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FUZZY SYSTEMS AND THEIR MATHEMATICS

General fractional interval-valued differential equations and Gronwall inequalities

Qin Fan[1,2] · Lan-Lan Huang[2] · Guo-Cheng Wu[2]

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Abstract
Interval-valued systems with the general fractional derivative are defined on closed intervals on the real line $\mathbb{R}$. Function spaces of the fractional integrals and derivatives are discussed. Then some fundamental theorems of the Caputo and Riemann–Liouville derivatives are provided, respectively. Finally, the interval-valued Gronwall inequalities are presented as one application.

Keywords General fractional calculus · Interval-valued function · Gronwall inequality · Interval-valued analysis

1 Introduction

Recently, a general fractional integral of real-valued functions (Osler 1970; Kilbas et al. 2006; Samko et al. 1993; Almeida 2017) was proposed as

$$I_{a+}^{\alpha,g} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (g(t) - g(s))^{\alpha-1} g'(s) f(s)\mathrm{d}s.$$

For some specific g, it can be reduced to various well-known fractional integrals:

- The classical Riemann–Liouville (R–L) integral (Kilbas et al. 2006) for $g(t) = t$;
- The fractional integral of Hadamard type (Kilbas et al. 2006) for $g(t) = \ln t$;
- The fractional integral of Katugampola type (Katugampola 2011) for $g(t) = \frac{t^{\sigma+1}}{\sigma+1}$;
- The fractional integral of exponential type (Fu et al. 2021a, b) for $g(t) = e^{\lambda t}$.

In addition, new fractional integrals can be obtained if they satisfy the boundedness (Fan et al. 2022). Numerical methods of general fractional differential equations (Wu et al. 2022), the physical meaning (Fu et al. 2021a, b) and their applications on time scales (Song et al. 2022; Wu et al. 2022) were then discussed. It can be concluded that the general fractional calculus is well-defined.

Fractional fuzzy and interval-valued equations can be used to describe nonlinear phenomena with uncertainties and memory effects. Much effort has been made and rich results are available now, for example, basic theory (Kara et al. 2022; Vu et al. 2018; Liu et al. 2017; Ho and Ngo 2021; Lupulescu 2015; Hoa et al. 2017; Shen 2016), numerical methods (Shiri et al. 2021; Alijani and Kangro 2022), discrete-time systems (Huang et al. 2021) et al. However, one important problem was still not addressed yet: Many interval-valued systems were proposed within different fractional derivatives in recent years. Which one is the best for the specific real-world application? So it is important to define a general fractional interval-valued system. But first and foremost, we need to consider the basics including definitions with function space and propositions.

We organize the paper as the following: Sect. 2 revisits calculus of interval-valued functions. Section 3 defines general fractional integral for interval-valued functions in $L([a, b], \mathcal{K})$ space. Section 4 gives definitions and properties of the general fractional derivatives in $AC_\delta([a, b], \mathcal{K})$

✉ Guo-Cheng Wu
wuguocheng@gmail.com

Qin Fan
mathqin@yeah.net

Lan-Lan Huang
mathlan@126.com

1 School of Science, Chongqing University of Posts and Telecommunications, Chongqing 400065, People's Republic of China

2 Data Recovery Key Laboratory of Sichuan Province, College of Mathematics and Information Science, Neijiang Normal University, Neijiang 641100, People's Republic of China

space. Section 5 uses these properties to give a general fractional Gronwall inequality for interval-valued functions. A conclusion is drawn at the end.

2 Preliminaries

In this section, we mainly introduce concepts of interval numbers and interval-valued functions (see Lupulescu 2015; Markov 1979 for more detail).

2.1 Definitions of interval numbers

First consider the set $\mathcal{K}$, which consists of all nonempty compact intervals on the real line $\mathbb{R}$. For interval numbers $A = [a_1, a_2]$ and $B = [b_1, b_2] \in \mathcal{K}$ $(a_1 \leq a_2,\ b_1 \leq b_2)$, the operators are defined by

$$A + B := [a_1 + b_1, a_2 + b_2]$$

and

$$\lambda A := \begin{cases} [\lambda a_1, \lambda a_2] & \text{if } \lambda > 0 \\ \{0\} & \text{if } \lambda = 0 \\ [\lambda a_2, \lambda a_1] & \text{if } \lambda < 0 \end{cases}$$

respectively.

Definition 1 (Hukuhara 1967) Let A and $B \in \mathcal{K}$. If there exists an interval number $C \in \mathcal{K}$ such that

$$A = B + C,$$

then C is called the Hukuhara difference (or H-difference) of A and B and it will be denoted by $A \ominus B$.

Although H-difference can satisfy $A \ominus A = 0$, it does not always exists for any two interval numbers. Thereafter, Stefanini (2010) introduced the following general Hukuhara difference.

Definition 2 (Stefanini 2010) The general Hukuhara difference (or gH-difference) of $A = [a_1, a_2]$ and $B = [b_1, b_2] \in \mathcal{K}$ is defined as

$$A \ominus_g B = [\min\{a_1 - b_1, a_2 - b_2\}, \max\{a_1 - b_1, a_2 - b_2\}]. \tag{1}$$

See (Stefanini 2008, 2010; Tao and Zhang 2015) for more basic properties of the gH-difference. If we define the width of an interval A as $w(A) = a_2 - a_1$ (Markov 1979). For all A and $B \in \mathcal{K}$, and $\lambda \in \mathbb{R}$, we have

$$w(A) \geq 0;\ w(\lambda A) = |\lambda| w(A);\ w(A + B) = w(A) + w(B);\ w(A \ominus_g B) = |w(A) - w(B)|.$$

Thus, it is obvious that

$$A \ominus_g B = \begin{cases} [a_1 - b_1, a_2 - b_2], & \text{if } w(A) \geq w(B), \\ [a_2 - b_2, a_1 - b_1], & \text{if } w(A) < w(B). \end{cases} \tag{2}$$

It can be seen that the H-difference must be the gH-difference, and reverse is not true. But in the case of $w(A) \geq w(B)$, there is $A \ominus_g B = A \ominus B$.

If $A,\ B$ and $C \in \mathcal{K}$, then

$$A \ominus_g B = C \Leftrightarrow \begin{cases} A = B + C, & \text{if } w(A) \geq w(B), \\ B = A + (-C), & \text{if } w(A) < w(B). \end{cases} \tag{3}$$

The Hausdorff–Pompeiu (Moore et al. 2009) metric $\mathcal{H}$ in quasi-linear space $\mathcal{K}$ is defined by

$$\mathcal{H}(A, B) = \max\{|a_1 - b_1|, |a_2 - b_2|\}. \tag{4}$$

Then $(\mathcal{K}, \mathcal{H})$ is a complete, separable and locally compact metric space (Li et al. 2013).

