepts of L-fuzzy topologies. Further, these algebraic structures provided several directions of study in mathematics as well as in logic and L-fuzzy topologies (cf., Fang 2010; Fang and Yue 2010; Koguep et al. 2008; Kubiak 1985; Kubiak and Šostak 1997; Chen and Zhang 2010; Ramadan et al. 2015; Ramadan and Kim 2018; Ramadan et al. 2022; Rodabaugh and Klement 2003; Šostak 1985, 1989; Tiwari et al. 2018; Yue 2007; Zhang 2007; Ramadan 1992; Liang and Shi 2014).

Many authors studied the relationship between fuzzy topologies and L-filters. In 1977, Lowen (1979) developed the idea of filters in I^X where $I = [0, 1]$ is the unit interval of real numbers, called prefilters to discuss convergence in fuzzy topological spaces. In 1999, Burton et al. (1999) introduced the concept of generalized filters as a mapping from 2^X to I. Subsequently, Höhle and Šostak (1999) developed the notion of L-filters and stratified L-filters on a complete quasi-monoidal lattice. Later, in Jäger (2013) developed the theory of stratified LM-filters which generalizes the theory of stratified L-filters by introducing stratification mapping, where L and M are frames (cf., Ko 2018; Koguep et al. 2008; Ramadan 1997; Liu et al. 2017; Tonga 2011). In Ramadan (2003), the authors introduced the concept of smooth ideal as a mapping from I^X to I which is the dual of a smooth filter (Ramadan 1997).

In this paper, we identify L-fuzzy topologies and L-fuzzy pre-proximities induced by L-fuzzy (prime) filters and study categorical relations between L-fuzzy (prime) filter spaces, L-fuzzy topological spaces and L-fuzzy pre-proximity spaces. The study obtains functors from the categories of L-fuzzy (prime) filter spaces, L-fuzzy topological spaces and L-fuzzy pre-proximity spaces.

2 Preliminaries

Definition 1 (Bělohlávek 2002; Hájek 1998; Höhle and Šostak 1999; Rodabaugh and Klement 2003; Turunen 1999) A complete residuated lattice is a pair $(L, \odot)$ which satisfies the following conditions:

(C1) $(L, \leq, \vee, \wedge, \perp, \top)$ is a complete lattice with the greatest element $\top$ and the least element $\perp$;
(C2) $(L, \odot, \top)$ is a commutative monoid;
(C3) $x \odot (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma} (x \odot y_i)$, for all $x \in L$ and $\{y_i\}_{i\in\Gamma} \subseteq L$. The binary relation $\odot$ induces another binary operation $\rightarrow$ on L which satisfies:
(C4) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ for $x, y, z \in L$.

In this paper, we always assume that $L = (L, \leq, \odot)$ is a complete residuated lattice unless otherwise specified.

✉ Ahmed A. Ramadan
ahmed.ramadan@science.bsu.edu.eg

[1] Department of Mathematics and Computer Science, Faculty of Science, Beni-Suef University, Beni-Suef 62511, Egypt

L is called idempotent if $x \odot x = x$, for $x \in L$.

Remark 1 The following lattices $(L, \leq, \odot)$ are complete residuated lattices.

(1) Complete locally finite BL-algebra.
(2) Any complete Boolean algebra where the operations $\odot$ and $\wedge$ coincide,
(3) Every left-continuous t-norm T on $([0,1], \leq, t)$ with $\odot = t$.
(4) Every GL-monoid.

Some basic properties of the binary operation $\odot$ and residuated operation $\rightarrow$ are collected in the following lemma, and they can be found in many works, for instance, (Bělohlávek 2002; Hájek 1998; Höhle and Šostak 1999; Rodabaugh and Klement 2003; Turunen 1999).

Lemma 1 *Let L be a complete residuated lattice. For each $x, y, z, x_i, y_i, w \in L, i \in \Gamma$, we have the following properties:*

(1) $x \rightarrow y = \bigvee\{z : z \odot x \leq y\}$,
(2) $\top \rightarrow x = x, \bot \odot x = \bot$, *and* $x \leq y$ *iff* $x \rightarrow y = \top$,
(3) *If* $y \leq z$, *then* $x \odot y \leq x \odot z, x \oplus y \leq x \oplus z, x \rightarrow y \leq x \rightarrow z$ *and* $z \rightarrow x \leq y \rightarrow x$,
(4) $x \odot (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \odot y_i)$, $x \rightarrow (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \rightarrow y_i)$,
(5) $(\bigvee_{i\in\Gamma} x_i) \rightarrow y = \bigwedge_{i\in\Gamma}(x_i \rightarrow y)$,
(6) $\bigvee_{i\in\Gamma} x_i \rightarrow \bigvee_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \rightarrow y_i)$, $\bigwedge_{i\in\Gamma} x_i \rightarrow \bigwedge_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \rightarrow y_i)$,
(7) $x \rightarrow (\bigvee_{i\in\Gamma} y_i) \geq \bigvee_{i\in\Gamma}(x \rightarrow y_i)$, $(\bigwedge_{i\in\Gamma} x_i) \rightarrow y \geq \bigvee_{i\in\Gamma}(x_i \rightarrow y)$,
(8) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$ *and* $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$,
(9) $(x \rightarrow y) \odot (z \rightarrow w) \leq (x \odot z) \rightarrow (y \odot w)$.

L is said to satisfy the double negation law if for any $x \in L, (x \rightarrow \bot) \rightarrow \bot = x$. In the following, we use x^* to denote $x \rightarrow \bot$. Furthermore, for any $x, y \in L$, we define $x \oplus y = (x^* \odot y^*)^*$.

Lemma 2 *If L satisfies the double negation law, then it satisfies moreover:*

(1) *If* $y \leq z$, *then* $x \oplus y \leq x \oplus z$,
(2) $(x \rightarrow y) \odot (z \rightarrow w) \leq (x \oplus z) \rightarrow (y \oplus w)$.
(3) $(x \odot y) \odot (z \oplus w) \leq (x \odot z) \oplus (y \odot w)$,
(4) $(x \oplus z) \odot (y \oplus w) \leq (x \oplus y) \oplus (z \odot w)$,
(5) $(\bigwedge_{i\in\Gamma} y_i)^* = \bigvee_{i\in\Gamma} y_i^*$ *and* $(\bigvee_{i\in\Gamma} y_i)^* = \bigwedge_{i\in\Gamma} y_i^*$,
(6) $x \rightarrow y = y^* \rightarrow x^*$ *and* $x \rightarrow y = (x \odot y^*)^*$,
(7) $\bigwedge_{i\in\Gamma} x_i \oplus \bigwedge_{j\in\Gamma} y_j = \bigwedge_{i\in\Gamma}\bigwedge_{j\in\Gamma}(x_i \oplus y_j)$.

Definition 2 (Bělohlávek 2002; Rodabaugh and Klement 2003) Let X be a set. A mapping $R_X : X \times X \rightarrow L$ is called L-fuzzy relation on X. Then, R is said to be

(1) reflexive if $R_X(x, x) = \top$ for all $x \in X$,
(2) transitive if $R_X(x, y) \odot R_X(y, z) \leq R_X(x, z)$ for all $x, y, z \in X$.

An L-fuzzy relation on X is called an L-fuzzy pre-order if it is reflexive and transitive.

All algebraic operation on L can be extended pointwise to L^X Goguen (1967). For $f, g \in L^X$, we denote $(f \rightarrow g), (f \odot g) \in L^X$ as $(f \rightarrow g)(x) = f(x) \rightarrow g(x), (f \odot g)(x) = f(x) \odot g(x)$,

$$\top_x(y) = \begin{cases} \top, & \text{if } y = x, \\ \bot, & \text{otherwise,} \end{cases} \quad \top_x^*(y) = \begin{cases} \bot, & \text{if } y = x, \\ \top, & \text{otherwise.} \end{cases}$$

Lemma 3 (Bělohlávek 2002; Fang 2010; Fang and Yue 2010) *Let X be a nonempty set, define a binary mapping $S : L^X \times L^X \rightarrow L$ of f, g by*

$$S(f, g) = \bigwedge_{x\in X}(f(x) \rightarrow g(x)).$$

Then, for each $f, g, f_i, g_i, h, l \in L^X$, $i \in \Gamma$, the following properties hold:

(1) $S(f, g) = \top \Leftrightarrow f \leq g$,
(2) $f \leq g \Rightarrow S(f, h) \geq S(g, h)$ *and* $S(h, f) \leq S(h, g)$,
(3) $S(f, g) \odot S(h, l) \leq S(f \odot h, g \odot l)$,
(4) $\bigwedge_{i\in\Gamma} \mathcal{S}(f_i, g_i) \leq \mathcal{S}(\bigvee_{i\in\Gamma} f_i, \bigvee_{i\in\Gamma} g_i)$ *and* $\bigwedge_{i\in\Gamma} \mathcal{S}(f_i, g_i) \leq \mathcal{S}(\bigwedge_{i\in\Gamma} f_i, \bigwedge_{i\in\Gamma} g_i)$,
(5) $S(f, g) \odot S(h, l) \leq S(f \oplus h, g \oplus l)$,
(6) *If L satisfies the double negation law, then* $S(f, g) = S(g^*, f^*)$.

