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APPLICATION OF SOFT COMPUTING

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Soft ideal topological spaces

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Abstract
The article deals with the correspondence between soft ideal topological spaces and ideal topological ones. Investigation of soft ideal topological spaces is based on methods of general topology, and the application of results for soft omega open and strongly soft omega open sets is given.

Keywords Relation · Set-valued mapping · Soft set · Soft topology · Soft ideal topological space · Soft topological sum

1 Introduction

Molodtsov (1999) initiated the concept of soft sets as a completely different approach for dealing with uncertainties, and over the past few years, the fundamentals of the soft set theory have been studied by many authors. Since the concept of soft topology was introduced by Shabir and Naz (2011), many terms of general topology have found their analogy in soft topological spaces.

There are several papers that document certain problems relating to the fundamentals of the soft set theory and soft topological spaces. (Shi and Pang 2014, 2015; Shi and Fan 2019) demonstrate the redundancies concerning the increasing popular soft set approaches to general topology, and they claim that soft topology is exactly a special subcase of general topology. Matejdes (2016) also states that a soft topology is nothing more than a topology on Cartesian product, and each soft topological concept has its topological equivalent. Some soft terms (for example, soft homogeneity, soft compactness, soft paracompactness, soft Lindelöfness, soft normality, soft connectedness, soft hyperconnectedness, soft topological sum; see Al Ghour and Bin-Saadon (2019); Al-shami et al. (2020); El-Shafei et al. (2018); Terepeta (2018)) correspond to known, commonly used and studied topological terms. Others (for example, soft separation axioms, see El-Shafei et al. (2018)) correspond to topological terms that bring new challenges to research. In principle, any soft concept can be studied by standard topological methods. Recently published works by Alcantud (2021); Matejdes (2021a); Matejdes (2021b); Matejdes (2021c) also document that soft topology is basically part of general topology, and concepts of soft topology can be reduced to the corresponding concepts in topology. More precisely, a soft topological space stemming from a topological space and vice versa is investigated by Alcantud (2021) (with application to base and separability), soft homogeneity is investigated by Matejdes (2021b), enriched soft topology, topological sum, soft regularity, soft compactness and soft Lindelöfness are studied by Matejdes (2021a), and soft continuity is studied in Matejdes (2021c). In all cases, topological counterparts were used as a substitute for soft methods. Based on these facts, we can say that the cumbersome methods used in the theory of soft topological spaces, which are often unnecessary imitations of topological methods, can be effectively overcome by identifying a set-valued mapping with its graph. From this point of view, it is a fundamental transformation of the existing methods used in soft topological spaces into corresponding topological methods. This is a major step forward in improving and simplifying the existing methods used in the current literature.

The aim of the article is to continue the above-mentioned transformations of topological methods into soft topological ones. Namely, the results of ideal topological spaces will be used in the field of soft ideal topological spaces. We especially focus on the results from Al Ghour and Hamed Worood (2020). It should be noted that many other results concerning soft ideal topological spaces (see, for example, Gharib and Abd El-latif (2019); Kandil et al. (2014) where one will find further references to the issue of soft ideal topological spaces)

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can be investigated within the framework of ideal topological spaces whose known results can be directly applied.

This research article is organized as follows: Sects. 2 and 3 are devoted to the basic concepts of the theory of ideal topological spaces. In order to achieve the specific objectives of the article by Al Ghour and Hamed Worood (2020), we only focus on some results of ideal topological spaces. One may recall these are known facts, but for the sake of completeness and purpose, we present them with proofs unless they are trivial. In Sects. 2 and 3, the choice of ideal topological terms, theorems and lemmas is determined by their use in Sects. 6 and 7 concerning soft omega open and strong soft omega open sets, where the results from Al Ghour and Hamed Worood (2020) are proved as direct consequences.

In Sects. 4 and 5, we show that soft topologies can be fully characterized in terms of standard topologies on a crisp set. This characterization is based on two constructions (see Definition 6, Theorem 4). The first one yields a soft topological space that is associated with any crisp topology on a certain Cartesian product of two sets. The second works in the opposite direction: With any soft topological space, it produces a crisp topology on the Cartesian product.