Now we define a functional $\|\cdot\| : \mathcal{K} \to [0, \infty)$ to be a norm on quasi-linear space $\mathcal{K}$ by $\|A\| = \max\{|a_1|, |a_2|\}$ for every $A = [a_1, a_2] \in \mathcal{K}$, and thus $(\mathcal{K}, \|\cdot\|)$ is a compete normed quasilinear space (Markov 2000; Tao 2016). Furthermore, the following relationships exist between the Hausdorff-Pompeiu metric $\mathcal{H}$ and the norm $\|\cdot\|$,

$$\|A\| = \mathcal{H}(A, \{0\}),\ \mathcal{H}(A, B) = \|A \ominus_g B\|. \tag{5}$$

2.2 Basics of interval-valued functions

Let $F(t) = [f_1(t), f_2(t)]$ be an interval-valued function, where f_1 and f_2 are real-valued functions defined on $[a, b]$, and for any $t \in [a, b]$, $f_1(t) \leq f_2(t)$ holds. Additionally, it is readily seen that the usual metric: $\mathcal{H}(F, G) = \sup_{a \leq t \leq b} \max\{|f_1(t) - g_1(t)|, |f_2(t) - g_2(t)|\}$ is associated with the norm by $\mathcal{H}(F, \{0\}) = \sup_{a \leq t \leq b} \|F(t)\|$ and $\mathcal{H}(F, G) = \sup_{a \leq t \leq b} \left\|F(t) \ominus_g G(t)\right\|$, which $\|F(t)\| = \max\{|f_1(t)|, |f_2(t)|\}$ is a function on $[a, b]$. We now can consider the concepts of limit, continuity, differentiability and integrability of interval-value functions by use of the metric $\mathcal{H}(\cdot, \cdot)$ as follows.

(i) (Lupulescu 2015) We recall that $\lim_{t \to t_0} F(t)$ exists if and only if $\lim_{t \to t_0} f_1(t)$ and $\lim_{t \to t_0} f_2(t)$ exist as finite numbers. In this case, we have

$$\lim_{t \to t_0} F(t) = \left[\lim_{t \to t_0} f_1(t), \lim_{t \to t_0} f_2(t)\right]. \tag{6}$$

In particular, F is continuous if and only if f_1 and f_2 are continuous.

It is easy to know that continuity of F and G imply continuity of $F + G$, λF, and also holds true for $F \ominus_g G$ from (Markov 1979).

(ii) (Definition 6 of Markov (1979)) If the functions f_1 and f_2 are Lebesgue integrable on $[a, b]$, then F is Lebesgue integrable on $[a, b]$. In this case we have

$$\int_a^b F(t)\mathrm{d}t = \left[\int_a^b f_1(t)\mathrm{d}t, \int_a^b f_2(t)\mathrm{d}t\right]. \tag{7}$$

(iii) (Proposition 4 of Lupulescu (2015)) $F(t)$ is absolutely continuous if and only if $f_1(t)$ and $f_2(t)$ are both absolutely continuous.

Definition 3 (Lupulescu 2015) The derivative of interval-valued function F on $t \in [a, b]$ (provided it exists) is

$$\frac{d}{dt}F(t) = \lim_{h\to 0} \frac{F(t+h) \ominus_g F(t)}{h}.$$

Remark 1 $\frac{d}{dt}F(t)$ is the general Hukuhara derivative (or gH-derivative) of F at $t \in [a, b]$, and at the end points of $[a, b]$, we consider only the one sided gH-derivatives. F is called general Hukuhara differentiable (or gH-differentiable) on $[a, b]$ if $\frac{d}{dt}F(t) \in \mathcal{K}$ exists at each point $t \in [a, b]$.

Next, we introduce δ derivative of interval-valued functions.

Definition 4 (Borges 2004; Cankaya 2021) (Interval-valued gH_δ-derivative) Suppose $g \in C^1([a, b], \mathbb{R})$ is a strictly increasing real-valued function with $g(a) > 0$ and $g'(t) > 0$ throughout this paper, and $F : [a, b] \to \mathcal{K}$ is an interval-valued function. The δ general Hukuhara derivative (gH_δ-derivative for short) of F on $t \in [a, b]$ is defined as follows:

$$\delta F(t) := \lim_{h\to 0} \frac{F(t+h) \ominus_g F(t)}{g(t+h) - g(t)}. \tag{8}$$

Remark 2 We say that F is gH_δ-differentiable on $[a, b]$ if $\delta F(t) \in \mathcal{K}$ exists at each point $t \in [a, b]$. It is easy to verify that if the interval-valued function F is gH-differentiable, then F is also gH_δ-differentiable, and we have

$$\delta F(t) := \frac{1}{g'(t)} \frac{d}{dt}F(t), \tag{9}$$

which $\frac{d}{dt}F(t)$ refers to the derivative of the interval-valued function F based on the general Hukuhara difference, i.e. $\frac{d}{dt}F(t) = \lim_{h\to 0} \frac{F(t+h)\ominus_g F(t)}{h}$. For a real-valued function f, if it is differentiable, then its δ derivative exists and f is said to be δ-differentiable.

Concerning the definition (8), there are the same way to define the q-derivative on time scales (Borges 2004; Cankaya 2021). So it is reasonable to define the δ derivative operator here.

Notice that if the real-valued function $w(F(t))$ is increasing (decreasing), that is $\delta w(F(t)) \geq 0$ ($\delta w(F(t)) \leq 0$), then the interval-valued function F is simply referred to as w_δ-increasing (w_δ-decreasing) and it is called as w_δ-monotone.

Theorem 1 *Let $F : [a, b] \to \mathcal{K}$ be an interval-valued function as $F(t) = [f_1(t), f_2(t)]$. If real-valued functions f_1 and f_2 are δ-differentiable for almost everywhere (a.e.) $t \in [a, b]$, then F is gH_δ-differentiable for a.e. $t \in [a, b]$ and*

$$\delta F(t) = [\min\{\delta f_1(t), \delta f_2(t)\}, \max\{\delta f_1(t), \delta f_2(t)\}]. \tag{10}$$

Moreover, this also has that

(i) $\delta F(t) = [\delta f_1(t), \delta f_2(t)]$ for a.e. $t \in [a, b]$, if F is w_δ-increasing;

(ii) $\delta F(t) = [\delta f_2(t), \delta f_1(t)]$ for a.e. $t \in [a, b]$, if F is w_δ-decreasing.

Proof By the definition of gH_δ-derivative, we have

$$\begin{aligned}
&\delta F(t)\\
&= \lim_{h\to 0} \frac{F(t+h) \ominus_g F(t)}{g(t+h) - g(t)}\\
&= \lim_{h\to 0} \left[\min\left\{\frac{f_1(t+h) - f_1(t)}{g(t+h) - g(t)}, \frac{f_2(t+h) - f_2(t)}{g(t+h) - g(t)}\right\},\right.\\
&\quad \left.\max\left\{\frac{f_1(t+h) - f_1(t)}{g(t+h) - g(t)}, \frac{f_2(t+h) - f_2(t)}{g(t+h) - g(t)}\right\}\right]\\
&= [\min\{\delta f_1(t), \delta f_2(t)\},\\
&\quad \max\{\delta f_1(t), \delta f_2(t)\}] \text{ for a.e. } t \in [a, b].
\end{aligned}$$

If F is w_δ-increasing, then $\delta w(F(t)) = \delta(f_2(t) - f_1(t)) \geq 0$, that is $\delta f_2(t) \geq \delta f_1(t)$. Therefore $\delta F(t) = [\delta f_1(t), \delta f_2(t)]$. Otherwise, if F is w_δ-decreasing, then $\delta F(t) = [\delta f_2(t), \delta f_1(t)]$ for a.e. $t \in [a, b]$. The proof is completed. □

Usually one only can obtain $\delta(F+G) \subseteq \delta F + \delta G$ when F and G are gH_δ-differentiable from (10). However, conditions are needed to guarantee that $\delta(F + G) = \delta F + \delta G$. For convenience, suppose $V_1(t, h) = F(t + h) \ominus_g F(t)$ and $V_2(t, h) = G(t + h) \ominus_g G(t)$.

