Soft Computing (2023) 27:11081–11089
https://doi.org/10.1007/s00500-023-07946-y

METHODOLOGIES AND APPLICATION

The investigation of *k*-fuzzy metric spaces with the first contraction principle in such spaces

Dhananjay Gopal[1] · Wutiphol Sintunavarat[2] · Abhay S. Ranadive[1] · Satish Shukla[3]

Accepted: 4 February 2023 / Published online: 3 March 2023

Abstract

This paper introduces the notion of k-fuzzy metric spaces, which generalizes and extends the concept of fuzzy metric spaces due to George and Veeramani in [A. George and P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets and Systems 64 (1994), 395-399.] for the fuzzy sets involving more than one (k) parameters. It is shown that the topology generated by the k-fuzzy metric is first countable, and the k-fuzzy metric space is Hausdorff. Finally, we prove a fixed point theorem, which generalizes and extends the results of Grabiec [M. Grabiec, Fixed points in fuzzy metric spaces, Fuzzy Sets and Systems, 27 (1988), 385-389.] into k-fuzzy metric spaces.

Keywords k-fuzzy metric spaces · Hausdorff spaces · Contractions · Fixed points

1 Introduction

The theory of fuzzy sets was introduced by Zadeh (1965) in 1965. This theory inspires several concepts in mathematics. One of these concepts is a fuzzy metric space close to the fixed point theory. The most popular definition of this space is due to Kramosil and Michalek (1975) ten years later the appearance of the fuzzy theory. The ideas of Cauchyness, completeness (now known as a G-completeness) and compactness in fuzzy metric spaces were introduced by Grabiec (1988). Furthermore, in his research, fuzzy versions of two classical results in fixed point theory, including Banach fixed point theorem and Edelstein fixed point theorem, were proved fuzzy metric spaces in the setting of Kramosil and Michálek. Based on the fact that $\mathbb{R}$ is not complete with the completeness due to Grabiec (1988), the idea of the Cauchy sequence in fuzzy metric spaces was modified by George and Veeramani (1994). They also slightly modified the definition of a fuzzy metric space introduced by Kramosil and Michalek (1975) and defined concepts of Hausdorff topology and first countable topology. In this setting, it brings to the impressive fact that fuzzy metrics appear more appropriate for studying induced topological structures.

The motivation in this paper for inventing a new space, which is more general than a fuzzy metric space due to George and Veeramani (1994), is given in this paragraph. In a fuzzy metric space, the fuzzy distance of two points is measured by the degree of the nearness of points with respect to a parameter $t \in (0, \infty)$. For instance, we can think of t as the time required to travel between two points x and y in a space. There is an interesting situation of the degree of nearness when we measure this degree with respect to different (more than one) parameters. For instance, suppose that we move from India, represented by x, to Thailand, represented by y, by a plane and measure the degree of the nearness of x and y with respect to time and fuel consumption with planes of different fuel efficiency. Then obviously, this degree will be different for distinct planes even for the same time t, as well as for the same plane but for different time intervals.

✉ Wutiphol Sintunavarat
wutiphol@mathstat.sci.tu.ac.th

Dhananjay Gopal
gopaldhananjay@yahoo.in

Abhay S. Ranadive
asranadive04@yahoo.co.in

Satish Shukla
satishmathematics@yahoo.co.in

1 Department of Mathematics, Guru Ghasidas Vishwavidyalya Bilaspur, Bilaspur, Chhattisgarh 495009, India

2 Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University Rangsit Center, Khlong Luang, Pathum Thani 12120, Thailand

3 Department of Mathematics, Shri Vaishnav Institute of Science, Shri Vaishnav Vidyapeeth Vishwavidyalaya, Indore, M.P. 453331, India

The mentioned situation in the previous paragraph brings the inspiration for introducing the notion of k-fuzzy metric spaces, where $k \in \{1, 2, 3, \ldots\}$, which is an extension and generalization of the concept of fuzzy metric spaces due to George and Veeramani (1994). In a k-fuzzy metric space, the fuzzy distance of two points is measured by the degree of nearness with respect to k parameter(s). Furthermore, fixed point results for contractive mappings in k-fuzzy metric spaces are proved. These results generalize the fixed point results of Grabiec (1988) into k-fuzzy metric spaces.

2 Preliminaries

This section presents the needed definitions in the next section.

Definition 1 (Schweizer and Sklar (1960)) A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (briefly, t-norm) if the following conditions are satisfied for all $a, b, c, d \in [0, 1]$:

1. $*(a, b) = *(b, a)$;
2. if $a \leq c$ and $b \leq d$, then $*(a, b) \leq *(c, d)$;
3. $*(*(a, b), c) = *(a, *(b, c))$;
4. $*(a, 1) = a$.

If $*$ is continuous, it is called a continuous t-norm.

