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FOUNDATION, ALGEBRAIC, AND ANALYTICAL METHODS IN SOFT COMPUTING

Operation properties and (α, β)-equalities of complex intuitionistic fuzzy sets

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Abstract
Complex intuitionistic fuzzy set, as an extension of intuitionistic fuzzy set, could describe the fuzzy characters of things more detail and comprehensively and is very useful in dealing with vagueness and uncertainty of problems that include the periodic or recurring phenomena. In this paper, various operation properties of complex intuitionistic fuzzy sets are investigated when the membership phase and non-membership phase are restricted to $[0, 2\pi]$. Meanwhile, consider that precise membership values and non-membership values should normally be of no practical significance, and there is no equality and proximity measure investigation on complex intuitionistic fuzzy sets. First of all, we proposed a new distance measure for complex intuitionistic fuzzy sets. Then this distance measure is used to define (α, β)-equalities of complex intuitionistic fuzzy sets. Two complex intuitionistic fuzzy sets are said to be (α, β)-equal if the distance between their membership degrees is less than $1-\alpha$ and the distance between their non-membership degrees is less than β. Furthermore, this paper shows how various operations between complex intuitionistic fuzzy sets affect given (α, β)-equalities of complex intuitionistic fuzzy sets. Finally, complex intuitionistic fuzzy relations are discussed and some examples are given to illuminate the results obtained in this paper.

Keywords Complex intuitionistic fuzzy set · Distance measure · (α, β)-equality · Complex intuitionistic fuzzy relations · Operation

1 Introduction

Since the concept of intuitionstic fuzzy sets was put forward by Atanassov (1986), the theories and applications of intuitionstic fuzzy sets have developed rapidly. It is well known that intuitionstic fuzzy set was a generalization of fuzzy set (Zadeh 1965). Meanwhile, the range of the membership function and non-membership function of intuitionstic fuzzy set are limited to the interval [0, 1], and their sum also belongs to the interval [0, 1], i.e., they all belong to the real numbers.

The question presented by Daniel Ramot and other researchers was, what will be the result, if change the co-domain in the fuzzy sets to complex numbers instead of real numbers? To discuss this issue, in 2002, Ramot et al. (2002) proposed the concept of complex fuzzy sets by considering both the membership degree and periodicity of uncertain problems. The membership function of a complex fuzzy set is given by a complex-valued function, which comprises an amplitude term and a phase term. However, this concept is different from the fuzzy complex set introduced and discussed by Buckley (1989, 1991, 1992), Zhang (1992) and Gong and Xiao (2021, 2022). Subsequently, to incorporate the hesitation degree and the periodicity information into the analysis, Alkouri and Salleh (2012) proposed a new innovative concept and called it complex intuitionistic fuzzy sets, where the membership function $\mu_A(x)$ and non-membership function $\nu_A(x)$ of a complex intuitionstic fuzzy set A instead of being real-valued functions with the rang of [0, 1] are replaced by complex-valued functions of the form

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$$\mu_A(x) = r_A(x) \cdot e^{i\bar{\omega}_{\mu A}(x)} \quad i = \sqrt{-1}$$

and

$$\nu_A(x) = s_A(x) \cdot e^{i\bar{\omega}_{\nu A}(x)} \quad i = \sqrt{-1},$$

where $r_A(x)$ and $s_A(x)$ are real-valued functions and both belong to the interval [0, 1] such that $0 \le r_A(x)+s_A(x) \le 1$, also $\bar{\omega}_{\mu A}(x)$ and $\bar{\omega}_{\nu A}(x)$ are real-valued functions. The novelty of complex intuitionistic fuzzy set lies in its ability for membership and non-membership functions to achieve more range of values. The ranges of values are extended to the unit circle in complex plane for both membership and non-membership functions instead of [0, 1] as in the conventional intuitionistic fuzzy functions. They also discussed the basic operations on complex intuitionistic fuzzy sets, developed a formula for calculating distance among complex intuitionistic fuzzy sets and gave its application in decision-making problems (Alkouri and Salleh 2013a, b). Meanwhile, some new types of fuzzy sets and their applications have been investigated by many researchers recently (Al-Shami 2022; Al-Shami et al. 2022; Ibrahim et al. 2021).

On the other hand, with an attempt to show that “precise membership values should normally be of no practical significance”, Pappis (1991) introduced firstly the notion of “proximity measure”. Hong and Hwang (1994) then discussed the value similarity of fuzzy systems variables. Further, Cai (1995, 2001) introduced and discussed δ-equalities of fuzzy sets and their properties. As the extension of the δ-equalities of fuzzy sets, the δ-equalities of complex fuzzy sets was discussed by Zhang et al. (2009). Meanwhile, in 2013, Gong et al. (2013) investigated the similarity and (α, β)-equalities of intuitionistic fuzzy choice functions based on triangular norms.

As a newly developed tool, complex intuitionistic fuzzy set can describe the fuzzy characters of things more detail and comprehensively and is very useful in dealing with vagueness and uncertainty of problems that include the periodic or recurring phenomena, which has been investigated systematically and exhaustively by many researchers and has successfully applied in actual decision-making problems and other areas (Garg and Rani 2019a, b, c, d, 2020a, b; Rani and Garg 2017). So in this paper, various operation properties of complex intuitionistic fuzzy sets are investigated when the membership phase and non-membership phase are restricted to $[0, 2\pi]$. Meanwhile, consider that precise membership values and non-membership values should normally be of no practical significance, and there is no equality and proximity measure investigation on complex intuitionistic fuzzy sets. First of all, we proposed a new distance measure for complex intuitionistic fuzzy sets. The distance of two complex intuitionistic fuzzy sets measures the difference between the grades of two complex intuitionistic fuzzy sets as well as that between the phases of the two complex intuitionistic fuzzy sets. Then this distance measure is used to define (α, β)-equalities of complex intuitionistic fuzzy sets. Two complex intuitionistic fuzzy sets are said to be (α, β)-equal if the distance between their membership degrees is less than $1-\alpha$ and the distance between their non-membership degrees is less than β. The concept of (α, β)-equalities of complex intuitionistic fuzzy sets allows us systematically develop the distance, equality and proximity measures for complex intuitionistic fuzzy sets, which not only deeply enrich the fundamental theory of complex intuitionistic fuzzy sets, but also provide a powerful tool to further investigate complex intuitionistic fuzzy sets.

The rest of this paper is organized as follows: In Sect. 2, after reviewing the concept of complex intuitionistic fuzzy set, some operations of complex intuitionistic fuzzy sets are introduced and their properties are discussed. Section 3 investigates distance measure and (α, β)-equalities of complex intuitionistic fuzzy sets and discusses (α, β)-equalities for various implication operators. Complex intuitionistic fuzzy relations are discussed in Sect. 4 and some examples are given to illuminate the results obtained in this paper in Sect. 5. Conclusion is given in Sect. 6.

2 Operations of complex intuitionistic fuzzy sets

After reviewing the concept of complex intuitionistic fuzzy sets, some operations of complex intuitionistic fuzzy sets are introduced and their properties are discussed in this section.

To distinguish complex intuitionistic fuzzy sets from intuitionistic fuzzy sets, we use $\bar{A}, \bar{B}, \ldots$ to denote intuitionistic fuzzy sets. And correspondingly, $A, B, C, D, \ldots$ are used to denote complex intuitionistic fuzzy sets. Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in U\}$, $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in U\}$, $C = \{\langle x, \mu_C(x), \nu_C(x)\rangle : x \in U\}$ and $D = \{\langle x, \mu_D(x), \nu_D(x)\rangle : x \in U\}$ be four complex intuitionistic fuzzy sets on U, then $\mu_A(x) = r_A(x) \cdot e^{i\omega_{\mu A}(x)}$, $\mu_B(x) = r_B(x) \cdot e^{i\omega_{\mu B}(x)}$, $\mu_C(x) = r_C(x) \cdot e^{i\omega_{\mu C}(x)}$, $\mu_D(x) = r_D(x) \cdot e^{i\omega_{\mu D}(x)}$, $\nu_A(x) = s_A(x) \cdot e^{i\omega_{\nu A}(x)}$, $\nu_B(x) = s_B(x) \cdot e^{i\omega_{\nu B}(x)}$, $\nu_C(x) = s_C(x) \cdot e^{i\omega_{\nu C}(x)}$ and $\nu_D(x) = s_D(x) \cdot e^{i\omega_{\nu D}(x)}$ denote their membership and non-membership functions, respectively. The collection of all complex intuitionistic fuzzy subsets is denoted by $\mathrm{CIF}^{\star}(U)$.

Definition 1 (Alkouri and Salleh 2012) A complex intuitionistic fuzzy set A, defined on an universe of discourse U, is characterized by membership and non-membership functions $\mu_A(x)$ and $\nu_A(x)$, respectively, that assign any element $x \in U$ a complex-valued grade of both membership and non-membership in A.

By Definition 1, the values of $\mu_A(x)$, $\nu_A(x)$ and their sum may receive all lying within the unit circle in the complex plane, and are on the form

$$\mu_A(x) = r_A(x) \cdot e^{i\bar{\omega}_{\mu A}(x)}$$

for membership function in A and

$$\nu_A(x) = s_A(x) \cdot e^{i\bar{\omega}_{\nu A}(x)}$$

for non-membership function in A, where $i = \sqrt{-1}$, each of $r_A(x)$ and $s_A(x)$ are real-valued functions and both belong to the interval $[0, 1]$ such that $0 \le r_A(x) + s_A(x) \le 1$, also $e^{i\bar{\omega}_{\mu A}(x)}$ and $e^{i\bar{\omega}_{\nu A}(x)}$ are periodic function whose periodic law and principal period are, respectively, 2π and $0 < \omega_{\mu A}(x), \omega_{\nu A}(x) \le 2\pi$, i.e., $\bar{\omega}_{\mu A}(x) = \omega_{\mu A}(x) + 2k\pi$, $\bar{\omega}_{\nu A}(x) = \omega_{\nu A}(x) + 2k\pi$, $k = 0, \pm 1, \pm 2, ...$, where $\omega_{\mu A}(x)$ and $\omega_{\nu A}(x)$ are the principal arguments. The principal arguments $\omega_{\mu A}(x)$ and $\omega_{\nu A}(x)$ will be used in the following text.

Let $\text{CIF}^\star(U)$ be the set of all complex intuitionistic fuzzy sets on U. The complex intuitionistic fuzzy set A may be represented as the set of ordered pairs

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in U\},$$

where $\mu_A(x) : U \to \{a | a \in C, |a| \le 1\}$, $\nu_A(x) : U \to \{a' | a' \in C, |a'| \le 1\}$ and $|\mu_A(x) + \nu_A(x)| \le 1$.

Definition 2 (1) A quasi-triangular norm T is a function $[0, 1]^2 \times [0, 1]^2 \to [0, 1]^2$ that satisfies the following conditions

(i) $T((1, 1), (1, 1)) = (1, 1)$;

(ii) $T((a, a'), (b, b')) = T((b, b'), (a, a'))$;

(iii) $T((a, a'), (b, b')) \le T((c, c'), (d, d'))$ whenever $a \le c$, $a' \le c'$ and $b \ge d$, $b' \ge d'$;

(iv) $T(T((a, a'), (b, b')), (c, c')) = T((a, a'), T((b, b'), (c, c')))$.

(2) A triangular norm T is a function $[0, 1]^2 \times [0, 1]^2 \to [0, 1]^2$ that satisfies the conditions (i)–(iv) and the following condition

(v) $T((0, 0), (0, 0)) = (0, 0)$.

We said T is an s-norm, if a triangular norm T satisfies

(vi) $T((a, a'), (0, 0)) = (a, a')$.

We said T is a t-norm, if a triangular norm T satisfies

(vii) $T((a, a'), (1, 1)) = (a, a')$.

(3) We said a binary function $\bar{T} : \text{CIF}^\star(U) \times \text{CIF}^\star(U) \to \text{CIF}^\star(U)$

$\bar{T}(A, B) \mapsto \langle \sup_{x \in U} T_1(\mu_A(x), \mu_B(x)) \cdot e^{i \sup_{x\in U} T_2(\omega_{\mu A}(x), \omega_{\mu B}(x))}$, $\inf_{x \in U} T_1(\nu_A(x), \nu_B(x)) \cdot e^{i \inf_{x\in U} T_2(\omega_{\nu A}(x), \omega_{\nu B}(x))}\rangle$ is a triangular norm if T_1 is a triangular norm and T_2 is a quasi-triangular norm; we said $\bar{T}$ is an s-norm if T_1 an s-norm; we said $\bar{T}$ is a t-norm if T_1 a t-norm.

Definition 3 (*Complex intuitionistic fuzzy union*) Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in U\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in U\}$ be two complex intuitionistic fuzzy sets on U. The complex intuitionistic fuzzy union of A and B, denoted by $A \cup B = \{\langle x, \mu_{A\cup B}(x), \nu_{A\cup B}(x)\rangle : x \in U\}$, where

$$\mu_{A\cup B}(x) = r_{A\cup B}(x) \cdot e^{i\omega_{\mu(A\cup B)}(x)} = \max(r_A(x), r_B(x)) \cdot e^{i \max(\omega_{\mu A}(x), \omega_{\mu B}(x))} \quad (1)$$

and

$$\nu_{A\cup B}(x) = s_{A\cup B}(x) \cdot e^{i\omega_{\nu(A\cup B)}(x)} = \min(s_A(x), s_B(x)) \cdot e^{i \min(\omega_{\nu A}(x), \omega_{\nu B}(x))}. \quad (2)$$

Example 1 Let $A = \frac{\langle 0.5\cdot e^{i1.2\pi}, 0.4\cdot e^{i0.8\pi}\rangle}{x} + \frac{\langle 0.4\cdot e^{i0.5\pi}, 0.6\cdot e^{i1.3\pi}\rangle}{y} + \frac{\langle 0.3\cdot e^{i2\pi}, 0.5\cdot e^{i1.5\pi}\rangle}{z}$, $B = \frac{\langle 0.6\cdot e^{i0.2\pi}, 0.3\cdot e^{i1.8\pi}\rangle}{x} + \frac{\langle 0.2\cdot e^{i0.5\pi}, 0.6\cdot e^{i0.5\pi}\rangle}{y} + \frac{\langle 0.7\cdot e^{i\pi}, 0.1\cdot e^{i0.9\pi}\rangle}{z}$, then $A \cup B = \frac{\langle 0.6\cdot e^{i1.2\pi}, 0.3\cdot e^{i0.8\pi}\rangle}{x} + \frac{\langle 0.4\cdot e^{i0.5\pi}, 0.6\cdot e^{i0.5\pi}\rangle}{y} + \frac{\langle 0.7\cdot e^{i2\pi}, 0.1\cdot e^{i0.9\pi}\rangle}{z}$.

Theorem 1 *The complex intuitionistic fuzzy union on CIF** (U) *is an s-norm.*

Proof Properties (i), (ii), (v) and (vi) can be easily verified from Definition 3. Here we only prove (iii) and (iv).

(iii) Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in U\}$, $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in U\}$, $C = \{\langle x, \mu_C(x), \nu_C(x)\rangle : x \in U\}$ and $D = \{\langle x, \mu_D(x), \nu_D(x)\rangle : x \in U\}$ be four complex intuitionistic fuzzy sets on U. Suppose $|\mu_A(x)| \le |\mu_C(x)|, \omega_{\mu A}(x) \le \omega_{\mu C}(x), |\nu_A(x)| \ge |\nu_C(x)|, \omega_{\nu A}(x) \ge \omega_{\nu C}(x)$ and $|\mu_B(x)| \le |\mu_D(x)|, \omega_{\mu B}(x) \le \omega_{\mu D}(x), |\nu_B(x)| \ge |\nu_D(x)|, \omega_{\nu B}(x) \ge \omega_{\nu D}(x)$. For any $x \in U$, we have

$$\begin{aligned} |\mu_{A\cup B}(x)| &= \max(r_A(x), r_B(x)) \\ &\le \max(r_C(x), r_D(x)) = |\mu_{C\cup D}(x)|, \\ \omega_{\mu(A\cup B)}(x) &= \max(\omega_{\mu A}(x), \omega_{\mu B}(x)) \\ &\le \max(\omega_{\mu C}(x), \omega_{\mu D}(x)) = \omega_{\mu(C\cup D)}(x), \end{aligned}$$

and

$$\begin{aligned} |\nu_{A\cup B}(x)| &= \min(s_A(x), s_B(x)) \\ &\ge \min(s_C(x), s_D(x)) = |\nu_{C\cup D}(x)|, \\ \omega_{\nu(A\cup B)}(x) &= \min(\omega_{\nu A}(x), \omega_{\nu B}(x)) \\ &\ge \min(\omega_{\nu C}(x), \omega_{\nu D}(x)) = \omega_{\nu(C\cup D)}(x). \end{aligned}$$

(iv) Suppose $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in U\}$, $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in U\}$ and $C = \{\langle x, \mu_C(x), \nu_C(x)\rangle :$

$x \in U\}$ be three complex intuitionistic fuzzy sets on U, then

$$\begin{aligned}
\mu_{(A\cup B)\cup C}(x) &= r_{(A\cup B)\cup C}(x)\cdot e^{i\omega_{\mu((A\cup B)\cup C)}(x)}\\
&= \max(r_{A\cup B}(x), r_C(x))\cdot e^{i\max(\omega_{\mu(A\cup B)}(x),\omega_{\mu C}(x))}\\
&= \max(\max(r_A(x), r_B(x)), r_C(x))\\
&\quad\cdot e^{i\max(\max(\omega_{\mu A}(x),\omega_{\mu B}(x)),\omega_{\mu C}(x))}\\
&= \max(r_A(x), \max(r_B(x), r_C(x))\\
&\quad\cdot e^{i\max(\omega_{\mu A}(x),\max(\omega_{\mu B}(x),\omega_{\mu C}(x))}\\
&= \mu_{A\cup(B\cup C)}(x).\\
\nu_{(A\cup B)\cup C}(x) &= s_{(A\cup B)\cup C}(x)\cdot e^{i\omega_{\nu((A\cup B)\cup C)}(x)}\\
&= \min(s_{A\cup B}(x), s_C(x))\cdot e^{i\min(\omega_{\nu(A\cup B)}(x),\omega_{\nu C}(x))}\\
&= \min(\min(s_A(x), s_B(x)), s_C(x))\\
&\quad\cdot e^{i\min(\min(\omega_{\nu A}(x),\omega_{\nu B}(x)),\omega_{\nu C}(x))}\\
&= \min(s_A(x), \min(s_B(x), s_C(x))\\
&\quad\cdot e^{i\min(\omega_{\nu A}(x),\min(\omega_{\nu B}(x),\omega_{\nu C}(x))}\\
&= \nu_{A\cup(B\cup C)}(x).
\end{aligned}$$

□

Corollary 1 *Let $C_\alpha \in CIF^\star(U), \alpha \in I$, $\mu_{C_\alpha}(x) = r_{C_\alpha}(x)\cdot e^{i\omega_{\mu C_\alpha}(x)}$ and $\nu_{C_\alpha}(x) = s_{C_\alpha}(x)\cdot e^{i\omega_{\nu C_\alpha}(x)}$ its membership and non-membership functions, respectively, where I is an arbitrary index set. Then $\cup_{\alpha\in I} C_\alpha \in CIF^\star(U)$, and its membership and non-membership functions are*

$$\mu_{\cup_{\alpha\in I} C_\alpha}(x) = \sup_{\alpha\in I} r_{C_\alpha}(x)\cdot e^{i\sup_{\alpha\in I}\omega_{\mu C_\alpha}(x)}$$

and

$$\nu_{\cup_{\alpha\in I} C_\alpha}(x) = \inf_{\alpha\in I} s_{C_\alpha}(x)\cdot e^{i\inf_{\alpha\in I}\omega_{\nu C_\alpha}(x)}.$$

Definition 4 (*Complex intuitionistic fuzzy intersection*) Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in U\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in U\}$ be two complex intuitionistic fuzzy sets on U. The complex intuitionistic fuzzy intersection of A and B, denoted by $A \cap B = \{\langle x, \mu_{A\cap B}(x), \nu_{A\cap B}(x)\rangle : x \in U\}$, where

$$\begin{aligned}
\mu_{A\cap B}(x) &= r_{A\cap B}(x)\cdot e^{i\omega_{\mu(A\cap B)}(x)}\\
&= \min(r_A(x), r_B(x))\cdot e^{i\min(\omega_{\mu A}(x),\omega_{\mu B}(x))} \qquad (3)
\end{aligned}$$

and

$$\begin{aligned}
\nu_{A\cap B}(x) &= s_{A\cap B}(x)\cdot e^{i\omega_{\nu(A\cap B)}(x)}\\
&= \max(s_A(x), s_B(x))\cdot e^{i\max(\omega_{\nu A}(x),\omega_{\nu B}(x))}. \qquad (4)
\end{aligned}$$

Example 2 Let $A = \frac{\langle 0.5\cdot e^{i1.2\pi}, 0.4\cdot e^{i0.8\pi}\rangle}{x} + \frac{\langle 0.4\cdot e^{i0.5\pi}, 0.6\cdot e^{i1.3\pi}\rangle}{y} + \frac{\langle 0.3\cdot e^{i2\pi}, 0.5\cdot e^{i1.5\pi}\rangle}{z}$, $B = \frac{\langle 0.6\cdot e^{i0.2\pi}, 0.3\cdot e^{i1.8\pi}\rangle}{x} + \frac{\langle 0.2\cdot e^{i0.5\pi}, 0.6\cdot e^{i0.5\pi}\rangle}{y} + \frac{\langle 0.7\cdot e^{i\pi}, 0.1\cdot e^{i0.9\pi}\rangle}{z}$, then $A \cap B = \frac{\langle 0.5\cdot e^{i0.2\pi}, 0.4\cdot e^{i1.8\pi}\rangle}{x} + \frac{\langle 0.2\cdot e^{i0.5\pi}, 0.6\cdot e^{i1.3\pi}\rangle}{y} + \frac{\langle 0.3\cdot e^{i\pi}, 0.5\cdot e^{i1.5\pi}\rangle}{z}$.