Theorem 2 *The following properties hold:*

(i) *If F and G are equally w_δ-monotonic, then*

$$\delta(F + G) = \delta F + \delta G \tag{11}$$

and

$$\delta(F \ominus_g G) = \delta F \ominus_g \delta G. \tag{12}$$

(ii) *If F and G are differently w_δ-monotonic, then*

$$\delta(F + G) = \delta F \ominus_g (-1)\delta G \tag{13}$$

and

$$\delta(F \ominus_g G) = \delta F + (-1)\delta G. \tag{14}$$

Proof (i) Suppose that F and G are w_δ-increasing. Hence, for $h > 0$, since $w(F(t+h)) \geq w(F(t))$ and $w(G(t+h)) \geq w(G(t))$. From (3) we get $F(t+h) = F(t) + V_1(t,h)$ and $G(t+h) = G(t) + V_2(t,h)$, and thus

$$F(t+h) + G(t+h) = F(t) + G(t) + V_1(t,h) + V_2(t,h).$$

Since $w(F(t+h) + G(t+h)) \geq w(F(t) + G(t))$, we have

$$(F(t+h) + G(t+h)) \ominus_g (F(t) + G(t)) \\ = V_1(t,h) + V_2(t,h).$$

For $h < 0$, since $w(F(t+h)) \leq w(F(t))$ and $w(G(t+h)) \leq w(G(t))$, we obtain $F(t) = F(t+h) + (-1)V_1(t,h)$ and $G(t) = G(t+h) + (-1)V_2(t,h)$, and thus

$$F(t) + G(t) = F(t+h) + G(t+h) \\ +(-1)(V_1(t,h) + V_2(t,h)).$$

Due to $w(F(t+h) + G(t+h)) \leq w(F(t) + G(t))$, it follows that

$$(F(t+h) + G(t+h)) \ominus_g (F(t) + G(t)) \\ = V_1(t,h) + V_2(t,h).$$

As a result, for $h > 0$ and $h < 0$, the following formula holds

$$\lim_{h\to 0^{+-}} \frac{(F(t+h) + G(t+h)) \ominus_g (F(t) + G(t))}{g(t+h) - g(t)} \\ = \lim_{h\to 0^{+-}} \frac{V_1(t,h) + V_2(t,h)}{g(t+h) - g(t)} \\ = \delta F + \delta G.$$

Thus, $F + G$ is gH_δ-differentiable and Eq. (11) is true. Similarly, if F and G are w_δ-decreasing, $F + G$ is gH_δ-differentiable and Eq. (11) also holds.

Let

$$M = (w(F(t+h)) - w(G(t+h)))(w(F(t)) - w(G(t))).$$

We can obtain $M \geq 0$. In fact, the condition $M < 0$ means that $w(F(t+h) - w(G(t+h))$ and $w(F(t) - w(G(t))$ have different signs, which is impossible for sufficiently small h and from the continuous function $w(F(t) - w(G(t)))$.

Since $M \geq 0$, we consider $w(F(t+h)) - w(G(t+h)) \geq 0$ and $w(F(t)) - w(G(t)) \geq 0$. In the case of $h > 0$. Because $w(F(t+h)) \geq w(G(t+h))$, from (i) of Lemma 2.3 of Tao (2016) we have

$$[F(t+h) \ominus_g G(t+h)] \ominus_g [F(t) \ominus_g G(t)] \\ = F(t+h) \ominus_g [G(t+h) + (F(t) \ominus_g G(t))]. \tag{15}$$

Since $F(t+h) = F(t) + V_1(t,h)$ and $G(t+h) = G(t) + V_2(t,h)$, and thus (15) is changed to

$$(F(t) + V_1(t,h)) \ominus_g [(G(t) + V_2(t,h)) \\ +(F(t) \ominus_g G(t))]. \tag{16}$$

Since $w(F(t)) \geq w(G(t))$. By means of the properties $(A \ominus_g B) + B = A$ if $w(A) \geq w(B)$ (Stefanini 2010), and $(A + B) \ominus_g (A + C) = B \ominus_g C$ (see (ii) of Lemma 2.2 of Tao (2016)). Thus (16) can be rewritten as

$$(F(t) + V_1(t,h)) \ominus_g (F(t) + V_2(t,h)) \\ = V_1(t,h) \ominus_g V_2(t,h).$$

In the case of $h < 0$, considering $w(F(t)) \geq w(G(t))$, from (ii) of Lemma 2.3 of Tao (2016) we have

$$[F(t+h) \ominus_g G(t+h)] \ominus_g [F(t) \ominus_g G(t)] \\ = [(F(t+h) \ominus_g G(t+h)) + G(t)] \ominus_g F(t), \tag{17}$$

and since $F(t) = F(t+h) + (-1)V_1(t,h)$ and $G(t) = G(t+h) + (-1)V_2(t,h)$, substituting them into (17), we give

$$[(F(t+h) \ominus_g G(t+h)) + G(t+h) + (-1)V_2(t,h)] \\ \ominus_g (F(t+h) + (-1)V_1(t,h)). \tag{18}$$

Since $w(F(t+h)) \geq w(G(t+h))$, from (vi) on pp. 5 of Tao (2016), (18) is improved as

$$(F(t+h) + (-1)V_2(t,h)) \ominus_g (F(t+h) + (-1)V_1(t,h)) \\ = [(-1)V_2(t,h)] \ominus_g [(-1)V_1(t,h)] \\ = V_1(t,h) \ominus_g V_2(t,h).$$

As a result, whether $h > 0$ or $h < 0$, there are

$$\lim_{h\to 0^{+-}} \frac{(F(t+h) \ominus_g G(t+h)) \ominus_g (F(t) \ominus_g G(t))}{g(t+h) - g(t)} \\ = \lim_{h\to 0^{+-}} \frac{V_1(t,h) \ominus_g V_2(t,h)}{g(t+h) - g(t)} \\ = \delta F \ominus_g \delta G.$$

Thus, $F \ominus_g G$ is gH_δ-differentiable and Eq. (12) is true. Similarly, when F and G are w_δ-decreasing, one can prove that $F \ominus_g G$ is gH_δ-differentiable and Eq. (12) is also true.

(ii) For the case of F and G are differently w_δ-monotonic, the proof can be completed in the same way as that in (i). □

3 Interval-valued functions' fractional integral

First, let us revisit the space $L^p([a,b],\mathcal{K})$.

Definition 5 (Lupulescu 2015) The space $L^p([a,b]),\mathcal{K})$ is defined to consist of those interval-valued functions $F=[f_1,f_2]:[a,b]\to\mathcal{K}$ for which $\|F\|_p<\infty$, with

$$\|F\|_p=\left(\int_a^b\|F(t)\|^p\mathrm{d}t\right)^{\frac{1}{p}}\quad(1\le p<\infty)$$

and

$$\|F\|_\infty=ess\sup_{a\le t\le b}\|F(t)\|,$$

where the real-valued function $\|F(t)\|=max\{|f_1(t)|,|f_2(t)|\}$.

Remark 3 An interval-valued function $F:[a,b]\to\mathcal{K}$ is said to be L^p integrable on $[a,b]$ if and only if f_1 and f_2 are L^p integrable on $[a,b]$.