Definition 3 (Adámek et al. 1990) A pair $(\mathcal{C}, U)$ is said to be a concrete category if $\mathcal{C}$ is a category and $U : \mathcal{C} \rightarrow$ Set is a faithful functor (or a forgetful functor). For each $\mathcal{C}$-object X, $U(X)$ is the underlying set of X. Thus, all objects in a concrete category can be taken as structured set. We write $\mathcal{C}$ for $(\mathcal{C}, U)$, if the concrete functor is clear. Categories presented in this paper are concrete categories. A concrete functor between two concrete categories $(\mathcal{C}, U)$ and $(\mathcal{D}, V)$ is a functor $G : \mathcal{C} \rightarrow D$ with $U = V \circ G$, which means that G only changes the structures on the underlying sets. Hence, in order to define a concrete functor $G : \mathcal{C} \rightarrow D$, we only consider the following two requirements. First, we assign to each $\mathcal{C}$-object X, a $\mathcal{D}$-object $G(X)$ such that $V(G(X)) = U(X)$. Second, we verify that if a function $f : U(X) \to U(Y)$ is a $\mathcal{C}$-morphism $X \to Y$, then it is also a $\mathcal{D}$-morphism $G(X) \to G(Y)$.

Definition 4 (Höhle and Šostak 1999; Rodabaugh and Klement 2003) A mapping $\mathcal{T} : L^X \to L$ is called L-fuzzy topology on X if it satisfies the following conditions:

(T1) $\mathcal{T}(\bot_X) = \mathcal{T}(\top_X) = \top$,
(T2) $\mathcal{T}(f \odot g) \geq \mathcal{T}(f) \odot \mathcal{T}(g)\ \forall f, g \in L^X$,
(T3) $\mathcal{T}(\bigvee_{i\in\Gamma} f_i) \geq \bigwedge_{i\in\Gamma} \mathcal{T}(f_i)$ for all $\{f_i : i \in \Gamma\} \subseteq L^X$.
The pair $(X, \mathcal{T})$ is called an L-fuzzy topological space. An L-fuzzy topological space is called
(AL) Alexandrov if $\mathcal{T}(\bigwedge_{i\in\Gamma} f_i) \geq \bigwedge_{i\in\Gamma} \mathcal{T}(f_i)\ \forall\ \{f_i : i \in \Gamma\} \subseteq L^X$,
(SE) discrete if $\mathcal{T}(\top_x) = \top$ for all $x \in X$.

Definition 5 (Chen and Zhang 2010; Xiu and Li 2019) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two L-fuzzy topological spaces and $\varphi : X \to Y$ be a mapping. Then, $D_{\mathcal{T}}(\varphi)$ defined by

$$D_{\mathcal{T}}(\varphi) = \bigwedge_{f\in L^Y} (\mathcal{T}_Y(f) \to \mathcal{T}_X(\varphi^{\leftarrow}(f)))$$

is the degree to which the map φ is an LF-continuous map.

If $D_{\mathcal{T}}(\varphi) = \top$, then $\mathcal{T}_Y(f) \leq \mathcal{T}_X(\varphi^{\leftarrow}(f))$ for all $f \in L^Y$, which is exactly the definition of LF-continuous map between L-fuzzy topological spaces.

The category of L-fuzzy topological spaces with LF-continuous mappings as morphisms is denoted by L-FTOP. Write AL-FTOP for the full subcategory of L-FTOP composed of objects of all Alexandrov L-fuzzy topological spaces.

Definition 6 (Ko 2018; Rodabaugh and Klement 2003) An L-fuzzy pre-filter on a set X is defined to be a mapping $\mathcal{F} : L^X \to L$ satisfying:

(LF1) $\mathcal{F}(\bot_X) = \bot$,
(LF2) $S(f, g) \leq \mathcal{F}(f) \to \mathcal{F}(g),\ \ \forall f, g \in L^X$, The pair $(X, \mathcal{F})$ is called an L-fuzzy pre-filter space. An L-fuzzy pre-filter is L-fuzzy filter if it satisfies
(LF3) $\mathcal{F}(f \odot g) \geq \mathcal{F}(f) \odot \mathcal{F}(g),\ \ \forall f, g \in L^X$. The pair $(X, \mathcal{F})$ is called an L-fuzzy filter space. An L-fuzzy pre-filter space is called
(AL) Alexandrov if $\mathcal{F}(\bigwedge_{i\in\Gamma} f_i) \geq \bigwedge_{i\in\Gamma} \mathcal{F}(f_i)\ \forall\ \{f_i : i \in \Gamma\} \subseteq 2^X$,
(SE) discrete if $\mathcal{F}(\top_x) = \top$ for all $x \in X$.

Definition 7 Let $(X, \mathcal{F}_X)$ and $(Y, \mathcal{F}_Y)$ be two L-fuzzy filter spaces and $\varphi : X \to Y$ be a mapping. Then, $D_{\mathcal{F}}(\varphi)$ defined by

$$D_{\mathcal{F}}(\varphi) = \bigwedge_{f\in L^Y} (\mathcal{F}_Y(f) \to \mathcal{F}_X(\varphi^{\leftarrow}(f)))$$

is the degree to which the map φ is an LF-filter map.

If $D_{\mathcal{F}}(\varphi) = \top$, then $\mathcal{F}_Y(f) \leq \mathcal{F}_X(\varphi^{\leftarrow}(f))$ for all $f \in L^Y$, which is exactly the definition of LF-filter map between L-fuzzy filter spaces.

Remark 2 In addition to the above axioms, if (LF4) $\mathcal{F}(\top_X) = \top$, then $(X, \mathcal{F})$ is called L-fuzzy prime filter space.

The category of L-fuzzy (prime) filter spaces with LF-filter mappings as morphisms is denoted by LF(P-LF). Write A-LF (AP-LF) for the full subcategory of LF(P-LF) composed of objects of all Alexandrov L-fuzzy (prime) filter spaces.

3 The relationships between *L*-fuzzy (prime) filter spaces and topological spaces

From the following theorems, we obtain the L-fuzzy topological spaces induced by an L-fuzzy prime filter spaces

Theorem 1 *Let $\mathcal{F}$ be an L-fuzzy (prime) filter on X and L satisfies the double negation law. Define $\mathcal{T}^{(1)}_{\mathcal{F}} : L^X \to L$ as follows:*

$$\mathcal{T}^{(1)}_{\mathcal{F}}(f) = \bigwedge_{x\in X} \Big(f^*(x) \oplus (f(x) \odot \mathcal{F}(f))\Big).$$

Then,

(1) *$(X, \mathcal{T}^{(1)}_{\mathcal{F}})$ is an L-fuzzy topological space.*
(2) *If $\mathcal{F}$ is discrete, then so is $\mathcal{T}^{(1)}_{\mathcal{F}}$.*
(3) *Let $\bigwedge_{i\in\Gamma} (x_i \odot y_i) = \bigwedge_{i\in\Gamma} x_i \odot \bigwedge_{i\in\Gamma} y_i$ for each $x_i, y_i \in L$. If $\mathcal{F}$ is Alexandrov, then so is $\mathcal{T}^{(1)}_{\mathcal{F}}$.*

Proof (1) (1)

(T1) Since $\mathcal{T}^{(1)}_{\mathcal{F}}(\bot_X) = \bigwedge_{x\in X} \Big(\top_X(x) \oplus (\bot_X(x) \odot \mathcal{F}(\bot_X))\Big) = \top$, $\mathcal{T}^{(1)}_{\mathcal{F}}(\top_X) = \bigwedge_{x\in X} \Big(\bot_X(x) \oplus (\top_X(x) \odot \mathcal{F}(\top_X))\Big) = \top$.