For each t-norm $* : [0, 1] \times [0, 1] \to [0, 1]$ and $a, b \in [0, 1]$, instead of $*(a, b)$ we will use the infix notation $a * b$. Three typical examples of continuous t-norms are a product t-norm $*_1$, a minimum t-norm $*_2$ and a Lukasiewicz t-norm $*_3$, which are defined for each $a, b \in [0, 1]$ by

$$a *_1 b = \min\{a, b\},$$
$$a *_2 b = ab$$
$$a *_3 b = \max\{a + b - 1, 0\}.$$

Remark 2 (George and Veeramani 1994) For each t-norm $* : [0, 1] \times [0, 1] \to [0, 1]$, the following assertions hold:

1. for each $a, b \in (0, 1)$ with $a > b$, there is $c \in (0, 1)$ such that $a * c \geq b$;
2. for each $d \in (0, 1)$, there is $e \in (0, 1)$ such that $e * e \geq d$.

Definition 3 (George and Veeramani (1994)) An ordered triple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, M is a fuzzy set on $X^2 \times (0, \infty)$, and the following conditions are satisfied for all $x, y, z \in X$ and $s, t > 0$:

(FM1) $M(x, y, t) > 0$;
(FM2) $M(x, y, t) = 1$ if and only if $x = y$;
(FM3) $M(x, y, t) = M(y, x, t)$;
(FM4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(FM5) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is a continuous mapping.

For a fuzzy metric space $(X, M, *)$, M with $*$ is called a fuzzy metric on X. Moreover, for each $x, y \in X$ and $t > 0$, $M(x, y, t)$ can be thought of as the definition of nearness between x and y with respect to t. It is also known that $M(x, y, \cdot)$ is non-decreasing. For various examples of fuzzy metric spaces, we refer to George and Veeramani (1994); Gregori et al. (2011); Sapena (2001). We state only a particular example of our interest.

Example 4 (Induced fuzzy metric) Let (X, d) be a metric space and $*$ be a product t-norm. Define a fuzzy set M on $X^2 \times (0, \infty)$ by

$$M(x, y, t) = \frac{kt^n}{kt^n + md(x, y)}$$

for all $x, y \in X$ and $t > 0$, where $k, m, n > 0$. Then, $(X, M, *)$ is a fuzzy metric space called the induced fuzzy metric (see George and Veeramani 1994).

In the above example, note that

$$\lim_{t\to\infty} M(x, y, t) = 1 \text{ for all } x, y \in X. \tag{1}$$

As $M(x, y, t)$ represents the degree of the nearness of points x and y with respect to the parameter t and it is a non-decreasing function of t for all $x, y \in X$; therefore, condition (1) is the most natural condition for the degree of the nearness to be perfect (that is, unity). Notice that this is a specific condition and may not hold in some particular fuzzy metric spaces, for instance, in stationary fuzzy metric spaces (see Gregori and Romaguera 2004). This brings to the following definition:

Definition 5 A fuzzy metric space $(X, M, *)$ is called a natural fuzzy metric space if and only if

$$\lim_{t\to\infty} M(x, y, t) = 1$$

for all $x, y \in X$.

3 *k*-fuzzy metric spaces

In this section, we introduce the idea of k-fuzzy metric spaces and investigate the properties of such spaces. We begin with the following definition:

Definition 6 Let X be a nonempty set, $*$ a continuous t-norm, k a positive integer and M be a fuzzy set on $X^2 \times (0, \infty)^k$. An ordered triple $(X, M, *)$ is called a k-fuzzy metric space if the following conditions are satisfied for all $x, y \in X$, $t, s > 0$ and $t_1, t_2, \ldots, t_k > 0$:

(kFM1) $M(x, y, t_1, t_2, \ldots, t_k) > 0$;
(kFM2) $M(x, y, t_1, t_2, \ldots, t_k) = 1$ if and only if $x = y$;
(kFM3) $M(\cdot, \cdot, t_1, t_2, \ldots, t_k)$ is symmetric;
(kFM4) for any $l \in \{1, 2, 3, \ldots, k\}$, we have

$$\begin{aligned} & M(y, z, t_1, t_2, \ldots, t_{l-1}, t, t_{l+1}, \ldots, t_{k-1}, t_k) \\ & *M(y, z, t_1, t_2, \ldots, t_{l-1}, s, t_{l+1}, \ldots, t_{k-1}, t_k) \\ & \leq M(x, z, t_1, t_2, \ldots, t_{l-1}, t+s, t_{l+1}, \ldots, t_k); \end{aligned}$$

(kFM5) $M(x, y, \cdot) : (0, \infty)^k \longrightarrow [0, 1]$ is a continuous mapping.

Remark 7 For $k = 1$, the k-fuzzy metric space reduces into the fuzzy metric space in the sense of George and Veeramani (1994).

Example 8 Let (X, d) be a metric space, $*$ the product (minimum) t-norm, $\omega > 0$ and k be a positive integer. Define a fuzzy set M on $X^2 \times (0, \infty)^k$ by

$$M(x, y, t_1, t_2, \ldots, t_k) = \frac{\omega t_1 t_2 \cdots t_k}{\omega t_1 t_2 \cdots t_k + d(x, y)}$$

for all $x, y \in X$ and and $t_1, t_2, \ldots, t_k > 0$. Then, $(X, M, *)$ is a k-fuzzy metric space.

