A note on *k*-cyclic modal pseudocomplemented De Morgan algebras

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Abstract

Symmetric and *k*-cyclic structure of modal pseudocomplemented De Morgan algebras was introduced previously. In this paper, we first present the construction of epimorphisms between finite symmetric (or 2-cyclic) modal pseudocomplemented De Morgan algebras. Furthermore, we compute the cardinality of the set of all epimorphism between finite structures. Secondly, we present the construction of finite free algebras on the variety of *k*-cyclic modal pseudocomplemented De Morgan algebras and display how our computations are in fact generalizations to others in the literature. Our work is strongly based on the properties of epimorphisms and automorphisms and the fact that the variety is finitely generated.

1 Introduction

A pseudocomplemented De Morgan algebra A is a De Morgan algebra with a unary operator $*$ such that every $a \in A$, the element a^* is the pseudocomplement of a, see for instance (Sankappanavar 1986), i.e.,

$a \wedge x = 0$ if and only if $x \leq a^*$.

A. V. Figallo considered the subvariety of pseudocomplemented De Morgan algebras (Figallo and sobre 1992) which verifies:

(tm)$x \vee \sim x \leq x \vee x^*$

This author called them modal pseudocomplemented De Morgan algebras (for short, $mpM-$algebras). In addition, we can define this class of algebras as follows:

Recall that A. Monteiro introduced tetravalent modal algebras (TMA-algebras) as algebras $\langle L, \wedge, \vee, \sim, \nabla, 0, 1 \rangle$ of type $(2, 2, 1, 1, 0, 0)$, such that $\langle L, \wedge, \vee, \sim, 0, 1 \rangle$ are De Morgan algebras which satisfy the following conditions:

$$\nabla x \vee \sim x = 1,$$
$$\nabla x \wedge \sim x = \sim x \wedge x.$$

These algebras arise as a generalization of three-valued Łukasiewicz algebras by omitting the identity $\nabla(x \wedge y) = \nabla x \wedge \nabla y$, and they were studied in Figallo and Ziliani (1991), Figallo and Landini (1995), Figallo (2021), Font and Rius (2000), Loureiro (1983). The variety of TMA-algebras is generated by the well-known four-element De Morgan algebra expanded with a simple modal operator ∇ (i.e., $\nabla 1 = 1$ and $\nabla x = 0$ for $x \neq 1$. Besides, $\sim 0 = 1$ and $\sim x = x$ for $x \neq 0, 1$). The operator ∇ has associated Δ operator through $\Delta x = \sim \nabla \sim x$, and when the TMA-algebra is a three-valued Łukasiewicz algebras, Δ coincides with Baaz's Δ operator, Baaz (1996). Baaz's Δ operator was intensive studied in the *Fuzzy Logic* area by Esteva, Godo, Hájek, Montagna, and others, see for instance (Esteva et al. 2001, 2000).

It is worth mentioning that the class of $mpM-$algebras constitutes a proper subvariety of the variety $\mathcal{V}_0$ studied by H. Sankappanavar in Sankappanavar (1986). More recently, the theory of operators over $mpM-$algebras was considered by Figallo-Orellano *et al.* in Figallo-Orellano and Pascual (2019), Figallo-Orellano et al. (2022), Figallo-Orellano et al. (2017). In particular, they studied the class of $mpM-$algebras enriched with an automorphism of period 2 and k, where k is an positive integer; in fact, this automorphism works as a new unary operator. Recently, the class of

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k-cyclic *mpM*-algebras has been related to paraconsistent logics and logics formal inconsistency (LFIs, see Carnielli and Marcos 2002) through degree-preserving construction in Figallo-Orellano et al. (2022); clearly, when we take $k = 2$, we obtain the class of symmetric *mpM*-algebras. On the other hand, monadic and modal operators over special Heyting algebras were studied in Almiñana and Pelaitay (2021), Cantú and Figallo (2020), Gallardo and Ziliani (2022).

In this paper, we will determine the condition to construct epimorphisms between finite symmetric *mpM*-algebras, with the purpose of computing them. To know how the epimorphisms can be built, it is useful, for instance, to determine the lattice of subvarieties of the variety of symmetric *mpM*-algebras, see (Figallo-Orellano et al. 2017, Section 7). Furthermore, the construction of epimorphisms between two finite algebra can be used to compute the cardinality of free algebras in finite generated varieties as it was the case of Figallo-Orellano et al. (2017), Section 6.

On the other hand, we will focus on the task of studying the structure of free *k*-cyclic *mpM*-algebras with a finite number of generators. The technique to be used in this section was applied recently in the papers (Figallo-Orellano and Gallardo 2015; Figallo-Orellano et al. 2017; Figallo et al. 2018). In particular, a technique to study a notion of free algebras over a poset, which is a generalization of the standard notion of free algebra, was presented by Figallo-Orellano and Gallardo (2015).
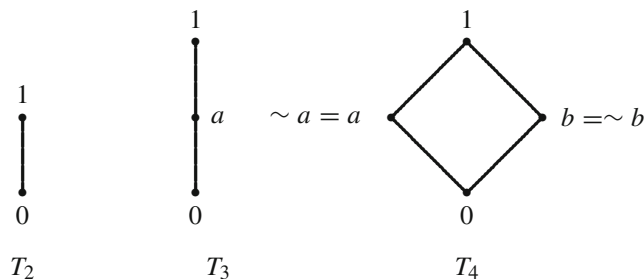
2 Preliminaries

Definition 2.1 We say that an algebra *A* is said to be an *mpM*-algebra if it is a De Morgan algebra for $\sim$ and $*$ verifies the following identities:

(P1) $x \wedge (x \wedge y)^* = x \wedge y^*$,
(P2) $x \wedge 0^* = x$,
(P3) $0^{**} = 0$,
(tm) $x \vee \sim x \leq x \vee x^*$.

In Figallo and sobre (1992), it was showed that every *mpM*−algebra is a TMA-algebra by defining $\nabla x = \sim (\sim x \wedge x^*)$ and $\triangle x = \sim \nabla \sim x$, but the varieties are not equivalent as it was shown in Figallo et al. (2018), Section 5. In Figallo et al. (2018) (see also Figallo et al. 2014), the authors proved that the subdirectly irreducible *mpM*−algebras are three as displayed in the following Remark.

Remark 2.2 Subdirectly irreducible *mpM*−algebras are the following: $T_2 = \{0, 1\}$ with $0 < 1$, $\sim 0 = 0^* = 1$, $\sim 1 = 1^* = 0$; $T_3 = \{0, a, 1\}$, with $0 < a < 1$, $\sim a = a$, $a^* = 0$, $\sim 0 = 0^* = 1$, $\sim 1 = 1^* = 0$; and $T_4 = \{0, a, b, 1\}$

with $a \nleq b$, $b \nleq a$ and $0 < a, b < 1$, $\sim b = a^* = b$, $\sim a = b^* = a$, $\sim 0 = 0^* = 1$, $\sim 1 = 1^* = 0$.

$T_2 \qquad\qquad T_3 \qquad\qquad\qquad T_4$

Definition 2.3 Figallo-Orellano et al. (2022) A *k*-cyclic *mpM*-algebra (for short $\mathcal{C}_k$-algebra) is a pair (A, t) where *A* is an *mpM*-algebra; i.e., *A* is a De Morgan algebra for $\sim$ and $*$ verifies the following identities:

(P1) $x \wedge (x \wedge y)^* = x \wedge y^*$,
(P2) $x \wedge 0^* = x$,
(P3) $0^{**} = 0$.
(tm) $x \vee \sim x \leq x \vee x^*$.

Besides, the function $t : A \rightarrow A$ is an automorphism for *mpM*-algebras such that $t^k(x) = x$ where *k* is an integer $k \geq 0$. Besides, we write $t^0(x) = x$ and $t^n(x) = (t^{n-1} \circ t)(x)$ if $n \geq 1$.

Sometimes we write "tx" instead of "$t(x)$." It is worth mentioning that the class $\mathcal{C}_k$-algebra is a variety, and as examples, we have the classes 1-cyclic *mpM*-algebra and 2-cyclic *mpM*-algebra were studied in Figallo et al. (2014) and Figallo-Orellano et al. (2017), respectively.