In fact, for necessity, since

$$\left(\int_a^b|f_1(t)|^p\mathrm{d}t\right)^{\frac{1}{p}}\le\left(\int_a^b\max\{|f_1(t)|,|f_2(t)|\}^p\mathrm{d}t\right)^{\frac{1}{p}}$$

and

$$\left(\int_a^b|f_2(t)|^p\mathrm{d}t\right)^{\frac{1}{p}}\le\left(\int_a^b\max\{|f_1(t)|,|f_2(t)|\}^p\mathrm{d}t\right)^{\frac{1}{p}},$$

thus from Definition 5 we know that

$$\|F\|_p=\left(\int_a^b\|F(t)\|^p\mathrm{d}t\right)^{\frac{1}{p}}=\left(\int_a^b\max\{|f_1(t)|,|f_2(t)|\}^p\mathrm{d}t\right)^{\frac{1}{p}}<\infty.$$

Therefore, f_1 and f_2 are L^p integrable on $[a,b]$.

Conversely, if f_1 and f_2 are L^p integrable on $[a,b]$, then by $\left(\int_a^b|f_1(t)|^p\mathrm{d}t\right)^{\frac{1}{p}}<\infty$ and $\left(\int_a^b|f_2(t)|^p\mathrm{d}t\right)^{\frac{1}{p}}<\infty$, we have

$$\left(\int_a^b\max\{|f_1(t)|,|f_2(t)|\}^p\mathrm{d}t\right)^{\frac{1}{p}}=\left(\int_{L_1}|f_1(t)|^p\mathrm{d}t+\int_{L_2}|f_2(t)|^p\mathrm{d}t\right)^{\frac{1}{p}}<\infty,$$

where L_1 and $L_2\subseteq[a,b]$.

In particular, when $p=1$, there is $L^p([a,b]),\mathcal{K})=L([a,b],\mathcal{K})$. It is a normed quasilinear space with respect to the norm $\|\cdot\|_p$ $(1\le p\le\infty)$.

Suppose $F\in L([a,b],\mathcal{K})$. Then the n-fold integral of F is given as

$$\begin{aligned}I_{a+}^{n,g}F(t)&=\int_a^t g'(t_1)\mathrm{d}t_1\int_a^{t_1}g'(t_2)\mathrm{d}t_2\cdots\int_a^{t_{n-1}}g'(s)F(s)\mathrm{d}s\\&=\int_a^t g'(s)F(s)\mathrm{d}s\int_s^t g'(t_1)\mathrm{d}t_1\int_s^{t_1}g'(t_2)\mathrm{d}t_2\cdots\\&\int_s^{t_{n-2}}g'(t_{n-1})\mathrm{d}t_{n-1}\\&=\frac{1}{\Gamma(n)}\int_a^t(g(t)-g(s))^{n-1}g'(s)F(s)\mathrm{d}s.\end{aligned}\tag{19}$$

Let n be a positive real number α, then an interval-valued general fractional integral is defined as follows.

Definition 6 Suppose $F\in L([a,b],\mathcal{K})$, and $g\in C^1[a,b]$ is a strictly increasing real-valued function with $g(a)\ge0$ and $g'(t)>0$. The interval-valued general fractional integral of order $\alpha>0$ is defined by

$$I_{a+}^{\alpha,g}F(t):=\frac{1}{\Gamma(\alpha)}\int_a^t(g(t)-g(s))^{\alpha-1}g'(s)F(s)\mathrm{d}s.\tag{20}$$

If $F=[f_1,f_2]\in L([a,b],\mathcal{K})$ and $\alpha>0$, then

$$I_{a+}^{\alpha,g}F(t)=[I_{a+}^{\alpha,g}f_1(t),I_{a+}^{\alpha,g}f_2(t)].$$

Lemma 1 (Corollary 2.5 of Fan et al. (2022)) *Let $f\in L^p[a,b]$ and $u=\frac{g(t)}{g(s)}$, $a\le s\le t$, $1\le p\le\infty$. If there is a function $J\in C[1,\frac{g(b)}{g(a)}]$ such that $g^{-1}(ug(s))\le J(u)s$ and $\frac{d(g^{-1}(ug(s)))}{ds}\le J(u)$, respectively, then*

$$\|I_{a+}^{\alpha,g}f(t)\|_p\le K\|f\|_p\tag{21}$$

where

$$K=\int_1^{\frac{g(b)}{g(a)}}\frac{g(b)^\alpha}{\Gamma(\alpha)}\frac{(u-1)^{\alpha-1}}{u^{\alpha+1}}J(u)^{\frac{1}{p}}du.\tag{22}$$

Theorem 3 (Boundedness theorem) *Let $F\in L([a,b],\mathcal{K})$. Then the interval-valued general fractional integral $I_{a+}^{\alpha,g}$ is bounded in the space $L([a,b],\mathcal{K})$:*

$$\|I_{a+}^{\alpha,g}F\|_{p=1}\le K\|F\|_{p=1},$$

where $K=\int_1^{\frac{g(b)}{g(a)}}\frac{g(b)^\alpha}{\Gamma(\alpha)}\frac{(u-1)^{\alpha-1}}{u^{\alpha+1}}J(u)du$ and J is a control function defined by Lemma 1.

Proof Since g is strictly increasing on $[a, b]$, we have

$$\begin{aligned}
&\|I_{a+}^{\alpha,g}F\| \\
&= \max\{|I_{a+}^{\alpha,g} f_1(t)|, |I_{a+}^{\alpha,g} f_2(t)|\} \\
&\le \max\{\frac{1}{\Gamma(\alpha)}\int_a^t (g(t)-g(s))^{\alpha-1}g'(s)|f_1(s)|\mathrm{d}s, \\
&\quad \frac{1}{\Gamma(\alpha)}\int_a^t (g(t)-g(s))^{\alpha-1}g'(s)|f_2(s)|\mathrm{d}s\} \\
&\le \frac{1}{\Gamma(\alpha)}\int_a^t (g(t)-g(s))^{\alpha-1}g'(s)\max\{|f_1(s)|, |f_2(s)|\}\mathrm{d}s \\
&= I_{a+}^{\alpha,g}\|F\|.
\end{aligned}$$

Since $\|F\|$ is a real Lebesgue integrable function on $[a, b]$, by Lemma 1, we get

$$\|I_{a+}^{\alpha,g}\|F\|\|_{p=1} \le K\|\|F\|\|_{p=1},$$

where $K = \int_1^{\frac{g(b)}{g(a)}} \frac{g(b)^\alpha}{\Gamma(\alpha)}\frac{(u-1)^{\alpha-1}}{u^{\alpha+1}}J(u)\mathrm{d}u$ is a positive constant (here $J(u)$ see Lemma 1). Therefore, we obtain

$$\begin{aligned}
\left\|I_{a+}^{\alpha,g}F\right\|_{p=1} &= \int_a^b \left\|I_{a+}^{\alpha,g}F\right\|\mathrm{d}t \\
&\le \int_a^b I_{a+}^{\alpha,g}\|F\|\mathrm{d}t \\
&\le K\int_a^b \|F\|\mathrm{d}t \\
&= K\|F\|_{p=1}.
\end{aligned}$$

As a result, the proof is completed. □

Theorem 4 (Semigroup property) *Let α and $\beta > 0$. For $F \in L([a, b], \mathcal{K})$, the semigroup property holds*

$$I_{a+}^{\alpha,g}I_{a+}^{\beta,g}F(t) = I_{a+}^{\alpha+\beta,g}F(t). \tag{23}$$

Proof Since $I_{a+}^{\beta,g}F(t) = \left[I_{a+}^{\beta,g}f_1(t), I_{a+}^{\beta,g}f_2(t)\right]$ for $t \in [a, b]$, by the semigroup property of real-valued functions (Fu et al. 2021a, b), we get

$$\begin{aligned}
I_{a+}^{\alpha,g}I_{a+}^{\beta,g}F(t) &= I_{a+}^{\alpha,g}\left[I_{a+}^{\beta,g}f_1(t), I_{a+}^{\beta,g}f_2(t)\right] \\
&= \left[I_{a+}^{\alpha,g}I_{a+}^{\beta,g}f_1(t), I_{a+}^{\alpha,g}I_{a+}^{\beta,g}f_2(t)\right] \\
&= \left[I_{a+}^{\alpha+\beta,g}f_1(t), I_{a+}^{\alpha+\beta,g}f_2(t)\right] \\
&= I_{a+}^{\alpha+\beta,g}F(t),
\end{aligned}$$

which completes the proof. □

Similar as Remark 5 and Theorem 1 of Lupulescu (2015), we give the following theorem without proof.