Theorem 2 *The complex intuitionistic fuzzy intersection on $CIF^\star(U)$ is a t-norm.*

Proof This proof is similar to the proof of Theorem 1. □

Corollary 2 *Let $C_\alpha \in CIF^\star(U), \alpha \in I$, $\mu_{C_\alpha}(x) = r_{C_\alpha}(x)\cdot e^{i\omega_{\mu C_\alpha}(x)}$ and $\nu_{C_\alpha}(x) = s_{C_\alpha}(x)\cdot e^{i\omega_{\nu C_\alpha}(x)}$ its membership and non-membership functions, where I is an arbitrary index set. Then $\cap_{\alpha\in I} C_\alpha \in CIF^\star(U)$, and its membership and non-membership functions are*

$$\mu_{\cap_{\alpha\in I} C_\alpha}(x) = \inf_{\alpha\in I} r_{C_\alpha}(x)\cdot e^{i\inf_{\alpha\in I}\omega_{\mu C_\alpha}(x)}$$

and

$$\nu_{\cap_{\alpha\in I} C_\alpha}(x) = \sup_{\alpha\in I} s_{C_\alpha}(x)\cdot e^{i\sup_{\alpha\in I}\omega_{\nu C_\alpha}(x)}.$$

Corollary 3 *Let $C_{\alpha\beta} \in CIF^\star(U), \alpha \in I_1, \beta \in I_2$, $\mu_{C_{\alpha\beta}}(x) = r_{C_{\alpha\beta}}(x)\cdot e^{i\omega_{\mu C_{\alpha\beta}}(x)}$ and $\nu_{C_{\alpha\beta}}(x) = s_{C_{\alpha\beta}}(x)\cdot e^{i\omega_{\nu C_{\alpha\beta}}(x)}$ its membership and non-membership functions, respectively, where I_1 and I_2 are two arbitrary index sets. Then $\cup_{\alpha\in I_1}\cap_{\beta\in I_2} C_{\alpha\beta}, \cap_{\alpha\in I_1}\cup_{\beta\in I_2} C_{\alpha\beta} \in CIF^\star(U)$, and their membership and non-membership functions are*

$$\mu_{\cup_{\alpha\in I_1}\cap_{\beta\in I_2} C_{\alpha\beta}}(x) = \sup_{\alpha\in I_1}\inf_{\beta\in I_2} r_{C_{\alpha\beta}}(x)\cdot e^{i\sup_{\alpha\in I_1}\inf_{\beta\in I_2}\omega_{\mu C_{\alpha\beta}}(x)},$$

$$\mu_{\cap_{\alpha\in I_1}\cup_{\beta\in I_2} C_{\alpha\beta}}(x) = \inf_{\alpha\in I_1}\sup_{\beta\in I_2} r_{C_{\alpha\beta}}(x)\cdot e^{i\inf_{\alpha\in I_1}\sup_{\beta\in I_2}\omega_{\mu C_{\alpha\beta}}(x)},$$

and

$$\nu_{\cup_{\alpha\in I_1}\cap_{\beta\in I_2} C_{\alpha\beta}}(x) = \inf_{\alpha\in I_1}\sup_{\beta\in I_2} s_{C_{\alpha\beta}}(x)\cdot e^{i\inf_{\alpha\in I_1}\sup_{\beta\in I_2}\omega_{\nu C_{\alpha\beta}}(x)},$$

$$\nu_{\cap_{\alpha\in I_1}\cup_{\beta\in I_2} C_{\alpha\beta}}(x) = \sup_{\alpha\in I_1}\inf_{\beta\in I_2} s_{C_{\alpha\beta}}(x)\cdot e^{i\sup_{\alpha\in I_1}\inf_{\beta\in I_2}\omega_{\nu C_{\alpha\beta}}(x)}.$$

Corollary 4 *Let $C_k \in CIF^\star(U), k = 1, 2, ...,$ $\mu_{C_k}(x) = r_{C_k}(x)\cdot e^{i\omega_{\mu C_k}(x)}$ and $\nu_{C_k}(x) = s_{C_k}(x)\cdot e^{i\omega_{\nu C_k}(x)}$ its membership and non-membership functions, respectively. Then*

$$\overline{\lim}_{n\to\infty} C_n = \cap_{n=1}^{\infty}\cup_{k=n}^{\infty} C_k,$$
$$\underline{\lim}_{n\to\infty} C_n = \cup_{n=1}^{\infty}\cap_{k=n}^{\infty} C_k \in CIF^\star(U),$$

and their membership and non-membership functions are

$$\mu_{\overline{\lim}_{n\to\infty} C_k}(x) = \inf_{n\geq 1}\sup_{k\geq n} r_{C_k}(x)\cdot e^{i\inf_{n\geq 1}\sup_{k\geq n}\omega_{\mu C_k}(x)},$$

$$\mu_{\underline{\lim}_{n\to\infty} C_k}(x) = \sup_{n\geq 1}\inf_{k\geq n} r_{C_k}(x)\cdot e^{i\sup_{n\geq 1}\inf_{k\geq n}\omega_{\mu C_k}(x)},$$

and

$$\nu_{\overline{\lim}_{n\to\infty} C_k}(x) = \sup_{n\geq 1}\inf_{k\geq n} s_{C_k}(x)\cdot e^{i\sup_{n\geq 1}\inf_{k\geq n}\omega_{\nu C_k}(x)},$$

$$\nu_{\underline{\lim}_{n\to\infty} C_k}(x) = \inf_{n\geq 1}\sup_{k\geq n} s_{C_k}(x)\cdot e^{i\inf_{n\geq 1}\sup_{k\geq n}\omega_{\nu C_k}(x)}.$$

Definition 5 (*Complex intuitionistic fuzzy complement*) Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in U\}$ be a complex intuitionistic fuzzy set on U. The complex intuitionistic fuzzy complement of A, denoted by $\overline{A}$ and defined as follows.

(i) $\overline{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle\}$;
(ii) $\overline{A} = \{\langle x, \mu_{\bar{A}}(x), \nu_{\bar{A}}(x)\rangle\}$, where $\mu_{\bar{A}}(x) = r_{\bar{A}}(x)\cdot e^{i\omega_{\mu\bar{A}}(x)}$, $\nu_{\bar{A}}(x) = s_{\bar{A}}(x)\cdot e^{i\omega_{\nu\bar{A}}(x)}$ and $r_{\bar{A}}(x) = 1 - r_A(x)$, $s_{\bar{A}}(x) = 1 - s_A(x)$,

$$\omega_{\mu\bar{A}}(x) = \begin{cases} \omega_{\mu A}(x), \\ 2\pi - \omega_{\mu A}(x) = -\omega_{\mu A}(x), \\ \omega_{\mu A}(x) + \pi, \end{cases}$$

and

$$\omega_{\nu\bar{A}}(x) = \begin{cases} \omega_{\nu A}(x), \\ 2\pi - \omega_{\nu A}(x) = -\omega_{\nu A}(x), \\ \omega_{\nu A}(x) + \pi. \end{cases}$$

The following example uses the first way of Definition 5 to calculate the complement of the complex intuitionistic fuzzy set A. Note that if the second way is used, the corresponding results also can be obtained.

Example 3 Let $A = \frac{\langle 0.5\cdot e^{i1.2\pi}, 0.4\cdot e^{i0.8\pi}\rangle}{x} + \frac{\langle 0.4\cdot e^{i0.5\pi}, 0.6\cdot e^{i1.3\pi}\rangle}{y} + \frac{\langle 0.3\cdot e^{i2\pi}, 0.5\cdot e^{i1.5\pi}\rangle}{z}$, then $\overline{A} = \frac{\langle 0.4\cdot e^{i0.8\pi}, 0.5\cdot e^{i1.2\pi}\rangle}{x} + \frac{\langle 0.6\cdot e^{i1.3\pi}, 0.4\cdot e^{i0.5\pi}\rangle}{y} + \frac{\langle 0.5\cdot e^{i1.5\pi}, 0.3\cdot e^{i2\pi}\rangle}{z}$.

Proposition 3 *Let A, B and C be three complex intuitionistic fuzzy sets on U, then the following propositions hold*

(i) $A\cup A = A$, $A\cap A = A$;
(ii) $A\cup B = B\cup A$, $A\cap B = B\cap A$;
(iii) $(A\cup B)\cap C = (A\cap C)\cup(B\cap C)$, $(A\cap B)\cup C = (A\cup C)\cap(B\cup C)$;
(iv) $A\cap(B\cap C) = (A\cap B)\cap C$, $A\cup(B\cup C) = (A\cup B)\cup C$;
(v) $\overline{(A\cap B)} = \overline{A}\cup\overline{B}$, $\overline{(A\cup B)} = \overline{A}\cap\overline{B}$;
(vi) $\overline{\overline{A}} = A$.

Proof Here we only prove (iii), (iv), (v) and (vi). Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x\in U\}$, $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in U\}$ and $C = \{\langle x, \mu_C(x), \nu_C(x)\rangle : x \in U\}$ be three complex intuitionistic fuzzy sets on U. The complement of A and B are $\overline{A} = \langle x, \nu_A(x), \mu_A(x)\rangle$ and $\overline{B} = \langle x, \nu_B(x), \mu_B(x)\rangle$, respectively. Then

(iii) First of all, we prove that $(A\cup B)\cap C = (A\cap C)\cup(B\cap C)$, since

$$\begin{aligned}
\mu_{(A\cup B)\cap C}(x) &= r_{(A\cup B)\cap C}(x)\cdot e^{i\omega_{\mu((A\cup B)\cap C)}(x)} \\
&= \min(r_{A\cup B}(x), r_C(x))\cdot e^{i\min(\omega_{\mu(A\cup B)}(x),\omega_{\mu C}(x))} \\
&= \min(\max(r_A(x), r_B(x)), r_C(x)) \\
&\quad\cdot e^{i\min(\max(\omega_{\mu A}(x),\omega_{\mu B}(x)),\omega_{\mu C}(x))} \\
&= \max(\min(r_A(x), r_C(x)), \min(r_B(x), r_C(x))) \\
&\quad\cdot e^{i\max(\min(\omega_{\mu A}(x),\omega_{\mu C}(x)),\min(\omega_{\mu B}(x),\omega_{\mu C}(x)))} \\
&= r_{(A\cap C)\cup(B\cap C)}(x)\cdot e^{i\omega_{\mu((A\cap C)\cup(B\cap C))}(x)} \\
&= \mu_{(A\cap C)\cup(B\cap C)}(x). \\
\nu_{(A\cup B)\cap C}(x) &= s_{(A\cup B)\cap C}(x)\cdot e^{i\omega_{\nu((A\cup B)\cap C)}(x)} \\
&= \max(s_{A\cup B}(x), s_C(x))\cdot e^{i\max(\omega_{\nu(A\cup B)}(x),\omega_{\nu C}(x))} \\
&= \max(\min(s_A(x), s_B(x)), s_C(x)) \\
&\quad\cdot e^{i\max(\min(\omega_{\nu A}(x),\omega_{\nu B}(x)),\omega_{\nu C}(x))} \\
&= \min(\max(s_A(x), s_C(x)), \max(s_B(x), s_C(x))) \\
&\quad\cdot e^{i\min(\max(\omega_{\nu A}(x),\omega_{\nu C}(x)),\max(\omega_{\nu B}(x),\omega_{\mu C}(x)))} \\
&= s_{(A\cap C)\cup(B\cap C)}(x)\cdot e^{i\omega_{\nu((A\cap C)\cup(B\cap C))}(x)} \\
&= \nu_{(A\cap C)\cup(B\cap C)}(x).
\end{aligned}$$

It implies that $(A\cup B)\cap C = (A\cap C)\cup(B\cap C)$.

Similarly, we can prove that $(A\cap B)\cup C = (A\cup C)\cap(B\cup C)$.

(iv) First of all, we prove that $A\cap(B\cap C) = (A\cap B)\cap C$, since

$$\begin{aligned}
\mu_{A\cap(B\cap C)}(x) &= r_{A\cap(B\cap C)}(x)\cdot e^{i\omega_{\mu(A\cap(B\cap C))}(x)} \\
&= \min(r_A(x), r_{B\cap C}(x))\cdot e^{i\min(\omega_{\mu A}(x),\omega_{\mu(B\cap C)}(x))} \\
&= \min(r_A(x), \min(r_B(x), r_C(x))) \\
&\quad\cdot e^{i\min(\omega_{\mu A}(x),\min(\omega_{\mu B}(x),\omega_{\mu C}(x)))} \\
&= \min(\min(r_A(x), r_B(x)), r_C(x)) \\
&\quad\cdot e^{i\min(\min(\omega_{\mu A}(x),\omega_{\mu B}(x)),\omega_{\mu C}(x))} \\
&= r_{(A\cap B)\cap C}(x)\cdot e^{i\omega_{\mu((A\cap B)\cap C)}(x)} \\
&= \mu_{(A\cap B)\cap C}(x). \\
\nu_{A\cap(B\cap C)}(x) &= s_{A\cap(B\cap C)}(x)\cdot e^{i\omega_{\nu(A\cap(B\cap C))}(x)} \\
&= \max(s_A(x), s_{B\cap C}(x))\cdot e^{i\max(\omega_{\nu A}(x),\omega_{\nu(B\cap C)}(x))} \\
&= \max(s_A(x), \max(s_B(x), s_C(x))) \\
&\quad\cdot e^{i\max(\omega_{\nu A}(x),\max(\omega_{\nu B}(x),\omega_{\nu C}(x)))} \\
&= \max(\max(s_A(x), s_B(x)), s_C(x))
\end{aligned}$$

$$\cdot e^{i\max(\max(\omega_{\nu A}(x),\omega_{\nu B}(x)),\omega_{\nu C}(x))}$$
$$= s_{(A\cap B)\cap C}(x)\cdot e^{i\omega_{\nu((A\cap B)\cap C)}(x)}$$
$$= \nu_{(A\cap B)\cap C}(x).$$

It implies that $A\cap(B\cap C)=(A\cap B)\cap C$.

Similarly, we can prove that $(A\cup B)\cup C = A\cup(B\cup C)$.

(v) First of all, we prove that $\overline{(A\cap B)}=\overline{A}\cup\overline{B}$, since

$$\begin{aligned}
\mu_{\overline{(A\cap B)}}(x) &= \nu_{A\cap B}(x)\\
&= s_{A\cap B}(x)\cdot e^{i\omega_{\nu(A\cap B)}(x)}\\
&= \max(s_A(x),s_B(x))\cdot e^{i\max(\omega_{\nu A}(x),\omega_{\mu B}(x))}\\
&= \mu_{\overline{A}\cup\overline{B}}(x).\\
\nu_{\overline{(A\cap B)}}(x) &= \mu_{A\cap B}(x)\\
&= r_{A\cap B}(x)\cdot e^{i\omega_{\mu(A\cap B)}(x)}\\
&= \min(r_A(x),r_B(x))\cdot e^{i\min(\omega_{\mu A}(x),\omega_{\mu B}(x))}\\
&= \nu_{\overline{A}\cup\overline{B}}(x).
\end{aligned}$$

It implies that $\overline{(A\cap B)}=\overline{A}\cup\overline{B}$.

Similarly, we can prove that $\overline{(A\cup B)}=\overline{A}\cap\overline{B}$.

(vi) $\mu_{\overline{\overline{A}}}(x)=\nu_{\overline{A}}(x)=\mu_A(x)$ and $\nu_{\overline{\overline{A}}}(x)=\mu_{\overline{A}}(x)=\nu_A(x)$, i.e., $\overline{\overline{A}}=A$. □

Definition 6 (*Complex intuitionistic fuzzy product*) Let $A=\{\langle x,\mu_A(x),\nu_A(x)\rangle : x\in U\}$ and $B=\{\langle x,\mu_B(x),\nu_B(x)\rangle : x\in U\}$ be two complex intuitionistic fuzzy sets on U. The complex intuitionistic fuzzy product of A and B, denoted by $A\circ B=\{\langle x,\mu_{A\circ B}(x),\nu_{A\circ B}(x)\rangle : x\in U\}$, where

$$\begin{aligned}
\mu_{A\circ B}(x) &= r_{A\circ B}(x)\cdot e^{i\omega_{\mu(A\circ B)}(x)}\\
&= (r_A(x)\cdot r_B(x))\cdot e^{i2\pi(\frac{\omega_{\mu A}(x)}{2\pi}\cdot\frac{\omega_{\mu B}(x)}{2\pi})} \qquad (5)
\end{aligned}$$

and

$$\begin{aligned}
\nu_{A\circ B}(x) &= s_{A\circ B}(x)\cdot e^{i\omega_{\nu(A\circ B)}(x)}\\
&= (s_A(x)+s_B(x)-s_A(x)\cdot s_B(x))\\
&\quad\cdot e^{i2\pi(\frac{\omega_{\nu A}(x)}{2\pi}+\frac{\omega_{\nu B}(x)}{2\pi}-\frac{\omega_{\nu A}(x)}{2\pi}\cdot\frac{\omega_{\nu B}(x)}{2\pi})}. \qquad (6)
\end{aligned}$$

Example 4 Let $A=\frac{\langle 0.5\cdot e^{i1.2\pi},0.4\cdot e^{i0.8\pi}\rangle}{x}+\frac{\langle 0.4\cdot e^{i0.5\pi},0.6\cdot e^{i1.3\pi}\rangle}{y}+\frac{\langle 0.3\cdot e^{i2\pi},0.5\cdot e^{i1.5\pi}\rangle}{z}$, $B=\frac{\langle 0.6\cdot e^{i0.2\pi},0.3\cdot e^{i1.8\pi}\rangle}{x}+\frac{\langle 0.2\cdot e^{i0.5\pi},0.6\cdot e^{i0.5\pi}\rangle}{y}+\frac{\langle 0.7\cdot e^{i\pi},0.1\cdot e^{i0.9\pi}\rangle}{z}$, then $A\circ B=\frac{\langle 0.3\cdot e^{i0.12\pi},0.58\cdot e^{i1.88\pi}\rangle}{x}+\frac{\langle 0.08\cdot e^{i0.125\pi},0.84\cdot e^{i1.475\pi}\rangle}{y}+\frac{\langle 0.21\cdot e^{i\pi},0.55\cdot e^{i1.725\pi}\rangle}{z}$.

Theorem 4 *The complex intuitionistic fuzzy product on $CIF^\star(U)$ is a t-norm.*

Proof Properties (i), (ii), (v) and (vii) can be easily verified from Definition 6. Here we only prove (iii) and (iv).