Now, we give a result to be used to characterize the prime spectrum for a given $\mathcal{C}_k$-algebra. First, recall that the notion of prime filter, ultrafilter, maximal and minimal is in the usual ones, see (Balbes and Dwinger 1974). Also, we will use the well-known Birula–Rasiowa transformation φ, see, for instance, Font and Rius (2000). Recall that, if *A* is a De Morgan algebra, the map φ is defined as follows: For every prime filter *P* of *A*

$$\varphi(P) = A \backslash \sim P = A \backslash \{\sim x : x \in P\}$$

This map has the following properties:

- $\varphi(P)$ is a prime filter of *A*,
- $\varphi(\varphi(P)) = P$,
- if *Q* is a prime filter of *A* such that $P \subseteq Q$ then $\varphi(Q) \subseteq \varphi(P)$.

Then, we have the following lemma:

Lemma 2.4 Figallo-Orellano et al. (2022) *Let* (A, t) *be a* $\mathcal{C}_k$-*algebra,* $P \subseteq A$ *a prime filter, and* φ *is the Birula–Rasiowa transformation on A. Then, the following properties hold:*

(a) $t^i(P)$ *is a prime filter for* $1 \leq i \leq k$;
(b) P *is minimal* (*maximal*) *iff* $t^i(P)$ *is minimal* (*maximal*) *for* $1 \leq i \leq k$;
(c) U *is an ultrafilter iff* $t^i(U)$ *is an ultrafilter for* $1 \leq i \leq k$;
(d) $\varphi(t^i(P)) = t^i\varphi(P)$, $1 \leq i \leq k$;
(e) *if F is a c-filter of A and* $F \subseteq P$, *then* $F \subseteq \varphi(P) \cap t^i(P)$;
(f) *if* $P \subseteq Q$ *and Q is a prime filter of A, then* $\varphi(P) = Q$, *or* $P = Q$.

Definition 2.5 For a given $\mathcal{C}_k$-algebra (A, t), we say that (A, t) is *r*-periodic if r is the smallest non-negative element such that $t^r(x) = x$ for every $x \in A$.

In what follows, we will present important examples of $\mathcal{C}_k$-algebra, in fact, as we will see they are the generating algebra of the variety.

Remark 2.6 Figallo-Orellano et al. (2022) Let T_2, T_3, and T_4 be the simple algebras of the variety of mpM-algebras, see Remark 2.2. In the following, we will consider the sets $T_{2,k}$, $T_{3,k}$, and $T_{4,k}$ of all sequences $x = (x_1, \ldots, x_k)$ with $x_i \in T_{s,k}$ ($s = 2, 3, 4$) and with the pointwise defined operations. Indeed, these algebras are also $\mathcal{C}_k$-algebras. Taking the function $t : T_{i,k} \to T_{i,k}$ defined as follows: $t(x_1, x_2, \ldots, x_k) = (x_k, x_1, x_2, \ldots, x_{k-1})$ where $(x_1, x_2, \ldots, x_k) \in T_{i,k}$ with $i = 2, 3, 4$. It is not hard to see that $\mathbf{T_{2,k}} = (T_{2,k}, t)$, $\mathbf{T_{3,k}} = (T_{3,k}, t)$ and $\mathbf{T_{4,k}} = (T_{4,k}, t)$ are *k*-periodic $\mathcal{C}_k$-algebras.

It is well-known that divides relation "/" between integers is a partial order. Then, for a given positive integer k, we have the set $Div(k) = \{z : z \text{ divisor of } k\}$ can be considered as distributive lattices; moreover, $k_1 \wedge k_2 = lcm(k_1, K_2)$ and $k_1 \vee k_2 = gcd(k_1, K_2)$ are the *least common multiple* and *greatest common divisor*, respectively. Now, it is possible to see that for every *k*-periodic Boolean algebra, we have $\mathcal{B}_k$ is lattice-isomorphic to the Boolean algebra $Div(k)$. Furthermore, for every $d \in Div(k)$ there is a unique $\mathcal{B}_g$ associated with d which is a subalgebra of $\mathcal{B}_k$. Besides, $\mathcal{B}_g$ is *d*-periodic characterized by $B_d = \{g \in B_k : t \text{ is an automorphism and } t^d g = g\}$.

Lemma 2.7 Figallo-Orellano et al. (2022) *The subalgebras of* $\mathbf{T_{i,k}} = (T_{i,k}, t)$ *are of the form* $\mathbf{T_{i,d}} = (T_{i,d}, t_d)$ *with* d/k *and* $i = \mathcal{C}, 3, 4$. *Besides,* $\mathbf{T_{2,d}}$ *is a* $\mathcal{C}_k$-*subalgebra of* $\mathbf{T_{3,d}}$ *and* $\mathbf{T_{4,d}}$, *but* $\mathbf{T_{3,d}}$ *is not subalgebra of* $\mathbf{T_{4,d}}$.

Lemma 2.8 Figallo-Orellano et al. (2022) $T_{i,d_1} \cap T_{i,d_2} = T_{i,gcd(d_1,d_2)}$, $T_{2,d_1} \cap T_{i,d_2} = T_{2,gcd(d_1,d_2)}$, $T_{3,d_1} \cap T_{4,d_2} = T_{2,gcd(d_1,d_2)}$ *with* $i = 2, 3, 4$

Theorem 2.9 Figallo-Orellano et al. (2022) *The variety of* $\mathcal{C}_k$-*algebras is finitely generated and locally finite. Besides, the only simple algebras are* $\mathbf{T_{i,k}} = (T_{i,k}, t)$ *with* $i = 3, 4$ *and their subalgebras.*

So, it is clear that the variety of $\mathcal{C}_k$-algebras is finitely generated and locally finite. Now, if we take $k = 2$, we obtain the simple algebras of the variety $\mathcal{S}$-algebras studied in Figallo-Orellano et al. (2017), Theorem 5.14. Indeed:

Definition 2.10 Figallo-Orellano et al. (2017) A symmetric modal pseudocomplemented De Morgan algebra (or $\mathcal{S}$-algebra) is a pair (A, T), where A is a mpM-algebra and t is an automorphism of period 2 defined on A, that is, $T(T(x)) = x$ for all $x \in A$.

Theorem 2.11 Figallo-Orellano et al. (2017) *Let* $\mathbf{A}$ *be an* $\mathcal{S}$-*algebra. Then,* $\mathbf{A}$ *is subdirectly irreducible if and only if* $\mathbf{A}$ *is isomorphic to a subalgebra of either* $\mathbf{T_4^2}$ *or* $\mathbf{T_3^2}$.

Next, we will display the $\mathcal{S}$-algebras $\mathbf{T_4^2}$ and $\mathbf{T_3^2}$:
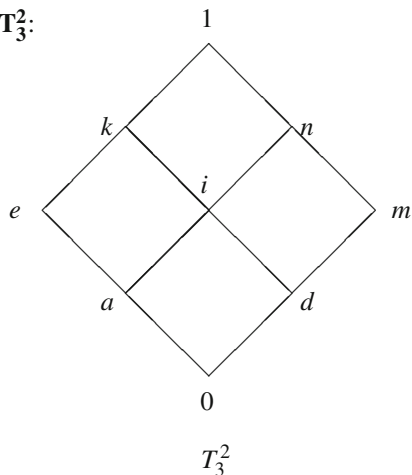(i) $\mathbf{T_4^2}$:
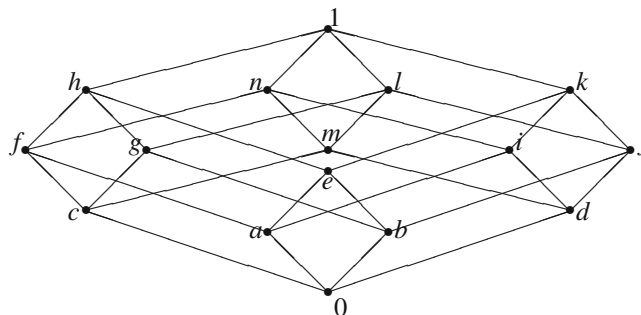
where the operations are given by:

x	$\sim x$	x^*	Tx
0	1	0	0
a	n	l	d
b	l	n	c
c	h	k	b
d	k	h	a
e	m	m	m
f	f	j	j
g	g	i	g

x	$\sim x$	x^*	Tx
h	c	d	l
i	i	g	i
j	j	f	f
k	d	c	n
l	b	a	h
m	e	c	e
n	a	b	k
1	0	0	1

(ii) $\mathbf{T_3^2}$:

T_3^2

x	$\sim x$	x^*	Tx
0	1	0	0
a	n	e	d
d	k	e	a
e	m	m	m
m	e	e	j
i	e	0	i
k	d	0	n
n	a	0	k
1	0	0	1

(iii) $\mathbf{T_{4,1}} = (A, t)$, where $A = \{0, a, b, 1\}$ and $t(x) = x$ and $a =\sim a$ for $x \in A$,

(iv) $\mathbf{T_{4,2}} = (A, t)$, where $A = \{0, f, j, 1\}$ and $t(f) = j$, $f =\sim j$,

(v) $\mathbf{T_{4,3}} = (A, t)$, where $A = \{0, e, m, 1\}$ and if $t(e) = m$, $e =\sim e$,
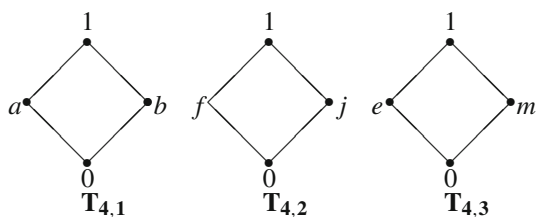
(vi) $A = \{0, c, 1\}$ and $t(x) = x$ and then we have $\mathbf{T_3} = (A, t)$,

(vii) $A = \{0, c, 1\}$, $t(c) =\sim c$ and so $\mathbf{T_3} = (A, t)$.