(T2) For $f, g \in L^X$,

$$\begin{aligned}
&\mathcal{T}_{\mathcal{F}}^{(1)}(f) \odot \mathcal{T}_{\mathcal{F}}^{(1)}(g) \\
&\quad = \bigwedge_{x\in X} \Big(f^*(x) \oplus (f(x) \odot \mathcal{F}(f))\Big) \\
&\quad\ \odot \bigwedge_{x\in X} \Big(g^*(x) \oplus (g(x) \odot \mathcal{F}(g))\Big) \\
&\le \bigwedge_{x\in X} \Big[\Big(f^*(x) \\
&\quad \oplus (f(x) \odot \mathcal{F}(f))\Big) \odot \Big(g^*(x) \oplus (g(x) \odot \mathcal{F}(g))\Big)\Big] \\
&\le \bigwedge_{x\in X} \Big[\Big(f^*(x) \oplus g^*(x)\Big) \\
&\quad \oplus \Big(f(x) \odot \mathcal{F}(f) \odot g(x) \odot \mathcal{F}(g)\Big)\Big] \\
&\quad \text{(by Lemma 2 (3))} \\
&\le \bigwedge_{x\in X} \Big[(f \odot g)^*(x) \oplus ((f \odot g)(x) \odot \mathcal{F}(f \odot g))\Big] \\
&= \mathcal{T}_{\mathcal{F}}^{(1)}(f \odot g).
\end{aligned}$$

(T3) For each family $\{f_i : i \in \Gamma\}$

$$\begin{aligned}
&\mathcal{T}_{\mathcal{F}}^{(1)}(\bigvee_{i\in\Gamma} f_i) \\
&\quad = \bigwedge_{x\in X} \Big((\bigvee_{i\in\Gamma} f_i)^*(x) \oplus (\bigvee_{i\in\Gamma} f_i(x) \odot \mathcal{F}(\bigvee_{i\in\Gamma} f_i))\Big) \\
&\quad \ge \bigwedge_{x\in X} \Big(\bigwedge_{i\in\Gamma} f_i^*(x) \oplus \Big(\bigwedge_{i\in\Gamma} \Big[f_i(x) \odot \mathcal{F}(f_i)\Big]\Big)\Big) \\
&\quad = \bigwedge_{x\in X} \bigwedge_{i\in\Gamma} \Big(f_i^*(x) \oplus [f_i(x) \odot \mathcal{F}(f_i)]\Big) \\
&\quad = \bigwedge_{i\in\Gamma} \bigwedge_{x\in X} \Big(f_i^*(x) \oplus (f_i(x) \odot \mathcal{F}(f_i))\Big) \\
&\quad = \bigwedge_{i\in\Gamma} \mathcal{T}_{\mathcal{F}}^{(1)}(f_i).
\end{aligned}$$

Hence, $\mathcal{T}_{\mathcal{F}}^{(1)}$ is an L-fuzzy topology on X.

(2)

$$\begin{aligned}
&\mathcal{T}_{\mathcal{F}}^{(1)}(\top_x) = \bigwedge_{y\in X} \Big(\top_x^*(y) \oplus (\top_x(y) \odot \mathcal{F}(\top_x)\Big) \\
&\quad = \Big(\top_x^*(x) \oplus (\top_x(x) \odot \mathcal{F}(\top_x)\Big) \\
&\quad\ \bigwedge \bigwedge_{y\in X, y\neq x} \Big(\top_x^*(y) \oplus (\top_x(y) \odot \mathcal{F}(\top_x)\Big) \\
&\quad = \Big(\bot \oplus (\top \odot \top)\Big) \bigwedge \bigwedge_{y\in X, y\neq x} \Big(\top \oplus (\bot \odot \top)\Big) \\
&\quad = \top.
\end{aligned}$$

(3) For each family $\{f_i : i \in \Gamma\}$

$$\begin{aligned}
&\bigwedge_{i\in\Gamma} \mathcal{T}_{\mathcal{F}}^{(1)}(f_i) \\
&\quad = \bigwedge_{i\in\Gamma} \bigwedge_{x\in X} \Big(f_i^*(x) \oplus (f_i(x) \odot \mathcal{F}(f_i))\Big) \\
&\quad = \bigwedge_{x\in X} \bigwedge_{i\in\Gamma} \Big(f_i^*(x) \oplus (f_i(x) \odot \mathcal{F}(f_i))\Big) \\
&\quad = \bigwedge_{x\in X} \Big((\bigwedge_{i\in\Gamma} f_i^*)(x) \oplus \bigwedge_{i\in\Gamma} (f_i(x) \odot \mathcal{F}(f_i))\Big) \\
&\quad = \bigwedge_{x\in X} \Big((\bigwedge_{i\in\Gamma} f_i^*)(x) \oplus (\bigwedge_{i\in\Gamma} f_i(x) \odot \bigwedge_{i\in\Gamma} \mathcal{F}(f_i))\Big) \\
&\quad \le \bigwedge_{x\in X} \Big((\bigvee_{i\in\Gamma} f_i^*)(x) \oplus (\bigwedge_{i\in\Gamma} f_i(x) \odot \bigwedge_{i\in\Gamma} \mathcal{F}(f_i))\Big) \\
&\quad \le \bigwedge_{x\in X} \Big(\bigwedge_{i\in\Gamma} f_i)^*(x) \oplus ((\bigwedge_{i\in\Gamma} f_i)(x) \odot \mathcal{F}(\bigwedge_{i\in\Gamma} f_i))\Big) \\
&\quad = \mathcal{T}_{\mathcal{F}}^{(1)}(\bigwedge_{i\in\Gamma} f_i).
\end{aligned}$$

□

Theorem 2 *Let $(X, \mathcal{F}_X)$ and $(Y, \mathcal{F}_Y)$ be L-fuzzy (prime) filter spaces and L satisfies the double negation law. Let $\varphi : X \to Y$ be a mapping, then $D_{\mathcal{F}}(\varphi) \le D_{\mathcal{T}_{\mathcal{F}}^{(1)}}(\varphi)$.*

Proof For any $f \in L^Y$,

$$\begin{aligned}
&D_{\mathcal{T}_{\mathcal{F}}^{(1)}}(\varphi) = \bigwedge_{f\in L^Y} \Big(\mathcal{T}_{\mathcal{F}_Y}^{(1)}(f) \to \mathcal{T}_{\mathcal{F}_X}^{(1)}(\varphi^{\leftarrow}(f))\Big) \\
&\quad = \bigwedge_{f\in L^Y} \Big[\bigwedge_{y\in Y} \Big(f^*(y) \oplus (f(y) \odot \mathcal{F}_Y(f))\Big) \\
&\quad \to \bigwedge_{x\in X} \Big(\varphi^{\leftarrow}(f^*)(x) \oplus (\varphi^{\leftarrow}(f)(x) \odot \mathcal{F}_X(\varphi^{\leftarrow}(f))\Big)\Big] \\
&\quad = \bigwedge_{f\in L^Y} \Big[\bigwedge_{y\in Y} \Big(f^*(y) \oplus (f(y) \odot \mathcal{F}_Y(f))\Big) \\
&\quad \to \bigwedge_{x\in X} \Big(f^*(\varphi(x)) \oplus (f(\varphi(x)) \odot \mathcal{F}_X(\varphi^{\leftarrow}(f))\Big)\Big] \\
&\quad \ge \bigwedge_{f\in L^Y} \bigwedge_{y\in Y} \Big[\Big(f^*(y) \oplus (f(y) \odot \mathcal{F}_Y(f))\Big) \\
&\quad \to \Big(f^*(y) \oplus (f(y) \odot \mathcal{F}_X(\varphi^{\leftarrow}(f))\Big)\Big] \\
&\ge \bigwedge_{f\in L^Y} \bigwedge_{y\in Y} \Big[\Big(f^*(y) \to f^*(y)\Big) \\
&\quad \odot \Big((f(y) \odot \mathcal{F}_Y(f)) \to (f(y) \odot \mathcal{F}_X(\varphi^{\leftarrow}(f)))\Big)\Big] \\
&\quad \text{(by Lemma 1 (9))} \\
&\quad \ge \bigwedge_{f\in L^Y} \Big(\mathcal{F}_Y(f) \to \mathcal{F}_X(\varphi^{\leftarrow}(f))\Big) = D_{\mathcal{F}}(\varphi)
\end{aligned}$$

□

From the above theorem, if $D_{\mathcal{F}}(\varphi) = \top$, then $\varphi : (X, \mathcal{T}_{\mathcal{F}_X}^{(1)}) \to (Y, \mathcal{T}_{\mathcal{F}_Y}^{(1)})$ is LF-continuous mapping.

By Theorems 1 and 2, we obtain the following corollary:

Corollary 1 *Υ : P-LF $\to$ L-FTOP is a functor defined by*

$$\Upsilon(X, \mathcal{F}) = (X, \mathcal{T}_{\mathcal{F}}^{(1)}), \quad \Upsilon(\varphi) = \varphi.$$

If we still write for the restriction of the functor Υ : P-LF $\to$ L-FTOP to the full subcategory AP-LF, then by Theorem 1, Υ : AP-LF $\to$ AL-FTOP forms a functor.

Theorem 3 *Let $\mathcal{F}$ be an L-fuzzy (prime) filter on X. Define $\mathcal{T}^{(2)}_{\mathcal{F}}: L^X \to L$ as follows:*

$$\mathcal{T}^{(2)}_{\mathcal{F}}(f) = S\Big(f, f \odot \mathcal{F}(f)\Big).$$

Then,

(1) *$(X, \mathcal{T}^{(2)}_{\mathcal{F}})$ is an L-fuzzy topological space.*
(2) *If $\mathcal{F}$ is discrete, then so is $\mathcal{T}^{(2)}_{\mathcal{F}}$.*
(3) *Let $\bigwedge_{i\in\Gamma}(x_i \odot y_i) = \bigwedge_{i\in\Gamma} x_i \odot \bigwedge_{i\in\Gamma} y_i$ for each $x_i, y_i \in L$. If $\mathcal{F}$ is Alexandrov, then so is $\mathcal{T}^{(2)}_{\mathcal{F}}$.*

Proof (1)(T1) $\mathcal{T}^{(2)}_{\mathcal{F}}(\bot_X) = S\Big(\bot_X, \bot_X \odot \mathcal{F}(\bot_X)\Big) = S(\bot_X, \bot_X) = \top$, $\mathcal{T}^{(2)}_{\mathcal{F}}(\top_X) = S\Big(\top_X, \top_X \odot \mathcal{F}(\top_X)\Big) = S(\top_X, \top_X) = \top$.