Theorem 5 *If F and $G \in L([a, b], \mathcal{K})$, $\alpha > 0$, then*

(a) $I_{a+}^{\alpha,g}(F(t) + G(t)) = I_{a+}^{\alpha,g}F(t) + I_{a+}^{\alpha,g}G(t)$ *for all* $t \in [a, b]$.

(b) $I_{a+}^{\alpha,g}(F(t) \ominus_g G(t)) \supseteq I_{a+}^{\alpha,g}F(t) \ominus_g I_{a+}^{\alpha,g}G(t)$ *for all* $t \in [a, b]$. *Moreover, if $w(F(t)) - w(G(t))$ has a constant sign, that is $w(F(t)) \ge w(G(t))$ or $w(F(t)) \le w(G(t))$ on $[a, b]$, then* $I_{a+}^{\alpha,g}\left(F(t) \ominus_g G(t)\right) = I_{a+}^{\alpha,g}F(t) \ominus_g I_{a+}^{\alpha,g}G(t)$.

4 Interval-valued functions' fractional derivatives

Definition 7 (Function space) In the δ derivative's sense, a function space is defined by

$$AC_\delta^n([a, b], \mathcal{K}) = \{F : [a, b] \to \mathcal{K} :\ \delta^{n-1}F \in AC[a, b]\}.$$

We use $AC_\delta([a, b], K) = AC_\delta^1([a, b], K)$. We note that $F \in AC_\delta([a, b], \mathcal{K})$ if and only if f_1 and $f_2 \in AC_\delta([a, b], \mathbb{R})$, where $F(t) = [f_1(t), f_2(t)]$, $t \in [a, b]$ (similar as that of Lupulescu (2015)). It is easy to show that if $F \in AC_\delta([a, b], \mathcal{K})$, then δF exists almost everywhere, and $\delta F \in L([a, b], \mathcal{K})$ (see pp. 2 of Kilbas et al. (2006)).

4.1 General Riemann–Liouville derivative

Definition 8 Suppose $F = [f_1, f_2] \in AC_\delta([a, b], \mathcal{K})$. The general fractional R–L derivative of F of $0 < \alpha < 1$ order is given by

$$\begin{aligned}
D_{a+}^{\alpha,g}F(t) :=& \delta I_{a+}^{1-\alpha,g}F(t) \\
=& \frac{\delta}{\Gamma(1-\alpha)}\int_a^t g'(s)(g(t)-g(s))^{-\alpha}F(s)\mathrm{d}s
\end{aligned} \tag{24}$$

for a.e. $t \in [a, b]$. For $\alpha = 1$, $D_{a+}^{1,g}F(t) = \delta F(t)$.

Lemma 2 *(Samko et al. 1993) If the real-valued function $f \in AC_\delta([a, b], \mathbb{R})$, then $I_{a+}^{1-\alpha,g}f \in AC_\delta([a, b], \mathbb{R})$.*

For the proof of this lemma, the reader can refer to Lemma 2.1 in Samko et al. (1993).

Remark 4 From Lemma 2, if $F = [f_1, f_2] \in AC_\delta([a, b], \mathcal{K})$, then $I_{a+}^{1-\alpha,g}F \in AC_\delta([a, b], \mathcal{K})$ and the general R–L derivative exists almost everywhere on $[a, b]$. So it is well-defined.

Theorem 6 *Let $F = [f_1, f_2] \in AC_\delta([a, b], \mathcal{K})$, then*

(a) $D_{a+}^{\alpha,g}F(t) = \left[\min\left\{D_{a+}^{\alpha,g}f_1(t), D_{a+}^{\alpha,g}f_2(t)\right\}, \max\left\{D_{a+}^{\alpha,g}f_1(t), D_{a+}^{\alpha,g}f_2(t)\right\}\right]$ *for a.e.* $t \in [a, b]$.

(b) *If $I_{a+}^{1-\alpha,g}F$ is w_δ-increasing on $[a, b]$, then*

$$D_{a+}^{\alpha,g}F(t) = \left[D_{a+}^{\alpha,g}f_1(t), D_{a+}^{\alpha,g}f_2(t)\right] \text{ for a.e. } t \in [a, b].$$

(c) *If* $I_{a+}^{1-\alpha,g}F$ *is* w_δ*-decreasing on* $[a,b]$*, then*

$$D_{a+}^{\alpha,g}F(t)=\left[D_{a+}^{\alpha,g}f_2(t),D_{a+}^{\alpha,g}f_1(t)\right] \text{ for a.e. } t\in[a,b].$$

Proof (a) Since $I_{a+}^{1-\alpha,g}F\in AC_\delta([a,b],\mathcal{K})$, $\delta I_{a+}^{1-\alpha,g}F(t)$ exists for a.e. $t\in[a,b]$, then $D_{a+}^{\alpha,g}F$ exists for a.e. $t\in[a,b]$. From (10), we obtain

$$\begin{aligned}D_{a+}^{\alpha,g}F(t)&=\delta I_{a+}^{1-\alpha,g}F(t)\\&=\Big[\min\left\{\delta I_{a+}^{1-\alpha,g}f_1(t),\delta I_{a+}^{1-\alpha,g}f_2(t)\right\},\\&\quad\max\left\{\delta I_{a+}^{1-\alpha,g}f_1(t),\delta I_{a+}^{1-\alpha,g}f_2(t)\right\}\Big]\\&=\left[\min\left\{D_{a+}^{\alpha,g}f_1(t),D_{a+}^{\alpha,g}f_2(t)\right\},\right.\\&\quad\left.\max\left\{D_{a+}^{\alpha,g}f_1(t),D_{a+}^{\alpha,g}f_2(t)\right\}\right] \text{ for a.e. } t\in[a,b].\end{aligned}$$

(b) Suppose $I_{a+}^{1-\alpha,g}F$ is w_δ-increasing on $[a,b]$. Then from Theorem 1 it follows that

$$\begin{aligned}D_{a+}^{\alpha,g}F(t)&=\delta I_{a+}^{1-\alpha,g}F(t)\\&=\left[\delta I_{a+}^{1-\alpha,g}f_1(t),\delta I_{a+}^{1-\alpha,g}f_2(t)\right]\\&=\left[D_{a+}^{\alpha,g}f_1(t),D_{a+}^{\alpha,g}f_2(t)\right] \text{ for a.e. } t\in[a,b].\end{aligned}$$

(c) The proof is similar to as that of (b). We do not give detail here. □

Next, we consider the following composite properties.