Example 9 Let (X, d) be a metric space, $*$ the product (minimum) t-norm, $\omega > 0$ and k be a positive integer. Define a fuzzy set M on $X^2 \times (0, \infty)^k$ by

$$M(x, y, t_1, t_2, \ldots, t_k) = \omega \left[\omega + \left(\sum_{i=1}^{k} \frac{1}{t_i} \right) d(x, y) \right]^{-1}$$

for all $x, y \in X$ and and $t_1, t_2, \ldots, t_k > 0$. Then, $(X, M, *)$ is a k-fuzzy metric space.

Example 10 Let $X = \mathbb{R}^k$, where k is a positive integer, $\omega > 0$ and $*$ be the product t-norm. Define a fuzzy set M on $X^2 \times (0, \infty)^k$ by

$$M(x, y, t_1, \ldots, t_k) = \omega \left[\omega + \sum_{i=1}^{k} \frac{|y_i - x_i|}{t_i} \right]^{-1}$$

for all $x = (x_1, x_2, \ldots, x_k), y = (y_1, y_2, \ldots, y_k) \in X$ and $t_1, t_2, \ldots, t_k > 0$. Then, $(X, M, *)$ is a k-fuzzy metric space.

From the application point of view, one should define the k-fuzzy metric with care to the physical nature of quantities. For instance, if one considers the degree of the nearness of two points x and y in a space with respect to time and fuel consumed in moving from x to y, one cannot use the formulae for the degree of the nearness as given in the above examples due to the different dimensions of these quantities. In the following example, one such case is presented.

Example 11 Let $X = \mathbb{R}^3$ be the Euclidean space with the usual distance d on X. Suppose that t is the time and f is the fuel consumed in moving from a point x to a point y in X. Then, the degree of the nearness of x and y with respect to t and f can be measured by the 2-fuzzy metric M on $X^2 \times (0, \infty)^2$ given by

$$M(x, y, t, f) = e^{-d(x,y)\left(\frac{\omega_1}{t} + \frac{\omega_2}{f}\right)},$$

for all $x, y \in X$ and $t > 0$, $f > 0$, where ω_1 and ω_2 are constants chosen with suitable physical dimensions.

In the present paper, we restrict ourselves to only mathematical properties of k-fuzzy metric spaces.

Definition 12 A k-fuzzy metric space $(X, M, *)$ is called l-natural k-fuzzy metric space if there exists $l \in \{1, 2, \ldots, k\}$ such that

$$\lim_{t_l \to \infty} M(x, y, t_1, \ldots, t_l, \ldots, t_k) = 1$$

for all $x, y \in X$.

For the rest of this paper, for a given k-fuzzy metric space $(X, M, *)$, $x, y \in X$ and $t_1, t_2, \ldots, t_k > 0$, for simplicity, we write $M(x, y, t_1^k)$ instead $M(x, y, t_1, t_2, \ldots, t_k)$.

Next, we discuss some properties k-fuzzy metric spaces and establish the topology of such spaces.

Proposition 13 *Let $(X, M, *)$ be a k-fuzzy metric space, $t, t_1, t_2, \ldots, t_k > 0$. Suppose that $t_l < t$ for some $l \in \{1, 2, \ldots, k\}$. Then,*

$$M(x, y, t_1^k) \leq M(x, y, t_1, \ldots t_{l-1}, t, t_{l+1}, \ldots, t_k)$$

for all $x, y \in X$.

Proof By using the property of $*$ and (kFM4), for each $x, y \in X$, we obtain

$$\begin{aligned} & M(x, y, t_1^k) \\ & = M(x, y, t_1^k) * 1 \\ & = M(x, y, t_1^k) * M(y, y, t_1 \ldots, t_{l-1}, t - t_l, t_{l+1}, t_k) \\ & \leq M(x, y, t_1, \ldots, t_{l-1}, t, t_{l-1}, \ldots, t_k). \end{aligned}$$

□

Remark 14 In a k-fuzzy metric space $(X, M, *)$, if

$$M(x, y, t_1^k) > 1 - \varepsilon,$$

where $x, y \in X, t_1, t_2, \ldots, t_k > 0$ and $0 < \varepsilon < 1$, then for each $l \in \{1, 2, \ldots, k\}$, we can find $t \in (0, t_l)$ such that

$$M(x, y, t_1, \ldots, t_{l-1}, t, t_l, \ldots, t_k) > 1 - \varepsilon.$$

Definition 15 Let $(X, M, *)$ be a k-fuzzy metric space. An open ball with center $x \in X$ and radius $\varepsilon \in (0, 1)$ with respect to parameters $t_1, t_2, \ldots, t_k > 0$, denoted by $B(x, \varepsilon; t_1, t_2, \ldots, t_k)$, is defined by

$$B(x, \varepsilon; t_1, t_2, \ldots, t_k) = \{y \in X : M\left(x, y, t_1^k\right) > 1 - \varepsilon\}.$$

Definition 16 Let $(X, M, *)$ be a k-fuzzy metric space. A subset A of X is called an open set if and only if there is an open ball B such that $B \subseteq A$. A subset C of X is called a closed set if and only if its complement is an open set.