(T2) For $f, g \in L^X$,

$$\begin{aligned}
&\mathcal{T}^{(2)}_{\mathcal{F}}(f) \odot \mathcal{T}^{(2)}_{\mathcal{F}}(g)\\
&= S\Big(f, f \odot \mathcal{F}(f)\Big) \odot S\Big(g, g \odot \mathcal{F}(g)\Big)\\
&\le S\Big(f \odot g, \mathcal{F}(f) \odot \mathcal{F}(g) \odot (f \odot g)\Big)\\
&\quad \text{(by Lemma 3 (3))}\\
&\le S\Big(f \odot g, \mathcal{F}(f \odot g) \odot (f \odot g)\Big)\\
&= \mathcal{T}^{(2)}_{\mathcal{F}}(f \odot g).
\end{aligned}$$

(T3) For each family $\{f_i : i \in \Gamma\}$, we have

$$\begin{aligned}
\mathcal{T}^{(2)}_{\mathcal{F}}(\bigvee_{i\in\Gamma} f_i) &= S\Big(\bigvee_{i\in\Gamma} f_i, \bigvee_{i\in\Gamma} f_i \odot \mathcal{F}(\bigvee_{i\in\Gamma} f_i)\Big)\\
&\ge S\Big(\bigvee_{i\in\Gamma} f_i, \bigvee_{i\in\Gamma}\Big(f_i \odot \mathcal{F}(f_i)\Big)\Big)\\
&\ge \bigwedge_{i\in\Gamma} S\Big(f_i, f_i \odot \mathcal{F}(f_i)\Big)\\
&= \bigwedge_{i\in\Gamma} \mathcal{T}^{(2)}_{\mathcal{F}}(f_i).
\end{aligned}$$

Hence, $\mathcal{T}^{(2)}_{\mathcal{F}}$ is an L-fuzzy topology on X.

(2) $\mathcal{T}^{(2)}_{\mathcal{F}}(\top_x) = S\Big(\top_x, \top_x \odot \mathcal{F}(\top_x))\Big) = S\Big(\top_x, \top_x \odot \top))\Big) = \top$.

(3) For each family $\{f_i : i \in \Gamma\}$, we have

$$\begin{aligned}
\bigwedge_{i\in\Gamma} \mathcal{F}^{(2)}_{\mathcal{F}}(f_i) &= \bigwedge_{i\in\Gamma} S\Big(f_i, f_i \odot \mathcal{F}(f_i))\Big)\\
&\le S\Big(\bigwedge_{i\in\Gamma} f_i, \bigwedge_{i\in\Gamma}\Big(f_i \odot \mathcal{F}(f_i)\Big)\Big)\\
&= S\Big(\bigwedge_{i\in\Gamma} f_i, \bigwedge_{i\in\Gamma} f_i \odot \bigwedge_{i\in\Gamma} \mathcal{F}(f_i)\Big)\\
&\le S\Big(\bigwedge_{i\in\Gamma} f_i, \bigwedge_{i\in\Gamma} f_i \odot \mathcal{F}(\bigwedge_{i\in\Gamma} f_i))\Big)\\
&= \mathcal{T}^{(2)}_{\mathcal{F}}(\bigwedge_{i\in\Gamma} f_i).
\end{aligned}$$

□

Theorem 4 *Let $(X, \mathcal{F}_X)$ and $(Y, \mathcal{F}_Y)$ be L-fuzzy (prime) filter spaces and $\varphi : X \to Y$ be a mapping, then $D_{\mathcal{F}}(\varphi) \le D_{\mathcal{T}^{(2)}_{\mathcal{F}}}(\varphi)$.*

Proof For any $f \in L^Y$,

$$\begin{aligned}
D_{\mathcal{T}^{(2)}_{\mathcal{F}}}(\varphi) &= \bigwedge_{f\in L^Y}\Big(\mathcal{T}^{(2)}_{\mathcal{F}_Y}(f) \to \mathcal{T}^{(2)}_{\mathcal{F}_X}(\varphi^{\leftarrow}(f))\Big)\\
&= \bigwedge_{f\in L^Y}\Big[S\Big(f, f \odot \mathcal{F}_Y(f)\Big)\\
&\to S\Big(\varphi^{\leftarrow}(f), \varphi^{\leftarrow}(f) \odot \mathcal{F}_X(\varphi^{\leftarrow}(f))\Big)\Big]\\
&= \bigwedge_{f\in L^Y}\Big[\bigwedge_{y\in Y}\Big(f(y) \to (f(y) \odot \mathcal{F}_Y(f))\Big)\\
&\to \bigwedge_{x\in X}\Big(f(\varphi(x)) \to (f(\varphi(x)) \odot \mathcal{F}_X(\varphi^{\leftarrow}(f)))\Big)\Big]\\
&\ge \bigwedge_{f\in L^Y}\Big[\bigwedge_{y\in Y}\Big(f(y) \to (f(y) \odot \mathcal{F}_Y(f))\Big)\\
&\to \bigwedge_{y\in X}\Big(f(y) \to (f(y) \odot \mathcal{F}_X(\varphi^{\leftarrow}(f)))\Big)\Big]\\
&= \bigwedge_{f\in L^Y}\bigwedge_{y\in Y}\Big[\Big(f(y) \to (f(y) \odot \mathcal{F}_Y(f))\Big)\\
&\to \Big(f(y) \to (f(y) \odot \mathcal{F}_X(\varphi^{\leftarrow}(f)))\Big)\Big]\\
&\quad \text{(by Lemma 1 (8))}\\
&\ge \bigwedge_{f\in L^Y}\Big(\mathcal{F}_Y(f) \to \mathcal{F}_X(\varphi^{\leftarrow}(f))\Big) = D_{\mathcal{F}}(\varphi).
\end{aligned}$$

□

From the above theorem, we deduce that if $\varphi : (X, \mathcal{F}_X) \to (Y, \mathcal{F}_Y)$ is an L-fuzzy filter mapping, then $\varphi : (X, \mathcal{T}^{(2)}_{\mathcal{F}_X}) \to (Y, \mathcal{T}^{(2)}_{\mathcal{F}_Y})$ is LF-continuous mapping.

By Theorems 3 and 4, we obtain the following corollary:

Corollary 2 *Ω : P-LF → L-FTOP is a functor defined by*

$$\Omega(X, \mathcal{F}) = (X, \mathcal{T}^{(2)}_{\mathcal{F}}), \qquad \Omega(\varphi) = \varphi.$$

If we still write for the restriction of the functor Ω : P-LF → L-FTOP to the full subcategory AP-LF, then by Theorem 3, Ω : AP-LF → AL-FTOP forms a functor.

Theorem 5 *Let $\mathcal{F}$ be an L-fuzzy prime filter on X. Define a mapping $\mathcal{T}^{(3)}_{\mathcal{F}}: L^X \to L$ by*

$$\mathcal{T}^{(3)}_{\mathcal{F}}(f) = \begin{cases} \mathcal{F}(f), & \text{if } f \ne \bot_X\\ \top, & \text{if } f = \bot_X. \end{cases}$$

Then,

(1) *$(X, \mathcal{T}^{(3)}_{\mathcal{F}})$ is an L-fuzzy topological space.*
(2) *If $\mathcal{F}$ is discrete(resp. Alexandrov), then so is $\mathcal{T}^{(3)}_{\mathcal{F}}$.*

Proof (1)(T1) By dentition $\mathcal{T}^{(3)}_{\mathcal{F}}(\bot_X) = \top$ and $\mathcal{T}^{(3)}_{\mathcal{F}}(\top_X) = \mathcal{F}(\top_X) = \top$.

(T2) For any $f, g \in L^X$.
Case 1 if $f \odot g = \bot_X$, then $\mathcal{T}_{\mathcal{F}}^{(3)}(f \odot g) = \top \geq \mathcal{T}_{\mathcal{F}}^{(3)}(f) \odot \mathcal{T}_{\mathcal{F}}^{(3)}(g)$
Case 2 if $f \odot g \neq \bot_X$, then $f \neq \bot_X$ and $g \neq \bot_X$. So,

$$\mathcal{T}_{\mathcal{F}}^{(3)}(f \odot g) = \mathcal{F}(f \odot g) \geq \mathcal{F}(f) \odot \mathcal{F}(g) = \mathcal{T}_{\mathcal{F}}^{(3)}(f) \odot \mathcal{T}_{\mathcal{F}}^{(3)}(g).$$

(T3) For each family $\{f_i : i \in \Gamma\}$.
Case 1 if $\bigvee_{i \in \Gamma} f_i = \bot_X$, then

$$\mathcal{F}_{\mathcal{F}}^{(3)}(\bigvee_{i \in \Gamma} f_i) = \top \geq \bigwedge_{i \in \Gamma} \mathcal{F}_{\mathcal{F}}^{(3)}(f_i).$$

Case 2 if $\bigvee_{i \in \Gamma} f_i \neq \bot_X$, then $f_i \neq \bot_X$ for each $i \in \Gamma$. So,

$$\mathcal{T}_{\mathcal{I}}^{(3)}(\bigvee_{i \in \Gamma} f_i) = \mathcal{F}(\bigvee_{i \in \Gamma} f_i) \geq \bigwedge_{i \in \Gamma} \mathcal{F}(f_i) = \bigwedge_{i \in \Gamma} \mathcal{F}_{\mathcal{F}}^{(3)}(f_i).$$

Hence, $\mathcal{T}_{\mathcal{F}}^{(3)}$ is an L-fuzzy topology on X.