(iii) Let $A=\{\langle x,\mu_A(x),\nu_A(x)\rangle : x\in U\}$, $B=\{\langle x,\mu_B(x),\nu_B(x)\rangle : x\in U\}$, $C=\{\langle x,\mu_C(x),\nu_C(x)\rangle : x\in U\}$ and $D=\{\langle x,\mu_D(x),\nu_D(x)\rangle : x\in U\}$ be four complex intuitionistic fuzzy sets on U. Suppose $|\mu_A(x)|\le|\mu_C(x)|$, $\omega_{\mu A}(x)\le\omega_{\mu C}(x)$, $|\nu_A(x)|\ge|\nu_C(x)|$, $\omega_{\nu A}(x)\ge\omega_{\nu C}(x)$ and $|\mu_B(x)|\le|\mu_D(x)|$, $\omega_{\mu B}(x)\le\omega_{\mu D}(x)$, $|\nu_B(x)|\ge|\nu_D(x)|$, $\omega_{\nu B}(x)\ge\omega_{\nu D}(x)$. For any $x\in U$, we have

$$\begin{aligned}
|\mu_{A\circ B}(x)| &= |r_A(x)|\cdot|r_B(x)|\le|r_C(x)|\cdot|r_D(x)|\\
&= |\mu_{C\circ D}(x)|,\\
\omega_{\mu(A\circ B)}(x) &= 2\pi\left(\frac{\omega_{\mu A}(x)}{2\pi}\cdot\frac{\omega_{\mu B}(x)}{2\pi}\right)\\
&\le 2\pi\left(\frac{\omega_{\mu C}(x)}{2\pi}\cdot\frac{\omega_{\mu D}(x)}{2\pi}\right)\\
&= \omega_{\mu(C\circ D)}(x),
\end{aligned}$$

and

$$\begin{aligned}
|\nu_{A\circ B}(x)| &= |s_A(x)|+|s_B(x)|-|s_A(x)|\cdot|s_B(x)|\\
&\ge |s_C(x)|+|s_D(x)|-|s_C(x)|\cdot|s_D(x)|\\
&= |\nu_{C\circ D}(x)|,\\
\omega_{\nu(A\circ B)}(x) &= 2\pi\left(\frac{\omega_{\nu A}(x)}{2\pi}+\frac{\omega_{\nu B}(x)}{2\pi}-\frac{\omega_{\nu A}(x)}{2\pi}\cdot\frac{\omega_{\nu B}(x)}{2\pi}\right)\\
&\ge 2\pi\left(\frac{\omega_{\nu C}(x)}{2\pi}+\frac{\omega_{\nu D}(x)}{2\pi}-\frac{\omega_{\nu C}(x)}{2\pi}\cdot\frac{\omega_{\nu D}(x)}{2\pi}\right)\\
&= \omega_{\nu(C\circ D)}(x).
\end{aligned}$$

(iv) Suppose $A=\{\langle x,\mu_A(x),\nu_A(x)\rangle : x\in U\}$, $B=\{\langle x,\mu_B(x),\nu_B(x)\rangle : x\in U\}$ and $C=\{\langle x,\mu_C(x),\nu_C(x)\rangle : x\in U\}$ be three complex intuitionistic fuzzy sets on U. Then

$$\begin{aligned}
\mu_{(A\circ B)\circ C}(x) &= r_{(A\circ B)\circ C}(x)\cdot e^{i\omega_{\mu((A\circ B)\circ C)}(x)}\\
&= (r_{A\circ B}(x)\cdot r_C(x))\cdot e^{i2\pi\left(\frac{\omega_{\mu(A\circ B)}(x)}{2\pi}\cdot\frac{\omega_{\mu C}(x)}{2\pi}\right)}\\
&= ((r_A(x)\cdot r_B(x))\cdot r_C(x))\\
&\quad\cdot e^{i2\pi\left(\frac{2\pi\left(\frac{\omega_{\mu A}(x)}{2\pi}\cdot\frac{\omega_{\mu B}(x)}{2\pi}\right)}{2\pi}\cdot\frac{\omega_{\mu C}(x)}{2\pi}\right)}\\
&= (r_A(x)\cdot(r_B(x)\cdot r_C(x)))\\
&\quad\cdot e^{i2\pi\left(\frac{\omega_{\mu A}(x)}{2\pi}\cdot\frac{2\pi\left(\frac{\omega_{\mu B}(x)}{2\pi}\cdot\frac{\omega_{\mu C}(x)}{2\pi}\right)}{2\pi}\right)}\\
&= \mu_{A\circ(B\circ C)}(x).
\end{aligned}$$

□

$$\begin{aligned}
\nu_{(A\circ B)\circ C}(x) &= s_{(A\circ B)\circ C}(x)\cdot e^{i\omega_{\nu((A\circ B)\circ C)}(x)}\\
&= (s_{A\circ B}(x)+s_C(x)-s_{A\circ B}(x)\cdot s_C(x))\cdot e^{i2\pi\left(\frac{\omega_{\nu(A\circ B)}(x)}{2\pi}+\frac{\omega_{\nu C}(x)}{2\pi}-\frac{\omega_{\nu(A\circ B)}(x)}{2\pi}\cdot\frac{\omega_{\nu C}(x)}{2\pi}\right)}\\
&= (s_A(x)+s_B(x)-s_A(x)\cdot s_B(x)+s_C(x)-(s_A(x)+s_B(x)-s_A(x)\cdot s_B(x))\cdot s_C(x))\\
&\quad\cdot e^{i2\pi\left(\frac{2\pi\left(\frac{\omega_{\nu A}(x)}{2\pi}+\frac{\omega_{\nu B}(x)}{2\pi}-\frac{\omega_{\nu A}(x)}{2\pi}\cdot\frac{\omega_{\nu B}(x)}{2\pi}\right)}{2\pi}+\frac{\omega_{\nu C}(x)}{2\pi}-\frac{2\pi\left(\frac{\omega_{\nu A}(x)}{2\pi}+\frac{\omega_{\nu B}(x)}{2\pi}-\frac{\omega_{\nu A}(x)}{2\pi}\cdot\frac{\omega_{\nu B}(x)}{2\pi}\right)}{2\pi}\cdot\frac{\omega_{\nu C}(x)}{2\pi}\right)}\\
&= (s_A(x)+s_B(x)+s_C(x)-s_B(x)\cdot s_C(x)-s_A(x)\cdot(s_B(x)+s_C(x)-s_B(x)\cdot s_C(x)))\\
&\quad\cdot e^{i2\pi\left(\frac{\omega_{\nu A}(x)}{2\pi}+\frac{2\pi\left(\frac{\omega_{\nu B}(x)}{2\pi}+\frac{\omega_{\nu C}(x)}{2\pi}-\frac{\omega_{\nu B}(x)}{2\pi}\cdot\frac{\omega_{\nu C}(x)}{2\pi}\right)}{2\pi}-\frac{\omega_{\nu A}(x)}{2\pi}\cdot\frac{2\pi\left(\frac{\omega_{\nu B}(x)}{2\pi}+\frac{\omega_{\nu C}(x)}{2\pi}-\frac{\omega_{\nu B}(x)}{2\pi}\cdot\frac{\omega_{\nu C}(x)}{2\pi}\right)}{2\pi}\right)}\\
&= s_{A\circ(B\circ C)}(x)\cdot e^{i\omega_{\nu(A\circ(B\circ C))}(x)}\\
&= \nu_{A\circ(B\circ C)}(x).
\end{aligned}$$

Corollary 5 *Let $C_\alpha \in CIF^\star(U), \alpha \in I$, $\mu_{C_\alpha}(x) = r_{C_\alpha}(x)\cdot e^{i\omega_{\mu C_\alpha}(x)}$ and $\nu_{C_\alpha}(x) = s_{C_\alpha}(x)\cdot e^{i\omega_{\nu C_\alpha}(x)}$ its membership and non-membership functions, where I is an arbitrary index set. Then $\prod_{\alpha\in I} C_\alpha = C_1\circ C_2\circ\cdots\circ C_\alpha \in CIF^\star(U)$, and its membership and non-membership functions are*

$$\mu_{\prod_{\alpha\in I} C_\alpha}(x) = (r_{C_1}(x)\cdot r_{C_2}(x)\cdots r_{C_\alpha}(x))\cdot e^{i2\pi\left(\frac{\omega_{\mu C_1}(x)}{2\pi}\cdot\frac{\omega_{\mu C_2}(x)}{2\pi}\cdots\frac{\omega_{\mu C_\alpha}(x)}{2\pi}\right)}$$

and

$$\begin{aligned}
&\nu_{\prod_{\alpha\in I} C_\alpha}(x)\\
&= [s_{C_1}(x)+s_{C_2}(x)+\cdots+s_{C_\alpha}(x)-\cdots+(-1)^{\alpha-1}(s_{C_1}(x)\cdot s_{C_2}(x)\cdots s_{C_\alpha}(x))]\\
&\quad\cdot e^{i2\pi\left[\left(\frac{\omega_{\mu C_1}(x)}{2\pi}+\frac{\omega_{\mu C_2}(x)}{2\pi}+\cdots+\frac{\omega_{\mu C_\alpha}(x)}{2\pi}\right)-\cdots+\frac{(-1)^{\alpha-1}}{(2\pi)^2}\left(\frac{\omega_{\nu C_1}(x)}{2\pi}\cdot\frac{\omega_{\nu C_2}(x)}{2\pi}\cdots\frac{\omega_{\nu C_\alpha}(x)}{2\pi}\right)\right]}.
\end{aligned}$$

Definition 7 (*Complex intuitionistic fuzzy Cartesian product*) Let $A_n, n = 1, 2, ..., N$ be N complex intuitionistic fuzzy sets on U, $\mu_{A_n}(x) = r_{A_n}(x)\cdot e^{i\omega_{\mu A_n}(x)}$, $\nu_{A_n}(x) = s_{A_n}(x)\cdot e^{i\omega_{\nu A_n}(x)}$ their membership and non-membership functions, respectively. The complex intuitionistic fuzzy Cartesian product of $A_n, n = 1, 2, ...N$, denoted by $A_1\times A_2\times\cdots\times A_N = \{\langle x, \mu_{A_1\times A_2\times\cdots\times A_N}(x), \nu_{A_1\times A_2\times\cdots\times A_N}(x)\rangle : x\in U\}$, where

$$\begin{aligned}
&\mu_{A_1\times A_2\times\cdots\times A_N}(x)\\
&= r_{A_1\times A_2\times\cdots\times A_N}(x)\cdot e^{i\omega_{\mu(A_1\times A_2\times\cdots\times A_N)}(x)}\\
&= \min(r_{A_1}(x), r_{A_2}(x), \ldots, r_{A_N}(x))\\
&\quad\cdot e^{i\min\left(\omega_{\mu A_1}(x),\omega_{\mu A_2}(x),\ldots,\omega_{\mu A_N}(x)\right)}
\end{aligned}\tag{7}$$

and

$$\begin{aligned}
&\nu_{A_1\times A_2\times\cdots\times A_N}(x)\\
&= s_{A_1\times A_2\times\cdots\times A_N}(x)\cdot e^{i\omega_{\nu(A_1\times A_2\times\cdots\times A_N)}(x)}\\
&= \max(s_{A_1}(x), s_{A_2}(x), \ldots, s_{A_N}(x))\\
&\quad\cdot e^{i\max\left(\omega_{\nu A_1}(x),\omega_{\nu A_2}(x),\ldots,\omega_{\nu A_N}(x)\right)}.
\end{aligned}\tag{8}$$

Example 5 Let $A = \frac{\langle 0.5\cdot e^{i1.2\pi}, 0.4\cdot e^{i0.8\pi}\rangle}{x} + \frac{\langle 0.4\cdot e^{i0.5\pi}, 0.6\cdot e^{i1.3\pi}\rangle}{y} + \frac{\langle 0.3\cdot e^{i2\pi}, 0.5\cdot e^{i1.5\pi}\rangle}{z}$, $B = \frac{\langle 0.6\cdot e^{i0.2\pi}, 0.3\cdot e^{i1.8\pi}\rangle}{x} + \frac{\langle 0.2\cdot e^{i0.5\pi}, 0.6\cdot e^{i0.5\pi}\rangle}{y} + \frac{\langle 0.7\cdot e^{i\pi}, 0.1\cdot e^{i0.9\pi}\rangle}{z}$, then $A\times B = \frac{\langle 0.5\cdot e^{i0.2\pi}, 0.4\cdot e^{i1.8\pi}\rangle}{(x,x)} + \frac{\langle 0.2\cdot e^{i0.5\pi}, 0.6\cdot e^{i0.8\pi}\rangle}{(x,y)} + \frac{\langle 0.5\cdot e^{i\pi}, 0.4\cdot e^{i0.9\pi}\rangle}{(x,z)} + \frac{\langle 0.4\cdot e^{i0.2\pi}, 0.6\cdot e^{i1.8\pi}\rangle}{(y,x)} + \frac{\langle 0.2\cdot e^{i0.5\pi}, 0.6\cdot e^{i1.3\pi}\rangle}{(y,y)} + \frac{\langle 0.4\cdot e^{i0.5\pi}, 0.6\cdot e^{i1.3\pi}\rangle}{(y,z)} + \frac{\langle 0.3e^{i0.2\pi}, 0.5e^{i1.8\pi}\rangle}{(z,x)} + \frac{\langle 0.2e^{i0.5\pi}, 0.6e^{i1.5\pi}\rangle}{(z,y)} + \frac{\langle 0.3e^{i\pi}, 0.5e^{i1.5\pi}\rangle}{(z,z)}$.

Definition 8 (*Complex intuitionistic fuzzy probabilistic sum*) Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x\in U\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x\in U\}$ be two complex intuitionistic fuzzy sets on U. The complex intuitionistic fuzzy probabilistic sum of A and B, denoted by $A\widehat{+}B = \{\langle x, \mu_{A\widehat{+}B}(x), \nu_{A\widehat{+}B}(x)\rangle : x \in U\}$, where

$$\mu_{A\widehat{+}B}(x) = r_{A\widehat{+}B}(x) \cdot e^{i\omega_{\mu(A\widehat{+}B)}(x)} = (r_A(x) + r_B(x) - r_A(x) \cdot r_B(x)) \cdot e^{i2\pi\left(\frac{\omega_{\mu A}(x)}{2\pi} + \frac{\omega_{\mu B}(x)}{2\pi} - \frac{\omega_{\mu A}(x)}{2\pi} \cdot \frac{\omega_{\mu B}(x)}{2\pi}\right)} \quad (9)$$

and

$$\nu_{A\widehat{+}B}(x) = s_{A\widehat{+}B}(x) \cdot e^{i\omega_{\nu(A\widehat{+}B)}(x)} = (s_A(x) \cdot s_B(x)) \cdot e^{i2\pi\left(\frac{\omega_{\nu A}(x)}{2\pi} \cdot \frac{\omega_{\nu B}(x)}{2\pi}\right)}. \quad (10)$$

Example 6 Let $A = \frac{\langle 0.5\cdot e^{i1.2\pi}, 0.4\cdot e^{i0.8\pi}\rangle}{x} + \frac{\langle 0.4\cdot e^{i0.5\pi}, 0.6\cdot e^{i1.3\pi}\rangle}{y} + \frac{\langle 0.3\cdot e^{i2\pi}, 0.5\cdot e^{i1.5\pi}\rangle}{z}$, $B = \frac{\langle 0.6\cdot e^{i0.2\pi}, 0.3\cdot e^{i1.8\pi}\rangle}{x} + \frac{\langle 0.2\cdot e^{i0.5\pi}, 0.6\cdot e^{i0.5\pi}\rangle}{y} + \frac{\langle 0.7\cdot e^{i\pi}, 0.1\cdot e^{i0.9\pi}\rangle}{z}$, then $A\widehat{+}B = \frac{\langle 0.8\cdot e^{i1.28\pi}, 0.12\cdot e^{i0.72\pi}\rangle}{x} + \frac{\langle 0.52\cdot e^{i0.875\pi}, 0.36\cdot e^{i0.325\pi}\rangle}{y} + \frac{\langle 0.79\cdot e^{i2\pi}, 0.05\cdot e^{i0.675\pi}\rangle}{z}$.

Theorem 5 *The complex intuitionistic fuzzy probabilistic sum on $CIF^\star(U)$ is an s-norm.*

Proof This proof is similar to the proof of Theorem 4. □

Corollary 6 *Let $C_\alpha \in CIF^\star(U), \alpha \in I$, $\mu_{C_\alpha}(x) = r_{C_\alpha}(x) \cdot e^{i\omega_{\mu C_\alpha}(x)}$ and $\nu_{C_\alpha}(x) = s_{C_\alpha}(x) \cdot e^{i\omega_{\nu C_\alpha}(x)}$ its membership and non-membership functions, where I is an arbitrary index set. Then $C_1\widehat{+}C_2\widehat{+}\cdots\widehat{+}C_\alpha \in CIF^\star(U)$, and its membership and non-membership functions are*

$$\mu_{C_1\widehat{+}C_2\widehat{+}\cdots\widehat{+}C_\alpha}(x) = [r_{C_1}(x) + r_{C_2}(x) + \cdots + r_{C_\alpha}(x) - \cdots + (-1)^{\alpha-1}(r_{C_1}(x) \cdot r_{C_2}(x) \cdots r_{C_\alpha}(x))] \cdot e^{i2\pi\left[\left(\frac{\omega_{\mu C_1}(x)}{2\pi} + \frac{\omega_{\mu C_2}(x)}{2\pi} + \cdots + \frac{\omega_{\mu C_\alpha}(x)}{2\pi}\right) - \cdots + \frac{(-1)^{\alpha-1}}{(2\pi)^2}\left(\frac{\omega_{\mu C_1}(x)}{2\pi} \cdot \frac{\omega_{\mu C_2}(x)}{2\pi} \cdots \frac{\omega_{\mu C_\alpha}(x)}{2\pi}\right)\right]}$$

and

$$\nu_{C_1\widehat{+}C_2\widehat{+}\cdots\widehat{+}C_\alpha}(x) = (s_{C_1}(x) \cdot s_{C_2}(x) \cdots s_{C_\alpha}(x)) \cdot e^{i2\pi\left(\frac{\omega_{\nu C_1}(x)}{2\pi} \cdot \frac{\omega_{\nu C_2}(x)}{2\pi} \cdots \frac{\omega_{\nu C_\alpha}(x)}{2\pi}\right)}.$$

Definition 9 (*Complex intuitionistic fuzzy bold sum*) Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in U\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in U\}$ be two complex intuitionistic fuzzy sets on U. The complex intuitionistic fuzzy bold sum of A and B, denoted by $A\dot{\cup}B = \{\langle x, \mu_{A\dot{\cup}B}(x), \nu_{A\dot{\cup}B}(x)\rangle : x \in U\}$, where

$$\mu_{A\dot{\cup}B}(x) = r_{A\dot{\cup}B}(x) \cdot e^{i\omega_{\mu(A\dot{\cup}B)}(x)} = \min(1, r_A(x) + r_B(x)) \cdot e^{i\min(2\pi, \omega_{\mu A}(x) + \omega_{\mu B}(x))} \quad (11)$$

and

$$\nu_{A\dot{\cup}B}(x) = s_{A\dot{\cup}B}(x) \cdot e^{i\omega_{\nu(A\dot{\cup}B)}(x)} = \max(0, r_A(x) + r_B(x) - 1) \cdot e^{i\max(0, \omega_{\mu A}(x) + \omega_{\mu B}(x) - 2\pi)}. \quad (12)$$

Example 7 Let $A = \frac{\langle 0.5\cdot e^{i1.2\pi}, 0.4\cdot e^{i0.8\pi}\rangle}{x} + \frac{\langle 0.4\cdot e^{i0.5\pi}, 0.6\cdot e^{i1.3\pi}\rangle}{y} + \frac{\langle 0.3\cdot e^{i2\pi}, 0.5\cdot e^{i1.5\pi}\rangle}{z}$, $B = \frac{\langle 0.6\cdot e^{i0.2\pi}, 0.3\cdot e^{i1.8\pi}\rangle}{x} + \frac{\langle 0.2\cdot e^{i0.5\pi}, 0.6\cdot e^{i0.5\pi}\rangle}{y} + \frac{\langle 0.7\cdot e^{i\pi}, 0.1\cdot e^{i0.9\pi}\rangle}{z}$, then $A\dot{\cup}B = \frac{\langle 1\cdot e^{i1.4\pi}, 0\cdot e^{i0.6\pi}\rangle}{x} + \frac{\langle 0.6 e^{i\pi}, 0.2\cdot e^{i0\pi}\rangle}{y} + \frac{\langle 1\cdot e^{i2\pi}, 0\cdot e^{i0.4\pi}\rangle}{z}$.

Theorem 6 *The complex intuitionistic fuzzy bold sum on $CIF^\star(U)$ is an s-norm.*

Proof Properties (i), (ii), (v) and (vi) can be easily verified from Definition 9. Here we only prove (iii) and (iv).