Both constructions are explicit and ensure a transition from one setting to the other. Their fundamental properties and mutual links are investigated in Theorem 5. We show that such notions as a soft subspace, a soft topological sum, a soft ideal topological space, a soft base and their properties can be transferred from crisp topologies to soft topologies or the other way round (Definition 7, Lemma 7, Lemma 8). This achievement has several remarkable consequences. Concepts from soft topology can be reduced to the corresponding concepts in topology, and results from topological spaces may be exported to soft topological spaces. We give examples of these fundamental advances, namely: soft ω-open sets, strong soft ω-open sets is identified with countable sets, sets with countable sections, respectively (Definition 8, Remark 5).

In Sect. 6, the results are specified into the soft ideals $\mathcal{I}^s$ and $\mathcal{I}^0$, and the last section summarizes the results from Al Ghour and Hamed Worood (2020), which are the direct consequence of the obtained results. One may recall that all the results of Al Ghour and Hamed Worood (2020) can be transformed into corresponding topological results and they can be extended for arbitrary soft ideal.

2 Ideal topological spaces

(X, τ) denotes a topological space, $cl_\tau(S)$, $int_\tau(S)$ the closure (the interior) of $S \subset X$, respectively. If $A \subset X$, then by (A, τ_A) we denote a topological subspace of (X, τ) where τ_A is a subspace topology.

An ideal $\mathcal{I}$ on X is a nonempty collection of subsets of X which satisfies the following properties: If $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$, and if $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$. An ideal topological space is a topological space (X, τ) with an ideal $\mathcal{I}$ on X, and it is denoted by $(X, \tau, \mathcal{I})$, see, for example, Al-Omari and Noiri (2013); Ekici and Noiri (2008); Kaniewski et al. (1998), where one can find rich references.

If $(X, \tau, \mathcal{I})$ is an ideal topological space and $S \subset X$, then the set of all points in which S is locally not in $\mathcal{I}$ with respect to τ, i.e., $\{x \in X : S \cap U \notin \mathcal{I}$ for every open set U containing $x\}$ is called the local function of S with respect to τ and $\mathcal{I}$, and it is denoted by $D_{\tau,\mathcal{I}}(S)$ (denoted also $S^*(\mathcal{I}, \tau)$, see, for example, Al-Omari and Noiri (2013); Ekici and Noiri (2008); Kaniewski et al. (1998)). Obviously $D_{\tau,\mathcal{I}}(S)$ is a closed subset of $cl_\tau(S)$. For a subset S of $A \subset X$ by $D_{\tau_A,\mathcal{I}_A}(S)$, we denote the set $\{x \in A : S \cap U \notin \mathcal{I}_A$ for every set $U \in \tau_A$ containing $x\}$ where $\mathcal{I}_A = \{A \cap I : I \in \mathcal{I}\}$. It is clear $\mathcal{I}_A$ is an ideal on A and $\mathcal{I}_A \subset \mathcal{I}$. An ideal $\mathcal{I}$ is called τ-codense, see Kaniewski et al. (1998) if $\mathcal{I} \cap \tau = \{\emptyset\}$. A subset A of X locally belongs to $\mathcal{I}$ if $A \cap D_{\tau,\mathcal{I}}(A) = \emptyset$, i.e., for any $x \in A$ there is $G \in \tau$ containing x such that $A \cap U \in \mathcal{I}$, see Kaniewski et al. (1998).

A topological space (X, τ) is Lindelöf (weakly Lindelöf, see Frolík (1959)) if every open cover $\mathcal{U}$ of X has a countable subfamily $\mathcal{V}$ such that $X = \cup\mathcal{V}$ $(X = cl_\tau(\cup\mathcal{V}))$. Note $\mathcal{U}$ can be replaced by a cover from a base of (X, τ).

Let $\mathcal{B}_{\tau,\mathcal{I}} = \{G \setminus I : G \in \tau, I \in \mathcal{I}\}$, $\mathcal{B}_{\tau_A,\mathcal{I}_A} = \{G \setminus I : G \in \tau_A, I \in \mathcal{I}_A\}$. By $(\tau)_\mathcal{I}$ (briefly $\tau_\mathcal{I}$), $(\tau_A)_{\mathcal{I}_A}$, we denote a topology on X, A generated by the base $\mathcal{B}_{\tau,\mathcal{I}}$, $\mathcal{B}_{\tau_A,\mathcal{I}_A}$, respectively. In the literature, $\tau_\mathcal{I}$ is usually denoted by $\tau^*(\mathcal{I})$ or briefly τ^*. By $co_{X,\mathcal{I}}$, we denote a family $\{X \setminus I : I \in \mathcal{I}\} \cup \{\emptyset\}$, which is a topology on X.