Finally,

(viii) $\mathbf{T_2}$.

Clearly $\mathbf{T_3^2}$ and $\mathbf{T_3}$ are not subalgebras of $\mathbf{T_4^2}$, but $\mathbf{T_{4,2}}$ and $\mathbf{T_2}$ are the only subalgebras of $\mathbf{T_3^2}$.

3 Epimorphism between finite $\mathcal{S}$-algebras

In this section, we will determine the condition to construct epimorphisms between finite $\mathcal{S}$-algebras, with the purpose of computing them. To know how the epimorphisms can be built, it is useful, for instance, to determine the lattice of sub-varieties of the variety of $\mathcal{S}$-algebra, see (Figallo-Orellano et al. 2017, Section 7). Furthermore, the construction of epimorphisms between two finite algebras will be used to compute the cardinality of free algebras in finite generated varieties as it was the case of Figallo-Orellano et al. (2017), Section 6.

3.1 Prime spectrum of a given algebra

It is well-known that for given a finite distributive lattice, every prime filter is a principal filter (i.e., generated by an element a) and so there is a bijection between the set of prime filters and the set of prime elements. In the following, we will refer to $\mathcal{S}$-algebra $\mathbf{A}$ where A is finite.

On the other hand, Birula–Rasiowa transformation can be used on the set of prime filter of A. So, we denote by $\Pi(A)$ the set of all prime filters of A and the transformation $\psi : \Pi(A) \to \Pi(A)$ can be defined by as $\psi(p) = q$ iff $\varphi([p)) = [q)$. So, we have that ψ verifies:

- $\psi(\psi(p)) = p$, for any $p \in \Pi(A)$,
- if $p_1, p_2 \in \Pi(A)$ are such that $p_1 \leq p_2$, then $\psi(p_2) \leq \psi(p_1)$.

As a particular case, we have that for every $x \in A$ with $x \neq 1$ the negation $\sim$ can be characterized by means of ψ as follows:

$$\sim x = \bigvee_{\{p \in \Pi(A):\psi(p) \not\leq x\}} p$$

From the result exposed in Figallo et al. (2014) and Theorem 2.11, we have the following:

Definition 3.1 Let $(\Pi(A), \psi)$ be the prime spectrum of a given $\mathcal{S}$-algebra $\mathbf{A}$ has the following connected components:

Type I : with $\psi(p) = p$, (1.1) $\psi = T$ and (1.2) $\psi \neq T$,

Type II : with $p < p'$, $\psi(p) = p'$ and $\psi(p') = p$, (2.1) $\{p, T(p)\}$ incomparable elements, (2.2) $p = T(p)$ and (2.3) $T = \psi$,

Type III : with p and p' incomparable elements, $\psi(p) = p'$ and $\psi(p') = p$, (3.1) $T = \psi$, (3.2) $p = T(p)$ and (3.3) $\{p, T(p), \psi(p), \psi(T(p))\}$ incomparable elements.

Thus, if $p \in \Pi(A)$ we say that p:

- is of type I iff $\psi(p) = p$,
- is of type II iff there is $q \in \Pi(A)$ such that if $p < q$, then $\psi(p) = q$, or if $q < p$, then $\psi(q) = p$,
- is of type III iff there exist $q \in \Pi(A)$ incomparable elements with p such that $\psi(p) = q$.

Let us observe that if A is a $mpM-$algebra, then the connected component has only type I, II or III, where the operator T does not play any role. In the next, we will display some technical properties that allow us to develop this section.

Lemma 3.2 *Let A be a finite mpM-algebra and $p \in \Pi(A)$. Then, we have that:*

(i) $p^* = \begin{cases} \bigvee\limits_{t \in \Pi_p^*} t \ si \ \Pi_p^* \neq \emptyset, \\ 0 \qquad si \ \Pi_p^* = \emptyset, \end{cases}$ *where* $\Pi_x^* = \{z \in \Pi(A) : z \wedge x = 0\}$,

(ii) $\sim p^* \in \Pi(A)$,

(iii) *if* $p = p_{c_i} = (0, \ldots, c_i, \ldots, 0)$ *with* $c_i \in \Pi(T_i)$ $i = 2, 3, 4$, *then the following hold:*

   1. $c_i \in \Pi(T_2)$ *implies* $\psi(p) = p$,
   2. $c_i \in \Pi(T_3)$ *implies* $\psi(p_c) = p_1$,
   3. $c_i \in \Pi(T_4)$ *implies* $\psi(p_a) = p_b$.

Proof It is not hard to see (i) holds. We prove that (ii) is verified: let $A \simeq \prod_{i=1}^{n} T_i$ where $i \in \{2, 3, 4\}$ and the mpM-algebra T_i are the generating algebras of the variety, see Remark 2.2. Then, we will identify the elements of A with n-tuples $(x_1, \ldots, x_n)$. It is possible to see that the prime elements of A are the form $p = (0, \ldots, p_i, \ldots, 0)$, where the coordinate i is p_i and the rest are 0, with $p_i \in \Pi(T_i)$. Thus, we have that $\sim p_i^* \in \Pi(T_i)$.

(iii) We know that if P is a prime filter of A, then the transformation verifies: $\varphi(P) = A\backslash \sim P$. Since A is finite, then $\psi(p) = q$ iff $\varphi([p)) = A\backslash \sim [p) = [q)$. So, let us observe that if $p = (0, \ldots, 1, \ldots, 0)$, then $\varphi([p)) = \varphi([(0, \ldots, 1, \ldots, 0))) = A\backslash \sim [(0, \ldots, 1, \ldots, 0)) = [(0, \ldots, 1, \ldots, 0)) = [p)$ and thus $\psi(p) = p$, what verifies 1. Let us now consider $p_c = (0, \ldots, c, \ldots, 0)$ and $p_1 = (0, \ldots, 1, \ldots, 0)$. Hence, $\varphi([p_1)) = \varphi([(0, \ldots, 1, \ldots, 0))) = A\backslash \sim [(0, \ldots, 1, \ldots, 0)) = A\backslash\{(x_1, \ldots, 0, \ldots, x_n) : x_i \in T_s\} = \{(w_1, \ldots, z, \ldots, w_n) : w_i \in T_s, z \in \{c, 1\}\} =$

$[(0, \ldots, c, \ldots, 0)) = p_c$ and therefore $\psi(p_c) = p_1$. In analogous way we can see that $\psi(p_a) = p_b$. $\square$

Corollary 3.3 *In every finite mpM-algebra A, we have:*

(i) *if* $p \in \Pi(A)$ *and* $\psi(p) = p$, *then* $\nabla p = p = \sim p^*$,

(ii) *Let* $p, q \in \Pi(A)$ *such that* $p < q$ *and* $\psi(q) = p$, *then* $\nabla p = q = \sim p^* = \nabla q = \sim q^*$,

(iii) *Let us suppose* $p, q \in \Pi(A)$ *are incomparable elements such that* $\psi(q) = p$ *and* $\psi(p) = q$, *then* $p < \nabla p = \nabla q > q$ *and* $\nabla p \notin \Pi(A)$.

Proof Taking into account the proof of Lemma 3.2, we will prove that (i) holds. Indeed, let $p = (0, \ldots, p_i, \ldots, 0)$ with $p_i \in \Pi(T_{\alpha_i})$ such that $\psi(p) = p$. Thus, it is clear that $p_i = 1 \in \Pi(T_2)$ and then $p = \sim p^* = \nabla p$.