(2) (SE) $\mathcal{T}_{\mathcal{F}}^{(3)}(\top_x) = \mathcal{F}(\top_x) = \top$.

(AL) *Case 1* if $\bigwedge_{i \in \Gamma} f_i = \bot_X$, then $f_i = \bot_X$ for each $i \in \Gamma$. So,

$$\mathcal{T}_{\mathcal{F}}^{(3)}(\bigwedge_{i \in \Gamma} f_i) = \top \geq \bigwedge_{i \in \Gamma} \mathcal{T}_{\mathcal{F}}^{(3)}(f_i).$$

Case 2 if $\bigwedge_{i \in \Gamma} f_i \neq \bot_X$, then $f_i \neq \bot_X$ for some $i \in \Gamma$. So,

$$\bigwedge_{i \in \Gamma} \mathcal{T}_{\mathcal{F}}^{(3)}(f_i) = \bigwedge_{i \in \Gamma} \mathcal{F}(f_i) \leq \mathcal{F}(\bigwedge_{i \in \Gamma} f_i) = \mathcal{T}_{\mathcal{F}}^{(3)}(\bigwedge_{i \in \Gamma} f_i).$$

□

Theorem 6 *Let $(X, \mathcal{F}_X)$ and $(Y, \mathcal{F}_Y)$ be L-fuzzy filter spaces such that $\varphi : (X, \mathcal{F}_X) \to (Y, \mathcal{F}_Y)$ be an L-fuzzy filter mapping. Then, $\varphi : (X, \mathcal{T}_{\mathcal{F}_X}^{(3)}) \to (Y, \mathcal{T}_{\mathcal{F}_Y}^{(3)})$ is a continuous mapping.*

Proof For any $f \in L^Y$.
Case 1 if $\varphi^{\leftarrow}(f) = \bot_X$, then $\mathcal{T}_{\mathcal{F}_X}^{(3)}(\varphi^{\leftarrow}(f)) = \top \geq \mathcal{T}_{\mathcal{F}_Y}^{(3)}(f)$.
Case 2 if $\varphi^{\leftarrow}(f) \neq \bot_X$, then $f \neq \bot_Y$. So,

$$\mathcal{T}_{\mathcal{F}_X}^{(3)}(\varphi^{\leftarrow}(f)) = \mathcal{F}_X(\varphi^{\leftarrow}(f)) \geq \mathcal{F}_Y(f)) = \mathcal{T}_{\mathcal{F}_Y}^{(3)}(f).$$

□

By Theorems 5 and 6, we obtain the following corollary:

Corollary 3 *Δ : P-LF → L-FTOP is a functor defined by*

$$\Delta(X, \mathcal{F}) = (X, \mathcal{T}_{\mathcal{F}}^{(3)}), \quad \Delta(\varphi) = \varphi.$$

If we still write for the restriction of the functor Δ : P-LF → L-FTOP to the full subcategory AP-LF, then by Theorem 5, Δ : AP-LF → AL-FTOP forms a functor.

4 The relationships between *L*-fuzzy pre-proximities and *L*-fuzzy filters

In this section, we introduce the relationship between L-fuzzy pre-proximity spaces and L-fuzzy filter spaces.

Definition 8 An L-fuzzy pre-proximity on X is a mapping $\delta : L^X \times L^X \to L$ such that for all $f, g, h, f_1, f_2, g_1, g_2 \in L^X$:

(P1) $\delta(f, \bot_X) = \bot$.
(P2) $\delta(f, g) \geq \bigvee_{x \in X} f(x) \odot g(x)$.
(P3) $S(f, g) \leq \delta(f, h) \to \delta(g, h)$ and $S(f, g) \leq \delta(h, f) \to \delta(h, g)$,
(P4) $\delta(f_1 \odot f_2, g_1 \oplus g_2) \leq \delta(f_1, g_1) \oplus \delta(f_2, g_2)$.
The pair (X, δ) is called L-fuzzy pre-proximity space.
An L-fuzzy pre-proximity δ on X is called
(SE) discrete if $\delta(\top_x, \top_x^*) = \bot$,
(AL) Alexandrov if $\delta(f, \bigvee_{i \in \Gamma} g_i) \leq \bigvee_{i \in \Gamma} \delta(f, g_i)$ for all $\{f_i, g_i : i \in \Gamma\} \subseteq L^X$.

Definition 9 Let (X, δ_X) and (Y, δ_Y) be two L-fuzzy pre-proximities and $\varphi : X \to Y$ be a mapping. Then, $D_\delta(\varphi)$ defined by

$$D_\delta(\varphi) = \bigwedge_{f, g \in L^Y} \big(\delta_X(\varphi^{\leftarrow}(f), \varphi^{\leftarrow}(g)) \to \delta_Y(f, g)\big)$$

is the degree to which the map φ is an LF-proximity map.
If $D_\delta(\varphi) = \top$, then $\delta_X(\varphi^{\leftarrow}(f), \varphi^{\leftarrow}(g)) \leq \delta_Y(f, g)$ for all $f, g \in L^Y$ which is exactly the definition of LF-proximity map between L-fuzzy pre-proximities.

The category of L-fuzzy pre-proximity spaces with LF-proximity mappings as morphisms is denoted by L-PROX. Write AL-PROX for the full subcategory of L-PROX composed of objects of all Alexandrov L-fuzzy pre-proximity spaces.

In the sequel, we assume that L satisfies the double negation law.

Theorem 7 *Let L be idempotent, δ be an L-fuzzy pre-proximity. Define a mapping $\mathcal{F}^k_\delta : L^X \longrightarrow L$ as follows:*

$$\mathcal{F}^k_\delta(f) = \begin{cases} \delta^*(k, f^*), & \text{if } f \neq \bot_X \\ \bot, & \text{if } f = \bot_X. \end{cases}$$

Then, $\mathcal{F}^k_\delta$ is L-fuzzy prime filter on X. Moreover, if δ is Alexandrov, then so is $\mathcal{F}^k_\delta$

Proof

(LF1) $\mathcal{F}^k_\delta(\bot_X) = \bot$ and $\mathcal{F}^k_\delta(\top_X) = \delta^*(k, \bot_X) = \top$.

(LF2) Let $f, g \in L^X$, then

$$\begin{aligned}\mathcal{F}^k_\delta(f) \to \mathcal{F}^k_\delta(g) &= \delta^*(k, f^*) \to \delta^*(k, g^*)\\ &= \delta(k, g^*) \to \delta(k, f^*)\\ &\geq S(g^*, f^*) = S(f, g).\end{aligned}$$

(LF3) Let $f, g \in L^X$ such that $f \odot g \neq \bot_X$, we have

$$\begin{aligned}\mathcal{F}^k_\delta(f \odot g) &= \delta^*(k, (f \odot g)^*) = \delta^*(k, f^* \oplus g^*)\\ &= \delta^*(k, f^*) \odot \delta^*(k, g^*) = \mathcal{F}^k_\delta(f) \odot \mathcal{F}^k_\delta(g).\end{aligned}$$

(AL) $\mathcal{F}^k_\delta(\bigwedge_{i\in\Gamma} f_i) = \delta^*(k, \bigvee_{i\in\Gamma} f_i^*) \geq \bigwedge_{i\in\Gamma} \delta^*(k, f^*) = \bigwedge_{i\in\Gamma} \mathcal{F}^k_\delta(f_i)$.

□

Now, let $\mathcal{F}(X)$ be the family of all L-fuzzy prime filter and $\mathcal{P}(X)$ be the family of all L-fuzzy pre-proximities on X.

Theorem 8 *Let L be idempotent, $\mathcal{H} : \mathcal{P}(X) \times \mathcal{F}(X) \to \mathcal{F}(X)$ be a mapping defined as follows:*

$$\mathcal{H}(\delta, \mathcal{F})(f) = \bigvee_{g\in L^X} \Big(\delta^*(g, f^*) \odot \mathcal{F}(f)\Big).$$

Then, we have the following properties:

(1) $\mathcal{H}(\delta, \mathcal{F}) \in \mathcal{F}(X)$,

(2) $\mathcal{H}(\delta, \mathcal{F}^k_\delta) = \mathcal{F}^k_\delta$.