Lemma 1 *(see Al-Omari and Noiri (2013); Ekici and Noiri (2008)) The operator $cl_{\tau_\mathcal{I}}(S) = S \cup D_{\tau,\mathcal{I}}(S)$ is a Kuratowski closure operator generating the topology $\tau_\mathcal{I}$. That means a set S is closed in $(X, \tau_\mathcal{I})$ if and only if $D_{\tau,\mathcal{I}}(S) \subset S$. Recall if $\mathcal{I} = \{\emptyset\}$, then $cl_{\tau_\mathcal{I}}(S) = cl_\tau(S)$.*

Remark 1 Let $\mathcal{I}$ and $\mathcal{J}$ be the ideals on X. Then

(1) $\tau \subset \mathcal{B}_{\tau,\mathcal{I}} \subset \tau_\mathcal{I}$, $co_{X,\mathcal{I}} \subset \mathcal{B}_{\tau,\mathcal{I}}$. Moreover, $(X, co_{X,\mathcal{I}})$, $(X, \tau_\mathcal{I})$ is a topological space in which any set $I \in \mathcal{I}$ is closed, respectively (see Lemma 1). It is clear if τ^{id} is the indiscrete topology on X, then $(\tau^{id})_\mathcal{I} = co_{X,\mathcal{I}}$.
(2) $(\tau_\mathcal{I})_\mathcal{I} = \tau_\mathcal{I}$.
(3) The next conditions are equivalent

   (a) $co_{X,\mathcal{I}} \subset \tau$,
   (b) $\tau = \mathcal{B}_{\tau,\mathcal{I}}$,
   (c) $\tau = \tau_\mathcal{I}$.

(4) $(co_{X,\mathcal{I}})_\mathcal{I} = co_{X,\mathcal{I}}$.
(5) If $\mathcal{I} \subset \mathcal{J}$, then

   (a) $co_{X,\mathcal{I}} \subset co_{X,\mathcal{J}}$,
   (b) $\tau \subset \mathcal{B}_{\tau,\mathcal{I}} \subset \mathcal{B}_{\tau,\mathcal{J}}$, so $\tau \subset \tau_\mathcal{I} \subset \tau_\mathcal{J}$,

(c) $\tau_A \subset \mathcal{B}_{\tau_A,\mathcal{I}_A} \subset \mathcal{B}_{\tau_A,\mathcal{J}_A}$, so $\tau_A \subset \tau_{A,\mathcal{I}_A} \subset \tau_{A,\mathcal{J}_A}$,
(d) $D_{\tau,\mathcal{J}}(S) \subset D_{\tau,\mathcal{I}}(S)$,
(e) $(\tau_{\mathcal{I}})_{\mathcal{J}} = (\tau_{\mathcal{J}})_{\mathcal{I}} = \tau_{\mathcal{J}}$,
(f) if $co_{X,\mathcal{J}} \subset \tau$, then $\tau_{\mathcal{I}} = \tau_{\mathcal{J}}$.

Proof We will prove the items (2), (3), (4), (5e) and (5f). The rest of the items are clear.

(2): Denote $\tau_{\mathcal{I}} = \theta$. Then $(\tau_{\mathcal{I}})_{\mathcal{I}} = \theta_{\mathcal{I}}$. Since $\mathcal{B}_{\tau,\mathcal{I}} \subset \mathcal{B}_{\theta,\mathcal{I}} \subset \tau_{\mathcal{I}}$, $\tau_{\mathcal{I}} = \theta_{\mathcal{I}}$ and $\tau_{\mathcal{I}} = (\tau_{\mathcal{I}})_{\mathcal{I}}$.

(3): $(a) \Rightarrow (b)$: The inclusion $\tau \subset \mathcal{B}_{\tau,\mathcal{I}}$ is clear. Let $S \in \mathcal{B}_{\tau,\mathcal{I}}$. Then $S = G \setminus I = G \cap (X \setminus I)$, $G \in \tau$, $I \in \mathcal{I}$. Since $X \setminus I \in co_{X,\mathcal{I}} \subset \tau$, $S \in \tau$.

$(b) \Rightarrow (c)$: The inclusion $\tau \subset \tau_{\mathcal{I}}$ is clear. Let $S \in \tau_{\mathcal{I}}$. Then $S = \cup_{t\in T} H_t$ where $H_t \in \mathcal{B}_{\tau,\mathcal{I}} = \tau$, so $S \in \tau$.