(iii) Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in U\}$, $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in U\}$, $C = \{\langle x, \mu_C(x), \nu_C(x)\rangle : x \in U\}$ and $D = \{\langle x, \mu_D(x), \nu_D(x)\rangle : x \in U\}$ be four complex intuitionistic fuzzy sets on U. Suppose $|\mu_A(x)| \le |\mu_C(x)|$, $\omega_{\mu A}(x) \le \omega_{\mu C}(x)$, $|\nu_A(x)| \ge |\nu_C(x)|$, $\omega_{\nu A}(x) \ge \omega_{\nu C}(x)$ and $|\mu_B(x)| \le |\mu_D(x)|$, $\omega_{\mu B}(x) \le \omega_{\mu D}(x)$, $|\nu_B(x)| \ge |\nu_D(x)|$, $\omega_{\nu B}(x) \ge \omega_{\nu D}(x)$. For any $x \in U$, we have

$$|\mu_{A\dot{\cup}B}(x)| = \min(1, r_A(x) + r_B(x)) \le \min(1, r_C(x) + r_D(x)) = |\mu_{C\dot{\cup}D}(x)|,$$

$$\omega_{\mu(A\dot{\cup}B)}(x) = \min(2\pi, \omega_{\mu A}(x) + \omega_{\mu B}(x)) \le \min(2\pi, \omega_{\mu C}(x) + \omega_{\mu D}(x)) = \omega_{\mu(C\dot{\cup}D)}(x),$$

and

$$|\nu_{A\dot{\cup}B}(x)| = \max(0, s_A(x) + s_B(x) - 1) \ge \max(0, s_C(x) + s_D(x) - 1) = |\nu_{C\dot{\cup}D}(x)|,$$

$$\omega_{\nu(A\dot{\cup}B)}(x) = \max(0, \omega_{\nu A}(x) + \omega_{\nu B}(x) - 2\pi) \ge \max(0, \omega_{\nu C}(x) + \omega_{\nu D}(x) - 2\pi) = \omega_{\nu(C\dot{\cup}D)}(x).$$

(iv) Suppose $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in U\}$, $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in U\}$ and $C = \{\langle x, \mu_C(x), \nu_C(x)\rangle : x \in U\}$ be three complex intuitionistic fuzzy sets on U. Then

$$\begin{aligned}\mu_{(A\dot{\cup}B)\dot{\cup}C}(x) &= r_{(A\dot{\cup}B)\dot{\cup}C}(x)\cdot e^{i\omega_{\mu((A\dot{\cup}B)\dot{\cup}C)}(x)}\\ &= \min(1, r_{A\dot{\cup}B}(x)+r_C(x))\\ &\quad\cdot e^{i\min(2\pi,\omega_{\mu(A\dot{\cup}B)}(x)+\omega_{\mu C}(x))}\\ &= \min(1,\min(1,r_A(x)+r_B(x))+r_C(x))\\ &\quad\cdot e^{i\min(2\pi,\min(2\pi,\omega_{\mu A}(x)+\omega_{\mu B}(x))+\omega_{\mu C}(x))}\\ &= \min(1,r_A(x)+\min(1,r_B(x)+r_C(x)))\\ &\quad\cdot e^{i\min(2\pi,\omega_{\mu A}(x)+\min(2\pi,\omega_{\mu B}(x)+\omega_{\mu C}(x)))}\\ &= \min(1,r_A(x)+r_{B\dot{\cup}C}(x))\cdot e^{i\min(2\pi,\omega_{\mu A}(x)+\omega_{\mu(B\dot{\cup}C)}(x))}\\ &= \mu_{A\dot{\cup}(B\dot{\cup}C)}(x)\\ \nu_{(A\dot{\cup}B)\dot{\cup}C}(x) &= s_{(A\dot{\cup}B)\dot{\cup}C}(x)\cdot e^{i\omega_{\nu((A\dot{\cup}B)\dot{\cup}C)}(x)}\\ &= \max(0,s_{A\dot{\cup}B}(x)+s_C(x)-1)\\ &\quad\cdot e^{i\max(0,\omega_{\nu(A\dot{\cup}B)}(x)+\omega_{\nu C}(x)-2\pi)}\\ &= \max(0,\max(0,s_A(x)+s_B(x)-1)+s_C(x)-1)\\ &\quad\cdot e^{i\max(0,\max(0,\omega_{\nu A}(x)+\omega_{\nu B}(x)-2\pi)+\omega_{\nu C}(x)-2\pi)}\\ &= \max(0,s_A(x)+\max(0,s_B(x)+s_C(x)-1)-1)\\ &\quad\cdot e^{i\max(0,\omega_{\nu A}(x)+\max(0,\omega_{\nu B}(x)+\omega_{\nu C}(x)-2\pi)-2\pi)}\\ &= \max(0,s_A(x)+s_{B\dot{\cup}C}(x)-1)\\ &\quad\cdot e^{i\max(2\pi,\omega_{\nu A}(x)+\omega_{\nu(B\dot{\cup}C)}(x)-2\pi)}\\ &= \nu_{A\dot{\cup}(B\dot{\cup}C)}(x).\end{aligned}$$

□

Corollary 7 *Let $C_\alpha \in CIF^\star(U), \alpha \in I$, $\mu_{C_\alpha}(x) = r_{C_\alpha}(x)\cdot e^{i\omega_{\mu C_\alpha}(x)}$ and $\nu_{C_\alpha}(x) = s_{C_\alpha}(x)\cdot e^{i\omega_{\nu C_\alpha}(x)}$ its membership and non-membership functions, where I is an arbitrary index set. Then $C_1\dot{\cup}C_2\dot{\cup}\cdots\dot{\cup}C_\alpha \in CIF^\star(U)$, and its membership and non-membership functions are*

$$\mu_{C_1\dot{\cup}C_2\dot{\cup}\cdots\dot{\cup}C_\alpha}(x) = \min(1, r_{C_1}(x)+r_{C_2}(x)+\cdots+r_{C_\alpha}(x))\cdot e^{i\min(2\pi,\omega_{\mu C_1}(x)+\omega_{\mu C_2}(x)+\cdots+\omega_{\mu C_\alpha}(x))}$$

and

$$\nu_{C_1\dot{\cup}C_2\dot{\cup}\cdots\dot{\cup}C_\alpha}(x) = \max(0, r_{C_1}(x)+r_{C_2}(x)+\cdots+r_{C_\alpha}(x)-1)\cdot e^{i\max(0,\omega_{\nu C_1}(x)+\omega_{\nu C_2}(x)+\cdots+\omega_{\nu C_\alpha}(x)-2\pi)}.$$

Definition 10 (*Complex intuitionistic fuzzy bold intersection*) Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in U\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in U\}$ be two complex intuitionistic fuzzy sets on U. The complex intuitionistic fuzzy bold intersection of A and B, denoted by $A\dot{\cap}B = \{\langle x, \mu_{A\dot{\cap}B}(x), \nu_{A\dot{\cap}B}(x)\rangle : x \in U\}$, where

$$\begin{aligned}\mu_{A\dot{\cap}B}(x) &= r_{A\dot{\cap}B}(x)\cdot e^{i\omega_{\mu(A\dot{\cap}B)}(x)}\\ &= \max(0, r_A(x)+r_B(x)-1)\cdot e^{i\max(0,\omega_{\mu A}(x)+\omega_{\mu B}(x)-2\pi)}\end{aligned} \quad (13)$$

and

$$\begin{aligned}\nu_{A\dot{\cap}B}(x) &= s_{A\dot{\cap}B}(x)\cdot e^{i\omega_{\nu(A\dot{\cap}B)}(x)}\\ &= \min(1, s_A(x)+s_B(x))\cdot e^{i\min(2\pi,\omega_{\nu A}(x)+\omega_{\nu B}(x))}.\end{aligned} \quad (14)$$

Example 8 Let $A = \frac{\langle 0.5\cdot e^{i1.2\pi}, 0.4\cdot e^{i0.8\pi}\rangle}{x} + \frac{\langle 0.4\cdot e^{i0.5\pi}, 0.6\cdot e^{i1.3\pi}\rangle}{y} + \frac{\langle 0.3\cdot e^{i2\pi}, 0.5\cdot e^{i1.5\pi}\rangle}{z}$, $B = \frac{\langle 0.6\cdot e^{i0.2\pi}, 0.3\cdot e^{i1.8\pi}\rangle}{x} + \frac{\langle 0.2\cdot e^{i0.5\pi}, 0.6\cdot e^{i0.5\pi}\rangle}{y} + \frac{\langle 0.7\cdot e^{i\pi}, 0.1\cdot e^{i0.9\pi}\rangle}{z}$, then $A\dot{\cap}B = \frac{\langle 0.1\cdot e^{i0\pi}, 0.7\cdot e^{i2\pi}\rangle}{x} + \frac{\langle 0\cdot e^{i0\pi}, 1\cdot e^{i1.8\pi}\rangle}{y} + \frac{\langle 0\cdot e^{i\pi}, 0.6\cdot e^{i2\pi}\rangle}{z}$.

Theorem 7 *The complex intuitionistic fuzzy bold intersection on $CIF^\star(U)$ is a t-norm.*

Proof This proof is similar to the proof of Theorem 6. □

Corollary 8 *Let $C_\alpha \in CIF^\star(U), \alpha \in I$, $\mu_{C_\alpha}(x) = r_{C_\alpha}(x)\cdot e^{i\omega_{\mu C_\alpha}(x)}$ and $\nu_{C_\alpha}(x) = s_{C_\alpha}(x)\cdot e^{i\omega_{\nu C_\alpha}(x)}$ its membership and non-membership functions, where I is an arbitrary index set. Then $C_1\dot{\cap}C_2\dot{\cap}\cdots\dot{\cap}C_\alpha \in CIF^\star(U)$, and its membership and non-membership functions are*

$$\mu_{C_1\dot{\cap}C_2\dot{\cap}\cdots\dot{\cap}C_\alpha}(x) = \max(0, r_{C_1}(x)+r_{C_2}(x)+\cdots+r_{C_\alpha}(x)-1)\cdot e^{i\max(0,\omega_{\mu C_1}(x)+\omega_{\mu C_2}(x)+\cdots+\omega_{\mu C_\alpha}(x)-2\pi)}$$

and

$$\nu_{C_1\dot{\cap}C_2\dot{\cap}\cdots\dot{\cap}C_\alpha}(x) = \min(1, r_{C_1}(x)+r_{C_2}(x)+\cdots+r_{C_\alpha}(x))\cdot e^{i\min(2\pi,\omega_{\mu C_1}(x)+\omega_{\mu C_2}(x)+\cdots+\omega_{\mu C_\alpha}(x))}.$$

Definition 11 (*Complex intuitionistic fuzzy bounded difference*) Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in U\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in U\}$ be two complex intuitionistic fuzzy sets on U. The complex intuitionistic fuzzy bounded difference of A and B, denoted by $A|-|B = \{\langle x, \mu_{A|-|B}(x), \nu_{A|-|B}(x)\rangle : x \in U\}$, where

$$\begin{aligned}\mu_{A|-|B}(x) &= r_{A|-|B}(x)\cdot e^{i\omega_{\mu(A|-|B)}(x)}\\ &= \max(0, r_A(x)-r_B(x))\cdot e^{i\max(0,\omega_{\mu A}(x)-\omega_{\mu B}(x))}\end{aligned} \quad (15)$$

and

$$\begin{aligned}\nu_{A|-|B}(x) &= s_{A|-|B}(x)\cdot e^{i\omega_{\nu(A|-|B)}(x)}\\ &= \min(1, 1-s_A(x)+s_B(x))\\ &\quad\cdot e^{i\min(2\pi,2\pi-\omega_{\nu A}(x)+\omega_{\nu B}(x))}.\end{aligned} \quad (16)$$

Example 9 Let $A = \frac{\langle 0.5\cdot e^{i1.2\pi}, 0.4\cdot e^{i0.8\pi}\rangle}{x} + \frac{\langle 0.4\cdot e^{i0.5\pi}, 0.6\cdot e^{i1.3\pi}\rangle}{y} + \frac{\langle 0.3\cdot e^{i2\pi}, 0.5\cdot e^{i1.5\pi}\rangle}{z}$, $B = \frac{\langle 0.6\cdot e^{i0.2\pi}, 0.3\cdot e^{i1.8\pi}\rangle}{x} + \frac{\langle 0.2\cdot e^{i0.5\pi}, 0.6\cdot e^{i0.5\pi}\rangle}{y}$

$+\frac{\langle 0.7\cdot e^{i\pi},0.1\cdot e^{i0.9\pi}\rangle}{z}$, then $A|-|B = \frac{\langle 0\cdot e^{i\pi},0.9\cdot e^{i2\pi}\rangle}{x} + \frac{\langle 0.2\cdot e^{i0\pi},1\cdot e^{i1.2\pi}\rangle}{y} + \frac{\langle 0\cdot e^{i\pi},0.6\cdot e^{i1.4\pi}\rangle}{z}$.

Definition 12 (*Complex intuitionistic fuzzy symmetrical difference*) Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in U\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in U\}$ be two complex intuitionistic fuzzy sets on U. The complex intuitionistic fuzzy symmetrical difference of A and B, denoted by $A\nabla B = \{\langle x, \mu_{A\nabla B}(x), \nu_{A\nabla B}(x)\rangle : x \in U\}$, where

$$\mu_{A\nabla B}(x) = r_{A\nabla B}(x)\cdot e^{i\omega_{\mu(A\nabla B)}(x)} = |r_A(x) - r_B(x)|\cdot e^{i|\omega_{\mu A}(x)-\omega_{\mu B}(x)|} \tag{17}$$

and

$$\nu_{A\nabla B}(x) = s_{A\nabla B}(x)\cdot e^{i\omega_{\nu(A\nabla B)}(x)} = |1 - s_B(x) - s_A(x)|\cdot e^{i|2\pi-\omega_{\nu B}(x)-\omega_{\nu A}(x)|}. \tag{18}$$

Example 10 Let $A = \frac{\langle 0.5\cdot e^{i1.2\pi},0.4\cdot e^{i0.8\pi}\rangle}{x} + \frac{\langle 0.4\cdot e^{i0.5\pi},0.6\cdot e^{i1.3\pi}\rangle}{y} + \frac{\langle 0.3\cdot e^{i2\pi},0.5\cdot e^{i1.5\pi}\rangle}{z}$, $B = \frac{\langle 0.6\cdot e^{i0.2\pi},0.3\cdot e^{i1.8\pi}\rangle}{x} + \frac{\langle 0.2\cdot e^{i0.5\pi},0.6\cdot e^{i0.5\pi}\rangle}{y} + \frac{\langle 0.7\cdot e^{i\pi},0.1\cdot e^{i0.9\pi}\rangle}{z}$, then $A\nabla B = \frac{\langle 0.1\cdot e^{i\pi},0.3\cdot e^{i0.6\pi}\rangle}{x} + \frac{\langle 0.2\cdot e^{i0\pi},0.2\cdot e^{i0.2\pi}\rangle}{y} + \frac{\langle 0.4\cdot e^{i\pi},0.4\cdot e^{i0.4\pi}\rangle}{z}$.

Definition 13 (*Complex intuitionistic fuzzy convex linear sum*) Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in U\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in U\}$ be two complex intuitionistic fuzzy sets on U. The complex intuitionistic fuzzy convex linear sum of min and max of A and B, denoted by $A||_\lambda B = \{\langle x, \mu_{A||_\lambda B}(x), \nu_{A||_\lambda B}(x)\rangle : x \in U\}$, $\lambda \in [0, 1]$, where

$$\begin{aligned}\mu_{A||_\lambda B}(x) &= r_{A||_\lambda B}(x)\cdot e^{i\omega_{\mu(A||_\lambda B)}(x)}\\ &= [\lambda \min(r_A(x), r_B(x)) + (1-\lambda)\max(r_A(x), r_B(x))]\\ &\quad \cdot e^{i[\lambda \min(\omega_{\mu A}(x),\omega_{\mu B}(x))+(1-\lambda)\max(\omega_{\mu A}(x),\omega_{\mu B}(x))]}\end{aligned} \tag{19}$$

and

$$\begin{aligned}\nu_{A||_\lambda B}(x) &= s_{A||_\lambda B}(x)\cdot e^{i\omega_{\nu(A||_\lambda B)}(x)}\\ &= [\lambda \max(s_A(x), s_B(x)) + (1-\lambda)\min(s_A(x), s_B(x))]\\ &\quad \cdot e^{i[\lambda \max(\omega_{\nu A}(x),\omega_{\nu B}(x))+(1-\lambda)\min(\omega_{\nu A}(x),\omega_{\nu B}(x))]}.\end{aligned} \tag{20}$$

Example 11 Let $A = \frac{\langle 0.5\cdot e^{i1.2\pi},0.4\cdot e^{i0.8\pi}\rangle}{x} + \frac{\langle 0.4\cdot e^{i0.5\pi},0.6\cdot e^{i1.3\pi}\rangle}{y} + \frac{\langle 0.3\cdot e^{i2\pi},0.5\cdot e^{i1.5\pi}\rangle}{z}$, $B = \frac{\langle 0.6\cdot e^{i0.2\pi},0.3\cdot e^{i1.8\pi}\rangle}{x} + \frac{\langle 0.2\cdot e^{i0.5\pi},0.6\cdot e^{i0.5\pi}\rangle}{y} + \frac{\langle 0.7\cdot e^{i\pi},0.1\cdot e^{i0.9\pi}\rangle}{z}$, then $A||_\lambda B = \frac{\langle 0.57\cdot e^{i0.9\pi},0.33\cdot e^{i1.1\pi}\rangle}{x} + \frac{\langle 0.34\cdot e^{i0.5\pi},0.6\cdot e^{i0.74\pi}\rangle}{y} + \frac{\langle 0.58\cdot e^{i1.7\pi},0.22\cdot e^{i1.08\pi}\rangle}{z}$ when $\lambda = 0.3$.

3 Distance measure and (α, β)-equalities of complex intuitionistic fuzzy sets

In this section, we define a new distance measure for complex intuitionistic fuzzy sets. The distance of two complex intuitionistic fuzzy sets measures the difference between the grades of two complex intuitionistic fuzzy sets as well as that between the phases of the two complex intuitionistic fuzzy sets. This distance measure is then used to define (α, β)-equalities of complex intuitionistic fuzzy sets which coincide with those of intuitionistic fuzzy sets already defined in the literature if complex intuitionistic fuzzy sets reduce to traditional intuitionistic fuzzy sets.

3.1 Distance measure for complex intuitionistic fuzzy sets

Definition 14 A distance between two complex intuitionistic fuzzy sets is a function $d : (\mathrm{CIF}^\star(U), \mathrm{CIF}^\star(U)) \to [0, 1]$, for any $A, B, C \in \mathrm{CIF}^\star(U)$, satisfying the following properties.

(i) $0 \le d(A, B) \le 1$, $d(A, B) = 0$ if and only if $A = B$;
(ii) $d(A, B) = d(B, A)$; and
(iii) $d(A, B) \le d(A, C) + d(C, B)$.

Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in U\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in U\}$ be two complex intuitionistic fuzzy sets on U. In the following, we introduce two functions $\rho(\mu_A, \mu_B)$ and $\rho(\nu_A, \nu_B)$ which play an important role in the remainder of this paper.

Definition 15 Let

$$\rho(\mu_A, \mu_B) = \max\left(\sup_{x\in U}|r_A(x) - r_B(x)|, \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A}(x) - \omega_{\mu B}(x)|\right) \tag{21}$$

and

$$\rho(\nu_A, \nu_B) = \max\left(\sup_{x\in U}|s_A(x) - s_B(x)|, \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A}(x) - \omega_{\nu B}(x)|\right), \tag{22}$$

then

$$d(A, B) = \frac{1}{2}\left(\rho(\mu_A, \mu_B) + \rho(\nu_A, \nu_B)\right). \tag{23}$$

Theorem 8 *$d(A, B)$ defined by Equation (23) is a distance function of complex intuitionistic fuzzy sets on U.*

Proof (i) and (ii) can be easily verified from Definition 15. Here we only prove (iii).