Proof

(1) (LF1) $\mathcal{H}(\delta, \mathcal{F})(\bot_X) = \bigvee_{g\in L^X} \Big(\delta^*(g, \top_X) \odot \mathcal{F}(\bot_X)\Big) = \bot$, $\mathcal{H}(\delta, \mathcal{F})(\top_X) = \bigvee_{g\in L^X} \Big(\delta^*(g, \bot_X) \odot \mathcal{F}(\top_X)\Big) = \top$.

(LF2) Let $f, g \in L^X$, then

$$\begin{aligned}&\mathcal{H}(\delta, \mathcal{F})(f) \to \mathcal{H}(\delta, \mathcal{F})(g)\\ &= \bigvee_{h\in L^X} \Big(\delta^*(h, f^*) \odot \mathcal{F}(f)\Big)\\ &\to \bigvee_{k\in L^X} \Big(\delta^*(k, g^*) \odot \mathcal{F}(g)\Big)\\ &= \bigwedge_{h\in L^X} \Big(\delta^*(h, f^* \odot \mathcal{F}(f) \to \bigvee_{k\in L^X} \Big(\delta^*(k, g^*) \odot \mathcal{F}(g)\Big)\\ &\geq \bigwedge_{h\in L^X} \Big((\delta^*(h, f^*) \odot \mathcal{F}(f) \to \Big(\delta^*(k, g^*) \odot \mathcal{F}(g)\Big)\\ &\geq \bigwedge_{h\in L^X} \Big((\delta^*(h, f^*) \to \delta^*(h, g^*)) \odot (\mathcal{F}(f) \to \mathcal{F}(g))\Big)\\ &= \bigvee_{h\in L^X} \Big((\delta(h, g^*) \to \delta(h, f^*)) \odot (\mathcal{F}(f) \to \mathcal{F}(g))\Big)\\ &\geq S(g^*, f^*) \odot S(f, g) = S(f, g) \odot S(f, g) = S(f, g).\end{aligned}$$

(LF3) Let $f, h \in L^X$, then

$$\begin{aligned}\mathcal{H}(\delta, \mathcal{F})(f \odot h) &= \bigvee_{g\in L^X} \Big(\delta^*(g, f^* \oplus h^*) \odot \mathcal{F}(f \odot h)\Big)\\ &\geq \bigvee_{g\in L^X} \Big((\delta^*(g, f^*) \odot \delta^*(g, h^*)) \odot (\mathcal{F}(f) \odot \mathcal{F}(h))\Big)\\ &= \bigvee_{g\in L^X} \Big(\delta^*(g, f^*) \odot \mathcal{F}(f)\Big)\\ &\quad\odot \bigvee_{g\in L^X} \Big(\delta^*(g, h^*)) \odot \mathcal{F}(h))\Big)\\ &= \mathcal{H}(\delta, \mathcal{F})(f) \odot \mathcal{H}(\delta, \mathcal{F})(h).\end{aligned}$$

(2) Let $f \in L^X$ such that $f \neq \bot_X$, then

$$\begin{aligned}\mathcal{H}(\delta, \mathcal{F}^k_\delta)(f) &= \bigvee_{g\in L^X} \Big(\delta^*(g, f^*) \odot \mathcal{F}^k_\delta(f)\Big)\\ &\leq \top \odot \mathcal{F}^k_\delta(f) = \mathcal{F}^k_\delta(f).\end{aligned}$$

Conversely,

$$\begin{aligned}\mathcal{H}(\delta, \mathcal{F}^k_\delta)(f) &= \bigvee_{g\in L^X} \Big(\delta^*(g, f^*) \odot \mathcal{F}^k_\delta(f)\Big)\\ &= \bigvee_{g\in L^X} \Big(\delta^*(g, f^*) \odot \delta^*(k, f^*)\Big)\\ &\geq \delta^*(k, f^*) \odot \delta^*(k, f^*) = \delta^*(k, f^*)\\ &= \mathcal{F}^k_\delta(f).\end{aligned}$$

Hence, $\mathcal{H}(\delta, \mathcal{F}^k_\delta) = \mathcal{F}^k_\delta$.

□

Theorem 9 *Let $\mathcal{F}$ be an L-fuzzy prime filter on X such that $\mathcal{F}(g) \leq g(x)$ for each $x \in X$ and $g \in L^X$. Define a mapping $\delta_\mathcal{F} : L^X \times L^X \to L$ by*

$$\delta_\mathcal{F}(f, g) = \bigvee_{x\in X} \Big(f(x) \odot \mathcal{F}^*(g^*)\Big).$$

Then, $\delta_\mathcal{F}$ is an L-fuzzy pre-proximity on X. Moreover, if $\mathcal{F}$ is discrete (resp., Alexandrov), then so is $\delta_\mathcal{F}$.

Proof

(P1) Since $\mathcal{F}(\top_X) = \top$, we have

$$\delta_{\mathcal{F}}(f, \perp_X) = \bigvee_{x \in X} f(x) \odot \mathcal{F}^*(\top_X) = \perp.$$

(P2) Since $\mathcal{F}(g) \le g(x)$, we have

$$\delta_{\mathcal{F}}(f, g) = \bigvee_{x \in X} \Big(f(x) \odot \mathcal{F}^*(g^*)\Big) \ge \bigvee_{x \in X} f(x) \odot g(x).$$

(P3) Let $f, g, h \in L^X$, we have

$$\begin{aligned}
&\delta_{\mathcal{F}}(h, f) \to \delta_{\mathcal{F}}(h, g) = \bigvee_{x \in X} \Big(h(x) \odot \mathcal{F}^*(f^*)\Big) \\
&\to \bigvee_{y \in X} \Big(h(y) \odot \mathcal{F}^*(g^*)\Big) \\
&\ge \bigvee_{x \in X} \Big[\Big(h(x) \to h(x)\Big) \\
&\quad \odot \Big(\mathcal{F}^*(f^*) \to \mathcal{F}^*(g^*)\Big)\Big] \\
&= \mathcal{F}^*(f^*) \to \mathcal{F}^*(g^*) = \mathcal{F}(g^*) \to \mathcal{F}(f^*) \\
&\ge S(g^*, f^*) = S(f, g).
\end{aligned}$$

Other case is similar.

(P4) For every $f_1, f_2, g_1, g_2 \in L^X$, we have by Lemma 2(3),

$$\begin{aligned}
&\delta_{\mathcal{F}}(f_1 \odot f_2, g_1 \oplus g_2) \\
&= \bigvee_{x \in X} \Big((f_1(x) \odot f_2(x)) \odot \mathcal{F}^*(g_1^* \odot g_2^*)\Big) \\
&\le \bigvee_{x \in X} \Big(f_1(x) \odot f_2(x)\Big) \odot \Big(\mathcal{F}^*(g_1^*) \oplus \mathcal{F}^*(g_2^*)\Big) \\
&\le \bigvee_{x \in X} \Big(f_1(x) \odot \mathcal{F}^*(g_1^*)\Big) \oplus \Big(f_2(x) \odot \mathcal{F}^*(g_2^*)\Big) \\
&\le \bigvee_{x \in X} \Big(f_1(x) \odot \mathcal{F}^*(g_1^*)\Big) \oplus \bigvee_{x \in X} \Big(f_2(x) \odot \mathcal{F}^*(g_2^*)\Big) \\
&= \delta_{\mathcal{F}}(f_1, g_1) \oplus \delta_{\mathcal{F}}(f_2, g_2).
\end{aligned}$$

□

Other cases are easily proven.

Theorem 10 *Let $(X, \mathcal{F}_X)$ and $(Y, \mathcal{F}_Y)$ be L-fuzzy filter spaces and $\varphi : X \to Y$ be a mapping. Then, $D_{\mathcal{F}}(\varphi) \le D_{\delta_{\mathcal{F}}}(\varphi)$.*

Proof For every $f, g \in L^Y$, we have

$$\begin{aligned}
D_{\delta_{\mathcal{F}}}(\varphi) &= \bigwedge_{f, g \in L^Y} \Big(\delta_{\mathcal{F}_X}(\varphi^{\leftarrow}(f), \varphi^{\leftarrow}(g)) \to \delta_{\mathcal{F}_Y}(f, g)\Big) \\
&= \bigwedge_{f, g \in L^Y} \Big[\bigvee_{x \in X} \Big(\varphi^{\leftarrow}(f)(x) \odot \mathcal{F}^*(\varphi^{\leftarrow}(g^*)\Big) \\
&\to \bigvee_{y \in Y} \Big(f(y) \odot \mathcal{F}_Y^*(g^*)\Big)\Big] \\
&= \bigwedge_{f, g \in L^Y} \Big[\bigvee_{x \in X} \Big(f(\varphi(x)) \odot \mathcal{F}^*(\varphi^{\leftarrow}(g^*)\Big) \\
&\to \bigvee_{y \in Y} \Big(f(y) \odot \mathcal{F}_Y^*(g^*)\Big)\Big] \\
&\ge \bigwedge_{f, g \in L^Y} \Big[\bigvee_{y \in X} \Big(f(y) \odot \mathcal{F}^*(\varphi^{\leftarrow}(g^*))\Big) \\
&\to \bigvee_{y \in Y} \Big(f(y) \odot \mathcal{F}_Y^*(g)\Big)\Big] \\
&\ge \bigwedge_{f, g \in L^Y} \bigwedge_{y \in X} \Big[\Big(f(y) \odot \mathcal{F}^*(\varphi^{\leftarrow}(g^*))\Big) \\
&\to \Big(f(y) \odot \mathcal{F}_Y^*(g^*)\Big)\Big] \\
&\quad \text{(by Lemma 1 (9))} \\
&\ge \bigwedge_{g \in L^Y} \Big(\mathcal{F}^*(\varphi^{\leftarrow}(g^*)) \to \mathcal{F}_Y^*(g^*)\Big) \\
&= \bigwedge_{g \in L^Y} \Big(\mathcal{F}_Y(g^*) \to \mathcal{F}(\varphi^{\leftarrow}(g^*))\Big) \\
&= D_{\mathcal{F}}(\varphi).
\end{aligned}$$