$(c) \Rightarrow (a)$: Let $S \in co_{X,\mathcal{I}}$. Then $S = X \setminus A$, $A \in \mathcal{I}$. Since $X \setminus A \in \tau_{\mathcal{I}} = \tau$, $S \in \tau$.

(4): Denote $\tau = co_{X,\mathcal{I}}$. Since $co_{X,\mathcal{I}} \subset \tau$, by (3) $\tau = \tau_{\mathcal{I}}$ so $co_{X,\mathcal{I}} = (co_{X,\mathcal{I}})_{\mathcal{I}}$.

(5e): Since $\mathcal{I} \subset \mathcal{J}$, $\tau_{\mathcal{I}} \subset \tau_{\mathcal{J}}$. So $(\tau_{\mathcal{I}})_{\mathcal{J}} \subset (\tau_{\mathcal{J}})_{\mathcal{J}} = \tau_{\mathcal{J}}$, by item (2). Since $\tau \subset \tau_{\mathcal{I}}$, $\tau_{\mathcal{J}} \subset (\tau_{\mathcal{I}})_{\mathcal{J}}$. That means $\tau_{\mathcal{J}} = (\tau_{\mathcal{I}})_{\mathcal{J}}$.

The inclusion $\tau_{\mathcal{J}} \subset (\tau_{\mathcal{J}})_{\mathcal{I}}$ is clear. Let $H \in (\tau_{\mathcal{J}})_{\mathcal{I}}$. Then $H = \cup_{t\in T}(G_t \setminus I_t)$ where $G_t \in \tau_{\mathcal{J}}$ and $I_t \in \mathcal{I} \subset \mathcal{J}$. Moreover, $G_t = \cup_{s\in S}(R_t^s \setminus I_t^s)$ where $R_t^s \in \tau$ and $I_t^s \in \mathcal{J}$. Then $H = \cup_{t\in T}(\cup_{s\in S}(R_t^s \setminus I_t^s) \setminus I_t) = \cup_{t\in T}(\cup_{s\in S}(R_t^s \setminus (I_t^s \cup I_t)))$. Since $I_t^s \cup I_t \in \mathcal{J}$, $H \in \tau_{\mathcal{J}}$. That means $\tau_{\mathcal{J}} = (\tau_{\mathcal{J}})_{\mathcal{I}}$.

(5f): The inclusion $\tau_{\mathcal{I}} \subset \tau_{\mathcal{J}}$ follows from the item (5b). Since $co_{X,\mathcal{J}} \subset \tau$, $\tau_{\mathcal{J}} = \tau \subset \tau_{\mathcal{I}}$, by (3). □

Lemma 2 *If $(X, \tau, \mathcal{I})$ is an ideal topological space, then $(\tau_A)_{\mathcal{I}_A} = (\tau_{\mathcal{I}})_A$ where $(\tau_{\mathcal{I}})_A$ is a subspace topology on $A \subset X$.*

Proof If $H \in (\tau_{\mathcal{I}})_A$, then $H = H_0 \cap A$ where $H_0 \in \tau_{\mathcal{I}}$, $H_0 = \cup_{t\in T} H_0^t$, $H_0^t = G_0^t \setminus I_0^t$, $G_0^t \in \tau$, $I_0^t \in \mathcal{I}$. Since $H = (\cup_{t\in T} H_0^t) \cap A = \cup_{t\in T}((G_0^t \cap A) \setminus (I_0^t \cap A))$ and $G_0^t \cap A \in \tau_A$, $I_0^t \cap A \in \mathcal{I}_A$, $H \in (\tau_A)_{\mathcal{I}_A}$.

If $H \in (\tau_A)_{\mathcal{I}_A}$, then $H = \cup_{t\in T} H_t$ where $H_t = G_t \setminus I_t$, $G_t = S_t \cap A \in \tau_A$, $S_t \in \tau$, $I_t \in \mathcal{I}_A \subset \mathcal{I}$. Since $H = \cup_{t\in T} H_t = \cup_{t\in T}(S_t \cap A \setminus I_t) = (\cup_{t\in T}(S_t \setminus I_t)) \cap A$ and $S_t \setminus I_t \in \tau_{\mathcal{I}}$, $H \in (\tau_{\mathcal{I}})_A$. □