(ii) : Let $p, q \in \Pi(A)$ such that $p < q$. So, it is clear that $p = (0, \ldots, c, \ldots, 0)$ and $q = (0, \ldots, 1, \ldots, 0)$ with $c, 1 \in \Pi(T_3)$. Hence, $q = \sim p^* = \sim q^*$ and besides $\nabla p = \nabla q = q$ as desired.

(iii) : If we take $t = (0, \ldots, t_i, \ldots, 0)$ of type III, then it is clear that $t_i \in \Pi(T_4)$ and therefore $t_i = a$, or $t_i = b$. Thus, if we take $p, q \in \Pi(A)$ incomparable elements such that $\psi(q) = p$ and $\psi(p) = q$, then $p = (0, \ldots, a, \ldots, 0)$ and $q = (0, \ldots, b, \ldots, 0)$. Therefore, $\nabla p = \nabla q = (0, \ldots, 1, \ldots, 0)$, which completes the proof. $\square$

Let us suppose that $(\Pi(A_1), \psi_1)$ and $(\Pi(A_2), \psi_2)$ are prime spectrum of given finite two mpM-algebras A_1 and A_2. So, we define $(\Pi(A_1), \psi_1) + (\Pi(A_2), \psi_2)$ as follows: $(\Pi(A_1) + \Pi(A_2), \psi)$, where $+$ is ordinal sum of the set of prime filters $\Pi(A_1)$ and $\Pi(A_2)$, and $\psi_{|\Pi(A_i)} = \psi_i$. $i = 1, 2$. Therefore, taking into account Lemma 3.2 and 3.3, we have proved the following theorem:

Theorem 3.4 *Let A be finite mpM-algebra and $(\Pi(A), \psi)$ its prime spectrum. Then,*

$$(\Pi(A), \psi) = \sum_{\alpha_2}(\Pi(T_2), \psi_2) + \sum_{\alpha_3}(\Pi(T_3), \psi_3)$$
$$+ \sum_{\alpha_4}(\Pi(T_4), \psi_4),$$

where $\alpha_i < \infty$ y ψ_i is the associated transformation of generating mpM-algebras T_i.

3.2 The construction of epimorphisms

Let **A** and **A**$'$ be finite $\mathcal{S}$-algebra, we denote by $Epi(A, A')$ the set of all $\mathcal{S}$-epimorphism between **A** and **A**$'$. Besides, we consider the set $R(A) = \Pi(A) \cup \{\nabla p : p \in \Pi(A)\}$.

Definition 3.5 Let **A** and **A**′ be $\mathcal{S}$-algebras. We say that the function $f : R(A_1) \to R(A_2)$ is said to be a $\mathcal{S}$-function if the following conditions hold:

$\mathcal{S}_1$. f is one-to-one;
$\mathcal{S}_2$. $f(\psi(p)) = \psi f(p)$, for every $p \in \Pi(A_1)$;
$\mathcal{S}_3$. $f(\nabla q) = \nabla f(q)$, for every $q \in \Pi(A_1)$;
$\mathcal{S}_4$. $f(T(q)) = T(f(q))$, for every $q \in \Pi(A_1)$.

We denote by $\mathcal{F}(A_2, A_1)$ the set of all $\mathcal{S}$-functions from **A₁** into **A₂**.

Lemma 3.6 *Let* **A₁** *and* **A₂** *be finite $\mathcal{S}$-algebras and* $f : R(A_1) \to R(A_2)$ *a $\mathcal{S}$-function. Then:*

(i) *f is increasing over $\Pi(A_1)$,*
(ii) *$f(p)$ and p are prime elements of the same type,*
(iii) *f is increasing over $R(A_1)$.*

Proof (i) Let $q_1, q_2 \in \Pi(A_1)$ such that $q_1 \leq q_2$. Then, by Corollary 3.3 (ii), we have that $\nabla q_1 = q_2$ and so $f(q_1) \leq \nabla f(q_1) = f(\nabla q_1) = f(q_2)$.

(ii) Let us suppose p is of type I, i.e., $\psi(p) = p$. Hence, $f(p) = f(\psi(p)) = \psi f(p)$ and then $f(p)$ is of type I. Furthermore, if $p \in \Pi(A)$ is of type II, then there is $q \in \Pi(A)$ such that $p < q$ (or $q < p$). Now, let us suppose that $p < q$, then $\psi(p) = q$ and $f(p) < f(q)$. Thus, $\psi(f(p)) = f(q)$ and therefore $f(p)$ is of type II. Now, if p is of type III, then there exist $q \in \Pi(A)$ incomparable elements with p such that $\psi(p) = q$. Thus, it is clear that $\psi(f(p)) = f(q)$, then $f(p)$ and $f(q)$ are incomparable elements. Indeed, if we suppose that $f(p) = f(q)$, then since f is one-to-one, we have $p = q$ which is a contradiction. Now, if we suppose that $f(p) < f(q)$, then—by Corollary 3.3 (ii)—we have $\nabla f(p) = f(q)$. Therefore, $\nabla p = q$ but using the same corollary item (iii), we have that ∇p is not a prime element which is a contradiction. The rest of the proof runs with a similar reasoning.

(iii) Taking into account (i), we only have to consider the case that $p, q \in R(A_1)$ such that $p = \nabla t_1$, or $q = \nabla t_2$ with $t_1, t_2 \in \Pi(A_1)$ and $p \leq q$. So, suppose that (1) $p \in \Pi(A_1)$ and $q = \nabla t_2$ with $t_2 \in \Pi(A_1)$ and consider the following sub-cases: (1.1) t_2 of type I or II and (1.2) t_2 of type III. To case (1.1), we have by Corollary 3.3 that $q = \nabla t_2 \in \Pi(A_1)$ and from (i) we have proved the case. To case (1.2), we have by Corollary 3.3 that $\nabla t_2 = (0, \ldots, 1, \ldots, 0) \notin \Pi(A_1)$. Thus, we have that $p = p_a$, or $p = p_b$ and then we need to consider the following two sub-cases (a) $p_a < \nabla p_a = \nabla p_b$, or (b) $p_b < \nabla p_a = \nabla p_b$. First, let us suppose that (a) holds, since $\psi(p_a) = p_b$, then $\psi(f(p_a)) = f(p_b)$ and $f(p_a)$, $f(p_b)$ are incomparable elements.

Hence, by Corollary 3.3 (iii), we can infer that $f(p_a) < \nabla f(p_b) = f(\nabla p_b)$ and therefore $f(p) < f(q)$. We can prove (b) in analogous way to (a).

Let us now suppose (2) $p = \nabla t_1$ and $q = \nabla t_2$ with $t_1, t_2 \in \Pi(A_1)$. Therefore, it is clear that $\nabla t_1 = (0, \ldots, 1_i, \ldots, 0)$ and $\nabla t_2 = (0, \ldots, 1_j, \ldots, 0)$. Since $p \leq q$, then $i = j$ which implies that $\nabla t_1 = \nabla t_2$ and so $f(p) = f(q)$.

On the other hand, let us suppose that (3) $p = \nabla t_1$ and $q, t_1 \in \Pi(A_1)$. Therefore, $p = \nabla t_1 = (0, \ldots, 1_i, \ldots, 0)$ and $q = (0, \ldots, q_j, \ldots, 0)$. From the latter and since $p \leq q$, we infer that $i = j$ and then $p = q$, which completes the proof. $\square$

Let **A₁** be a $\mathcal{S}$-algebra, we denote with $A_x^r = \{q \in R(A_1) : f(q) \leq x\}$ and $A_x^\pi = \{p \in \Pi(A_1) : f(p) \leq x\} \neq \emptyset$, for every $x \in A_1$. Now, Lemma 3.6 allows us to prove the following lemma:

Lemma 3.7 *Let $f \in \mathcal{F}(A_1, A_2)$. Then, the following holds:*

$$\bigvee_{q \in A_x^r} q = \bigvee_{p \in A_x^\pi} p.$$

Proof First, consider $N_x = A_x^r - A_x^\pi$, $a = \bigvee_{p \in A_x^\pi} p$, $b = \bigvee_{q \in A_x^r} q$ and $c = \bigvee_{r \in N_x} r$, then it is clear that $b = a \vee c$. Now, let us suppose that $N_x = \emptyset$, then $c \leq a$. Indeed, let $r \in N_x$ and so there is $p_0 \in \Pi(A_1)$ such that $r = \nabla p_0 = \sim p_0^* \vee p_0$. From the latter, we have that $\sim p_0^* \leq r$ and $p_0 \leq r$. Taking into account Lemma 3.7, we conclude that $f(\sim p_0^*) \leq f(r)$ and $f(p_0) \leq f(r)$. Since $f(r) \leq x$ and by Lemma 3.2, we have that $\sim p_0^* \in \Pi(A_1)$. Therefore, $\sim p_0^*, p_0 \in A_x^\pi$ as desired. $\square$

Now, we will consider a new kind of function associated with f as follows:

Definition 3.8 Let $f \in \mathcal{F}(A_1, A_2)$, then we say that the function $F_f : A_2 \to A_1$ is a Π-function associated with f if the following hold for each $x \in A$:

(i) $F_f(x) = \bigvee_{q \in A_x^r} q$, if $A_x^\pi \neq \emptyset$;
(ii) $F_f(x) = 0$, if $A_x^\pi = \emptyset$, where $A_x^\pi = \{p \in \Pi(A_1) : f(p) \leq x\}$.