Theorem 7 *Let* $0<\alpha\le 1$*. If* $F=[f_1,f_2]\in L([a,b],\mathcal{K})$*, then*

$$D_{a+}^{\alpha,g}I_{a+}^{\alpha,g}F(t)=F(t) \text{ for } t\in[a,b]. \tag{25}$$

Proof For $\alpha=1$. It is obvious that $I_{a+}^{1,g}F(t)=[\int_a^t g'(s)f_1(s)\,ds,\int_a^t g'(s)f_2(s)ds]$, then $\delta w(I_{a+}^{1,g}F(t))=\delta\{\int_a^t g'(s)(f_2(s)-f_1(s))\}ds=f_2(t)-f_1(t)\ge 0$. Thus $I_{a+}^{1,g}F(t)$ is w_δ-increasing.

For $0<\alpha<1$. From Theorem 4 and Theorem 1, we have

$$\begin{aligned}D_{a+}^{\alpha,g}I_{a+}^{\alpha,g}F(t)&=\delta I_{a+}^{1-\alpha,g}I_{a+}^{\alpha,g}F(t)=\delta I_{a+}^{1,g}F(t)\\&=\delta[\int_a^t g'(s)f_1(s)ds,\int_a^t g'(s)f_2(s)ds]\\&=[f_1(t),f_2(t)]\\&=F(t)\end{aligned}$$

for $t\in[a,b]$. This completes the proof. □

Remark 5 More generally, let $\beta>\alpha$ and $0<\alpha\le 1$. By Theorem 4 and Theorem 7, then

$$\begin{aligned}D_{a+}^{\alpha,g}I_{a+}^{\beta,g}F(t)&=D_{a+}^{\alpha,g}I_{a+}^{\alpha,g}I_{a+}^{\beta-\alpha,g}F(t)\\&=I_{a+}^{\beta-\alpha,g}F(t)\end{aligned}$$

for $t\in[a,b]$.

Theorem 8 *Let* $F\in AC_\delta([a,b],\mathcal{K})$ *and* $0<\alpha\le 1$*. If* $I_{a+}^{1-\alpha,g}F(t)$ *is* w_δ*-monotone on* $[a,b]$*, then*

$$I_{a+}^{\alpha,g}D_{a+}^{\alpha,g}F(t)=F(t)\ominus_g\frac{I_{a+}^{1-\alpha,g}F(a)}{\Gamma(\alpha)}(g(t)-g(a))^{\alpha-1} \tag{26}$$

for a.e. $t\in[a,b]$.

Proof Suppose $f\in AC_\delta([a,b],\mathbb{R})$. By Theorem 2.6 of Jarad and Abdeljawad (2020), we have $I_{a+}^{\alpha,g}D_{a+}^{\alpha,g}f(t)=f(t)-\frac{I_{a+}^{1-\alpha,g}f(a)}{\Gamma(\alpha)}(g(t)-g(a))^{\alpha-1}$ for a.e. $t\in[a,b]$. If $I_{a+}^{1-\alpha,g}F$ is w_δ-increasing, from (b) of Theorem 6, then

$$\begin{aligned}&I_{a+}^{\alpha,g}D_{a+}^{\alpha,g}F(t)\\&=I_{a+}^{\alpha,g}\left[D_{a+}^{\alpha,g}f_1(t),D_{a+}^{\alpha,g}f_2(t)\right]\\&=\left[I_{a+}^{\alpha,g}D_{a+}^{\alpha,g}f_1(t),I_{a+}^{\alpha,g}D_{a+}^{\alpha,g}f_2(t)\right]\\&=\Bigg[f_1(t)-\frac{I_{a+}^{1-\alpha,g}f_1(a)}{\Gamma(\alpha)}(g(t)-g(a))^{\alpha-1},\\&\quad f_2(t)-\frac{I_{a+}^{1-\alpha,g}f_2(a)}{\Gamma(\alpha)}(g(t)-g(a))^{\alpha-1}\Bigg]\\&=F(t)\ominus_g\frac{I_{a+}^{1-\alpha,g}F(a)}{\Gamma(\alpha)}(g(t)\\&-g(a))^{\alpha-1} \text{ for a.e. } t\in[a,b].\end{aligned}$$

A similar proof can be given if $I_{a+}^{1-\alpha,g}F$ is w_δ-decreasing. So Eq. (26) holds. □

4.2 General Caputo derivative

Definition 9 Suppose $F=[f_1,f_2]\in AC_\delta([a,b],\mathcal{K})$. The general Caputo fractional derivative of F of order $0<\alpha<1$ is given by

$$\begin{aligned}{}^C D_{a+}^{\alpha,g}F(t)&:=I_{a+}^{1-\alpha,g}\delta F(t)\\&=\frac{1}{\Gamma(1-\alpha)}\int_a^t g'(s)(g(t)-g(s))^{-\alpha}\delta F(s)ds\end{aligned} \tag{27}$$

for a.e. $t\in[a,b]$. For $\alpha=1$, ${}^C D_{a+}^{1,g}F(t)=\delta F(t)$ for a.e. $t\in[a,b]$.

Remark 6 Suppose $F\in AC_\delta([a,b],\mathcal{K})$. δF exists for a.e. $t\in[a,b]$ and $\delta F\in L([a,b],\mathcal{K})$. Thus $I_{a+}^{1-\alpha,g}\delta F<\infty$ and the Caputo derivative of F exists a.e. $t\in[a,b]$.

Theorem 9 *(Hoa et al. 2017) The following properties holds.*

(1) *If* $F \in AC_\delta([a,b],\mathcal{K})$ *and* $0<\alpha\le 1$, *then*

$$ {}^C D^{\alpha,g}_{a+}F(x) \supseteq \left[\min\left\{{}^C D^{\alpha,g}_{a+}f_1(t), {}^C D^{\alpha,g}_{a+}f_2(t)\right\}, \max\left\{{}^C D^{\alpha,g}_{a+}f_1(t), {}^C D^{\alpha,g}_{a+}f_2(t)\right\}\right] $$

for a.e. $t\in[a,b]$.

(2) *If* F *is* w_δ*-increasing, then* ${}^C D^{\alpha,g}_{a+}F(t) = \left[{}^C D^{\alpha,g}_{a+}f_1(t), {}^C D^{\alpha,g}_{a+}f_2(t)\right]$ *for a.e.* $t\in[a,b]$.

(3) *If* F *is* w_δ*-decreasing, then* ${}^C D^{\alpha,g}_{a+}F(t) = \left[{}^C D^{\alpha,g}_{a+}f_2(t), {}^C D^{\alpha,g}_{a+}f_1(t)\right]$ *for a.e.* $t\in[a,b]$.

Theorem 10 *Let* F *and* $G \in AC_\delta([a,b],\mathcal{K})$ *be* w_δ*-monotone, and* $0<\alpha\le 1$. *Then the following properties hold.*

(a) *If* F *and* G *are equally* w_δ*-monotonic on* $[a,b]$, *then*

$$ {}^C D^{\alpha,g}_{a+}(F(t)+G(t)) = {}^C D^{\alpha,g}_{a+}F(t) + {}^C D^{\alpha,g}_{a+}G(t) $$

and

$$ {}^C D^{\alpha,g}_{a+}(F(t)\ominus_g G(t)) \supseteq {}^C D^{\alpha,g}_{a+}F(t) \ominus_g {}^C D^{\alpha,g}_{a+}G(t) $$

for a.e. $t\in[a,b]$. *Moreover, if* $w(\delta F(t)) - w(\delta G(t))$ *has a constant sign, then*

$$ {}^C D^{\alpha,g}_{a+}(F(t)\ominus_g G(t)) = {}^C D^{\alpha,g}_{a+}F(t) \ominus_g {}^C D^{\alpha,g}_{a+}G(t) \ \textit{for a.e. } t\in[a,b]. \tag{28} $$