Corollary 1 *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If G is open in $(X, \tau_{\mathcal{I}})$ and $A \subset X$, then $G \cap A \in (\tau_A)_{\mathcal{I}_A}$.*

Proof Since $G \in \tau_{\mathcal{I}}$, $G \cap A \in (\tau_{\mathcal{I}})_A = (\tau_A)_{\mathcal{I}_A}$, by Lemma 2. □

Lemma 3 *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $\mathcal{I} \cap \tau = \{\emptyset\}$ (i.e., if $\mathcal{I}$ is τ-codense), then $cl_\tau(G) = cl_{\tau_{\mathcal{I}}}(G) = D_{\tau,\mathcal{I}}(G)$ for any $G \in \tau_{\mathcal{I}}$.*

Proof The inclusion $cl_{\tau_{\mathcal{I}}}(G) \subset cl_\tau(G)$ is clear. Suppose there is $x \in cl_\tau(G) \setminus cl_{\tau_{\mathcal{I}}}(G)$. Then there is $H \in \tau$, $x \in H$ and $I \in \mathcal{I}$ such that $(H \setminus I) \cap G = (H \cap G) \setminus (I \cap G) = \emptyset$. Since $H \cap G \subset I \cap G \in \mathcal{I}$, $H \cap G \in \mathcal{I}$. On the other hand, $x \in cl_\tau(G)$ and $x \in H$, so $H \cap G \neq \emptyset$. Since $G \in \tau_{\mathcal{I}}$, there are $H_0 \in \tau$, $H_0 \subset H$ and $I_0 \in \mathcal{I}$ such that $H_0 \setminus I_0 \subset G \cap H \in \mathcal{I}$. Since $\mathcal{I} \cap \tau = \{\emptyset\}$, $H_0 \setminus I_0 \notin \mathcal{I}$, a contradiction.

The inclusion $D_{\tau,\mathcal{I}}(G) \subset cl_\tau(G)$ is clear. Let $x \in cl_\tau(G)$ and $x \in H \in \tau$. Then $H \cap G \neq \emptyset$. Since $G \in \tau_{\mathcal{I}}$, there are $H_0 \in \tau$, $H_0 \subset H$ and $I_0 \in \mathcal{I}$ such that $H_0 \setminus I_0 \subset G \cap H$. Since $\mathcal{I} \cap \tau = \{\emptyset\}$, $H_0 \setminus I_0 \notin \mathcal{I}$; consequently, $G \cap H \notin \mathcal{I}$, so $x \in D_{\tau,\mathcal{I}}(G)$. That means $D_{\tau,\mathcal{I}}(G) = cl_\tau(G)$. □

Lemma 4 *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then*

(1) $\mathcal{I} \cap \tau = \{\emptyset\}$ if and only if $\mathcal{I} \cap \tau_{\mathcal{I}} = \{\emptyset\}$,
(2) if $\mathcal{I}$ contains all singletons, then
  (a) X locally belongs to $\mathcal{I}$ if and only if $D_{\tau,\mathcal{I}}(X) = \emptyset$ if and only if $(X, \tau_{\mathcal{I}})$ is a discrete space if and only if $\{x\} \in \mathcal{B}_{\tau,\mathcal{I}}$ for any $x \in X$,
  (b) $(X, \tau_{\mathcal{I}})$ is a T_1-space.
(3) If $(X, \tau_{\mathcal{I}})$ is Lindelöf, then (X, τ) is Lindelöf.
(4) If $\mathcal{I} \cap \tau = \{\emptyset\}$, then (X, τ) is weakly Lindelöf if and only if $(X, \tau_{\mathcal{I}})$ is weakly Lindelöf.
(5) If for any $A \in \mathcal{I}$ and any cover $\mathcal{U} \subset \mathcal{B}_{\tau,\mathcal{I}}$ of A contains a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $A \subset \cup\mathcal{V}$, then (X, τ) is Lindelöf if and only if $(X, \tau_{\mathcal{I}})$ is Lindelöf.
(6) If $X = \cup_{i=1}^{\infty} X_i$ and (X_i, τ_{X_i}) is weakly Lindelöf for any i, then (X, τ) is weakly Lindelöf.
(7) If (X, τ) is separable, then (X, τ) is weakly Lindelöf.
(8) If (X, τ) is Lindelöf, then (X, τ) is weakly Lindelöf.

Proof The items (1), (2), (3), (7), (8) are trivial.