We denote by $\Pi_\mathcal{S}(A_2, A_1)$ the set of all Π-functions associated with $\mathcal{S}$-functions.

Lemma 3.9 *If $p \in \Pi(A_2)$, then $F_f(p) = 0$, or $F_f(p) \in \Pi(A)$.*

Proof Let us suppose $p \in \Pi(A_2)$. So, we can consider the following cases:

(i): p is a minimal element of $\Pi(A_2)$. If we suppose that $F_f(p) \neq 0$, then there exists $p_1 \in \Pi(A_1)$ such that $f(p_1) \leq p$. Now, since $f(p_1) \in \Pi(A_1)$, then $f(p_1) = p$. Hence, it is not hard to see that $A_p^\pi = \{p_1\}$ and therefore $F_f(p) = p_1$.

(ii): p is a maximal element but not minimal of $\Pi(A_2)$. Then, there is $q \in \Pi(A_2)$ such that $q < p$ and $\nabla q = p$. Now, if we suppose that $F_f(p) \neq 0$, then there is $p' \in \Pi(A_1)$ such that $f(p') \leq p$. If $F_f(q) = 0$, then we have there is not $t' \in \Pi(A_1)$ such that $f(t') \leq q$, then $f(p') = p$ and $A_p^\pi = \{p'\}$. Hence, $F_f(p) = p'$. On the other hand, if $F_f(q) \neq 0$ there is a unique element $q' \in \Pi(A_1)$ such that $f(q') = q$ and $q' < \nabla q'$. Now, if we have $q' = \nabla q'$ would have $f(q') = f(\nabla q') = \nabla f(q') = \nabla q = p$ which would be a contradiction. As consequence, we infer that $A_p^\pi = \{q', \nabla q'\}$. Indeed, it is clear that $f(q') = q$ is of type II, and by Lemma 3.6, we have also that q' is of type II. Then, $\nabla q' \in \Pi(A_1)$ and $f(\nabla q') = p$. Let us suppose that there is $h' \in \Pi(A)$ such that $f(h') \leq p$. Since $f(h') \in \Pi(A_1)$, then $f(h') = q$, or $f(h') = p$. From the latter, we obtain that $h' = q'$, or $h' = \nabla q'$ and therefore $F_f(p) = \nabla q'$ as desired. □

Theorem 3.10 *Every Π-function F_f associated with an $\mathcal{S}$-function f is a $\mathcal{S}$-epimorphism.*

Proof Let f be an $\mathcal{S}$-function of $\mathbf{A}$ en $\mathbf{A}'$ and let $F_f : A' \to A$ be the Π-function associated. Then, it is not hard to see that $F_f(x \vee y) = F_f(x) \vee F_f(y)$ for every $x, y \in A'$. So, we will see that $F_f(\sim x) = \sim F_f(x)$. Indeed:

(i) Let us suppose $A_{\sim x}^\pi \neq \emptyset$ because in the contrary it would be verified without any difficulty that $F_f(\sim x) \leq \sim F_f(x)$. Now, let $q_0' \in A_{\sim x}^\pi$, then $q_0' \in \Pi(A')$ and so $f(q_0') \leq \sim x = \bigvee_{\{p \in \Pi(A):\psi(p) \not\leq x\}} p$. Hence, there is $p_0 \in \Pi(A)$ such that $f(q_0') \leq p_0$ and $\psi(p_0) \not\leq x$. So, we have that $\psi(p_0) \leq \psi(f(q_0'))$ and therefore $f(\psi(q_0')) \not\leq x$. As consequence of these assertions, we infer that $\psi(q_0') \not\leq F_f(x)$ and so $q_0' \leq \sim F_f(x)$. Thus, by Lemma 3.7 we have that we wanted. Conversely, let us suppose that $\sim F_f(x) \neq 0$; on the contrary, we would have trivially that $\sim F_f(x) \leq F_f(\sim x)$. Therefore, let $p' \in \Pi(A')$ such that $\psi(p') \not\leq F_f(x)$. From the latter, we obtain that $\psi(f(p')) \not\leq x$ and therefore $p' \leq F_f(\sim x)$, which completes the proof.

(ii) We will see that F_f respects the operation $*$. To this end, it is enough to prove that it respects $*$ for prime elements; i.e., $F_f(p^*) \leq (F_f(p))^*$ for every $p \in \Pi(A')$. Indeed, let us suppose that $F_f(p^*) \neq 0$ and $F_f(p) \neq 0$. Since $F_f(p^*) = \bigvee_{\{q' \in \Pi(A_1), f(q') \leq p^*\}} q'$, then we can suppose that $q_0' \in \Pi(A)$ such that $f(q_0') \leq p^*$. From

Lemma 3.2 (i), we have that $p^* = \bigvee_{\{t \in \Pi(A'):t \wedge p=0\}} t$. Therefore, $f(q_0') \not\leq p$ and so $q_0' \not\leq F_f(p)$. Taking into account Lemma 3.9, we can assert that q_0' and $F_f(p)$ are incomparable prime elements. Thus, $q_0' \wedge F_f(p) = 0$ and therefore $q_0' \leq (F_f(p))^*$. On the other hand, we will see that $F_f(p)^* \leq F_f(p^*)$. First, let us consider $F_f(p)^* \neq 0$ because the contrary we would have the property holds trivially. Let $q' \in \Pi(A_1)$ such that $q' \wedge F_f(p) = 0$. Thus, $q' \not\leq F_f(p)$ and so we conclude that $f(q') \not\leq p$. From the latter and Lemma 3.9, we have that $f(q')$ and p are incomparable prime elements. Therefore, $f(q') \wedge p = 0$ and $f(q') \leq p^*$ what prove that we wanted. Since it is not hard to see that $F_f(x^*) = F_f(x)^*$ for every $x \in A'$, then we have proved that (ii) holds.

(iii) Finally, it is clear that $T(F_f(p)) = T(\bigvee_{q' \in A_p^\pi} q') = \bigvee_{q' \in A_p^\pi} T(q')$. Now, let $q_0' \in \Pi(A')$ such that $f(q_0') \leq p$, then $T(f(q_0')) = f(T(q_0')) \leq T(p)$. Since $T(q_0') \in \Pi(A')$, we have that $T(q_0') \leq \bigvee_{t' \in A_{T(p)}^\pi} T(t') \leq F_f(T(p))$. Conversely, since $F_f(T(p)) = \bigvee_{h' \in A_{T(p)}^\pi} h'$ and let $h_0' \in \Pi(A')$ such that $f(h_0') \leq T(p)$, then $T(f(h_0')) = f(T(h_0')) \leq T^2(p) = p$. Therefore, $T(h_0') \leq F_f(p)$. So, $h_0' \leq T(F_f(p))$. From the last assertions, it is clear that $F_f(T(x)) = T(F_f(x))$ with $x \in A$. Therefore, $F_f : A \to A'$ is an $\mathcal{S}$-homomorphism.

To see that F_f is onto, let $b \in \Pi(A')$. Then, $f(b) = a \in \Pi(A)$ and so $F_f(a) \neq 0$. Recall by Lemma 3.9, we have that $F_f(a) \in \Pi(A')$. Now, let $w = F_f(a)$. So, $f(b) \leq f(w)$ and taking into account that $f(w) \leq a = f(b)$ holds, we have that $b = w$ and then $F_f(a) = b$. Finally, it is not difficult to see that if $w \in A'$, then there is $z \in A$ such that $F_f(z) = w$ as desired. □

In the following, we will see the reciprocal of Lemma 3.10, before we see the following technical result that we need.