(iii) Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in U\}$, $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in U\}$ and $C = \{\langle x, \mu_C(x), \nu_C(x)\rangle : x \in U\}$ be three complex intuitionistic fuzzy sets on U. According to Definition 15, we have

$$\begin{aligned}
d(A, B) &= \frac{1}{2}\left(\rho(\mu_A, \mu_B) + \rho(\nu_A, \nu_B)\right)\\
&= \frac{1}{2}\left(\max\left(\sup_{x\in U}|r_A(x) - r_B(x)|, \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A}(x) - \omega_{\mu B}(x)|\right)\right.\\
&\quad \left. + \max\left(\sup_{x\in U}|s_A(x) - s_B(x)|, \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A}(x) - \omega_{\nu B}(x)|\right)\right)\\
&\le \frac{1}{2}\left(\max\left(\sup_{x\in U}(|r_A(x) - r_C(x)| + |r_C(x) - r_B(x)|),\right.\right.\\
&\quad \left.\frac{1}{2\pi}\sup_{x\in U}(|\omega_{\mu A}(x) - \omega_{\mu C}(x)| + |\omega_{\mu C}(x) - \omega_{\mu B}(x)|)\right)\\
&\quad + \max(\sup_{x\in U}(|s_A(x) - s_C(x)| + |s_C(x) - s_B(x)|),\\
&\quad \left.\frac{1}{2\pi}\sup_{x\in U}(|\omega_{\nu A}(x) - \omega_{\nu C}(x)| + |\omega_{\nu C}(x) - \omega_{\nu B}(x)|))\right)\\
&= \frac{1}{2}(\max(\sup_{x\in U}|r_A(x) - r_C(x)|,\\
&\quad \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A}(x) - \omega_{\mu C}(x)|) \max(\sup_{x\in U}|s_A(x) - s_C(x)|,\\
&\quad \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A}(x) - \omega_{\nu C}(x)|) + \frac{1}{2}(\max(\sup_{x\in U}|r_C(x) - r_B(x)|,\\
&\quad \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu C}(x) - \omega_{\mu B}(x)|)\\
&\quad + \max(\sup_{x\in U}|s_C(x) - s_B(x)|, \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu C}(x) - \omega_{\nu B}(x)|))\\
&= \frac{1}{2}(\rho(\mu_A, \mu_B) + \rho(\nu_A, \nu_B))\\
&= d(A, C) + d(C, B).
\end{aligned}$$

□

Example 12 Let $A = \frac{\langle 0.5\cdot e^{i1.2\pi}, 0.4\cdot e^{i0.8\pi}\rangle}{x} + \frac{\langle 0.4\cdot e^{i0.5\pi}, 0.6\cdot e^{i1.3\pi}\rangle}{y} + \frac{\langle 0.3\cdot e^{i2\pi}, 0.5\cdot e^{i1.5\pi}\rangle}{z}$, $B = \frac{\langle 0.6\cdot e^{i0.2\pi}, 0.3\cdot e^{i1.8\pi}\rangle}{x} + \frac{\langle 0.2\cdot e^{i0.5\pi}, 0.6\cdot e^{i0.5\pi}\rangle}{y} + \frac{\langle 0.7\cdot e^{i\pi}, 0.1\cdot e^{i0.9\pi}\rangle}{z}$. Since $\sup_{x\in U}|r_A(x) - r_B(x)| = 0.4$, $\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A}(x) - \omega_{\mu B}(x)| = 0.5$, $\sup_{x\in U}|s_A(x) - s_B(x)| = 0.4$, and $\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A}(x) - \omega_{\nu B}(x)| = 0.5$, therefore $\rho(\mu_A, \mu_B) = 0.5$ and $\rho(\nu_A, \nu_B) = 0.5$, so $d(A, B) = \frac{1}{2}(\rho(\mu_A, \mu_B) + \rho(\nu_A, \nu_B)) = 0.5$.

Remark 1 It is easy to see that, if A and B are two intuitionistic fuzzy sets on U, then

$$\rho(\mu_A, \mu_B) = \sup_{x\in U}|\mu_A(x) - \mu_B(x)|,$$
$$\rho(\nu_A, \nu_B) = \sup_{x\in U}|\nu_A(x) - \nu_B(x)|$$

and

$$d(A, B) = \frac{1}{2}(\rho(\mu_A, \mu_B) + \rho(\nu_A, \nu_B)).$$

3.2 (α, β)-equalities of complex intuitionistic fuzzy sets

Definition 16 (Gong et al. 2013) Let U be an universe of discourse, $\bar{A}$ and $\bar{B}$ be two intuitionistic fuzzy sets on U, $\mu_{\bar{A}}(x)$, $\mu_{\bar{B}}(x)$, $\nu_{\bar{A}}(x)$ and $\nu_{\bar{B}}(x)$ their membership and non-membership functions, respectively. Then $\bar{A}$ and $\bar{B}$ are said to be (α, β)-equal, if and only if

$$\sup_{x\in U}|\mu_{\bar{A}}(x) - \mu_{\bar{B}}(x)| \le 1 - \alpha,\ \sup_{x\in U}|\nu_{\bar{A}}(x) - \nu_{\bar{B}}(x)| \le \beta,$$

where $0 \le \alpha \le 1, 0 \le \beta \le 1$ and $\alpha + \beta \le 1$. Symbolically, we denote $\bar{A} = (\alpha, \beta)\bar{B}$. In this way we say $\bar{A}$ and $\bar{B}$ construct a (α, β)-equality.

Lemma 1 *Let*

$$\alpha_1 * \alpha_2 = \max(0, \alpha_1 + \alpha_2 - 1) \tag{24}$$

and

$$\beta_1 * \beta_2 = \min(1, \beta_1 + \beta_2), \tag{25}$$

where $0 \le \alpha_1, \alpha_2 \le 1, 0 \le \beta_1, \beta_2 \le 1$ *and* $\alpha_1 + \beta_1 \le 1, \alpha_2 + \beta_2 \le 1$. *Then*

(i) $0 * \alpha_1 = 0, 0 * \beta_1 = \beta_1, \forall \alpha_1 \in [0, 1], \beta_1 \in [0, 1]$;
(ii) $1 * \alpha_1 = \alpha_1, 1 * \beta_1 = 1, \forall \alpha_1 \in [0, 1], \beta_1 \in [0, 1]$;
(iii) $0 \le \alpha_1 * \alpha_2 \le 1, 0 \le \beta_1 * \beta_2 \le 1, \forall \alpha_1, \alpha_2 \in [0, 1], \beta_1, \beta_2 \in [0, 1]$;
(iv) $\alpha_1 \le \alpha \Rightarrow \alpha_1 * \alpha_2 \le \alpha * \alpha_2, \beta_1 \le \beta \Rightarrow \beta_1 * \beta_2 \le \beta * \beta_2, \forall \alpha_1, \alpha, \alpha_2 \in [0, 1], \beta_1, \beta, \beta_2 \in [0, 1]$;
(v) $\alpha_1 * \alpha_2 = \alpha_2 * \alpha_1, \beta_1 * \beta_2 = \beta_2 * \beta_1, \forall \alpha_1, \alpha_2 \in [0, 1], \beta_1, \beta_2 \in [0, 1]$;
(vi) $(\alpha_1 * \alpha_2) * \alpha_3 = \alpha_1 * (\alpha_2 * \alpha_3), (\beta_1 * \beta_2) * \beta_3 = \beta_1 * (\beta_2 * \beta_3), \forall \alpha_1, \alpha_2, \alpha_3 \in [0, 1], \beta_1, \beta_2, \beta_3 \in [0, 1]$.

Definition 17 Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in U\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in U\}$ be two complex intuitionistic fuzzy sets on U. Then A and B are said to be (α, β)-equal, if and only if

$$\rho(\mu_A, \mu_B) \le 1 - \alpha,\ \rho(\nu_A, \nu_B) \le \beta, \tag{26}$$

where $0 \le \alpha \le 1, 0 \le \beta \le 1$ and $\alpha + \beta \le 1$. Symbolically, we denote $A = (\alpha, \beta)B$. In this way we say A and B construct a (α, β)-equality.

Remark 2 Two complex intuitionistic fuzzy sets A and B are said to build a (α, β)-equality if $\rho(\mu_A, \mu_B) \leq 1-\alpha$ and $\rho(\nu_A, \nu_B) \leq \beta$. An advantage of using $1-\alpha$ rather than α is that the interpretation of α can comply with common sense. That is, the greater α is, the more equal the two complex intuitionistic fuzzy sets are; the smaller β is, the more equal the two complex intuitionistic fuzzy sets are; and if $\alpha = 1$ or $\beta = 0$, then the two complex intuitionistic fuzzy sets are strictly equal.

Theorem 9 *Let A and B be two complex intuitionistic fuzzy sets on U. Then*

(i) $A = (0, 1)B$;
(ii) $A = (1, 0)B \Leftrightarrow A = B$;
(iii) $A = (\alpha, \beta)B \Leftrightarrow B = (\alpha, \beta)A$;
(iv) $A = (\alpha_1, \beta_1)B$ *and* $\alpha_1 \geq \alpha_2,\ \beta_1 \leq \beta_2 \Rightarrow A = (\alpha_2, \beta_2)B$;
(v) *If* $\forall\ i \in I,\ A = (\alpha_i, \beta_i)B$, *where I is an index set and* $\sup_{i\in I}\alpha_i + \sup_{i\in I}\beta_i \leq 1$, *then* $A = (\sup_{i\in I}\alpha_i, \sup_{i\in I}\beta_i)B$;
(vi) *Let* $A = (\alpha_1, \beta_1)B$. *If there exists an unique α and β, such $A = (\alpha, \beta)B$ for any A and B, then* $\alpha \leq \alpha_1,\ \beta \geq \beta_1$.

Proof Properties (i)–(iv) can be easily proved. Here we only prove properties (v) and (vi).

(v) Since $A = (\alpha_i, \beta_i)B$, for any $i \in I$, we have

$$\rho(\mu_A, \mu_B) = \max\left(\sup_{x\in U}|r_A(x) - r_B(x)|, \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A}(x) - \omega_{\mu B}(x)|\right) \leq 1-\alpha_i,$$

and

$$\rho(\nu_A, \nu_B) = \max\left(\sup_{x\in U}|s_A(x) - s_B(x)|, \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A}(x) - \omega_{\nu B}(x)|\right) \leq \beta_i,$$

therefore

$$\sup_{x\in U}|r_A(x) - r_B(x)| \leq 1 - \sup_{i\in I}\alpha_i,\ \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A}(x) - \omega_{\mu B}(x)| \leq 1 - \sup_{i\in I}\alpha_i,$$

and

$$\sup_{x\in U}|s_A(x) - s_B(x)| \leq \sup_{i\in I}\beta_i,\ \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A}(x) - \omega_{\nu B}(x)| \leq \sup_{i\in I}\beta_i,$$

hence

$$\rho(\mu_A, \mu_B) = \max\left(\sup_{x\in U}|r_A(x) - r_B(x)|, \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A}(x) - \omega_{\mu B}(x)|\right) \leq 1 - \sup_{i\in I}\alpha_i,$$

and

$$\rho(\nu_A, \nu_B) = \max(\sup_{x\in U}|s_A(x) - s_B(x)|, \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A}(x) - \omega_{\nu B}(x)|) \leq \sup_{i\in I}\beta_i.$$

It implies that $A = (\sup_{i\in I}\alpha_i, \sup_{i\in I}\beta_i)B$.

(vi) Let $\alpha_1 = 1 - \rho(\mu_A, \mu_B),\ \beta_1 = \rho(\nu_A, \nu_B)$. Then $A = (\alpha_1, \beta_1)B$. Obviously, if $A = (\alpha, \beta)B$, we have $1 - \alpha_1 = \rho(\mu_A, \mu_B) \leq 1 - \alpha$ and $\beta_1 = \rho(\mu_A, \mu_B) \leq \beta$. There must be $\alpha \leq \alpha_1,\ \beta \geq \beta_1$. □

Theorem 10 *Let A, B and C be three complex intuitionistic fuzzy sets on U. If $A = (\alpha_1, \beta_1)B$ and $B = (\alpha_2, \beta_2)C$, then $A = (\alpha, \beta)C$, where $\alpha = \alpha_1 * \alpha_2,\ \beta = 1 - \alpha_1 * \alpha_2$.*

Proof Since $A = (\alpha_1, \beta_1)B$ and $B = (\alpha_2, \beta_2)C$, we have

$$\rho(\mu_A, \mu_B) = \max(\sup_{x\in U}|r_A(x) - r_B(x)|, \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A}(x) - \omega_{\mu B}(x)|) \leq 1 - \alpha_1,$$
$$\rho(\nu_A, \nu_B) = \max(\sup_{x\in U}|s_A(x) - s_B(x)|, \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A}(x) - \omega_{\nu B}(x)|) \leq \beta_1,$$

and

$$\rho(\mu_B, \mu_C) = \max(\sup_{x\in U}|r_B(x) - r_C(x)|, \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu B}(x) - \omega_{\mu C}(x)|) \leq 1 - \alpha_2,$$
$$\rho(\nu_B, \nu_C) = \max(\sup_{x\in U}|s_B(x) - s_C(x)|, \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu B}(x) - \omega_{\nu C}(x)|) \leq \beta_2,$$

therefore

$$\sup_{x\in U}|r_A(x) - r_B(x)| \leq 1 - \alpha_1,\ \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A}(x) - \omega_{\mu B}(x)| \leq 1 - \alpha_1,$$
$$\sup_{x\in U}|s_A(x) - s_B(x)| \leq \beta_1,\ \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A}(x) - \omega_{\nu B}(x)| \leq \beta_1,$$

and

$$\sup_{x\in U}|r_B(x) - r_C(x)| \leq 1-\alpha_2,\ \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu B}(x) -\omega_{\mu C}(x)| \leq 1-\alpha_2,$$

$$\sup_{x\in U}|s_B(x) - s_C(x)| \leq \beta_2,\ \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu B}(x) -\omega_{\nu C}(x)| \leq \beta_2,$$

consequently, we have

$$\begin{aligned}\rho(\mu_A,\mu_C) &= \max(\sup_{x\in U}|r_A(x)-r_C(x)|, \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A}(x)-\omega_{\mu C}(x)|)\\ &\leq \max(\sup_{x\in U}|r_A(x)-r_B(x)| + \sup_{x\in U}|r_B(x)-r_C(x)|,\\ &\quad \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A}(x)-\omega_{\mu B}(x)| + \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu B}(x)-\omega_{\mu C}(x)|)\\ &\leq \max((1-\alpha_1)+(1-\alpha_2),(1-\alpha_1)+(1-\alpha_2))\\ &= (1-\alpha_1)+(1-\alpha_2)\\ &= 1-(\alpha_1+\alpha_2-1),\end{aligned}$$

furthermore, note that $\rho(\mu_A,\mu_C) \leq 1$. Hence

$$\rho(\mu_A,\mu_C) \leq 1-\max(0,\alpha_1+\alpha_2-1) = 1-\alpha_1 * \alpha_2 = 1-\alpha.$$

$$\begin{aligned}\rho(\nu_A,\nu_C) &= \max(\sup_{x\in U}|s_A(x)-s_C(x)|, \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A}(x)-\omega_{\nu C}(x)|)\\ &\leq \max(\sup_{x\in U}|s_A(x)-s_B(x)| + \sup_{x\in U}|s_B(x)-s_C(x)|,\\ &\quad \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A}(x)-\omega_{\nu B}(x)| + \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu B}(x)-\omega_{\nu C}(x)|)\\ &\leq \max(\beta_1+\beta_2,\beta_1+\beta_2)\\ &= \beta_1+\beta_2.\end{aligned}$$

That is to say

$$\begin{aligned}&\rho(\nu_A,\nu_C) \leq \beta_1+\beta_2 \leq 1-\alpha_1+1-\alpha_2 = 1-(\alpha_1+\alpha_2-1)\\ &= 1-\max(0,\alpha_1+\alpha_2-1) = 1-\alpha_1 * \alpha_2 = \beta.\end{aligned}$$

It implies that $A = (\alpha,\beta)C$. □

Theorem 11 *If $A_1 = (\alpha_1,\beta_1)B_1$ and $A_2 = (\alpha_2,\beta_2)B_2$, then $A_1 \cup A_2 = (\min(\alpha_1,\alpha_2), \max(\beta_1,\beta_2))(B_1 \cup B_2)$.*

Proof According to Definition 15, we have

$$\begin{aligned}&\rho(\mu_{A_1\cup A_2},\mu_{B_1\cup B_2})\\ &= \max(\sup_{x\in U}|r_{A_1\cup A_2}(x)-r_{B_1\cup B_2}(x)|,\\ &\quad \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\cup A_2)}(x)-\omega_{\mu(B_1\cup B_2)}(x)|)\end{aligned}$$

and

$$\begin{aligned}&\rho(\nu_{A_1\cup A_2},\nu_{B_1\cup B_2})\\ &= \max(\sup_{x\in U}|s_{A_1\cup A_2}(x)-s_{B_1\cup B_2}(x)|,\\ &\quad \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\cup A_2)}(x)-\omega_{\nu(B_1\cup B_2)}(x)|).\end{aligned}$$

Based on Definitions 3 and 17, we can obtain that

$$\begin{aligned}&\sup_{x\in U}|r_{A_1\cup A_2}(x)-r_{B_1\cup B_2}(x)|\\ &= \sup_{x\in U}|\max(r_{A_1}(x),r_{A_2}(x))-\max(r_{B_1}(x),r_{B_2}(x))|\\ &\leq \sup_{x\in U}\max(|r_{A_1}(x)-r_{B_1}(x)|,|r_{A_2}(x)-r_{B_2}(x)|)\\ &\leq \sup_{x\in U}\max(1-\alpha_1,1-\alpha_2)\\ &\leq 1-\min(\alpha_1,\alpha_2).\end{aligned}$$

$$\begin{aligned}&\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\cup A_2)}(x)-\omega_{\mu(B_1\cup B_2)}(x)|\\ &= \frac{1}{2\pi}\sup_{x\in U}|\max(\omega_{\mu A_1}(x),\omega_{\mu A_2}(x))-\max(\omega_{\mu B_1}(x),\omega_{\mu B_2}(x))|\\ &\leq \frac{1}{2\pi}\sup_{x\in U}\max(|\omega_{\mu A_1}(x)-\omega_{\mu B_1}(x)|,|\omega_{\mu A_2}(x)-\omega_{\mu B_2}(x)|)\\ &\leq \frac{1}{2\pi}\sup_{x\in U}\max(1-\alpha_1,1-\alpha_2)\\ &\leq 1-\min(\alpha_1,\alpha_2).\end{aligned}$$

$$\begin{aligned}&\sup_{x\in U}|s_{A_1\cup A_2}(x)-s_{B_1\cup B_2}(x)|\\ &= \sup_{x\in U}|\min(s_{A_1}(x),s_{A_2}(x))-\min(s_{B_1}(x),s_{B_2}(x))|\\ &\leq \sup_{x\in U}\min(|s_{A_1}(x)-s_{B_1}(x)|,|s_{A_2}(x)-s_{B_2}(x)|)\\ &\leq \sup_{x\in U}\min(\beta_1,\beta_2)\\ &\leq \max(\beta_1,\beta_2).\end{aligned}$$

$$\begin{aligned}&\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\cup A_2)}(x)-\omega_{\nu(B_1\cup B_2)}(x)|\\ &= \frac{1}{2\pi}\sup_{x\in U}|\min(\omega_{\nu A_1}(x),\omega_{\nu A_2}(x))-\min(\omega_{\nu B_1}(x),\omega_{\nu B_2}(x))|\\ &\leq \frac{1}{2\pi}\sup_{x\in U}\min(|\omega_{\nu A_1}(x)-\omega_{\nu B_1}(x)|,|\omega_{\nu A_2}(x)-\omega_{\nu B_2}(x)|)\\ &\leq \frac{1}{2\pi}\sup_{x\in U}\min(\beta_1,\beta_2)\\ &\leq \max(\beta_1,\beta_2).\end{aligned}$$

It implies that

$$A_1\cup A_2 = (\min(\alpha_1,\alpha_2),\max(\beta_1,\beta_2))(B_1\cup B_2).$$

□

Corollary 9 *If $A_i = (\alpha_i,\beta_i)B_i,\ i\in I$, where I is an index set, then $\cup_{i\in I}A_i = (\inf_{i\in I}\alpha_i, \sup_{i\in I}\beta_i)\cup_{i\in I}B_i$.*

Theorem 12 *If* $A_1 = (\alpha_1, \beta_1)B_1$ *and* $A_2 = (\alpha_2, \beta_2)B_2$, *then* $A_1 \cap A_2 = (\min(\alpha_1, \alpha_2), \max(\beta_1, \beta_2))(B_1 \cap B_2)$.

Proof According to Definition 15, we have

$$\begin{aligned}\rho(\mu_{A_1\cap A_2}, \mu_{B_1\cap B_2}) &= \max(\sup_{x\in U}|r_{A_1\cap A_2}(x) - r_{B_1\cap B_2}(x)|,\\ &\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\cap A_2)}(x) - \omega_{\mu(B_1\cap B_2)}(x)|)\end{aligned}$$

and

$$\begin{aligned}\rho(\nu_{A_1\cap A_2}, \nu_{B_1\cap B_2}) &= \max(\sup_{x\in U}|s_{A_1\cap A_2}(x) - s_{B_1\cap B_2}(x)|,\\ &\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\cap A_2)}(x) - \omega_{\nu(B_1\cap B_2)}(x)|).\end{aligned}$$

Based on Definitions 4 and 17, similar to the proof of Theorem 11, we can obtain that

$$A_1 \cap A_2 = (\min(\alpha_1, \alpha_2), \max(\beta_1, \beta_2))(B_1 \cap B_2).$$

□

Corollary 10 *If* $A_i = (\alpha_i, \beta_i)B_i,\ i \in I$, *where* I *is an index set, then* $\cap_{i\in I} A_i = (\inf_{i\in I} \alpha_i, \sup_{i\in I} \beta_i) \cap_{i\in I} B_i$.