Theorem 17 *Every open ball in a k-fuzzy metric space is an open set.*

Proof Let $(X, M, *)$ be a k-fuzzy metric space, $x \in X$, $t_1, t_2, \ldots, t_k > 0$ and $\varepsilon \in (0, 1)$. Assume that

$$y \in B(x, \varepsilon; t_1, t_2 \ldots, t_k).$$

Then, we have $M\left(x, y, t_1^k\right) > 1 - \varepsilon$. Therefore, we can find $l \in \{1, 2, \ldots, k\}$ and $t \in (0, t_l)$ such that

$$\varepsilon_0 := M(x, y, t_1, \ldots, t_{l-1}, t, t_l, \ldots, t_k) > 1 - \varepsilon.$$

Then, we can find $\delta \in (0, 1)$ such that

$$\varepsilon_0 > \delta > 1 - \varepsilon.$$

By Remark 2, there is $\varepsilon_1 \in (0, 1)$ such that $\varepsilon_0 * \varepsilon_1 \geq \delta$. Now, we will claim that

$$\begin{aligned} &B(y, 1 - \varepsilon_1; t_1, t_2, \ldots, t_l, t - t_l, t_{l+1}, \ldots, t_k) \\ &\quad \subseteq B(x, \varepsilon; t_1, t_2, \ldots, t_k). \end{aligned}$$

Assume that

$$z \in B(y, 1 - \varepsilon_1; t_1, t_2, \ldots, t_l, t - t_l, t_{l+1}, \ldots, t_k).$$

Then,

$$M(y, z, t_1, t_2, \ldots, t_{l-1}, t - t_l, t_l, \ldots, t_k) > \varepsilon_1.$$

This implies that

$$\begin{aligned} &M\left(x, z, t_1^k\right) \\ &\geq M(x, y, t_1, t_2, \ldots, t_{l-1}, t, t_{l+1}, \ldots, t_k) \\ &*M(y, z, t_1, t_2, \ldots, t_{l-1}, t - t_l, t_{l+1}, \ldots, t_k) \\ &\geq \varepsilon_0 * \varepsilon_1 \\ &\geq \delta \\ &> 1 - \varepsilon, \end{aligned}$$

which proves the result. □

From the above theorem, we can directly get the following result:

Theorem 18 *Let $(X, M, *)$ be a k-fuzzy metric space and*

$$\begin{aligned} \tau := \{&A \subseteq X : x \in X \text{ if and only if there exist} \\ &t_1, t_2, \ldots, t_k > 0 \text{ and } \varepsilon \in (0, 1) \text{ such that} \\ &B(x, \varepsilon; t_1, t_2 \ldots, t_k) \subseteq A\}. \end{aligned}$$

Then, τ is a topology on X.

Remark 19 Let $(X, M, *)$ be a k-fuzzy metric space and $x \in X$. Since

$$B_x := \left\{B\left(x, \frac{1}{n}; t_1, t_2 \ldots, t_k\right) : n \in \mathbb{N}\right\},$$

where $t_1 = t_2 = \cdots = t_k = \frac{1}{n}$, is a local base at a point x, the topology τ given in Theorem 18 is first countable.

Theorem 20 *Every k-fuzzy metric space is Hausdorff.*

Proof Let $(X, M, *)$ be a k-fuzzy metric space and $x, y \in X$ with $x \neq y$. Then, for given $t_1, t_2, \ldots, t_k > 0$, we have

$$0 < M\left(x, y, t_1^k\right) < 1.$$

Let $\varepsilon := M\left(x, y, t_1^k\right) \in (0, 1)$. For each $\varepsilon_0 \in (\varepsilon, 1)$, we can choose ε_1 such that $\varepsilon_1 * \varepsilon_1 \geq \varepsilon_0$. We will claim that

$$\begin{aligned} B_{xy} := &B(x, 1 - \varepsilon_1; t_1, t_2, \ldots, t_l/2, \ldots, t_k) \\ &\cap B(y, 1 - \varepsilon_1; t_1, t_2, \ldots, t_l/2, \ldots, t_k) \\ &= \emptyset. \end{aligned}$$

Assume that $B_{xy} \neq \emptyset$, that is, there is $z \in B_{x,y}$. Then, we have

$$\begin{aligned} \varepsilon &= M(x, y, t_1^k) \\ &\geq M(x, z, t_1, \ldots, t_l/2, \ldots, t_k) \\ &\quad * M(z, y, t_1, \ldots, t_l/2, \ldots, t_k) \\ &\geq \varepsilon_1 * \varepsilon_1 \\ &\geq \varepsilon_0 \\ &> \varepsilon. \end{aligned}$$

This contradiction proves the claim and so the result. □

Definition 21 Let $(X, M, *)$ be a k-fuzzy metric space. A sequence $\{x_n\}$ in X is said to be convergent and converges to $x \in X$ if and only if for every real $\epsilon \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that

$$M(x_n, x, t_1^k) > 1 - \epsilon$$

for all $n > n_0$ and $t_1, t_2, \ldots, t_k > 0$.

The proof of the following lemma is straightforward, so we will omit the proof.