(4) Let (X, τ) be weakly Lindelöf and $\mathcal{U} = \{U_t \setminus I_t : t \in T\}$ be an open cover of $(X, \tau_{\mathcal{I}})$ where $U_t \in \tau$, $I_t \in \mathcal{I}$. Since (X, τ) is weakly Lindelöf and $\{U_t : t \in T\}$ is open cover of (X, τ), there is a countable set $T_0 \subset T$ such that $\cup_{t\in T_0} U_t$ is dense in (X, τ). We will show $S := \cup_{t\in T_0}(U_t \setminus I_t)$ is dense in $(X, \tau_{\mathcal{I}})$. Let $G \setminus I \in \mathcal{B}_{\tau,\mathcal{I}}$. Then $G \cap (\cup_{t\in T_0} U_t) \neq \emptyset$, so $G \cap U_{t_0} \neq \emptyset$ for some $t_0 \in T_0$. Since $\mathcal{I} \cap \tau = \{\emptyset\}$, $G \cap U_{t_0} \notin \mathcal{I}$. That means $(G \setminus I) \cap (U_{t_0} \setminus I_{t_0}) \notin \mathcal{I}$, so $(G \setminus I) \cap S \neq \emptyset$ and S is dense in $(X, \tau_{\mathcal{I}})$. We have proved $(X, \tau_{\mathcal{I}})$ is weakly Lindelöf.

Suppose $(X, \tau_{\mathcal{I}})$ is weakly Lindelöf. If $\mathcal{U} = \{U_t : t \in T\}$ is an open cover of (X, τ), then there is a countable set $T_0 \subset T$ such that $\cup_{t\in T_0} U_t$ is dense in $(X, \tau_{\mathcal{I}})$. That means $\cup_{t\in T_0} U_t$ is also dense in (X, τ), so (X, τ) is weakly Lindelöf.

(5) Let (X, τ) be Lindelöf and $\mathcal{U} = \{U_t \setminus I_t : t \in T\}$ be an open cover of $(X, \tau_{\mathcal{I}})$ where $U_t \in \tau$, $I_t \in \mathcal{I}$. Since (X, τ) is Lindelöf and $\{U_t : t \in T\}$ is an open cover of (X, τ), there is a countable set $T_0 \subset T$ such that $\mathcal{V} = \{U_t : t \in T_0\}$ is a cover of (X, τ). For any I_t, $t \in T_0$ there is a countable subfamily $\mathcal{V}_t$ of $\mathcal{U}$ such that $I_t \subset \cup\mathcal{V}_t$. It is clear that $\mathcal{V} \cup (\cup_{t\in T_0} \mathcal{V}_t)$ is a countable subfamily of $\mathcal{U}$ and $\mathcal{V} \cup (\cup_{t\in T_0} \mathcal{V}_t)$ is a cover

of $(X, \tau_{\mathcal{I}})$. That means $(X, \tau_{\mathcal{I}})$ is Lindelöf. The opposite implication is clear, see item (3).

(6) Let $\mathcal{U} = \{U_t : t \in T\}$ be open cover of (X, τ). Then $\mathcal{U}_i = \{U_t \cap X_i : U_t \in \mathcal{U}\}$ is an open cover of (X_i, τ_{X_i}), $i = 1, 2, 3, ...$ Since (X_i, τ_{X_i}) is weakly Lindelöf for any i, there is a countable subfamily $\mathcal{V}_i$ of $\mathcal{U}_i$, such that $A_i := \cup\mathcal{V}_i$ is dense in (X_i, τ_{X_i}). Let $\mathcal{V} = \cup_{i=1}^{\infty}\{U_t : U_t \cap X_i \in \mathcal{V}_i\}$. It is clear $\mathcal{V}$ is a countable subfamily of $\mathcal{U}$. Let G be a nonempty open set from τ. Then $G \cap X_i \neq \emptyset$ for some i. Since A_i is dense in (X_i, τ_{X_i}), $A_i \cap G \cap X_i \neq \emptyset$, so $G \cap (\cup\mathcal{V}) \neq \emptyset$. That means $\cup\mathcal{V}$ is dense in (X, τ), so (X, τ) is weakly Lindelöf. □

3 Ideals and topologies on the Cartesian product, topological sum

Definition 1 Let E, U be two nonempty sets. A nonempty family $\mathcal{I} \subset 2^{E\times U}$ is called an ideal on $E \times U$ if

(1) $A \cup B \in \mathcal{I}$ for any $A, B \in \mathcal{I}$,

(2) if $B \in \mathcal{I}$ and $A \subset B$, then $A \in \mathcal{I}$.