Theorem 13 *If* $A = (\alpha, \beta)B$, *then* $\overline{A} = (\alpha, \beta)\overline{B}$.

Proof According to Definitions 5 and 17, we have

$$\begin{aligned}\rho(\mu_{\bar{A}}, \mu_{\bar{B}}) &= \max(\sup_{x\in U}|r_{\bar{A}}(x) - r_{\bar{B}}(x)|, \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu\bar{A}}(x) - \omega_{\mu\bar{B}}(x)|)\\ &= \max(\sup_{x\in U}|(1 - r_A(x)) - (1 - r_B(x))|,\\ &\frac{1}{2\pi}\sup_{x\in U}|(2\pi - \omega_{\mu A}(x)) - (2\pi - \omega_{\mu B}(x))|)\\ &= \max(\sup_{x\in U}|r_A(x) - r_B(x)|, \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A}(x) - \omega_{\mu B}(x)|)\\ &= \rho(\mu_A, \mu_B)\\ &\le 1 - \alpha.\end{aligned}$$

$$\begin{aligned}\rho(\nu_{\bar{A}}, \nu_{\bar{B}}) &= \max(\sup_{x\in U}|s_{\bar{A}}(x) - s_{\bar{B}}(x)|, \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu\bar{A}}(x) - \omega_{\nu\bar{B}}(x)|)\\ &= \max(\sup_{x\in U}|(1 - s_A(x)) - (1 - s_B(x))|,\\ &\frac{1}{2\pi}\sup_{x\in U}|(2\pi - \omega_{\nu A}(x)) - (2\pi - \omega_{\nu B}(x))|)\\ &= \max(\sup_{x\in U}|s_A(x) - s_B(x)|, \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A}(x) - \omega_{\nu B}(x)|)\\ &= \rho(\nu_A, \nu_B)\\ &\le \beta.\end{aligned}$$

It implies that $\overline{A} = (\alpha, \beta)\overline{B}$. □

Corollary 11 *If* $A_{ij} = (\alpha_{ij}, \beta_{ij})B_{ij},\ i \in I_1,\ j \in I_2$, *where* I_1 *and* I_2 *are two index sets, then*

$$\cup_{i\in I_1} \cap_{j\in I_2} A_{ij} = (\inf_{i\in I_1}\inf_{j\in I_2} \alpha_{ij}, \sup_{i\in I_1}\sup_{j\in I_2} \beta_{ij}) \cup_{i\in I_1} \cap_{j\in I_2} B_{ij}$$

and

$$\cap_{i\in I_1} \cup_{j\in I_2} A_{ij} = (\inf_{i\in I_1}\inf_{j\in I_2} \alpha_{ij}, \sup_{i\in I_1}\sup_{j\in I_2} \beta_{ij}) \cap_{i\in I_1} \cup_{j\in I_2} B_{ij}.$$

Corollary 12 *Suppose* $A_i = (\alpha_i, \beta_i)B_i,\ i = 1, 2, \ldots$. *Let*

$$\lim_{n\to\infty} \sup A_i = \cap_{n=1}^{\infty} \cup_{i=n}^{\infty} A_i,\ \lim_{n\to\infty} \inf A_i = \cup_{n=1}^{\infty} \cap_{i=n}^{\infty} A_i,$$

and

$$\lim_{n\to\infty} \sup B_i = \cap_{n=1}^{\infty} \cup_{i=n}^{\infty} B_i,\ \lim_{n\to\infty} \inf B_i = \cup_{n=1}^{\infty} \cap_{i=n}^{\infty} B_i,$$

then

$$\lim_{n\to\infty} \sup A_n = (\inf_{n\ge1} \alpha_n, \sup_{n\ge1} \beta_n) \lim_{n\to\infty} \sup B_n$$

and

$$\lim_{n\to\infty} \inf A_n = (\inf_{n\ge1} \alpha_n, \sup_{n\ge1} \beta_n) \lim_{n\to\infty} \inf B_n.$$

Theorem 14 *If* $A_1 = (\alpha_1, \beta_1)B_1$ *and* $A_2 = (\alpha_2, \beta_2)B_2$, *then* $A_1 \circ A_2 = (\alpha_1 * \alpha_2, 1 - \alpha_1 * \alpha_2)(B_1 \circ B_2)$.

Proof According to Definition 15, we have

$$\begin{aligned}\rho(\mu_{A_1\circ A_2}, \mu_{B_1\circ B_2}) &= \max(\sup_{x\in U}|r_{A_1\circ A_2}(x) - r_{B_1\circ B_2}(x)|,\\ &\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\circ A_2)}(x) - \omega_{\mu(B_1\circ B_2)}(x)|)\end{aligned}$$

and

$$\begin{aligned}\rho(\nu_{A_1\circ A_2}, \nu_{B_1\circ B_2}) &= \max(\sup_{x\in U}|s_{A_1\circ A_2}(x) - s_{B_1\circ B_2}(x)|,\\ &\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\circ A_2)}(x) - \omega_{\nu(B_1\circ B_2)}(x)|)\end{aligned}$$

Base on Definitions 6 and 17, we can obtain that

$$\sup_{x\in U}|r_{A_1\circ A_2}(x) - r_{B_1\circ B_2}(x)|$$

$$
\begin{aligned}
&= \sup_{x\in U}|r_{A_1}(x)\cdot r_{A_2}(x) - r_{B_1}(x)\cdot r_{B_2}(x)| \\
&= \sup_{x\in U}|r_{A_1}(x)\cdot r_{A_2}(x) - r_{A_2}(x)\cdot r_{B_1}(x) \\
&\quad + r_{A_2}(x)\cdot r_{B_1}(x) - r_{B_1}(x)\cdot r_{B_2}(x)| \\
&= \sup_{x\in U}|r_{A_2}(x)(r_{A_1}(x) - r_{B_1}(x)) + r_{B_1}(x)(r_{A_2}(x) - r_{B_2}(x))| \\
&\le \sup_{x\in U}|r_{A_1}(x) - r_{B_1}(x)| + \sup_{x\in U}|r_{A_2}(x) - r_{B_2}(x)| \\
&\le 1-\alpha_1+1-\alpha_2 \\
&= 1-(\alpha_1+\alpha_2-1).
\end{aligned}
$$

We note that $\sup_{x\in U}|r_{A_1\circ A_2}(x) - r_{B_1\circ B_2}(x)| \le 1$, so we have $\sup_{x\in U}|r_{A_1\circ A_2}(x) - r_{B_1\circ B_2}(x)| \le 1-(\alpha_1+\alpha_2-1) = 1-\max(0, \alpha_1+\alpha_2-1) = 1-\alpha_1 * \alpha_2$.

$$
\begin{aligned}
&\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\circ A_2)}(x) - \omega_{\mu(B_1\circ B_2)}(x)| \\
&= \frac{1}{2\pi}\sup_{x\in U}\left|2\pi\left(\frac{\omega_{\mu A_1}(x)}{2\pi}\cdot\frac{\omega_{\mu A_2}(x)}{2\pi}\right) - 2\pi\left(\frac{\omega_{\mu B_1}(x)}{2\pi}\cdot\frac{\omega_{\mu B_2}(x)}{2\pi}\right)\right| \\
&= \frac{1}{2\pi}\sup_{x\in U}\left|\frac{\omega_{\mu A_1}(x)\cdot\omega_{\mu A_2}(x)}{2\pi} - \frac{\omega_{\mu A_2}(x)\cdot\omega_{\mu B_1}(x)}{2\pi}\right. \\
&\quad \left. + \frac{\omega_{\mu A_2}(x)\cdot\omega_{\mu B_1}(x)}{2\pi} - \frac{\omega_{\mu B_1}(x)\cdot\omega_{\mu B_2}(x)}{2\pi}\right| \\
&= \frac{1}{2\pi}\sup_{x\in U}\left|\frac{\omega_{\mu A_2}(x)(\omega_{\mu A_1}(x) - \omega_{\mu B_1}(x))}{2\pi}\right. \\
&\quad \left. + \frac{\omega_{\mu B_1}(x)(\omega_{\mu A_2}(x) - \omega_{\mu B_2}(x))}{2\pi}\right| \\
&\le \frac{1}{2\pi}(\sup_{x\in U}|\omega_{\mu A_1}(x) - \omega_{\mu B_1}(x)| + \sup_{x\in U}|\omega_{\mu A_2}(x) - \omega_{\mu B_2}(x)|) \\
&\le 1-\alpha_1+1-\alpha_2 \\
&= 1-(\alpha_1+\alpha_2-1).
\end{aligned}
$$

We note that $\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\circ A_2)}(x) - \omega_{\mu(B_1\circ B_2)}(x)| \le 1$, so we have $\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\circ A_2)}(x) - \omega_{\mu(B_1\circ B_2)}(x)| \le 1-(\alpha_1+\alpha_2-1) = 1-\max(0,\alpha_1+\alpha_2-1) = 1-\alpha_1*\alpha_2$.

In the similar way, we can prove that $\sup_{x\in U}|s_{A_1\circ A_2}(x) - s_{B_1\circ B_2}(x)| \le \beta_1+\beta_2 \le 1-\alpha_1+1-\alpha_2 = 1-(\alpha_1+\alpha_2-1) = 1-\max(0,\alpha_1+\alpha_2-1) = 1-\alpha_1*\alpha_2$ and $\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\circ A_2)}(x) - \omega_{\nu(B_1\circ B_2)}(x)| \le \beta_1+\beta_2 \le 1-\alpha_1+1-\alpha_2 = 1-(\alpha_1+\alpha_2-1) = 1-\max(0,\alpha_1+\alpha_2-1) = 1-\alpha_1*\alpha_2$.

It implies that

$$A_1\circ A_2 = (\alpha_1*\alpha_2, 1-\alpha_1*\alpha_2)(B_1\circ B_2).$$

□

Corollary 13 *Suppose $A_i = (\alpha_i, \beta_i)B_i$, $i\in I$, where I is an index set, then $A_1\circ A_2\circ\cdots\circ A_i = (\alpha_1*\alpha_2*\cdots*\alpha_i, 1-\alpha_1*\alpha_2*\cdots*\alpha_i)(B_1\circ B_2\circ\cdots\circ B_i)$.*

Theorem 15 *If $A_n = (\alpha_n, \beta_n)B_n$, $n = 1, 2, ..., N$, then $A_1\times A_2\times\cdots\times A_n = (\inf_{1\le n\le N}\alpha_n, \sup_{1\le n\le N}\beta_n)(B_1\times B_2\times\cdots\times B_n)$.*

Proof Trivial from Definitions 7 and 17. □

Theorem 16 *If $A_1 = (\alpha_1, \beta_1)B_1$ and $A_2 = (\alpha_2, \beta_2)B_2$, then $A_1\hat{+}A_2 = (\alpha_1*\alpha_2, 1-\alpha_1*\alpha_2)(B_1\hat{+}B_2)$.*

Proof According to Definition 15, we have

$$
\begin{aligned}
&\rho(\mu_{A_1\hat{+}A_2}, \mu_{B_1\hat{+}B_2}) \\
&\quad = \max(\sup_{x\in U}|r_{A_1\hat{+}A_2}(x) - r_{B_1\hat{+}B_2}(x)|, \\
&\qquad \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\hat{+}A_2)}(x) - \omega_{\mu(B_1\hat{+}B_2)}(x)|)
\end{aligned}
$$

and

$$
\begin{aligned}
&\rho(\nu_{A_1\hat{+}A_2}, \nu_{B_1\hat{+}B_2}) \\
&\quad = \max(\sup_{x\in U}|s_{A_1\hat{+}A_2}(x) - s_{B_1\hat{+}B_2}(x)|, \\
&\qquad \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\hat{+}A_2)}(x) - \omega_{\nu(B_1\hat{+}B_2)}(x)|).
\end{aligned}
$$

Based on Definitions 8 and 17, we can obtain that

$$
\begin{aligned}
&\sup_{x\in U}|r_{A_1\hat{+}A_2}(x) - r_{B_1\hat{+}B_2}(x)| \\
&\quad = \sup_{x\in U}|r_{A_1}(x) + r_{A_2}(x) - r_{A_1}(x)\cdot r_{A_2}(x) \\
&\qquad - (r_{B_1}(x) + r_{B_2}(x) - r_{B_1}(x)\cdot r_{B_2}(x))| \\
&\quad = \sup_{x\in U}|(1-r_{B_2}(x))(r_{A_1}(x) - r_{B_1}(x)) \\
&\qquad + (1-r_{A_1}(x))(r_{A_2}(x) - r_{B_2}(x))| \\
&\quad \le \sup_{x\in U}|r_{A_1}(x) - r_{B_1}(x)| + \sup_{x\in U}|r_{A_2}(x) - r_{B_2}(x)| \\
&\quad \le 1-\alpha_1+1-\alpha_2 \\
&\quad = 1-(\alpha_1+\alpha_2-1).
\end{aligned}
$$

We note that $\sup_{x\in U}|r_{A_1\hat{+}A_2}(x) - r_{B_1\hat{+}B_2}(x)| \le 1$, so we have $\sup_{x\in U}|r_{A_1\hat{+}A_2}(x) - r_{B_1\hat{+}B_2}(x)| \le 1-(\alpha_1+\alpha_2-1) = 1-\max(0,\alpha_1+\alpha_2-1) = 1-\alpha_1*\alpha_2$.

$$
\begin{aligned}
&\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\hat{+}A_2)}(x) - \omega_{\mu(B_1\hat{+}B_2)}(x)| \\
&\quad = \frac{1}{2\pi}\sup_{x\in U}\left|2\pi\left(\frac{\omega_{\mu A_1}(x)}{2\pi} + \frac{\omega_{\mu A_2}(x)}{2\pi} - \frac{\omega_{\mu A_1}(x)}{2\pi}\cdot\frac{\omega_{\mu A_2}(x)}{2\pi}\right)\right. \\
&\qquad \left. - 2\pi\left(\frac{\omega_{\mu B_1}(x)}{2\pi} + \frac{\omega_{\mu B_2}(x)}{2\pi} - \frac{\omega_{\mu B_1}(x)}{2\pi}\cdot\frac{\omega_{\mu B_2}(x)}{2\pi}\right)\right| \\
&\quad = \sup_{x\in U}\left|\left(1-\frac{\omega_{\mu A_2}(x)}{2\pi}\right)\left(\frac{\omega_{\mu A_1}(x)}{2\pi} - \frac{\omega_{\mu B_1}(x)}{2\pi}\right)\right. \\
&\qquad \left. + \left(1-\frac{\omega_{\mu B_1}(x)}{2\pi}\right)\left(\frac{\omega_{\mu A_2}(x)}{2\pi} - \frac{\omega_{\mu B_2}(x)}{2\pi}\right)\right| \\
&\quad \le \sup_{x\in U}\left(|1-\frac{\omega_{\mu A_2}(x)}{2\pi}||\frac{\omega_{\mu A_1}(x)}{2\pi} - \frac{\omega_{\mu B_1}(x)}{2\pi}|\right.
\end{aligned}
$$

$$+\left|1-\frac{\omega_{\mu B_1}(x)}{2\pi}\right|\left|\frac{\omega_{\mu A_2}(x)}{2\pi}-\frac{\omega_{\mu B_2}(x)}{2\pi}\right|\Big)$$
$$\le \frac{1}{2\pi}\left(\sup_{x\in U}\left|\omega_{\mu A_1}(x)-\omega_{\mu B_1}(x)\right|+\left|\omega_{\mu A_2}(x)-\omega_{\mu B_2}(x)\right|\right)$$
$$\le 1-\alpha_1+1-\alpha_2=1-(\alpha_1+\alpha_2-1).$$

We note that $\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\hat{+}A_2)}(x)-\omega_{\nu(B_1\hat{+}B_2)}(x)|\le 1$, so we have $\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\hat{+}A_2)}(x)-\omega_{\nu(B_1\hat{+}B_2)}(x)|\le 1-(\alpha_1+\alpha_2-1)=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2$.

In the similar way, we can prove that $\sup_{x\in U}|s_{\nu_{(A_1\hat{+}A_2)}}(x)-s_{\nu_{(B_1\hat{+}B_2)}}(x)|\le\beta_1+\beta_2\le 1-\alpha_1+1-\alpha_2=1-(\alpha_1+\alpha_2-1)=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2$ and $\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\hat{+}A_2)}(x)-\omega_{\nu(B_1\hat{+}B_2)}(x)|\le\beta_1+\beta_2\le 1-\alpha_1+1-\alpha_2=1-(\alpha_1+\alpha_2-1)=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2$.

It implies that

$$A_1\hat{+}A_2=(\alpha_1*\alpha_2,1-\alpha_1*\alpha_2)(B_1\hat{+}B_2).$$

□

Corollary 14 *Suppose $A_i=(\alpha_i,\beta_i)B_i$, $i\in I$, where I is an index set, then $A_1\hat{+}A_2\hat{+}\cdots\hat{+}A_i=(\alpha_1*\alpha_2*\cdots*\alpha_i,1-\alpha_1*\alpha_2*\cdots*\alpha_i)(B_1\hat{+}B_2\hat{+}\cdots\hat{+}B_i)$.*

Theorem 17 *If $A_1=(\alpha_1,\beta_1)B_1$ and $A_2=(\alpha_2,\beta_2)B_2$, then $A_1\dot{\cup}A_2=(\alpha_1*\alpha_2,1-\alpha_1*\alpha_2)(B_1\dot{\cup}B_2)$.*

Proof According to Definition 15, we have

$$\rho(\mu_{A_1\dot{\cup}A_2},\mu_{B_1\dot{\cup}B_2})=\max(\sup_{x\in U}|r_{A_1\dot{\cup}A_2}(x)-r_{B_1\dot{\cup}B_2}(x)|,\ \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\dot{\cup}A_2)}(x)-\omega_{\mu(B_1\dot{\cup}B_2)}(x)|)$$

and

$$\rho(\nu_{A_1\dot{\cup}A_2},\nu_{B_1\dot{\cup}B_2})=\max(\sup_{x\in U}|s_{A_1\dot{\cup}A_2}(x)-s_{B_1\dot{\cup}B_2}(x)|,\ \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\dot{\cup}A_2)}(x)-\omega_{\nu(B_1\dot{\cup}B_2)}(x)|).$$

Based on Definitions 9 and 17, we can obtain that

$$\sup_{x\in U}|r_{A_1\dot{\cup}A_2}(x)-r_{B_1\dot{\cup}B_2}(x)|$$
$$=\sup_{x\in U}|\min(1,r_{A_1}(x)+r_{A_2}(x))-\min(1,r_{B_1}(x)+r_{B_2}(x))|$$
$$\le\sup_{x\in U}|r_{A_1}(x)+r_{A_2}(x)-r_{B_1}(x)-r_{B_2}(x)|$$
$$\le\sup_{x\in U}|r_{A_1}(x)-r_{B_1}(x)|+\sup_{x\in U}|r_{A_2}(x)-r_{B_2}(x)|$$
$$\le 1-\alpha_1+1-\alpha_2$$
$$=1-(\alpha_1+\alpha_2-1).$$

We note that $\sup_{x\in U}|r_{A_1\dot{\cup}A_2}(x)-r_{B_1\dot{\cup}B_2}(x)|\le 1$, so we have $\sup_{x\in U}|r_{A_1\dot{\cup}A_2}(x)-r_{B_1\dot{\cup}B_2}(x)|\le 1-(\alpha_1+\alpha_2-1)=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2$

$$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\dot{\cup}A_2)}(x)-\omega_{\mu(B_1\dot{\cup}B_2)}(x)|$$
$$=\frac{1}{2\pi}\sup_{x\in U}|\min(2\pi,\omega_{\mu A_1}(x)+\omega_{\mu A_2}(x))-\min(2\pi,\omega_{\mu B_1}(x)+\omega_{\mu B_2}(x))|$$
$$\le\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A_1}(x)+\omega_{\mu A_2}(x)-\omega_{\mu B_1}(x)-\omega_{\mu B_2}(x)|$$
$$\le\frac{1}{2\pi}(\sup_{x\in U}|\omega_{\mu A_1}(x)-\omega_{\mu B_1}(x)|+\sup_{x\in U}|\omega_{\mu A_2}(x)-\omega_{\mu B_2}(x)|)$$
$$\le 1-\alpha_1+1-\alpha_2$$
$$=1-(\alpha_1+\alpha_2-1).$$

We note that $\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\dot{\cup}A_2)}(x)-\omega_{\mu(B_1\dot{\cup}B_2)}(x)|\le 1$, so we have $\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\dot{\cup}A_2)}(x)-\omega_{\mu(B_1\dot{\cup}B_2)}(x)|\le 1-(\alpha_1+\alpha_2-1)=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2$.