Lemma 22 *Let $(X, M, *)$ be a k-fuzzy metric space. A sequence $\{x_n\}$ in X converges to $x \in X$ if and only if*

$$\lim_{n\to\infty} M\left(x_n, x, t_1^k\right) = 1$$

for all $t_1, t_2, \ldots, t_k > 0$.

Definition 23 Let $(X, M, *)$ be a k-fuzzy metric space and $\{x_n\}$ be a sequence in X.

1. $\{x_n\}$ is called an M-Cauchy sequence if for every $\epsilon \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that

   $$M\left(x_n, x_m, t_1^k\right) > 1 - \epsilon$$

   for all $n, m > n_0$ and $t_1, t_2, \ldots, t_k > 0$.
2. $\{x_n\}$ is called a G-Cauchy sequence if

   $$\lim_{n\to\infty} M\left(x_n, x_{n+p}, t_1^k\right) = 1$$

   for all $t_1, t_2, \ldots, t_k > 0$ and $p > 0$.

Note that the above definitions of Cauchy sequences are different (for the case $k = 1$, see Vasuki and Veeramani 2003).

Definition 24 Let $(X, M, *)$ be a k-fuzzy metric space.

1. $(X, M, *)$ is said to be M-complete if every M-Cauchy sequence in X converges to some $x \in X$.
2. $(X, M, *)$ is said to be G-complete if every G-Cauchy sequence in X converges to some $x \in X$.

Remark 25 For $k = 1$, the M-completeness and the G-completeness of a k-fuzzy metric space reduce to the M-completeness in the sense of George and Veeramani (1997) and the G-completeness in the sense of Grabiec (1988) of a fuzzy metric space, respectively.

4 Fixed point theorems

In this section, we prove many fixed point results in k-fuzzy metric spaces. For simplicity, for a given k-fuzzy metric space $(X, M, *)$, $l \in \{1, 2, \ldots, k\}$, $a > 0$, $x, y \in X$ and $t_1, t_2, \ldots, t_k > 0$, we write $M_l^a\left(x, y, t_1^k\right)$ instead

$$M\left(x, y, t_1, \ldots, t_{l-1}, t_l/a, t_{l+1}, \ldots, t_k\right).$$

Theorem 26 *Let $(X, M, *)$ be a G-complete k-fuzzy metric space and $T : X \to X$ be a mapping satisfying the following condition:*

$$M_l^{1/\lambda}\left(Tx, Ty, t_1^k\right) \geq M\left(x, y, t_1^k\right) \tag{2}$$

for all $x, y \in X$ and $t_1, t_2, \ldots, t_k > 0$, where $l \in \{1, 2, \ldots, k\}$ and $\lambda \in (0, 1)$ is a constant. Suppose that $(X, M, *)$ is an l-natural k-fuzzy metric space. Then, T has a unique fixed point.

Proof First, we will show that if a fixed point of T exists, then it is unique. Suppose that u and v are fixed points of T. By (2), we have

$$\begin{aligned} M\left(u, v, t_1^k\right) &= M\left(Tu, Tv, t_1^k\right) \\ &\geq M\left(u, v, t_1, \ldots, t_{l-1}, t_l/\lambda, t_{l+1}, \ldots, t_k\right) \\ &= M_l^{\lambda}\left(u, v, t_1^k\right). \end{aligned}$$

By repeating this process, we obtain

$$M(u, v, t_1^k) \geq M_l^{\lambda^n}\left(u, v, t_1^k\right) \tag{3}$$

for all $n \in \mathbb{N}$. Note that, if $\{a_n\}$ be any sequence such that $a_n > 0$ and $\lim_{n\to\infty} a_n = 0$, then since $(X, M, *)$ is l-natural, we have

$$\lim_{n\to\infty} M_l^{a_n}\left(x, y, t_1^k\right) = 1$$

for all $t_1, t_2, \ldots, t_k > 0$. Using this fact in (3), we obtain $M(u, v, t_1^k) = 1$ for all $t_1, t_2, \ldots, t_k > 0$, that is, $u = v$. Therefore, the fixed point of T is unique.

For the existence of a fixed point of T, we choose $x_0 \in X$ and define an iterative sequence $\{x_n\}$ by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. If $x_n = x_{n-1}$ for some $n \in \mathbb{N}$, then x_n is the unique fixed point of T. Therefore, we may assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. For any $n \in \mathbb{N}$ and $t_1, t_2, \ldots, t_k > 0$, we have

$$\begin{aligned} & M\left(x_n, x_{n+1}, t_1^k\right) \\ & = M\left(Tx_{n-1}, Tx_n, t_1^k\right) \\ & \geq M\left(x_{n-1}, x_n, t_1, \ldots, t_{l-1}, t_l/\lambda, t_{l+1}, \ldots, t_k\right) \\ & = M_l^{\lambda}\left(x_{n-1}, x_n, t_1^k\right). \end{aligned}$$