It is clear that if $\varphi : (X, \mathcal{F}_X) \to (Y, \mathcal{F}_Y)$ is L-fuzzy filter mapping, then $\varphi : (X, \delta_{\mathcal{F}_X}) \to (Y, \delta_{\mathcal{F}_Y})$ is an LF-proximity mapping. □

By Theorems 9 and 10, we obtain the following corollary:

Corollary 4 *Φ : P-LF $\to$ L-PROX is a functor defined by*

$$\Phi(X, \mathcal{F}) = (X, \delta_{\mathcal{F}}), \ \ \Phi(\varphi) = \varphi.$$

If we still write for the restriction of the functor Φ : P-LF $\to$ L-PROX to the full subcategory AP-LF, then by Theorem 9, Δ : AP-LF $\to$ AL-PROX forms a functor.

Let L-FRR be a category with object (X, R_X), where R_X is a reflexive L-fuzzy relation with an order preserving map $\varphi : (X, R_X) \to (Y, R_Y)$ such that $R_X(x, y) \le R_Y(\varphi(x), \varphi(y))$ for all $x, y \in X$.

Theorem 11 *Let R_X be a reflexive L-fuzzy relation. Define a mapping $\mathcal{F}_R^x : L^X \to L$ as follows:*

$$\mathcal{F}_R^x(f) = \bigwedge_{y \in X} \Big(R(x, y) \to f(y)\Big), \quad \forall\, x \in X, f \in L^X.$$

Then,

(1) *$\mathcal{F}_R^x$ is an Alexandrov L-fuzzy filter on X,*

(2) *If $\varphi : (X, R_X) \to (Y, R_Y)$ is an order preserving mapping, then $\varphi : (X, \mathcal{F}^x_{R_X}) \to (Y, \mathcal{F}^x_{R_Y})$ is L-fuzzy filter map.*

Proof (1)

(LF1)

$$\mathcal{F}^x_{R_X}(\perp_X) = \bigwedge_{y\in X}\Big(R_X(x,y) \to \perp_X(y)\Big) \le R_X(x,x) \to \perp_X(x) = \top \to \perp = \perp.$$

(LF2)

$$\begin{aligned}\mathcal{F}^x_{R_X}(f) \to \mathcal{F}^x_{R_X}(g) &= \bigwedge_{y\in X}\Big(R_X(x,y) \to f(y)\Big)\\ &\to \bigwedge_{z\in X}\Big(R_X(x,z) \to g(z)\Big)\\ &\ge \bigwedge_{y\in X}\Big((R_X(x,y) \to f(y)) \to (R_X(x,y) \to g(y))\Big)\\ &\ge \bigwedge_{y\in X}(f(y) \to g(y)) = S(f,g).\end{aligned}$$

(AL)

$$\begin{aligned}\mathcal{F}^x_{R_X}(\bigwedge_{i\in\Gamma} f_i) &= \bigwedge_{y\in X}\Big(R_X(x,y) \to (\bigwedge_{i\in\Gamma} f_i)(y)\Big)\\ &= \bigwedge_{y\in X}\Big(\bigwedge_{i\in\Gamma}\Big(R_X(x,y) \to f_i(y)\Big)\Big)\\ &\ge \bigwedge_{i\in\Gamma}\Big(\bigwedge_{y\in X}\Big(R_X(x,y) \to f_i(y)\Big)\Big)\\ &= \bigwedge_{i\in\Gamma}\mathcal{F}^x_{R_X}(f_i).\end{aligned}$$

(2)

$$\begin{aligned}\mathcal{F}^x_{R_X}(\varphi^{\leftarrow}(f)) &= \bigwedge_{y\in X}\Big(R_X(x,y) \to \varphi^{\leftarrow}(f)(y)\Big)\\ &= \bigwedge_{y\in X}\Big(R_X(x,y) \to f(\varphi(y))\Big)\\ &\ge \bigwedge_{y\in X}\Big(R_Y(\varphi(x),\varphi(y)) \to f(\varphi(y))\Big)\\ &\ge \bigwedge_{z\in Y}\Big(R_Y(\varphi(x),z) \to f(z)\Big) = \mathcal{F}^{\varphi(x)}_{R_Y}(f)\end{aligned}$$

.

□

By Theorem 11, we obtain the following corollary:

Corollary 5 *$\Psi : L$-FRR $\to$ A-LF is a functor defined by*

$$\Psi(X, \mathcal{F}^x) = (X, \delta_{\mathcal{F}^x}),\ \Psi(\varphi) = \varphi.$$

As an information system and an extension of Pawlak's rough set (Pawlak 1982, 1991), we give the following example for L-fuzzy pre-proximities and L-fuzzy filters.

Example 1 (1) Define $\mathcal{F}_1 : L^X \to L$ as $\mathcal{F}_1(f) = \bigwedge_{x\in X} f(x)$. Hence, $\mathcal{F}_1$ is Alexandrov L-fuzzy filter on X. Since $\mathcal{F}_1(\top_x) = \bigwedge_{y\in X}\top_x(y) = \top_x(x) \wedge \bigwedge_{y\neq x}\top_x(y) = \perp$, $\mathcal{F}_1$ is not discrete. By Theorem 9, we have

$$\begin{aligned}\delta_{\mathcal{F}_1}(f,g) &= \textstyle\bigvee_{x\in X} f(x) \odot \mathcal{F}^*_1(g^*_2)\\ &= \bigvee_{x\in X} f(x) \odot \bigvee_{y\in X} g(y).\end{aligned}$$

(2) Define $\mathcal{F}_2 : L^X \to L$ as $\mathcal{F}_2(f) = f(x)$. Hence, $\mathcal{F}_1$ is a discrete and Alexandrov L-fuzzy filter on X. By Theorem 9, we have

$$\delta_{\mathcal{F}_2}(f,g) = \bigvee_{x\in X} f(x) \odot \mathcal{F}^*_2(g^*) = \bigvee_{x\in X} f(x) \odot g(x).$$

Example 2 (1) Let $X = \{h_i \mid i = \{1,2,3\}\}$ with h_i=house and $Y = \{e, b, w, c, i\}$ with e=expensive, b= beautiful, w=wooden, c= creative, i=in the green surroundings. Let $([0,1], \odot, \to, ^*, 0, 1)$ be a complete residuated lattice as

$$x \odot y = \max\{0, x + y - 1\},$$
$$x \to y = \min\{1 - x + y, 1\},\ x^* = 1 - x.$$

Let $R \in [0,1]^{X\times Y}$ be a fuzzy information as follows:

R	e	b	w	c	i
h_1	0.7	0.6	0.5	0.9	0.2
h_2	0.6	0.8	0.4	0.3	0.5
h_3	0.4	0.9	0.8	0.6	0.6

Define a mapping $\mathcal{F}^x_R : L^Y \to L$ as follows:

$$\mathcal{F}^x_R(f) = \bigwedge_{y\in Y}\Big(R(x,y) \to f(y)\Big),$$

for each $x \in X$ and $f \in L^Y$. From Theorem 11, $\mathcal{F}_R$ is an Alexandrov L-fuzzy filter on X. For $f = (0.3, 0.5, 0.6, 0.1, 0.1)$, we obtain $\mathcal{F}^{h_1}_R(f) = 0.2$, $\mathcal{F}^{h_2}_R(f) = 0.6$, and $\mathcal{F}^{h_3}_R(f) = 0.5$. From Theorem 9, we obtain

$$\begin{aligned}\delta_{\mathcal{F}_R}(f,g) &= \bigvee_{x\in X}\Big(f(x) \odot \mathcal{F}^*_R(g^*)\Big)\\ &= \bigvee_{x\in X}\Big(f(x) \odot \bigvee_{y\in X} R(x,y) \odot g(y)\Big)\\ &= \bigvee_{x,y\in X}\Big(R(x,y) \odot f(x) \odot g(y)\Big).\end{aligned}$$

(i) Let $R = \top_{X\times X}$ be given, then $\delta_{\mathcal{F}_R}(f,g) = \bigvee_{x,y\in X}\Big(f(x) \odot g(y)\Big)$. Hence, $\delta_{\mathcal{F}_R}$ is an L-fuzzy pre-proximity on X. Moreover, $\delta_{\mathcal{F}_R}$ is Alexandrov. Since $\delta_{\mathcal{F}_R}(\top_x, \top^*_x) = \top$, $\delta_{\mathcal{F}_R}$ is not discrete.