In the similar way, we can prove that $\sup_{x\in U}|s_{A_1\dot{\cup}A_2}(x)-s_{B_1\dot{\cup}B_2}(x)|\le\beta_1+\beta_2\le 1-\alpha_1+1-\alpha_2=1-(\alpha_1+\alpha_2-1)=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2$ and $\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\dot{\cup}A_2)}(x)-\omega_{\nu(B_1\dot{\cup}B_2)}(x)|\le\beta_1+\beta_2\le 1-\alpha_1+1-\alpha_2=1-(\alpha_1+\alpha_2-1)=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2$.

It implies that

$$A_1\dot{\cup}A_2=(\alpha_1*\alpha_2,1-\alpha_1*\alpha_2)(B_1\dot{\cup}B_2).$$

□

Corollary 15 *Suppose $A_i=(\alpha_i,\beta_i)B_i$, $i\in I$, where I is an index set, then $A_1\dot{\cup}A_2\dot{\cup}\cdots\dot{\cup}A_i=(\alpha_1*\alpha_2*\cdots*\alpha_i,1-\alpha_1*\alpha_2*\cdots*\alpha_i)(B_1\dot{\cup}B_2\dot{\cup}\cdots\dot{\cup}B_i)$.*

Theorem 18 *If $A_1=(\alpha_1,\beta_1)B_1$ and $A_2=(\alpha_2,\beta_2)B_2$, then $A_1\dot{\cap}A_2=(\alpha_1*\alpha_2,1-\alpha_1*\alpha_2)(B_1\dot{\cap}B_2)$.*

Proof According to Definition 15, we have

$$\rho(\mu_{A_1\dot{\cap}A_2},\mu_{B_1\dot{\cap}B_2})=\max(\sup_{x\in U}|r_{A_1\dot{\cap}A_2}(x)-r_{B_1\dot{\cap}B_2}(x)|,\ \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\dot{\cap}A_2)}(x)-\omega_{\mu(B_1\dot{\cap}B_2)}(x)|)$$

and

$$\rho(\nu_{A_1\dot{\cap}A_2},\nu_{B_1\dot{\cap}B_2})=\max(\sup_{x\in U}|s_{A_1\dot{\cap}A_2}(x)-s_{B_1\dot{\cap}B_2}(x)|,$$

$$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\dot{\cap}A_2)}(x)-\omega_{\nu(B_1\dot{\cap}B_2)}(x)|).$$

Based on Definitions 10 and 17, we can obtain that

$$\begin{aligned}&\sup_{x\in U}|r_{A_1\dot{\cap}A_2}(x)-r_{B_1\dot{\cap}B_2}(x)|\\&=\sup_{x\in U}|\max(0,r_{A_1}(x)+r_{A_2}(x)-1)\\&\quad-\max(0,r_{B_1}(x)+r_{B_2}(x)-1)|\\&\le\sup_{x\in U}|r_{A_1}(x)+r_{A_2}(x)-r_{B_1}(x)-r_{B_2}(x)|\\&\le\sup_{x\in U}|r_{A_1}(x)-r_{B_1}(x)|+\sup_{x\in U}|r_{A_2}(x)-r_{B_2}(x)|\\&\le1-\alpha_1+1-\alpha_2\\&=1-(\alpha_1+\alpha_2-1).\end{aligned}$$

We note that $\sup_{x\in U}|r_{A_1\dot{\cap}A_2}(x)-r_{B_1\dot{\cap}B_2}(x)|\le 1$, so we have $\sup_{x\in U}|r_{A_1\dot{\cap}A_2}(x)-r_{B_1\dot{\cap}B_2}(x)|\le 1-(\alpha_1+\alpha_2-1)=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2$.

$$\begin{aligned}&\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\dot{\cap}A_2)}(x)-\omega_{\mu(B_1\dot{\cap}B_2)}(x)|\\&=\frac{1}{2\pi}\sup_{x\in U}|\max(0,\omega_{\mu A_1}(x)+\omega_{\mu A_2}(x)-2\pi)\\&\quad-\max(0,\omega_{\mu B_1}(x)+\omega_{\mu B_2}(x)-2\pi)|\\&\le\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A_1}(x)+\omega_{\mu A_2}(x)-\omega_{\mu B_1}(x)-\omega_{\mu B_2}(x)|\\&\le\frac{1}{2\pi}(\sup_{x\in U}|\omega_{\mu A_1}(x)-\omega_{\mu B_1}(x)|+\sup_{x\in U}|\omega_{\mu A_2}(x)-\omega_{\mu B_2}(x)|)\\&\le1-\alpha_1+1-\alpha_2\\&=1-(\alpha_1+\alpha_2-1)\\&=1-\alpha_1*\alpha_2.\end{aligned}$$

We note that $\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\dot{\cap}A_2)}(x)-\omega_{\mu(B_1\dot{\cap}B_2)}(x)|\le 1$, so we have $\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\dot{\cap}A_2)}(x)-\omega_{\mu(B_1\dot{\cap}B_2)}(x)|\le 1-(\alpha_1+\alpha_2-1)=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2$.

In the similar way, we can prove that $\sup_{x\in U}|s_{A_1\dot{\cap}A_2}(x)-s_{B_1\dot{\cap}B_2}(x)|\le\beta_1+\beta_2\le 1-\alpha_1+1-\alpha_2=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2$ and $\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\dot{\cap}A_2)}(x)-\omega_{\nu(B_1\dot{\cap}B_2)}(x)|\le\beta_1+\beta_2\le 1-\alpha_1+1-\alpha_2=1-(\alpha_1+\alpha_2-1)=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2$.

It implies that

$$A_1\dot{\cap}A_2=(\alpha_1*\alpha_2,1-\alpha_1*\alpha_2)(B_1\dot{\cap}B_2).$$

□

Corollary 16 *Suppose $A_i=(\alpha_i,\beta_i)B_i$, $i\in I$, where I is an index set, then $A_1\dot{\cap}A_2\dot{\cap}\cdots\dot{\cap}A_i=(\alpha_1*\alpha_2*\cdots*\alpha_i,1-\alpha_1*\alpha_2*\cdots*\alpha_i)(B_1\dot{\cap}B_2\dot{\cap}\cdots\dot{\cap}B_i)$.*

Theorem 19 *If $A_1=(\alpha_1,\beta_1)B_1$ and $A_2=(\alpha_2,\beta_2)B_2$, then $A_1|-|A_2=(\alpha_1*\alpha_2,1-\alpha_1*\alpha_2)(B_1|-|B_2)$.*

Proof According to Definition 15, we have

$$\begin{aligned}&\rho(\mu_{A_1|-|A_2},\mu_{B_1|-|B_2})\\&=\max(\sup_{x\in U}|r_{A_1|-|A_2}(x)-r_{B_1|-|B_2}(x)|,\\&\quad\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1|-|A_2)}(x)-\omega_{\mu(B_1|-|B_2)}(x)|)\end{aligned}$$

and

$$\begin{aligned}&\rho(\nu_{A_1|-|A_2},\nu_{B_1|-|B_2})\\&=\max(\sup_{x\in U}|s_{A_1|-|A_2}(x)-s_{B_1|-|B_2}(x)|,\\&\quad\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1|-|A_2)}(x)-\omega_{\nu(B_1|-|B_2)}(x)|).\end{aligned}$$

Based on Definitions 11 and 17, we can obtain that

$$\begin{aligned}&\sup_{x\in U}|r_{A_1|-|A_2}(x)-r_{B_1|-|B_2}(x)|\\&=\sup_{x\in U}|\max(0,r_{A_1}(x)-r_{A_2}(x))-\max(0,r_{B_1}(x)-r_{B_2}(x))|\\&\le\sup_{x\in U}|r_{A_1}(x)-r_{A_2}(x)-r_{B_1}(x)+r_{B_2}(x)|\\&\le\sup_{x\in U}|r_{A_1}(x)-r_{B_1}(x)|+\sup_{x\in U}|r_{A_2}(x)-r_{B_2}(x)|\\&\le1-\alpha_1+1-\alpha_2\\&=1-(\alpha_1+\alpha_2-1).\end{aligned}$$

We note that $\sup_{x\in U}|r_{A_1|-|A_2}(x)-r_{B_1|-|B_2}(x)|\le 1$, so we have $\sup_{x\in U}|r_{A_1|-|A_2}(x)-r_{B_1|-|B_2}(x)|\le 1-(\alpha_1+\alpha_2-1)=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2$.

$$\begin{aligned}&\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1|-|A_2)}(x)-\omega_{\mu(B_1|-|B_2)}(x)|\\&=\frac{1}{2\pi}\sup_{x\in U}|\max(0,\omega_{\mu A_1}(x)-\omega_{\mu A_2}(x))\\&\quad-\max(0,\omega_{\mu B_1}(x)-\omega_{\mu B_2}(x))|\\&\le\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A_1}(x)-\omega_{\mu A_2}(x)-\omega_{\mu B_1}(x)+\omega_{\mu B_2}(x)|\\&\le\frac{1}{2\pi}(\sup_{x\in U}|\omega_{\mu A_1}(x)-\omega_{\mu B_1}(x)|+\sup_{x\in U}|\omega_{\mu A_2}(x)-\omega_{\mu B_2}(x)|)\\&\le1-\alpha_1+1-\alpha_2\\&=1-(\alpha_1+\alpha_2-1).\end{aligned}$$

We note that $\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1|-|A_2)}(x)-\omega_{\mu(B_1|-|B_2)}(x)|\le 1$, so we have $\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1|-|A_2)}(x)-\omega_{\mu(B_1|-|B_2)}(x)|\le 1-(\alpha_1+\alpha_2-1)=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2$.

In the similar way, we can prove that $\sup_{x\in U}|s_{A_1|-|A_2}(x)-s_{B_1|-|B_2}(x)|\le\beta_1+\beta_2\le 1-\alpha_1+1-\alpha_2=1-(\alpha_1+\alpha_2-1)=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2$ and $\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1|-|A_2)}(x)-\omega_{\nu(B_1|-|B_2)}(x)|\le\beta_1+\beta_2\le 1-\alpha_1+1-\alpha_2=1-(\alpha_1+\alpha_2-1)=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2$.

It implies that

$$A_1| - |A_2 = (\alpha_1 * \alpha_2, 1 - \alpha_1 * \alpha_2)(B_1| - |B_2).$$

□

Corollary 17 *Suppose* $A_i = (\alpha_i, \beta_i)B_i$, $i \in I$, *where* I *is an index set, then* $A_1| - |A_2| - | \cdots | - |A_i = (\alpha_1 * \alpha_2 * \cdots * \alpha_i, 1 - \alpha_1 * \alpha_2 * \cdots * \alpha_i)(B_1| - |B_2| - | \cdots | - |B_i)$.

Theorem 20 *If* $A_1 = (\alpha_1, \beta_1)B_1$ *and* $A_2 = (\alpha_2, \beta_2)B_2$, *then* $A_1 \nabla A_2 = (\alpha_1 * \alpha_2, 1 - \alpha_1 * \alpha_2)(B_1 \nabla B_2)$.

Proof According to Definition 15, we have

$$\begin{aligned}
&\rho(\mu_{A_1 \nabla A_2}, \mu_{B_1 \nabla B_2}) \\
&\quad = \max(\sup_{x \in U} |r_{A_1 \nabla A_2}(x) - r_{B_1 \nabla B_2}(x)|, \\
&\quad\quad \frac{1}{2\pi} \sup_{x \in U} |\omega_{\mu(A_1 \nabla A_2)}(x) - \omega_{\mu(B_1 \nabla B_2)}(x)|)
\end{aligned}$$

and

$$\begin{aligned}
&\rho(\nu_{A_1 \nabla A_2}, \nu_{B_1 \nabla B_2}) \\
&\quad = \max(\sup_{x \in U} |s_{A_1 \nabla A_2}(x) - s_{B_1 \nabla B_2}(x)|, \\
&\quad\quad \frac{1}{2\pi} \sup_{x \in U} |\omega_{\nu(A_1 \nabla A_2)}(x) - \omega_{\nu(B_1 \nabla B_2)}(x)|)
\end{aligned}$$

Based on Definitions 12 and 17, we can obtain that

$$\begin{aligned}
&\sup_{x \in U} |r_{A_1 \nabla A_2}(x) - r_{B_1 \nabla B_2}(x)| \\
&\quad = \sup_{x \in U} ||r_{A_1}(x) - r_{A_2}(x)| - |r_{B_1}(x) - r_{B_2}(x)|| \\
&\quad = \sup_{x \in U} |\max(r_{A_1}(x) - r_{A_2}(x), r_{A_2}(x) - r_{A_1}(x)) \\
&\quad\quad - \max(r_{B_1}(x) - r_{B_2}(x), r_{B_2}(x) - r_{B_1}(x))| \\
&\quad \le \sup_{x \in U} |r_{A_1}(x) - r_{B_1}(x)| + \sup_{x \in U} |r_{A_2}(x) - r_{B_2}(x)| \\
&\quad \le 1 - \alpha_1 + 1 - \alpha_2 \\
&\quad = 1 - (\alpha_1 + \alpha_2 - 1).
\end{aligned}$$

We note that $\sup_{x \in U} |r_{A_1 \nabla A_2}(x) - r_{B_1 \nabla B_2}(x)| \le 1$, so we have $\sup_{x \in U} |r_{A_1 \nabla A_2}(x) - r_{B_1 \nabla B_2}(x)| \le 1 - (\alpha_1 + \alpha_2 - 1) = 1 - \max(0, \alpha_1 + \alpha_2 - 1) = 1 - \alpha_1 * \alpha_2$.

$$\begin{aligned}
&\frac{1}{2\pi} \sup_{x \in U} |\omega_{\mu(A_1 \nabla A_2)}(x) - \omega_{\mu(B_1 \nabla B_2)}(x)| \\
&\quad = \frac{1}{2\pi} \sup_{x \in U} ||\omega_{\mu A_1}(x) - \omega_{\mu A_2}(x)| - |\omega_{\mu B_1}(x) - \omega_{\mu B_2}(x)|| \\
&\quad = \frac{1}{2\pi} \sup_{x \in U} |\max(\omega_{\mu A_1}(x) - \omega_{\mu A_2}(x), \omega_{\mu A_2}(x) - \omega_{\mu A_1}(x)) \\
&\quad\quad - \max(\omega_{\mu B_1}(x) - \omega_{\mu B_2}(x), \omega_{\mu B_2}(x) - \omega_{\mu B_1}(x))| \\
&\quad \le \frac{1}{2\pi} (\sup_{x \in U} |\omega_{\mu A_1}(x) - \omega_{\mu B_1}(x)| + \sup_{x \in U} |\omega_{\mu A_2}(x) - \omega_{\mu B_2}(x)|) \\
&\quad \le 1 - \alpha_1 + 1 - \alpha_2 \\
&\quad = 1 - (\alpha_1 + \alpha_2 - 1).
\end{aligned}$$

We note that $\frac{1}{2\pi} \sup_{x \in U} |\omega_{\mu(A_1 \nabla A_2)}(x) - \omega_{\mu(B_1 \nabla B_2)}(x)| \le 1$, so we have $\frac{1}{2\pi} \sup_{x \in U} |\omega_{\mu(A_1 \nabla A_2)}(x) - \omega_{\mu(B_1 \nabla B_2)}(x)| \le 1 - (\alpha_1 + \alpha_2 - 1) = 1 - \max(0, \alpha_1 + \alpha_2 - 1) = 1 - \alpha_1 * \alpha_2$.

In the similar way, we can prove that $\sup_{x \in U} |s_{A_1 \nabla A_2}(x) - s_{B_1 \nabla B_2}(x)| \le \beta_1 + \beta_2 \le 1 - \alpha_1 + 1 - \alpha_2 = 1 - (\alpha_1 + \alpha_2 - 1) = 1 - \max(0, \alpha_1 + \alpha_2 - 1) = 1 - \alpha_1 * \alpha_2$ and $\frac{1}{2\pi} \sup_{x \in U} |\omega_{\nu(A_1 \nabla A_2)}(x) - \omega_{\nu(B_1 \nabla B_2)}(x)| \le \beta_1 + \beta_2 \le 1 - \alpha_1 + 1 - \alpha_2 = 1 - (\alpha_1 + \alpha_2 - 1) = 1 - \max(0, \alpha_1 + \alpha_2 - 1) = 1 - \alpha_1 * \alpha_2$.

It implies that

$$A_1 \nabla A_2 = (\alpha_1 * \alpha_2, 1 - \alpha_1 * \alpha_2)(B_1 \nabla B_2).$$

□

Corollary 18 *Suppose* $A_i = (\alpha_i, \beta_i)B_i$, $i \in I$, *where* I *is an index set, then* $A_1 \nabla A_2 \nabla \cdots \nabla A_i = (\alpha_1 * \alpha_2 * \cdots * \alpha_i, 1 - \alpha_1 * \alpha_2 * \cdots * \alpha_i)(B_1 \nabla B_2 \nabla \cdots \nabla B_i)$.

Theorem 21 *If* $A_1 = (\alpha_1, \beta_1)B_1$ *and* $A_2 = (\alpha_2, \beta_2)B_2$, *then* $A_1 ||_\lambda A_2 = (\min(\alpha_1, \alpha_2), \max(\beta_1, \beta_2))(B_1 ||_\lambda B_2)$.

Proof According to Definition 15, we have

$$\begin{aligned}
&\rho(\mu_{A_1 ||_\lambda A_2}, \mu_{B_1 ||_\lambda B_2}) \\
&\quad = \max(\sup_{x \in U} |r_{A_1 ||_\lambda A_2}(x) - r_{B_1 ||_\lambda B_2}(x)|, \\
&\quad\quad \frac{1}{2\pi} \sup_{x \in U} |\omega_{\mu(A_1 ||_\lambda A_2)}(x) - \omega_{\mu(B_1 ||_\lambda B_2)}(x)|)
\end{aligned}$$

and

$$\begin{aligned}
&\rho(\nu_{A_1 ||_\lambda A_2}, \nu_{B_1 ||_\lambda B_2}) \\
&\quad = \max(\sup_{x \in U} |s_{A_1 ||_\lambda A_2}(x) - s_{B_1 ||_\lambda B_2}(x)|, \\
&\quad\quad \frac{1}{2\pi} \sup_{x \in U} |\omega_{\nu(A_1 ||_\lambda A_2)}(x) - \omega_{\nu(B_1 ||_\lambda B_2)}(x)|).
\end{aligned}$$

Based on Definitions 13 and 17, we can obtain that

$$\begin{aligned}
&\sup_{x \in U} |r_{A_1 ||_\lambda A_2}(x) - r_{B_1 ||_\lambda B_2}(x)| \\
&\quad = \sup_{x \in U} [\lambda |\min(r_{A_1}(x), r_{A_2}(x)) - \min(r_{B_1}(x), r_{B_2}(x))| \\
&\quad\quad + (1 - \lambda) |\max(r_{A_1}(x), r_{A_2}(x)) - \max(r_{B_1}(x), r_{B_2}(x))|] \\
&\quad \le \sup_{x \in U} [\lambda \max(|r_{A_1}(x) - r_{A_2}(x)|, |r_{B_1}(x) - r_{B_2}(x)|) \\
&\quad\quad + (1 - \lambda) \max(|r_{A_1}(x) - r_{A_2}(x)|, |r_{B_1}(x) - r_{B_2}(x)|)] \\
&\quad \le \max(1 - \alpha_1, 1 - \alpha_2) = 1 - \min(\alpha_1, \alpha_2).
\end{aligned}$$

$$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1||_\lambda A_2)}(x)-\omega_{\mu(B_1||_\lambda B_2)}(x)|$$
$$=\frac{1}{2\pi}\sup_{x\in U}[\lambda|\min(\omega_{\mu_{A_1}}(x),\omega_{\mu_{A_2}}(x))-\min(\omega_{\mu_{B_1}}(x),\omega_{\mu_{B_2}}(x))|$$
$$+(1-\lambda)|\max(\omega_{\mu_{A_1}}(x),\omega_{\mu_{A_2}}(x))-\max(\omega_{\mu_{B_1}}(x),\omega_{\mu_{B_2}}(x))|]$$
$$\leq\frac{1}{2\pi}\sup_{x\in U}[\lambda\max(|\omega_{\mu_{A_1}}(x)-\omega_{\mu_{A_2}}(x)|,|\omega_{\mu_{B_1}}(x)-\omega_{\mu_{B_2}}(x)|)$$
$$+(1-\lambda)\max(|\omega_{\mu_{A_1}}(x)-\omega_{\mu_{A_2}}(x)|,|\omega_{\mu_{B_1}}(x)-\omega_{\mu_{B_2}}(x)|)]$$
$$\leq\max(1-\alpha_1,1-\alpha_2)$$
$$=1-\min(\alpha_1,\alpha_2).$$
$$\sup_{x\in U}|s_{A_1||_\lambda A_2}(x)-s_{B_1||_\lambda B_2}(x)|$$
$$=\sup_{x\in U}[\lambda|\max(s_{A_1}(x),s_{A_2}(x))-\max(s_{B_1}(x),s_{B_2}(x))|$$
$$+(1-\lambda)|\min(s_{A_1}(x),s_{A_2}(x))-\min(s_{B_1}(x),s_{B_2}(x))|]$$
$$\leq\sup_{x\in U}[\lambda\max(|s_{A_1}(x)-s_{A_2}(x)|,|s_{B_1}(x)-s_{B_2}(x)|)$$
$$+(1-\lambda)\max(|s_{A_1}(x)-s_{A_2}(x)|,|s_{B_1}(x)-s_{B_2}(x)|]$$
$$\leq\max(\beta_1,\beta_2).$$
$$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1||_\lambda A_2)}(x)-\omega_{\nu(B_1||_\lambda B_2)}(x)|$$
$$=\frac{1}{2\pi}\sup_{x\in U}[\lambda|\max(\omega_{\nu_{A_1}}(x),\omega_{\nu_{A_2}}(x))-\max(\omega_{\nu_{B_1}}(x),\omega_{\nu_{B_2}}(x))|$$
$$+(1-\lambda)|\min(\omega_{\nu_{A_1}}(x),\omega_{\nu_{A_2}}(x))-\min(\omega_{\nu_{B_1}}(x),\omega_{\nu_{B_2}}(x))|]$$
$$\leq\frac{1}{2\pi}\sup_{x\in U}[\lambda\max(|\omega_{\nu_{A_1}}(x)-\omega_{\nu_{A_2}}(x)|,|\omega_{\nu_{B_1}}(x)-\omega_{\nu_{B_2}}(x)|)$$
$$+(1-\lambda)\max(|\omega_{\nu_{A_1}}(x)-\omega_{\nu_{A_2}}(x)|,|\omega_{\nu_{B_1}}(x)-\omega_{\nu_{B_2}}(x)|)]$$
$$\leq\max(\beta_1,\beta_2).$$

It implies that

$$A_1||_\lambda A_2=(\min(\alpha_1,\alpha_2),\max(\beta_1,\beta_2))(B_1||_\lambda B_2).$$

□

Corollary 19 *Suppose $A_i=(\alpha_i,\beta_i)B_i$, $i\in I$, where I is an index set, then $A_1||_\lambda A_2||_\lambda\cdots||_\lambda A_i=(\inf(\alpha_1,\alpha_2,\cdots,\alpha_i),\sup(\beta_1,\beta_2,\cdots,\beta_i))(B_1||_\lambda B_2||_\lambda\cdots||_\lambda B_i)$.*

4 Complex intuitionistic fuzzy relations

In this section, complex intuitionistic fuzzy relations are discussed.