Lemma 3.11 *Let A_1 and A_2 be finite $\mathcal{S}$-algebras and let $h : A_2 \to A_1$ be an $\mathcal{S}$-epimorphism and let $p' \in \Pi(A_1)$. Then, there is a unique $p_0 \in \Pi(A_2)$ such that $h(p_0) = p'$.*

Proof We know that there is $x \in A_2$ such that $h(x) = p'$. Let us suppose that $h^{-1}(p') = \{x_1, \ldots, x_t\}$ and consider the element $p_0 = \bigwedge_{i=1}^{t} x_i$. It is clear that $h(p_0) = p'$ and $p_0 \neq 0$. So, we have to see that it is a prime element. Indeed, let us suppose that $p_0 = a \vee b$, then we have that $a \leq p_0$ and $b \leq p_0$. Besides, $p' = h(p_0) = h(a) \vee h(b)$ and therefore $p_0 = a$, or $p_0 = b$.

Now, we will see that p_0 is unique. Indeed, let $p_1 \in \Pi(A)$ and $p_1 \in h^{-1}(p')$. Thus, $p_0 \leq p_1$, and if p_1 is a prime element of type I or III, we have that $p_0 = p_1$. Now, let us observe that if p_1 is of type II and we suppose $p_0 < p_1$, then $\nabla p_0 = p_1$ and $\triangle p_0 = (\sim p_0)^* \wedge p_0 = 0$. On the other hand, since h is a $\nabla, \triangle$-morphism we have that $p' = h(p_1) = h(\nabla p_0) = \nabla h(p_0) = \nabla p'$. Since $\triangle p' = \triangle \nabla p' = \triangle \sim \triangle \sim p' = \sim \nabla \triangle \sim p' = \sim \triangle \sim p' = \nabla p'$, we infer that $0 = h(\triangle p_0) = \triangle h(p_0) = \triangle p'$, which is a contradiction. Therefore, we have that $p_0 = p_1$ as desired. □

For a given $\mathcal{S}$-epimorphism $h : A_2 \to A_1$, we will say that the function $f : \Pi(A_1) \to \Pi(A_2)$ is induced by h if the following condition holds: $f(q) = p$ iff $h(p) = q$. The existence of f is insured by Lemma 3.11; furthermore, f is one-to-one. On the other hand, f can be extended for a unique $\mathcal{S}$-function $f_s : R(A_1) \to R(A_2)$ in the following way: if $q \in \Pi(A_1)$, then $f_s(q) = f(q)$, $f_s(\psi(q)) = \psi f(q)$ and $f_s(T(q)) = f(T(q))$; if $q \in R(A_1)/\Pi(A)$, then $f_s(\nabla q) = \nabla f(q)$. Thus, we will say that f_s is the $\mathcal{S}$-function induced by h.

Lemma 3.12 *If f is the function induced by the $\mathcal{S}$-epimorphism h, then f is a Π-function.*

Proof Let $h : A_1 \to A_2$ be an $\mathcal{S}$-epimorphism and let g be the $\mathcal{S}$-function induced by h. Besides, let us consider the Π-function F_g associated with g. We will see that $F_g = h$. Indeed, let first $p \in \Pi(A_2)$ and suppose that $F_g(p) \neq 0$. From Lemma 3.9, we have that $F_g(p) = p'$ with $f(p') = p$ and $p' \in \Pi(A_1)$. By definition of f, we can infer that $h(p) = p'$ and so $F_g(p') = h(p)$. Let us now suppose that $F_g(p) = 0$, then we have to see that $h(p) = 0$. So, let us first suppose that $h(p) \neq 0$ by contradiction, then there is $p' \in \Pi(A_1)$ such that $p' \leq h(p)$. From Lemma 3.11, there is a unique $q \in \Pi(A_1)$ such that $h(q) = p'$. Since f is induced by h, we have that $f(p') = q$. On the other hand, we have that $p' = p' \wedge h(p) = h(p \wedge q)$. By Lemma 3.11, we obtain that $q = p \wedge q$. So, $q = f(p') \leq p$ and therefore $A_p^\pi \neq \emptyset$, which is not possible. □

Taking into account the previous Lemma and Theorem 3.10, we can observe that for every epimorphism has Π-function induced by it and for every Π-function has associated a unique $\mathcal{S}$-epimorphism. On the other hand, it is not hard to see that the set of all $\mathcal{S}$-epimorphisms between **A** en **A$'$**, denoted by $Epi(A, A')$, is equipotent with the set of all Π-functions from **A$'$** into **A**, denoted by $\mathcal{F}(A', A)$, i.e.,

$$|Epi(A, A')| = |\mathcal{F}(A', A)|,$$

where $|X|$ denotes the number of elements of X.

For a given finite $\mathcal{S}$-algebras A_1. Let us now consider $|\Pi(A_1)| = m$ and suppose A_1 has $t_{1,1} + 2t_{1,2}$ prime elements

of type I; $2(t_{2,2} + t_{2,3}) + 4t_{2,1}$ of type II; and $2(t_{3,1} + t_{3,2}) + 4t_{3,3}$ of type III. Therefore,

$$t_{1,1} + 2t_{1,2} + 2(t_{2,2} + t_{2,3}) + 4t_{2,1} + 2(t_{3,1} + t_{3,2}) + 4t_{3,3} = m.$$

An in this case we denote $A_1 = A_{t_{1,i};t_{2,j};t_{3,k}}$ with the propose to mark what kind of prime element has A. Now, let consider a finite $\mathcal{S}$-algebras $A_2 = A_{s_{1,i};s_{2,j};s_{3,k}}$ and suppose there is an $\mathcal{S}$-epimorphism from A_1 into A_2. It is clear that $s_{i,j} \leq t_{i,j}$ and then the set $\mathcal{F}(A_2, A_1)$ is non-empty. Besides, if we denote by $\mathcal{F}_{ij}(A_2, A_1)$ the set of all one-to-one $\mathcal{S}$-functions between the set of all prime elements of i and subtype j of A_2 and A_1, respectively. So, we have:

$$|\mathcal{F}(A', A)| = \prod_{1 \leq i \leq 2} |\mathcal{F}_{1,i}(A', A)| \cdot \prod_{\substack{2 \leq s \leq 3 \\ 1 \leq j \leq 3}} |\mathcal{F}_{s,j}(A', A)|.$$

From all results exposed up to here, we have the following central theorem:

Theorem 3.13

$$|Epi(A_2, A_1)| = \frac{s_{1,1}!}{(s_{1,1} - t_{1,1})!} \cdot \frac{(2s_{1,2})!}{(2s_{1,2} - t_{1,2})!} \cdot \frac{s_{2,1}!}{(s_{2,1} - t_{2,1})!}$$
$$\cdot \frac{s_{2,2}!}{(s_{2,2} - t_{2,2})!} \cdot \frac{(2s_{2,3})!}{(2s_{2,3} - t_{2,3})!} \cdot \frac{(2s_{3,1})!}{(2s_{3,1} - t_{3,1})!} \cdot \frac{(2s_{3,2})!}{(2s_{3,2} - t_{3,2})!}$$
$$\cdot \frac{(4s_{3,3})!}{(4s_{3,3} - t_{3,3})!}$$
$$|Aut(A_2, A_2)| = s_{1,1}! \cdot (2s_{1,2})! \cdot s_{2,1}! \cdot s_{2,2}!(2s_{2,3})! \cdot (2s_{3,1})!$$
$$\cdot (2s_{3,2})! \cdot (4s_{3,3})!,$$

where we denote by $Aut(A_2, A_1)$ the set of all automorphisms over A_2.

Proof To compute the number of epimorphisms, we have to calculate the number of one-to-one functions from the set of prime elements of A_1 into the one of prime elements of A_2. On the other hand, if we take $h \in Epi(A, A)$ such that h is one-to-one, then the $\mathcal{S}$-function f (function induced by h) is onto. Since $f/_{\Pi(A)} : \Pi(A) \to \Pi(A)$ is one-to-one, we have proved the theorem. □

3.3 Examples: *mpM*-algebras, ukasiewicz algebras of order 3 and Boole algebras

In this part of the paper, we will display some particular cases. To this end, we take $T(x) = x$ (i.e., the identity function) and let A_1 and A_2 be finite *mpM*-algebras. So, we have:

$$|Epi_\mathcal{S}(A_2, A_1)| = |Epi_{mpM}(A_2, A_1)| = \frac{s_{1,1}!}{(s_{1,1} - t_{1,1})!}$$
$$\cdot \frac{s_{2,1}!}{(s_{2,1} - t_{2,1})!} \cdot \frac{(2s_{3,1})!}{(2s_{3,1} - t_{3,1})}!$$

Now, if A_1 and A_2 are 3-valued Łukasiewicz algebras, then:

$$|Epi_{Ł_3}(A_2, A_1)| = \frac{s_{1,1}!}{(s_{1,1} - t_{1,1})!} \cdot \frac{s_{2,1}!}{(s_{2,1} - t_{2,1})!}.$$

Besides, if A_1 and A_2 are Boolean algebras, then:

$$|Epi_B(A_2, A_1)| = \frac{s_{1,1}!}{(s_{1,1} - t_{1,1})!}.$$

4 Free *k*-cyclic modal pseudocomplemented De Morgan algebras

In this section, we will focus on the task of studying the structure of free $\mathcal{C}_k$-algebras with a finite number of generators. The technique that will use in this section was applied recently in the papers (Figallo-Orellano and Gallardo 2015; Figallo-Orellano et al. 2017; Figallo et al. 2018; Gallardo and Ziliani 2021). In particular, it was presented a technique to study a notion of free algebras over a poset in Figallo-Orellano and Gallardo (2015), which is a generalization the standard notion of free algebra.