(b) *If* F *and* G *are differently* w_δ*-monotonic on* $[a,b]$, *then*

$$ {}^C D^{\alpha,g}_{a+}(F(t)\ominus_g G(t)) = {}^C D^{\alpha,g}_{a+}F(t) + (-{}^C D^{\alpha,g}_{a+}G(t)) $$

and

$$ {}^C D^{\alpha,g}_{a+}(F(t)+G(t)) \supseteq {}^C D^{\alpha,g}_{a+}F(t) \ominus_g (-{}^C D^{\alpha,g}_{a+}G(t)) $$

for a.e. $t\in[a,b]$. *Moreover, if* $w(\delta F(t)) - w(\delta G(t))$ *has a constant sign, then*

$$ {}^C D^{\alpha,g}_{a+}(F(t)+G(t)) = {}^C D^{\alpha,g}_{a+}F(t) \ominus_g (-{}^C D^{\alpha,g}_{a+}G(t)) \ \textit{for a.e. } t\in[a,b]. $$

Proof (a) If F and G are equally w_δ-monotonic on $[a,b]$. By Theorem 2, it follows that $\delta(F+G)=\delta F+\delta G$ and $\delta(F\ominus_g G)=\delta F\ominus_g \delta G$, and from Theorem 5, we obtain

$$ \begin{aligned} {}^C D^{\alpha,g}_{a+}(F(t)+G(t)) &= I^{1-\alpha,g}_{a+}\delta(F(t)+G(t)) \\ &= I^{1-\alpha,g}_{a+}(\delta F(t)+\delta G(t)) \\ &= {}^C D^{\alpha,g}_{a+}F(t) + {}^C D^{\alpha,g}_{a+}G(t) \end{aligned} $$

and

$$ \begin{aligned} D^{\alpha,g}_{a+}(F(t)\ominus_g G(t)) &= I^{1-\alpha,g}_{a+}\delta(F(t)\ominus_g G(t)) \\ &= I^{1-\alpha,g}_{a+}(\delta F(t)\ominus_g \delta G(t)) \\ &\supseteq I^{1-\alpha,g}_{a+}\delta F(t) \ominus_g I^{1-\alpha,g}_{a+}\delta G(t) \\ &= {}^C D^{\alpha,g}_{a+}F(t) \ominus_g {}^C D^{\alpha,g}_{a+}G(t) \end{aligned} $$

for a.e. $t\in[a,b]$. Moreover, if $w(\delta F(t)) \ge w(\delta G(t))$ or $w(\delta F(t)) \le w(\delta G(t))$, by Theorem 5, we have

$$ {}^C D^{\alpha,g}_{a+}(F(t)\ominus_g G(t)) = {}^C D^{\alpha,g}_{a+}F(t) \ominus_g {}^C D^{\alpha,g}_{a+}G(t). $$

(b) For the case, F and G are different w_δ-monotonic on $[a,b]$, the proof can be completed similarly. □

Next, we give the following composite properties.

Theorem 11 *Let* $0<\alpha\le 1$ *and* $F\in L([a,b],\mathcal{K})$. *If* $I^{\alpha,g}_{a+}F$ *is* w_δ*-increasing, then*

$$ {}^C D^{\alpha,g}_{a+} I^{\alpha,g}_{a+}F(t) = F(t) \ \textit{for a.e. } t\in[a,b]. \tag{29} $$

Proof It is known that by Theorem 9, since $I^{\alpha,g}_{a+}F$ is w_δ-increasing on $[a,b]$ and from Corollary 1 of Jarad and Abdeljawad (2020), we have

$$ \begin{aligned} {}^C D^{\alpha,g}_{a+} I^{\alpha,g}_{a+}F(t) &= {}^C D^{\alpha,g}_{a+}\left[I^{\alpha,g}_{a+}f_1(t), I^{\alpha,g}_{a+}f_2(t)\right] \\ &= \left[{}^C D^{\alpha,g}_{a+} I^{\alpha,g}_{a+}f_1(t), {}^C D^{\alpha,g}_{a+} I^{\alpha,g}_{a+}f_2(t)\right] \\ &= [f_1(t), f_2(t)]. \end{aligned} $$

□

Remark 7 More generally, let $\beta>\alpha$, $0<\alpha\le 1$ and $F\in L([a,b],\mathcal{K})$ such that $I^{\alpha,g}_{a+}F$ is w_δ-increasing. By Theorem 4 and Theorem 11, then

$$ \begin{aligned} {}^C D^{\alpha,g}_{a+} I^{\beta,g}_{a+}F(t) &= {}^C D^{\alpha,g}_{a+} I^{\alpha,g}_{a+} I^{\beta-\alpha,g}_{a+}F(t) \\ &= I^{\beta-\alpha,g}_{a+}F(t) \end{aligned} $$

for $t\in[a,b]$.

Theorem 12 *Let* $0<\alpha\le 1$ *and* $F\in AC_\delta([a,b],\mathcal{K})$ *is a* w_δ*-monotone interval-valued function. Then*

$$ I^{\alpha,g}_{a+}\,{}^C D^{\alpha,g}_{a+}F(t) = F(t)\ominus_g F(a) \ \textit{for a.e. } t\in[a,b]. \tag{30} $$

Proof According to Definition 9 and Proposition 6 of Lupulescu (2015), for a.e. $t \in [a,b]$, we obtain

$$\begin{aligned} I_{a+}^{\alpha,g}\, {}^{C}D_{a+}^{\alpha,g} F(t) &= I_{a+}^{\alpha,g} I_{a+}^{1-\alpha,g} \delta F(t) \\ &= I_{a+}^{1,g} \delta F(t) \\ &= \int_a^t g'(s)\delta F(s)\mathrm{d}s \\ &= F(t) \ominus_g F(a). \end{aligned}$$

□

5 Gronwall inequalities

Consider the following interval-valued linear initial value problem

$$\begin{cases} {}^{C}D_{a+}^{\alpha,g} X(t) = \lambda X(t), \\ X(a) = X_a, \end{cases} \tag{31}$$

where $X = [x_1, x_2] \in AC_\delta([a,b], \mathcal{K})$ is w_δ-monotone, $0 < \alpha \leq 1$, $\lambda \in \mathbb{R}$ is a constant, and $X_a = [x_1(a), x_2(a)]$ is the initial value.

From Theorem 12, we obtain the following integral equation

$$X(t) \ominus_g X_a = I_{a+}^{\alpha,g} \lambda X(t). \tag{32}$$

We can say X is a solution of Eq. (31), if and only if it solves Eq. (32).

If X is w_δ-increasing on $[a,b]$, then Eq. (32) can be written as

$$X(t) = X_a + I_{a+}^{\alpha,g} \lambda X(t). \tag{33}$$

If X is w_δ-decreasing on $[a,b]$, then Eq. (32) can be written as

$$X(t) = X_a \ominus (-1) I_{a+}^{\alpha,g} \lambda X(t). \tag{34}$$

Thus, we will discuss two fractional Gronwall inequalities for interval-valued functions. Some necessary definitions are given first.

Definition 10 (Tao 2016) If an interval-valued inequality $X(t) \leq Y(t)$ for any $t \in [a,b]$, this means

$$x_1(t) \leq y_1(t),\ x_2(t) \leq y_2(t) \ \text{ for any } t \in [a,b].$$

Theorem 13 *Suppose* $0 \leq \lambda \leq 1$, *an interval number* $K = [k_1, k_2] \geq 0$, *and the interval-valued function* $I_{a+}^{\alpha,g} X$ *is* w_δ-*increasing on* $[a,b]$.