Definition 18 (Atanassov 1986, 1999) Let U and W be two arbitrary finite non-empty sets. An intuitionistic fuzzy relation $\bar{R}(U,W)$ is an intuitionistic fuzzy subset of the product space $U\times W$. The relation $\bar{R}(U,W)$ is characterized by the membership function $\mu_{\bar{R}}(x,y):U\times W\to[0,1]$ and the non-membership function $\nu_{\bar{R}}(x,y):U\times W\to[0,1]$ with the condition

$$0\leq\mu_{\bar{R}}(x,y)+\nu_{\bar{R}}(x,y)\leq 1$$

for all $x\in U$ and $y\in W$.

Like any intuitionistic fuzzy set, $\bar{R}(U,W)$ may be represented as the set of ordered pairs

$$\bar{R}(U,W)=\{((x,y),\mu_{\bar{R}}(x,y),\nu_{\bar{R}}(x,y))\mid(x,y)\in U\times W\}.$$

Definition 19 Let U and W be two arbitrary finite non-empty sets. A complex intuitionistic fuzzy relation $R(U,W)$ is a complex intuitionistic fuzzy subset of the product space $U\times W$. The relation $R(U,W)$ is characterized by the membership function $\mu_R(x,y):U\times W\to\{a|a\in C,|a|\leq 1\}$ and the non-membership function $\nu_R(x,y):U\times W\to\{a'|a'\in C,|a'|\leq 1\}$ with the condition

$$|\mu_R(x,y)+\nu_R(x,y)|\leq 1,$$

where $x\in U$ and $y\in W$, $\mu_R(x,y)$ and $\nu_R(x,y)$ assign each pair (x,y) a complex-valued grade of membership and a complex-valued grade of non-membership to the set $R(U,W)$.

Like any complex intuitionistic fuzzy set, $R(U,W)$ may be represented as the set of ordered pairs

$$R(U,W)=\{((x,y),\mu_R(x,y),\nu_R(x,y))\mid(x,y)\in U\times W\}.$$

The value $\mu_R(x,y)$ and $\nu_R(x,y)$ may receive lie within the unit circle in the complex plane, and are on the form $\mu_R(x,y)=r_R(x)\cdot e^{i\bar{\omega}_{\mu R}(x)}$ and $\nu_R(x,y)=s_R(x)\cdot e^{i\bar{\omega}_{\nu R}(x)}$, where $i=\sqrt{-1}$, each of $r_R(x)$ and $s_R(x)$ are real-valued and both belong to the interval [0, 1] such that $0\leq r_R(x)+s_R(x)\leq 1$, also $\bar{\omega}_{\mu R}(x)$ and $\bar{\omega}_{\nu R}(x)$ are periodic function whose periodic law and principal period are, respectively, 2π and $0<\omega_{\mu R}(x),\omega_{\nu R}(x)\leq 2\pi$.

The complex membership function $\mu_R(x,y)$ and the complex non-membership function $\nu_R(x,y)$ are to be interpreted in the following manner.

(i) $r_R(x)$ represents a degree of interaction or interconnectedness between the elements of U and W; Correspondingly $s_R(x)$ represents a degree of no connection or no interaction between the elements of U and W;

(ii) $\bar{\omega}_{\mu R}(x)$ represents the phase of association, interaction, or interconnectedness between the elements of U and W; Correspondingly $\bar{\omega}_{\nu R}(x)$ represents the phase of no connection or no interaction between the elements of U and W.

Remark 3 Without the phase terms $\bar{\omega}_{\mu R}(x)$ and $\bar{\omega}_{\nu R}(x)$, a complex intuitionistic fuzzy relation $R(U,W)$ reduces to a traditional intuitionistic fuzzy relation $\bar{R}(U,W)$.

5 Examples

As is well known, in the practice of financial work, we can make accurate evaluation and judgment on the advantages and disadvantages of the economic benefits of enterprises by dissecting and analyzing the financial situation and operating results of enterprises. The selection and application of financial indicators as evaluation and judgment standards is particularly important. In this section, we consider financial indicators selection and application between two companies below which involves the significance of the phase terms of a complex intuitionistic fuzzy relation and the application of operation of complex intuitionistic fuzzy set. Meanwhile, an example of "therapeutic effects of drugs" is given to illuminate (α, β)-equality between two complex intuitionistic fuzzy sets.

Example 13 Let U be the set of financial indicators of the **A** company. Possible elements of this set are "return on equity", "total asserts turnover", "current asserts turnover", "asset-liability rate", "quick rate" and "capital accumulation rate", etc. Let W be the set of financial indicators of the **B** company. Suppose the complex intuitionistic fuzzy relation $R(U, W)$ represents the impact of company **A**'s financial indicators on company **B**'s financial indicators, i.e., y is influenced by x, where $x \in U$ and $y \in W$.

The membership function $\mu_R(x, y)$ of complex intuitionistic fuzzy relation $R(U, W)$ is a complex value function, with an amplitude term and a phase term. The amplitude term indicates the degree of influence of an **A** company indicator on a **B** company indicator. An amplitude term with a value close to 0 implies a small degree of influence, while a value close to 1 suggests a large degree of influence. The phase term indicates the "phase" of influence, or time lag that characterizes the influence of an **A** company indicator on a **B** company indicator.

The non-membership function $\nu_R(x, y)$ of the complex intuitionistic fuzzy relation $R(U, W)$ is also a complex value function, with an amplitude term and a phase term. The amplitude term indicates the degree of uninfluence of an **A** company indicator on a **B** company indicator. An amplitude term with a value close to 0 implies a small degree of no influence, while a value close to 1 suggests a large degree of no influence. The phase term indicates the "phase" of no influence, or time lag that characterizes the no influence of an **A** company indicator on a **B** company indicator.

For example, let x ="asset-liability rate" and y="capital accumulation rate". Then $\mu_R(x, y)$ and $\nu_R(x, y)$ are the degrees of membership and non-membership of the influence of "asset-liability rate" of company **A** on "capital accumulation rate" of company **B**. The value of $\mu_R(x, y)$ and $\nu_R(x, y)$ may be obtained from an expert.

Suppose an expert states that "**A** company's 'asset-liability rate' has a great influence on **B** company's 'capital accumulation rate,' and the effect of a decline or increase of **A** company's 'asset-liability rate' is evident on **B** company's 'capital accumulation rate' in two-four months. While the degrees to which **A** company's 'asset-liability rate' has no influence on **B** company's 'capital accumulation rate' is small, and the no influence of a decline or increase of **A** company's 'asset-liability rate' is evident on **B** company's 'capital accumulation rate' in two-four months." If $R(U, W)$ is a traditional intuitionistic fuzzy relation, according to the expert's statement, let membership degree $\mu_R(x, y) = 0.85$ and non-membership degree $\nu_R(x, y) = 0.1$. Then we notice that the time information provided by the expert will be lost. However, if a complex intuitionistic fuzzy relation is used to express $R(U, W)$, i.e. $\mu_R(x, y)$ and $\nu_R(x, y)$ are assigned two complex values, then it would include all of the information provided by the expert.

Assume $R(U, W)$ is used to measure interactions between **A** company and **B** company financial indexes in the limited time frame of 12 months. Then let

$$\mu_R(x, y) = 0.85 \cdot e^{i2\pi \frac{3}{12}}$$

and

$$\nu_R(x, y) = 0.1 \cdot e^{i2\pi \frac{3}{12}},$$

thus

$$R(x, y) = (0.85 \cdot e^{i2\pi \frac{3}{12}}, 0.1 \cdot e^{i2\pi \frac{3}{12}}).$$

Note that the amplitude of $\mu_R(x, y)$ and $\nu_R(x, y)$ were selected to be 0.85 and 0.1, respectively. They are similar to the degrees of membership and the degree of non-membership of a intuitionistic fuzzy set. The phase term was chosen to be $2\pi \frac{3}{12}$ as an average of "two-four months", normalized by 12 months the maximum time frame the relation was designed to take into account.

Complex intuitionistic fuzzy relation, as an extension of intuitionistic fuzzy relation, could describe the fuzzy characters of things more detail and comprehensively and is very useful in dealing with vagueness and uncertainty of problems that include the periodic or recurring phenomena. Similar to complex intuitionistic fuzzy set, the novelty of complex intuitionistic fuzzy relation lies in its ability for membership and non-membership functions to achieve more range of values and contains more information.

Example 14 In this example, we will continue discuss Example 13, let V be the set of development indicators of the city, such as "consumer price index", "producer price index", etc. Now, consider the following two complex intuitionistic fuzzy relations.

(i) The relation $R(U, W)$ discussed in detail in Example 13 represents the relation of influence of **A** company's "financial indexes" on **B** company's "financial indexes".

(ii) The relation $R(W, V)$ representing the relation of influence of **B** company's "financial indexes" on city development indicators.

Let x="return on equity", y ="total asserts turnover" and z ="producer price index", where $x \in U$, $y \in W$ and $z \in V$.

Suppose the following information is available from an expert.

(i) The influence of **A** company's "return on equity" on **B** company's "total asserts turnover" is medium, and its effect is evident in four-six months, while no influence of **A** company's "return on equity" on **B** company's "total asserts turnover" is medium, and its effect is not evident in four-six months. According to Definition 19, we can describe the information provided by the expert as follows.

$$R(x, y) = (0.55 \cdot e^{i2\pi \frac{5}{12}}, 0.4 \cdot e^{i2\pi \frac{5}{12}}).$$

(ii) The influence of **B** company's "total asserts turnover" on city development "producer price index" is verge large, and its effect is evident in nine-ten months, while no influence of **B** company's "total asserts turnover" on city development "producer price index" is very small, and its effect is not evident in nine-ten months. According to Definition 19, we can describe the information provided by the expert as follows.

$$R(y, z) = (0.9 \cdot e^{i2\pi \frac{8}{12}}, 0.05 \cdot e^{i2\pi \frac{8}{12}})).$$

The two relations defined above may be combined in order to produce a third relation, $R(U, V)$, the relation of influence of **A** company's "return on equity" on city development "producer price index". The relation $R(U, V)$ is obtained through the composition of relation $R(x, y)$ and $R(y, z)$. It is possible to provide a general and rigorous definition for the composition of complex intuitionistic fuzzy relation. In this example, we consider the composition of the two degrees of membership and the two degrees of non-membership derived above: $\mu_R(x, y)$, $\mu_R(y, z)$, $\nu_R(x, y)$, and $\nu_R(y, z)$.

The result of this composition is the degree of membership and non-membership $\mu_R(x, z)$ and $\nu_R(x, z)$. From intuitive consideration, we suggest that the value of $\mu_R(x, z)$ and $\nu_R(x, z)$ should equal the product of $\mu_R(x, y)$ and $\mu_R(y, z)$ and the product of $\nu_R(x, y)$ and $\nu_R(y, z)$, i.e., $R(x, z)$ equal the product of $R(x, y)$ and $R(y, z)$. According to Definition 6, we have

$$R(x, z) = R(x, y) \circ R(y, z),$$

where

$$\mu_R(x, z) = \mu_R(x, y) \circ \mu_R(y, z) = 0.55 \cdot e^{i2\pi \frac{5}{12}} \circ 0.9 \cdot e^{i2\pi \frac{8}{12}} = 0.495 \cdot e^{i2\pi \frac{3.3}{12}}$$

and

$$\nu_R(x, z) = \nu_R(x, y) \circ \nu_R(y, z) = 0.4 \cdot e^{i2\pi \frac{5}{12}} \circ 0.05 \cdot e^{i2\pi \frac{8}{12}} = 0.38 \cdot e^{i2\pi \frac{9.7}{12}},$$

thus

$$R(x, z) = (0.495 \cdot e^{i2\pi \frac{3.3}{12}}, 0.43 \cdot e^{i2\pi \frac{9.7}{12}}).$$

Note that for the membership function, the amplitude term of $\mu_R(x, z)$ is derived by intersecting the amplitudes of $\mu_R(x, y)$ and $\mu_R(y, z)$, with product used as the intersection function of choice. The phase term of $\mu_R(x, z)$ is also derived by intersecting the amplitudes of $\mu_R(x, y)$ and $\mu_R(y, z)$, with product used as the intersection function of choice. While for the non-membership function, the amplitude term of $\nu_R(x, z)$ is derived by intersecting the amplitudes of $\nu_R(x, y)$ and $\nu_R(y, z)$, with probabilistic sum used as the union function of choice. The phase term of $\nu_R(x, z)$ is also derived by intersecting the amplitudes of $\nu_R(x, y)$ and $\nu_R(y, z)$, probabilistic sum used as the union function of choice.

Hence, the use of multiplication in this example makes good intuitive sense. Note that the product operation emphasizes a unique property of complex intuitionistic fuzzy setsthe complex algebra of its grades of membership and non-membership. It is a feature of complex intuitionistic fuzzy sets that is difficult to reproduce using traditional intuitionistic fuzzy sets.

Example 15 Consider the problem of " therapeutic effects of drugs". Let 1 represents 100 percent of the treatment effects and 2π represents 12 months. Suppose $A, B \in \mathrm{CIF}^{\star}(U)$, where $U = \{x_1, x_2, x_3\}$ denotes three drugs. Two experts evaluated the therapeutic effect of three drugs and described them by two complex intuitionistic fuzzy sets A and B as follows.

$$A = \frac{\langle 0.6 \cdot e^{i\pi}, 0.3 \cdot e^{i0.8\pi} \rangle}{x_1} + \frac{\langle 0.5 \cdot e^{i1.2\pi}, 0.45 \cdot e^{i\pi} \rangle}{x_2} + \frac{\langle 0.3 \cdot e^{i2\pi}, 0.5 \cdot e^{i1.5\pi} \rangle}{x_3},$$

$$B = \frac{\langle 0.4 \cdot e^{i1.2\pi}, 0.5 \cdot e^{i0.6\pi} \rangle}{x_1} + \frac{\langle 0.2 \cdot e^{i0.8\pi}, 0.6 \cdot e^{i1.5\pi} \rangle}{x_2} + \frac{\langle 0.5 \cdot e^{i\pi}, 0.3 \cdot e^{i1.2\pi} \rangle}{x_3}.$$

Take complex intuitionistic fuzzy set A as an example, for x_1 drug, the membership function $\mu_A(x_1) = 0.6 \cdot e^{i\pi}$ indicates that x_1 drug reached 60 percent effective in treating a disease within about 6 months and non-membership function

$\nu_A(x_1) = 0.3 \cdot e^{i0.8\pi}$ indicates that x_1 drug reached 30 percent no effective in treating a disease within about 5 months. Similar explanations can be made for the treatment effects of the other two drugs. Then the (α, β)-equality between two complex intuitionistic fuzzy sets A and B will be discussed.

According to Definition 15, we can obtain that

$$\sup_{x\in U}|r_A(x) - r_B(x)| = 0.3, \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A}(x) - \omega_{\mu B}(x)| = 0.5,$$

and

$$\sup_{x\in U}|s_A(x) - s_B(x)| = 0.2, \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A}(x) - \omega_{\nu B}(x)| = 0.25,$$

therefore

$$\rho(\mu_A, \mu_B) = 0.5, \rho(\nu_A, \nu_B) = 0.25,$$

and

$$d(A, B) = \frac{1}{2}(\rho(\mu_A, \mu_B) + \rho(\nu_A, \nu_B)) = 0.375.$$

Let $\rho(\mu_A, \mu_B) = 0.5 \le 1-\alpha$ and $\rho(\nu_A, \nu_B) = 0.25 \le \beta$, then A and B are said (α, β)-equal if and only if $0 \le \alpha \le 0.5$, $0.25 \le \beta \le 1$ and satisfy $\alpha + \beta \le 1$.

According to Example 15, we note that the distance measure between two complex intuitionistic fuzzy sets A and B is 0.375, and when $\alpha = 0.5$ and $\beta = 0.25$, the more equal the two complex intuitionistic fuzzy sets are. Meanwhile, the (α, β)-equality between two complex intuitionistic fuzzy sets can also be used to describe the proximity between two experts' evaluation for three drugs. Therefore, the concept of (α, β)-equalities of complex intuitionistic fuzzy sets has very good practical significance and use value.

6 Conclusion

Consider that complex intuitionistic fuzzy set is very useful in dealing with vagueness and uncertainty of problems that include the periodic or recurring phenomena. So in this paper, various operation properties of complex intuitionistic fuzzy sets are investigated when the membership phase and non-membership phase are restricted to $[0, 2\pi]$. Meanwhile, we notice that the precise membership values and non-membership values should normally be of no practical significance, and there is no equality and proximity measure investigation on complex intuitionistic fuzzy sets. First of all, we proposed a new distance measure for complex intuitionistic fuzzy sets. The distance of two complex intuitionistic fuzzy sets measures the difference between the grades of two complex intuitionistic fuzzy sets as well as that between the phases of the two complex intuitionistic fuzzy sets. Then this distance measure is used to define (α, β)-equalities of complex intuitionistic fuzzy sets. Two complex intuitionistic fuzzy sets are said to be (α, β)-equal if the distance between their membership degrees is less than $1 - \alpha$ and the distance between their non-membership degrees is less than β. The concept of (α, β)-equalities of complex intuitionistic fuzzy sets allows us systematically develop the distance, equality and proximity measures for complex intuitionistic fuzzy sets. Finally, complex intuitionistic fuzzy relations are discussed and two examples are given to illuminate the importance of the phase term of intuitionistic fuzzy relation and the application of operation of complex intuitionistic fuzzy set. Furthermore, the problem of "therapeutic effects of drugs" is given to illuminate (α, β)-equality between two complex intuitionistic fuzzy sets. Note that the operations discussed in this paper makes good intuitive sense. Some operations emphasize a unique property of complex intuitionistic fuzzy sets, the complex algebra of its grades of membership and non-membership. It is a feature of complex intuitionistic fuzzy sets that is difficult to reproduce using traditional intuitionistic fuzzy sets. All these conclusions not only deeply enrich the fundamental theory of complex intuitionistic fuzzy sets, but also provide a powerful tool to investigate complex intuitionistic fuzzy sets.

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