For $\mathcal{A} \subset 2^U$, $\mathcal{B} \subset 2^{E\times U}$, $A \subset U$, $B \subset E \times U$ and $e \in E$ we denote

$\mathcal{I}_B = \{B \cap I : I \in \mathcal{I}\}$ (it is an ideal on B).

$\varphi_e : U \to E \times U$ where $\varphi_e(u) = (e, u)$ for any $u \in U$,

$A[e] = \varphi_e(A) = \{e\} \times A$,

$\mathcal{A}[e] = \{\varphi_e(A) = A[e] : A \in \mathcal{A}\}$,

$B_e = {\varphi_e}^{-1}(B) = \{u \in U : (e, u) \in B\}$,

$\mathcal{B}_e = \{{\varphi_e}^{-1}(B) = B_e : B \in \mathcal{B}\}$,

$\mathcal{A}_E = \{S \subset E \times U : S_e \in \mathcal{A}$ for any $e \in E\}$.

In some cases, we use the notation $A[e] = (A)[e]$, $\mathcal{A}[e] = (\mathcal{A})[e]$, $B_e = (B)_e$, $\mathcal{B}_e = (\mathcal{B})_e$, $\mathcal{A}_E = (\mathcal{A})_E$. If $\mathcal{I}_e = \mathcal{I}_f$ for any $e, f \in E$, then $\mathcal{I}$ is called a constant ideal.

Remark 2 In this remark, we specify the families $\mathcal{A}$ and $\mathcal{B}$. Let τ be a topology on U, $U[e]$, $E \times U$, respectively, and $\mathcal{J}$, $\mathcal{I}$ be an ideal on U, $E \times U$, respectively. Then

(1) if τ is a topology on U, then $\tau[e]$ is a topology on $U[e]$ and (U, τ) is homeomorphic to $(U[e], \tau[e])$ (the function $\varphi_e : U \to U[e]$ is a homeomorphism, i.e., $\varphi_e(G) \in \tau[e]$ if and only if $G \in \tau$),

(2) if τ is a topology on $U[e]$, then τ_e is a topology on U and $\varphi_e : U \to U[e]$ is a homeomorphism from (U, τ_e) to $(U[e], \tau)$, i.e., $\varphi_e(G) \in \tau$ if and only if $G \in \tau_e$.

(3) if τ is a topology on $E \times U$, then τ_e is a topology on U for any $e \in E$. For a subspace topology $\tau_{U[e]}$ on $U[e]$, $(U, (\tau_{U[e]})_e)$ and $(U[e], \tau_{U[e]})$ are homeomorphic (see item (2)). Since $(\tau_{U[e]})_e = \tau_e$, (U, τ_e) and $(U[e], \tau_{U[e]})$ are homeomorphic, i.e., $\varphi_e(G) \in \tau_{U[e]}$ if and only if $G \in \tau_e$ for any $e \in E$.

(4) If τ is a topology on U, $E \times U$, then $(\tau[e])_e = \tau$, $(\tau_e)[e] = \tau_{U[e]}$ for any $e \in E$, respectively.

(5) $\mathcal{J}[e], \mathcal{I}_e$ is an ideal on $U[e]$, U, respectively. Moreover, $(\mathcal{J}[e])_e = \mathcal{J}$, $(\mathcal{I}_e)[e] = \mathcal{I}_{U[e]}$ and $I \in \mathcal{I}_e$ if and only if $\varphi_e(I) \in \mathcal{I}_{U[e]}$ for any $e \in E$.

(6) $\mathcal{J}_E$ is a constant ideal on $E \times U$ and $(\mathcal{J}_E)_e = \mathcal{J}$ for any $e \in E$.

Definition 2 Let $\{(U, \sigma_e) : e \in E\}$ be an indexed family of topological spaces. By $(E \times U, \oplus_{e\in E}\sigma_e)$ we denote a topological sum of $\{(U, \sigma_e) : e \in E\}$. Note that $\oplus_{e\in E}\sigma_e$ is a topology defined as the finest topology on $\oplus_{e\in E}U = \cup_{e\in E}\{e\} \times U = \cup_{e\in E}U[e] = E \times U$ for which all canonical injections $\varphi_e : (U, \sigma_e) \to (E \times U, \oplus_{e\in E}\sigma_e)$ defined by $\varphi_e(u) = (e, u)$ for $u \in U$ are continuous.