By repeating this process, we obtain

$$M(x_n, x_{n+1}, t_1^k) \geq M_l^{\lambda^n}\left(x_0, x_1, t_1^k\right) \tag{4}$$

for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, $t_1, t_2, \ldots, t_k > 0$ and $p > 0$, we have

$$\begin{aligned} & M\left(x_n, x_{n+p}, t_1^k\right) \\ & \geq M\left(x_n, x_{n+1}, t_1, \ldots, t_{l-1}, t_l/2, t_{l-1}, \ldots, t_k\right) \\ & \quad * M\left(x_{n+1}, x_{n+p}, t_1, \ldots, t_{l-1}, t_l/2, t_l, \ldots, t_k\right) \\ & \geq M_l^2\left(x_n, x_{n+1}, t_1^k\right) \\ & \quad * M\left(x_{n+1}, x_{n+2}, t_1, \ldots, t_{l-1}, t_l/2^2, t_{l-1}, \ldots, t_k\right) \\ & \quad * M\left(x_{n+2}, x_{n+p}, t_1, \ldots, t_{l-1}, t_l/2^2, t_l, \ldots, t_k\right) \\ & \geq M_l^2\left(x_n, x_{n+1}, t_1^k\right) * M_l^{2^2}\left(x_{n+1}, x_{n+2}, t_1^k\right) \\ & \quad * \cdots * M_l^{2^{p-1}}\left(x_{n+p-2}, x_{n+p-1}, t_1^k\right) \\ & \quad * M_l^{2^{p-1}}\left(x_{n+p-1}, x_{n+p}, t_1^k\right). \end{aligned}$$

By using (4), we obtain

$$\begin{aligned} M(x_n, x_{n+p}, t_1^k) \geq & M_l^{2\lambda^n}\left(x_0, x_1, t_1^k\right) \\ & * M_l^{2^2\lambda^{n+1}}\left(x_0, x_1, t_1^k\right) \\ & * \cdots * M_l^{2^{p-1}\lambda^{n+p-1}}\left(x_0, x_1, t_1^k\right). \end{aligned}$$

Since $(X, M, *)$ is l-natural, it follows from the above inequality that

$$\lim_{n\to\infty} M\left(x_n, x_{n+p}, t_1^k\right) = 1.$$

Therefore, $\{x_n\}$ is a G-Cauchy sequence. By the G-completeness of $(X, M, *)$, there exists $u \in X$ such that

$$\lim_{n\to\infty} M\left(x_n, u, t_1^k\right) = 1 \tag{5}$$

for all $t_1, t_2, \ldots, t_k > 0$. We will show that u is a fixed point of T. For each $t_1, t_2, \ldots, t_k > 0$, we have

$$\begin{aligned} M(u, Tu, t_1^k) & \geq M_l^2\left(u, x_n, t_1^k\right) * M_l^2\left(x_n, Tu, t_1^k\right) \\ & = M_l^2\left(u, x_n, t_1^k\right) * M_l^2\left(Tx_{n-1}, Tu, t_1^k\right) \\ & \geq M_l^2\left(u, x_n, t_1^k\right) * M_l^{2\lambda}\left(x_{n-1}, u, t_1^k\right). \end{aligned}$$

By using (5) in the above inequality, we obtain $M(u, Tu, t_1^k) = 1$ for all $t_1, t_2, \ldots, t_k > 0$, that is, $Tu = u$. Thus, u is the unique fixed point of T. □

For $k = 1$, the above theorem reduces to the following result of Grabiec (1988).

Corollary 27 *(Grabiec (1988)) Let $(X, M, *)$ be a G-complete fuzzy metric space such that*

$$\lim_{t\to\infty} M(x, y, t) = 1 \text{ for all } x, y \in X \tag{6}$$

and $T : X \to X$ be a mapping. Suppose that there exists $\lambda \in (0, 1)$ such that

$$M(Tx, Ty, \lambda t) \geq M(x, y, t) \tag{7}$$

for all $x, y \in X$. Then, T has a unique fixed point.

Remark 28 Let $(X, M, *)$ be a fuzzy metric space and $T : X \to X$ be a mapping. The contractive condition (7) tells that the mapping T contract the space with respect to the parameter t in the sense that the degree of the nearness of images of any two points under T is not less than the degree of the nearness of corresponding points (obviously in case of stationery fuzzy metric spaces (see Gregori and Romaguera 2004) it is not applicable). In Theorem 26, the mapping contracts the space with respect to only parameter t_l for some $l \in \{1, 2, \ldots, k\}$ and it may not be contractive with respect to other parameters. Similarly, $(X, M, *)$ is assumed l-natural k-fuzzy metric space for at least one $l \in \{1, 2, \ldots, k\}$ only.

The following example verifies the above remark.