(ii) Let $R = \triangle_{X\times X}$ be given, where

$$\triangle_{X\times X}(x, y) = \begin{cases} \top, & \text{if } y = x, \\ \bot, & \text{otherwise.} \end{cases}$$

Then, $\delta_{\mathcal{F}_R}(f, g) = \bigvee_{x\in X} \Big(f(x) \odot g(x) \Big)$. Hence, $\delta_{\mathcal{F}_R}$ is an L-fuzzy pre-proximity on X. Moreover,$\delta_{\mathcal{F}_R}$ is Alexandrov. Since $\delta_{\delta_{\mathcal{F}_R}}(\top_x, \top_x^*) = \bot$, $\delta_{\mathcal{F}_R}$ is a discrete.

(2) Define [0, 1]-fuzzy pre-orders R_X^Y, $R_X^{\{b,w\}} \in [0, 1]^{X\times X}$ by

$$R_X^Y(h_i, h_j) \;= \bigwedge\nolimits_{y\in Y} \Big(R(h_i, y) \to R(h_j, y)) \Big),$$
$$R_X^{\{b,w\}}(h_i, h_j) = \bigwedge\nolimits_{y\in \{b,w\}} \Big(R(h_i, y) \to R(h_j, y) \Big).$$
$$R_X^Y = \begin{pmatrix} 1 & 0.4 & 0.7 \\ 0.7 & 1 & 0.8 \\ 0.6 & 0.6 & 1 \end{pmatrix}, \quad R_X^{\{b,w\}} = \begin{pmatrix} 1 & 0.9 & 1 \\ 0.8 & 1 & 1 \\ 0.7 & 0.6 & 1 \end{pmatrix}.$$

(i) For each $R \in \{R_X^Y, R_X^{\{b,w\}}\}$, we obtain Alexandrov L-fuzzy filter $\mathcal{F}_R : [0, 1]^X \to [0, 1]$ as

$$\mathcal{F}_R(f) = \bigwedge_{h_j\in X} \Big(R_X^Y(h_i, h_j) \to f(h_j) \Big).$$

By Theorem 9, we obtain Alexandrov [0, 1]-fuzzy pre-proximity $\delta_{\mathcal{F}_R} : [0, 1]^X \times [0, 1]^X \to [0, 1]$ as

$$\delta_{\mathcal{F}_R}(f, g) = \bigvee\nolimits_{h_i,h_j\in X} \Big(R_X^Y(h_i, h_j) \odot f(h_i) \odot g(h_j) \Big).$$

(ii) For each $R \in \{R_X^Y, R_X^{\{b,w\}}\}$, we obtain Alexandrov [0, 1]-fuzzy filter $\mathcal{F}_R : [0, 1]^X \to [0, 1]$ as

$$\mathcal{F}_R(f) = \bigwedge\nolimits_{h_j\in X} \Big(R(h_j, h_i) \to f(h_j) \Big).$$

By Theorem 9, we obtain Alexandrov [0, 1]-fuzzy quasi-proximity $\delta_{\mathcal{F}_R} : [0, 1]^X \times [0, 1]^X \to [0, 1]$ as

$$\begin{aligned} \delta_{\mathcal{F}_R}(f, g) &= \bigvee\nolimits_{h_i\in X} f(h_i) \\ &\quad \odot \Big(\bigvee\nolimits_{h_j\in X} R(h_j, h_i) \odot g(h_j) \Big) \\ &= \bigvee\nolimits_{h_i,h_j\in X} \Big(R(h_j, h_i) \odot f(h_i) \odot g(h_j) \Big). \end{aligned}$$

5 Conclusion

In complete residuated lattices, this study identified some functors from the category of L-fuzzy (prime) filter spaces to the category of L-fuzzy topological spaces and the category of L-fuzzy pre-proximity spaces. As a unified structure of extension of Pawlak's rough set (Pawlak 1982, 1991), we presented example 2 through fuzzy information system which confirmed the feasibility of using the proposed approaches to solve real-world problems.

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References

Adámek J, Herrlich H, Strecker GE (1990) Abstract and concrete categories. Wiley, New York

Bělohlávek R (2002) Fuzzy relational systems. Kluwer Academic Publishers, New York

Burton MH, Muraleetharan M, Gutierrez Garcia J (1999) Generalised filters 2. Fuzzy Sets Syst 106:393–400

Fang J (2010) The relationship between L-ordered convergence structures and strong L-topologies. Fuzzy Sets Syst 161:2923–2944

Fang J, Yue Y (2010) L-fuzzy closure systems. Fuzzy Sets Syst 161:1242–1252

Goguen JA (1967) L-fuzzy sets. J Math Anal Appl 145–174

Hájek P (1998) Metamathematices of fuzzy logic. Kluwer Academic Publishers, Dordrecht

Höhle U, Šostak A (1999) Axiomatic foundations of fixed-basis fuzzy topology. In: Höhle U, Rodabaugh SE (eds) Mathematics of fuzzy sets. Logic, Topology and Measure Theory, Kluwer, Boston/Dordrecht/London, pp 123–272

Jäger G (2013) A note on stratified LM-filters. Iran J Fuzzy Syst 10:135–142

Ko JM, Kim YC (2018) Alexandrov L-filters and Alexandrov L-convergence spaces. J Intell Fuzzy Syst 33(3):3255–3266

Koguep BBN, Nkuimi C, Lele C (2008) On fuzzy ideals of hyperlattice. Int J Algebra 2:739–750

Kubiak T (1985) On fuzzy topologies, Ph.D. Thesis, Adam Mickiewicz, Poznan, Poland

Kubiak T, Šostak A (1997) Lower set-valued fuzzy topologies. Quaestiones Mathematicae 20(3):423–430

Liang CY, Shi FG (2014) Degree of continuity for mappings of (L, M)-fuzzy topological spaces. J Intell Fuzzy Syst 27:2665–2677
Pawlak Z (1982) Rough set. Int J Comput Inf Sci 11:341–356
Pawlak Z (1991) Rough set: theoretical aspects of reasoning about data. Kluwer Academic Publishers, Boston
Chen Piwei, Zhang Dexue (2010) Alexandroff L-co-topological spaces. Fuzzy Sets Syst 161:2505–2514
Ramadan AA (1992) Smooth topological spaces. Fuzzy Sets Syst 48:371–375
Ramadan AA (1997) Smooth filter structures. J Fuzzy Math 5(2):297–308
Ramadan AA, Abdel-Sattar MA, Kim YC (2003) Some properties of smooth ideals. Indian J Pure Appl Math 34:247–264
Ramadan AA, Kim YC (2018) Alexandroff L-fuzzy topological spaces and reflexive L-fuzzy relations. J Math Comput Sci 8(3):437–453
Ramadan AA, Elkordy EH, Kim YC (2015) Perfect L-fuzzy topogenous spaces, L-fuzzy quasi-proximities and L-fuzzy quasi-uniform spaces. J Intell Fuzzy Syst 28:2591–2604
Ramadan AA, Elkordy E, Ahmed RM (2022) On Alexandrov L-fuzzy nearness (II). Soft Comput. https://doi.org/10.1007/s00500-022-07548-0
Rodabaugh SE, Klement EP (2003) Topological and algebraic structures in fuzzy sets. In: The handbook of recent developments in the mathematics of fuzzy sets. Kluwer Academic Publishers, Boston
Liu Y, Qin Y, Qin X, Xu Y (2017) Ideals and fuzzy ideals on residuated lattices. Int J Mach Learn Cybern 8:239–253
Lowen R (1979) Convergence in fuzzy topological spaces. Gen Topol Appl 10(2):147–160
Šostak A (1985) On a fuzzy topological structure. Suppl Rend Circ Mat Palermo Ser II(11):89–103
Šostak A (1989) Two decades of fuzzy topology: basic ideas, notions and results. Russ Math Surv 44:125–186
Tiwari SP, Yadav VK, Davvaz B (2018) A categorical approach to minimal realization for a fuzzy language. Fuzzy Sets Syst 351:122–137
Turunen E (1999) Mathematics behind fuzzy logic. Springer, Heidelberg
Tonga M (2011) Maximality on fuzzy filters of lattice. Afrika Math 22:105–114
Ward M, Dilworth RP (1939) Residuated lattices. Trans Am Math Soc 45:335–354
Xiu ZY, Li QG (2019) Degrees of L-continuity for mappings between L-topological spaces. Mathematics 7:10–13
Yue Y (2007) Lattice-valued induced fuzzy topological spaces. Fuzzy Sets Syst 158:1461–1471
Zhang D (2007) An enriched category approach to many valued topology. Fuzzy Sets Syst 158:349–366