(i) *If* $X(t) \leq K + I_{a+}^{\alpha,g} \lambda X(t)$, *then*

$$X(t) \leq K E_\alpha(\lambda, (g(t) - g(a))^\alpha).$$

(ii) *If* $X(t) \leq K \ominus (-1) I_{a+}^{\alpha,g} \lambda X(t)$, *then*

$$\begin{aligned} X(t) \leq [&k_1 E_\alpha(\lambda^2, (g(t) - g(a))^{2\alpha}) + k_2 \lambda (g(t) \\ &- g(a))^\alpha E_{2\alpha,\alpha+1}(\lambda^2, (g(t) \\ &- g(a))^{2\alpha}), k_1 \lambda (g(t) - g(a))^\alpha E_{2\alpha,\alpha+1}(\lambda^2, (g(t) \\ &- g(a))^{2\alpha}) + k_2 E_\alpha(\lambda^2, (g(t) - g(a))^{2\alpha})]. \end{aligned}$$

Proof (i) Let $U(t) = K + I_{a+}^{\alpha,g} \lambda X(t)$, then

$$X(t) \leq U(t),\ \ X(a) \leq K. \tag{35}$$

Since $I_{a+}^{\alpha,g} X$ is w_δ-increasing, we obtain $I_{a+}^{\alpha,g} \lambda X$ is w_δ-increasing. Further U is also w_δ-increasing. According to Theorems 10 and 11, from $U(t) = K + I_{a+}^{\alpha,g} \lambda X(t)$ we derive ${}^{C}D_{a+}^{\alpha,g} U = \lambda X \leq \lambda U$. Therefore, we obtain

$${}^{C}D_{a+}^{\alpha,g} u_1 \leq \lambda u_1 \text{ and } {}^{C}D_{a+}^{\alpha,g} u_2 \leq \lambda u_2.$$

Let $c_1(t) \geq 0$ and $c_2(t) \geq 0$ for any $t \in [a,b]$. Then we consider the following real-valued equations

$$\begin{cases} {}^{C}D_{a+}^{\alpha,g} u_1(t) = \lambda u_1(t) - c_1(t) \\ u_1(a) = k_1 \end{cases} \tag{36}$$

and

$$\begin{cases} {}^{C}D_{a+}^{\alpha,g} u_2(t) = \lambda u_2(t) - c_2(t), \\ u_2(a) = k_2. \end{cases} \tag{37}$$

Then we have

$$\begin{aligned} &u_1(t) \\ &\quad = k_1 E_\alpha(\lambda, (g(t) - g(a))^\alpha) \\ &\qquad - \int_a^t E_{\alpha,\alpha}(\lambda, ((g(t) - g(s))^\alpha))(g(t) - g(s))^{\alpha-1} \\ &\quad g'(s) c_1(s)\mathrm{d}s \leq k_1 E_\alpha(\lambda, (g(t) - g(a))^\alpha). \end{aligned}$$

Similarly, $u_2(t) \leq k_2 E_\alpha(\lambda, (g(t) - g(a))^\alpha)$. Therefore, $U(t) \leq K E_\alpha(\lambda, (g(t) - g(a))^\alpha)$, we arrive at the desired result.

(ii) Let $U(t) = K \ominus (-1) I_{a+}^{\alpha,g} \lambda X$, then

$$X(t) \leq U(t) \text{ and } X(a) \leq K. \tag{38}$$

Thus we have $U(t) = [k_1 + I_{a+}^{\alpha,g} \lambda x_2, k_2 + I_{a+}^{\alpha,g} \lambda x_1]$. Since $I_{a+}^{\alpha,g} X$ is w_δ-increasing, we obtain that U is w_δ-decreasing.

Thus according to Theorem 9, we derive

$$\begin{aligned} {}^{C}D_{a+}^{\alpha,g}U &= {}^{C}D_{a+}^{\alpha,g}[k_1 + I_{a+}^{\alpha,g}\lambda x_2, k_2 + I_{a+}^{\alpha,g}\lambda x_1] \\ &= [{}^{C}D_{a+}^{\alpha,g}(k_2 + I_{a+}^{\alpha,g}\lambda x_1), {}^{C}D_{a+}^{\alpha,g}(k_1 + I_{a+}^{\alpha,g}\lambda x_2)] \\ &= [\lambda x_1, \lambda x_2] \\ &= \lambda X \leq \lambda U. \end{aligned}$$

Therefore, we have

$${}^{C}D_{a+}^{\alpha,g}u_1 \leq \lambda u_2 \text{ and } {}^{C}D_{a+}^{\alpha,g}u_2 \leq \lambda u_1.$$

Suppose $c_1(t) \geq 0$ and $c_2(t) \geq 0$ for any $t \in [a, b]$. Then we consider the following real-valued equation

$$\begin{cases} {}^{C}D_{a+}^{\alpha,g}u_1(t) = \lambda u_2(t) - c_1(t), \\ {}^{C}D_{a+}^{\alpha,g}u_2(t) = \lambda u_1(t) - c_2(t), \\ u_1(a) = k_1, \ u_2(a) = k_2. \end{cases} \tag{39}$$

It can be rewritten as

$$\begin{cases} {}^{C}D_{a+}^{\alpha,g}\boldsymbol{u}(t) = A\boldsymbol{u}(t) - \boldsymbol{c}(t), \\ \boldsymbol{u}(a) = \boldsymbol{k}, \end{cases} \tag{40}$$

where vector $\boldsymbol{u}(t) = (u_1(t), u_2(t))^{\mathrm{T}}$, $\boldsymbol{c}(t) = (c_1(t), c_2(t))^{\mathrm{T}}$, $\boldsymbol{k} = (k_1, k_2)^{\mathrm{T}}$, and matrix $A = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}$.

It is clear that the solution of Eq. (40) has the following relationship

$$\boldsymbol{u}(t) \leq E_\alpha(A, (g(t) - g(a)^\alpha))\boldsymbol{k}.$$

It is easy to verify that matrix A can be diagonalized, that is, there is an invertible matrix $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, such that $P^{-1}AP = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$. Thus $A = P\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}P^{-1}$. Furthermore, we can obtain

$$\begin{aligned} &E_\alpha(A, (g(t) - g(a)^\alpha)) \\ &= P\begin{pmatrix} E_\alpha(\lambda, (g(t) - g(a)^\alpha)) & 0 \\ 0 & E_\alpha(-\lambda, (g(t) - g(a)^\alpha)) \end{pmatrix}P^{-1}. \end{aligned}$$

Then

$$\begin{aligned} \boldsymbol{u}(t) &\leq P\begin{pmatrix} E_\alpha(\lambda, (g(t) - g(a)^\alpha)) & 0 \\ 0 & E_\alpha(-\lambda, (g(t) - g(a)^\alpha)) \end{pmatrix}P^{-1}\boldsymbol{k} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &\begin{pmatrix} E_\alpha(\lambda, (g(t) - g(a)^\alpha)) & 0 \\ 0 & E_\alpha(-\lambda, (g(t) - g(a)^\alpha)) \end{pmatrix} \\ &\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \end{aligned}$$

Thus, we get the desired result. □

Conclusion

The general fractional calculus for interval-valued functions is developed in this study. A general fractional Gronwall inequalities are given. The general fractional calculus theory of interval-valued functions in this paper will be used in the existence, uniqueness and stability of solutions to general fractional interval-valued differential equations. We will consider this aspect in future research.

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Declarations

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