Example 29 Let $X = [0, 1] \times [0, 1]$ and $*$ be the product t-norm and the fuzzy set M on $X^2 \times (0, \infty)^2$ be defined by

$$M(x, y, t_1, t_2) = \left[1 + \frac{|y_1 - x_1| + |y_2 - x_2|}{t_1}\right]^{-1}$$

for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in X$ and $t_1, t_2 > 0$. Then, $(X, M, *)$ is a G-complete 2-fuzzy metric space ($k = 2$). Moreover,

$$\lim_{t_1 \to \infty} M(x, y, t_1, t_2) = 1 \text{ for all } x, y \in X, t_2 > 0,$$

that is, $(X, M, *)$ is a 1-natural 2-fuzzy metric space. Define a mapping $T : X \longrightarrow X$ by

$$T(x_1, x_2) = \left(\frac{x_1}{2}, \frac{x_2}{2}\right) \text{ for all } (x_1, x_2) \in X.$$

For $x = (x_1, x_2)$, $y = (y_1, y_2) \in X$ and $t_1, t_2 > 0$, we have

$$\begin{aligned} M(Tx, Ty, \lambda t_1, t_2) &= \left[1 + \frac{|y_1 - x_1| + |y_2 - x_2|}{2\lambda t_1}\right]^{-1} \\ &\geq \left[1 + \frac{|y_1 - x_1| + |y_2 - x_2|}{t_1}\right]^{-1} \\ &= M(x, y, t_1, t_2) \end{aligned}$$

for $\lambda \in [1/2, 1)$. By Theorem 26, T has a unique fixed point. In this case, a point $(0, 0) \in X$ is a fixed point of T.

In Theorem 26, corresponding to condition (2), we assume that the space $(X, M, *)$ is l-natural. Notice that, for the existence of a fixed point, the l-naturalness cannot be replaced by the m-naturalness with $m \neq l$. The following example verifies this fact.

Example 30 Let $X = [0, 1] \times [0, 1]$ and $*$ be the product t-norm and the fuzzy set M on $X^2 \times (0, \infty)^2$ be defined by

$$M(x, y, t_1, t_2) = \left[1 + \frac{|y_1 - x_1| + |y_2 - x_2|}{t_2}\right]^{-1}$$

for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in X$ and $t_1, t_2 > 0$. Then, $(X, M, *)$ is a G-complete 2-fuzzy metric space ($k = 2$). Moreover,

$$\lim_{t_2 \to \infty} M(x, y, t_1, t_2) = 1 \text{ for all } x, y \in X, t_1 > 0,$$

that is, $(X, M, *)$ is a 2-natural 2-fuzzy metric space. Define a mapping $T : X \longrightarrow X$ by

$$T(x_1, x_2) = (x_1, x_2) \text{ for all } (x_1, x_2) \in X.$$

Notice that, for any arbitrary $\lambda \in (0, 1)$

$$M(Tx, Ty, \lambda t_1, t_2) \geq M(x, y, t_1, t_2)$$

for all $x, y \in X$, $t_1, t_2 > 0$. But the fixed point of T is not unique. Indeed, every point $(x_1, x_2) \in X$ is a fixed point of T.

Finally, we will prove a fixed point result for a k-fuzzy contraction mapping. We begin with the definition of a k-fuzzy contraction mapping as follows:

Definition 31 Let $(X, M, *)$ be a k-fuzzy metric space. A mapping $T : X \to X$ is called a k-fuzzy contraction mapping if

$$\frac{1}{M\left(Tx, Ty, t_1^k\right)} - 1 \leq \lambda \left[\frac{1}{M\left(x, y, t_1^k\right)} - 1\right] \tag{8}$$

for all $x, y \in X$ and $t_1, t_2, \ldots, t_k > 0$, where $\lambda \in [0, 1)$ is a constant.

Theorem 32 *Let $(X, M, *)$ be a G-complete k-fuzzy metric space and $T : X \to X$ be a k-fuzzy contraction mapping. Then, T has a unique fixed point.*

Proof Let $x_0 \in X$ and define a sequence $\{x_n\}$ by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. We will show that this sequence is a G-Cauchy sequence. For any $n \in \mathbb{N}$, we have

$$\begin{aligned} \frac{1}{M\left(x_n, x_{n+1}, t_1^k\right)} - 1 &= \frac{1}{M\left(Tx_{n-1}, Tx_n, t_1^k\right)} - 1 \\ &\leq \lambda \left[\frac{1}{M\left(x_{n-1}, x_n, t_1^k\right)} - 1\right]. \end{aligned}$$

By repeating in this manner, we obtain

$$\frac{1}{M\left(x_n, x_{n+1}, t_1^k\right)} - 1 \leq \lambda^n \left[\frac{1}{M\left(x_0, x_1, t_1^k\right)} - 1\right] \tag{9}$$

for all $n \in \mathbb{N}$. Since $\lambda \in [0, 1)$, we conclude from (9) that

$$\lim_{n \to \infty} \left[\frac{1}{M\left(x_n, x_{n+1}, t_1^k\right)} - 1\right] \leq 0,$$

that is,

$$\lim_{n \to \infty} M\left(x_n, x_{n+1}, t_1^k\right) = 1 \tag{10}$$

for all $t_1, t_2, \ldots, t_k > 0$. For each $n \in \mathbb{N}$, $p > 0$ and $t_1, t_2, \ldots, t_k > 0$, we have

$$\begin{aligned} & M\left(x_n, x_{n+p}, t_1^k\right) \\ & \geq M_l^2\left(x_n, x_{n+1}, t_1^k\right) * M_l^2\left(x_{n+1}, x_{n+p}, t_1^k\right) \\ & \geq M_l^2\left(x_n, x_{n+1}, t_1^k\right) * M_l^{2^2}\left(x_{n+1}, x_{n+2}, t_1^k\right) \\ & \quad * \cdots * M_l^{2^{p-1}}\left(x_{n+p-2}, x_{n+p-1}, t_1^k\right) \\ & \quad * M_l^{2^{p-1}}\left(x_{n+p-1}, x_{n+p}, t_1^k\right). \end{aligned} \tag{11}$$