We denote by $F_{\mathcal{C}_k}(n)$ this algebra where n is a positive integer. From the results shown in Sect. 2 and the fact that the variety of $\mathcal{C}_k$-algebras is finitely generated and locally finite, we have that for every d divisor of k, the family $\mathcal{C}$ of the maximal *c*-filters of $F_{\mathcal{C}_k}(n)$ can be partitioned in the following way:

$$N_{i,d} = \{N \in \mathcal{C} : F_{\mathcal{C}_k}(n)/N \simeq T_{i,d}\}.$$

Thus, from Theorem 2.9 we have:

$$F_{\mathcal{C}_k}(n) \simeq \prod_{d/k} T_{2,d}^{\alpha_{2,d}} \times \prod_{d/k} T_{3,d}^{\alpha_{3,d}} \times \prod_{d/k} T_{4,d}^{\alpha_{4,d}},$$

where $\alpha_{i,d} = |N_{i,d}|$ with $i = 2, 3, 4$.

Now, let us consider $Epi(F_{\mathcal{C}_k}(n), T_{i,d})$ the set of all epimorphisms from $F_{\mathcal{C}_k}(n)$ into $T_{i,d}$ and we denote by $Aut(T_{i,d})$ the set of all automorphisms over the algebra $T_{i,d}$ and $|X|$ denotes the number of elements of X. It is not hard to see that $Aut(T_{l,d})$ (with $l = 2, 3$) has only d automorphisms and they are $t, t^2, \ldots, t^{d-1}, t^d$. Besides, it is possible to prove that $|Aut(T_{4,d})| = 2d$. Taking the function $s : Epi(F_{\mathcal{C}_k}(n), T_{i,d}) \to N_{i,d}$ defined by $s(h) = ker(h) = h^{-1}(\{1\})$ for every $h \in Epi(F_{\mathcal{C}_k}(n), T_{i,d})$, we can see that s is onto and $s^{-1}(N) = \{\alpha \circ h : \alpha \in Aut(T_{i,d})\}$. Therefore,

$$\alpha_{i,d} = \frac{|Epi(F_{\mathcal{C}_k}(n), T_{i,d})|}{|Aut(T_{l,d})|}$$

On the other hand, for every $h \in Epi(F_{\mathcal{C}_k}(n), T_{i,d})$ there is a function $f : G \to T_{i,d}$ such that $f = h|_G$. Now, if $F^*(G, T_{i,d}) = \{f : G \to T_{i,d}$ such that $[f(G)]_{\mathcal{C}_k} = T_{i,d}\}$, then we have:

$$|Epi(F_{\mathcal{C}_k}(n), T_{i,d})| = |F^*(G, T_{i,d})|.$$

where $[f(G)]_{\mathcal{C}_k}$ denotes the $\mathcal{C}_k$-algebra generated by $f(G)$. Let us observe that the condition $[f(G)]_{\mathcal{C}_k} = T_{i,d}$ is equivalent to asking $f(G) \subseteq T_{i,d}$ and $f(G) \nsubseteq S$ for every maximal subalgebra S of $T_{i,d}$. If we denote $M(d)$ the set of maximal divisors of d different of d. Then, for every maximal subalgebra of $T_{i,d}$ are of the form $T_{i,x}$ with $x \in M(d)$. Thus, we can write:

$$F^*(G, T_{i,d}) = F_{i,d} \backslash \bigcup_{l \leq i} \bigcup_{x \in M(d)} F_{l,x},$$

where $F_{i,d}$ is the set of all functions from G into $T_{i,d}$. Therefore,

$$
\begin{aligned}
|F^*(G, T_{i,d})| &= |F_{i,d}| - |\bigcup_{l \leq i} \bigcup_{x \in M(d)} F_{l,x}| \\
&= (i^d)^n - |\bigcup_{l \leq i} \bigcup_{x \in M(d)} F_{l,x}|.
\end{aligned}
$$

On the other hand, it is well-known that every finite set $\mathcal{J}$ and the family of finite sets $\{A_j\}_{j \in \mathcal{J}}$ verify:

$$|\bigcup_{j \in \mathcal{J}} A_i| = \sum_{X \subseteq \mathcal{J}, X \neq \emptyset} (-1)^{|X|-1} |\bigcap_{j \in X} A_j|.$$

Now, let us observe that if we take $I = \{w : w \leq i\}$, then:

$$
\begin{aligned}
|\bigcup_{l \leq i} \bigcup_{x \in M(d)} F_{l,x}| &= |\bigcup_{(z,t) \in I \times M(d)} F_{z,t}| \\
&= \sum_{X \subseteq I \times M(d)} (-1)^{|X|} |\bigcap_{(j_1, j_2) \in X} F_{j_1, j_2}|,
\end{aligned}
$$

where

$$\bigcap_{(j_1, j_2) \in X} F_{j_1, j_2} = \{f \in F_{i,d} : f : G \to \bigcap_{(j_1, j_2) \in X} T_{j_1, j_2}\}.$$

Calculating $\alpha_{i,d}$

If $i = 2$, then $I = \{2\}$ and therefore the following holds:

$$\alpha_{2,d} = \frac{(2^d)^n - |\bigcup_{x \in M(d)} F_{2,x}|}{d}$$

$$= \frac{(2^d)^n - \sum\limits_{Z \subseteq M(d), Z \neq \emptyset} (-1)^{|Z|-1} |\bigcap\limits_{w \in Z} F_{2,w}|}{d}$$

$$\alpha_{3,d} = \frac{(3^d)^n - (2^d)^n - \sum\limits_{W \subseteq M(d), W \neq \emptyset} (-1)^{|W|-1} (3^{gcd(W)})^n + \sum\limits_{H \subseteq M(d) \times M(d), H \neq \emptyset} (-1)^{|H|-1} (2^{gcd(H_1 \cup H_2)})^n}{d}$$

$$= \frac{(2^d)^n - \sum\limits_{Z \subseteq M(d), Z \neq \emptyset} (-1)^{|Z|-1} (2^{mcd(Z)})^n}{d},$$

where $gcd(Z)$ is the set of greatest common divisors of Z. If $i = 3$, then:

$$\alpha_{3,d} = \frac{(3^d)^n - |\bigcup\limits_{(j,x) \in \{2,3\} \times M(d)} F_{j,x} \cup F_{2,d}|}{d}$$

$$\alpha_{4,d} = \frac{(4^d)^n - (2^d)^n - \sum\limits_{W \subseteq M(d), W \neq \emptyset} (-1)^{|W|-1} (4^{gcd(W)})^n + \sum\limits_{H \subseteq M(d) \times M(d), H \neq \emptyset} (-1)^{|H|-1} (2^{gcd(H_1 \cup H_2)})^n}{2d}.$$

$$= \frac{(3^d)^n - |F_{2,d} \cup \bigcup\limits_{y \in M(d)} F_{3,y}|}{d}.$$

Since

$$|F_{2,d} \cup \bigcup\limits_{y \in M(d)} F_{3,y}| = |F_{2,d}| + |\bigcup\limits_{y \in M(d)} F_{3,y}| - |F_{2,d} \cap \bigcup\limits_{y \in M(d)} F_{3,y}|$$

and

$$\alpha_{2,d} = \frac{(2^d)^n - \sum\limits_{Z \subseteq M(d), Z \neq \emptyset} (-1)^{|Z|-1} (2^{gcd(Z)})^n}{d},$$

$$\alpha_{3,d} = \frac{(3^d)^n - (2^d)^n - \sum\limits_{W \subseteq M(d), W \neq \emptyset} (-1)^{|W|-1} (3^{gcd(W)})^n + \sum\limits_{H \subseteq M(d) \times M(d), H \neq \emptyset} (-1)^{|H|-1} (2^{gcd(H_1 \cup H_2)})^n}{d},$$

$$\alpha_{4,d} = \frac{(4^d)^n - \sum\limits_{W \subseteq M(d), W \neq \emptyset} (-1)^{|W|-1} (4^{gcd(W)})^n - (2^d)^n + \sum\limits_{H \subseteq M(d) \times M(d), H \neq \emptyset} (-1)^{|H|-1} (2^{gcd(H_1 \cup H_2)})^n}{2d}.$$

$$|F_{2,d} \cap \bigcup\limits_{y \in M(d)} F_{3,y}| = |\bigcup\limits_{y \in M(d)} (F_{2,d} \cap F_{3,y})|,$$

I have that

where $H_1 = \{x : (x, y) \in H\}$ and $H_2 = \{y : (x, y) \in H\}$. Finally, for $i = 4$ and by Lemma 2.7, we have:

$$\alpha_{4,d} = \frac{(4^d)^n - |\bigcup\limits_{(j,x) \in \{2,4\} \times M(d)} F_{j,x} \cup F_{2,d}|}{2d}$$

$$= \frac{(4^d)^n - |F_{2,d} \cup \bigcup\limits_{y \in M(d)} F_{4,y}|}{2d}.$$

By analogous reasoning to the last case, we have:

From the last assertions, we have proved the following theorem:

Theorem 4.1 *Let $F_{\mathcal{C}_k}(n)$ be the free $\mathcal{C}_k$-algebra with n generators, then:*

$$|F_{\mathcal{C}_k}(n)| = \prod\limits_{d/k} 2^{\alpha_{2,d}} \cdot \prod\limits_{d/k} 3^{\alpha_{3,d}} \cdot \prod\limits_{d/k} 4^{\alpha_{4,d}}$$

with

4.1 Particular cases

In this subsection, we will show that Theorem 4.1 has particular cases that was previously obtained in the literature in Figallo et al. (2018), Figallo-Orellano et al. (2017).

4.2 $\mathcal{C}_1$-algebras or *mpM*-algebras

If $k = 1$, the class of $\mathcal{C}_1$-algebras coincides with the one of mpM-algebras. According to Theorem 4.1, we have that the free $\mathcal{C}_1$-algebra with r generators is:

$$F_{\mathcal{C}_1}(r) \simeq T_{2,1}^{\alpha_{2,1}} \times T_{3,1}^{\alpha_{3,1}} \times T_{4,1}^{\alpha_{4,1}}.$$

Since $M(1) = \emptyset$ we have that $\sum_{Z \subseteq M(1)} (-1)^{|Z|} (2^{gcd(Z)})^r = 0$ and therefore $\alpha_{2,1} = 2^r$. Besides, it is possible to see that

$$\sum_{H \subseteq M(1) \times M(1), H \neq \emptyset} (-1)^{|H|} (2^{gcd(H_1 \cup H_2)})^r = 0,$$
$$\sum_{W \subseteq M(d), W \neq \emptyset} (-1)^{|W|-1} (4^{gcd(W)})^r = 0$$
$$= \sum_{W \subseteq M(d), W \neq \emptyset} (-1)^{|W|-1} (3^{gcd(W)})^r,$$

and thus, $\alpha_{3,1} = 3^r - 2^r$ and $\alpha_{4,1} = \frac{4^r - 2^r}{2} = 2^{r-1}(2^r - 1)$. Therefore,

$$|F_{\mathcal{C}_1}(n)| = 2^{2^r} \times 3^{3^r - 2^r} \times 4^{2^{r-1}(2^r-1)}.$$

This number was presented by A. V. Figallo et al. in Figallo et al. (2018), Theorem 4.5.

4.3 $\mathcal{C}_2$-algebras or $\mathcal{S}$-algebras

For the $k = 2$, we have the class of $\mathcal{C}_2$-algebras coincides with the one of $\mathcal{S}$-algebras introduced in Figallo-Orellano et al. (2017). Then, the free $\mathcal{C}_2$-algebra with r generators is:

$$F_{\mathcal{C}_2}(r) \simeq \prod_{d=1}^{2} T_{2,d}^{\alpha_{2,d}} \times \prod_{d=1}^{2} T_{3,d}^{\alpha_{3,d}} \times \prod_{d=1}^{2} T_{4,d}^{\alpha_{4,d}}.$$

Since $|T_{i,d}| = i^d$, $\alpha_{2,2} = \frac{4^r - 2^r}{2}$, $\alpha_{3,2} = \frac{9^r - 3^r - 4^r + 2^r}{2}$ and $\alpha_{4,2} = \frac{4^{2r} - 4^r - 2^{2r} + 2^r}{4}$, then

$$|F_{\mathcal{C}_2}(r)| = |T_{2,1}|^{2^r} \times |T_{2,2}|^{\frac{4^r - 2^r}{2}} \times |T_{3,1}|^{3^r - 2^r}$$
$$\times |T_{3,2}|^{\frac{3^{2r} - 3^r - 2^{2r} + 2^r}{2}} \times |T_{4,1}|^{\frac{4^r - 2^r}{2}} \times |T_{4,2}|^{\frac{4^{2r} - 4^r - 2^{2r} + 2^r}{4}}$$
$$= 2^{2^r} \times 4^{\frac{4^r - 2^r}{2}} \times 3^{3^r - 2^r} \times 9^{\frac{3^{2r} - 3^r - 2^{2r} + 2^r}{2}}$$
$$\times 4^{\frac{4^r - 2^r}{2}} \times 16^{\frac{4^{2r} - 4^r - 2^{2r} + 2^r}{4}}$$

$$= 2^{2^r} \times 4^{2 \cdot \frac{4^r - 2^r}{2}} \times 3^{3^r - 2^r} \times 9^{\frac{9^r - 3^r - 4^r + 2^r}{2}}$$
$$\times 16^{\frac{16^r - 4^r - 4^r + 2^r}{4}}$$
$$= 2^{2^r} \times 4^{2 \cdot \frac{4^r - 2^r}{2}} \times 3^{3^r - 2^r} \times 9^{\frac{9^r - 3^r - 4^r + 2^r}{2}}$$
$$\times 16^{\frac{16^r - 3 \cdot 4^r + 2 \cdot 2^r + 4^r - 2^r}{4}}$$
$$= 2^{2^r} \times 4^{2 \cdot \frac{4^r - 2^r}{2}} \times 3^{3^r - 2^r} \times 9^{\frac{9^r - 3^r - 4^r + 2^r}{2}}$$
$$\times 16^{\frac{16^r - 3 \cdot 4^r + 2 \cdot 2^r}{4}} \times 16^{\frac{4^r - 2^r}{4}}$$
$$= 2^{2^r} \times 4^{3 \cdot \frac{4^r - 2^r}{2}} \times 3^{3^r - 2^r} \times 9^{\frac{9^r - 3^r - 4^r + 2^r}{2}}$$
$$\times 16^{\frac{16^r - 3 \cdot 4^r + 2 \cdot 2^r}{4}}.$$

This number was obtained by Figallo-Orellano *et al.* in Figallo-Orellano et al. (2017), Theorem 6.4.

4.4 $\mathcal{C}_k$-algebras with k are prime number

$$|F_{\mathcal{C}_k}(r)| = \prod_{N \in N_{2,d}, d/k} |T_{2,d}|^{|N_{2,d}|} \times \prod_{N \in N_{3,d}, d/k} |T_{3,d}|^{|N_{3,d}|}$$
$$\times \prod_{N \in N_{4,d}, d/k} |T_{4,d}|^{|N_{4,d}|}$$
$$= \prod_{d/k} (2^d)^{\alpha_{2,d}} \times \prod_{d/k} (3^d)^{\alpha_{3,d}} \times \prod_{d/k} (4^d)^{\alpha_{4,d}}$$
$$= 2^{\alpha_{2,1}} \times (2^k)^{\alpha_{2,k}} \times 3^{\alpha_{3,1}} \times (3^k)^{\alpha_{3,k}} \times 4^{\alpha_{4,1}} \times (4^k)^{\alpha_{4,k}}$$
$$= 2^{2^r} \times (2^k)^{\frac{2^{kr} - 2^r}{k}} \times 3^{3^r - 2^r} \times (3^k)^{\frac{3^{kr} - 3^r - 2^{kr} + 2^r}{k}} \times 4^{\frac{4^r - 2^r}{2}}$$
$$\times (4^k)^{\frac{4^{kr} - 4^r - 2^{kr} + 2^r}{2k}}$$

Declarations

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