From (10), we have

$$\lim_{n\to\infty} M_a^l\left(x_n, x_{n+1}, t_1^k\right) = 1$$

for all $t_1, t_2, \ldots, t_k > 0$ and $a > 0$, which together with inequality (11) yields

$$\lim_{n\to\infty} M\left(x_n, x_{n+p}, t_1^k\right) \geq 1 * 1 * \cdots * 1 = 1$$

for all $t_1, t_2, \ldots, t_k > 0$ and $p > 0$. Therefore, the sequence $\{x_n\}$ is a G-Cauchy sequence in X. By the G-completeness of X, there exists $u \in X$ such that the sequence $\{x_n\}$ converges to u, that is,

$$\lim_{n\to\infty} M\left(x_n, u, t_1^k\right) = 1 \tag{12}$$

for all $t_1, t_2, \ldots, t_k > 0$.

Now, we will show that u is a fixed point of T. For each $n \in \mathbb{N}$, we have

$$\frac{1}{M\left(x_{n+1}, Tu, t_1^k\right)} - 1 = \frac{1}{M\left(Tx_n, Tu, t_1^k\right)} - 1 \leq \lambda\left[\frac{1}{M\left(x_n, u, t_1^k\right)} - 1\right].$$

By using (12), we have

$$\lim_{n\to\infty}\left[\frac{1}{M\left(x_{n+1}, Tu, t_1^k\right)} - 1\right] = 0,$$

that is,

$$\lim_{n\to\infty} M\left(x_{n+1}, Tu, t_1^k\right) = 1 \tag{13}$$

for all $t_1, t_2, \ldots, t_k > 0$. For any $n \in \mathbb{N}$, we have

$$M\left(u, Tu, t_1^k\right) \geq M_2^l\left(u, x_{n+1}, t_1^k\right) * M_2^k\left(x_{n+1}, Tu, t_1^k\right),$$

which together with (12) and (13) yields

$$M\left(u, Tu, t_1^k\right) = 1$$

for all $t_1, t_2, \ldots, t_k > 0$, that is, $Tu = u$. Thus, u is a fixed point of T.

For the uniqueness, we suppose v is another fixed point of T distinct from u. Then, there exist $r_1, r_2, \ldots, r_k > 0$ such that

$$M\left(u, v, r_1^k\right) < 1,$$

that is,

$$\frac{1}{M\left(u, v, r_1^k\right)} - 1 > 0.$$

Now, we have

$$\frac{1}{M\left(u, v, r_1^k\right)} - 1 = \frac{1}{M\left(Tu, Tv, r_1^k\right)} - 1 \leq \lambda\left[\frac{1}{M\left(u, v, r_1^k\right)} - 1\right].$$

Since $\lambda < 1$, the above inequality yields a contradiction. Therefore, we must have $u = v$. Thus, the fixed point of T is unique. □

Acknowledgements We would like to thank the referee for his/her great efforts in proofreading the manuscript.

Author Contributions DG, AR and SS conceived of the presented idea. WS developed theorems and the proof. All the authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Funding This work was supported by the Thailand Science Research and Innovation Fundamental Fund fiscal year 2023 (TUFF 20/2566). Dhananjay Gopal and Abhay S. Ranadive thank the Department of Science and Technology, New Delhi, India, for approving the proposal under the FIST Program scheme (Ref. No. SR/FST/MS/2022/122 dated 19/12/2022).

Code availability Not applicable.

Declarations

Conflict of interest All the authors declare that they have no competing interests.

References

George A, Veeramani P (1994) On some results in fuzzy metric spaces. Fuzzy Sets Syst 64:395–399

George A, Veeramani P (1997) On some results of analysis for fuzzy metric spaces. Fuzzy Sets Syst 90:365–368

Grabiec M (1988) Fixed points in fuzzy metric spaces. Fuzzy Sets Syst 27:385–389

Gregori V, Morillas S, Sapena A (2011) Examples of fuzzy metrics and applications. Fuzzy Sets Syst 170:95–111

Gregori V, Romaguera S (2004) Characterizing completable fuzzy metric spaces. Fuzzy Sets Syst 144:411–420

Kramosil I, Michalek J (1975) Fuzzy metric and statistical metric spaces. Kybernetica 15:326–334

Sapena A (2001) A contribution to the study of fuzzy metric spaces. Appl Gen Topol 2(1):63–75

Schweizer B, Sklar A (1960) Statistical metric spaces. Pac J Math 10:313–334

Vasuki R, Veeramani P (2003) Fixed point theorems and Cauchy sequences in fuzzy metric spaces. Fuzzy Sets Syst 135:415–417
Zadeh LA (1965) Fuzzy sets. Inf Control 8:338–353