The following lemma will be useful for further investigation, see Engelking (1977).

Lemma 5 *Let $\{(U, \sigma_e) : e \in E\}$ be an indexed family of topological spaces. Then*

(1) a canonical injection φ_e is a continuous, open and closed map for any $e \in E$, so it is a homeomorphic embedding, i.e., $\varphi_e : (U, \sigma_e) \to (U[e], \sigma_e[e])$ is a homeomorphism.

(2) $S \subset E \times U$ is closed (open, dense) in $(E \times U, \oplus_{e\in E}\sigma_e)$ if and only if S_e is closed (open, dense) in (U, σ_e) for any $e \in E$ if and only if $S \cap U[e]$ is closed (open, dense) in $(U[e], \sigma_e[e])$ for any $e \in E$,

(3) $(E \times U, \oplus_{e\in E}\sigma_e)$ is compact (Lindelöf, weakly Lindelöf) if and only if E is finite (E is countable) and (U, σ_e) is compact (Lindelöf, weakly Lindelöf) for any $e \in E$.

Remark 3 If $\{(U, \sigma_e) : e \in E\}$ is an indexed family of topological spaces and $\mathcal{I}$ is an ideal on $E \times U$, then

(1) By Remark 2 item (1), (U, σ_e) is homeomorphic to $(U[e], \sigma_e[e])$. Since $\varphi_e(G) = \varphi_e(H) \setminus \varphi_e(I)$ for any base element $G = H \setminus I$ of $(\sigma_e)_{\mathcal{I}_e}$ where $\varphi_e(H) \in \sigma_e[e]$ (see Remark 2 item (1)) and $\varphi_e(I) \in \mathcal{I}_{U[e]}$ (see Remark 2 item (5)), φ_e is a homeomorphism from $(U, (\sigma_e)_{\mathcal{I}_e})$ to $(U[e], (\sigma_e[e])_{\mathcal{I}_{U[e]}})$.

(2) $D_{\oplus_{e\in E}\sigma_e,\mathcal{I}}(S)$, $D_{\sigma_e[e],\mathcal{I}_{U[e]}}(S \cap U[e])$, $D_{\sigma_e,\mathcal{I}_e}(S_e)$ is the set of all points in which S, $S \cap U[e]$, S_e is locally not in $\mathcal{I}$, $\mathcal{I}_{U[e]}$, $\mathcal{I}_e$ with respect to $\oplus_{e\in E}\sigma_e$, $\sigma_e[e]$, σ_e, respectively.

Theorem 1 *Let $(E \times U, \tau, \mathcal{I})$ be an ideal topological space. Then*

(1) for any set $G \in \tau_{\mathcal{I}}$ and any $e \in E$, $G_e \in (\tau_e)_{\mathcal{I}_e}$,

(2) $(\tau_{\mathcal{I}})_e = (\tau_e)_{\mathcal{I}_e}$.

Proof (1): Put $A = U[e]$. Then $G \cap U[e] = G \cap A \in (\tau_{\mathcal{I}})_A = (\tau_A)_{\mathcal{I}_A} = (\tau_{U[e]})_{\mathcal{I}_{U[e]}}$, by Lemma 2. That means $G \cap U[e] = \cup_{t\in T}(H_t \setminus I_t)$ where $H_t \in \tau_{U[e]}$ and $I_t \in \mathcal{I}_{U[e]}$.

By Remark 2 item (3), item (5), $\varphi_e^{-1}(H_t) = (H_t)_e \in \tau_e$ and $\varphi_e^{-1}(I_t) = (I_t)_e \in \mathcal{I}_e$, respectively. So $(H_t)_e \setminus (I_t)_e \in (\tau_e)_{\mathcal{I}_e}$ and $G_e = (G \cap U[e])_e = \cup_{t\in T}((H_t)_e \setminus (I_t)_e) \in (\tau_e)_{\mathcal{I}_e}$.

(2): If $H \in (\tau_{\mathcal{I}})_e$, then $H = G_e$ for some $G \in \tau_{\mathcal{I}}$. By (1), $H = G_e \in (\tau_e)_{\mathcal{I}_e}$, so $(\tau_{\mathcal{I}})_e \subset (\tau_e)_{\mathcal{I}_e}$.

Let $H \in (\tau_e)_{\mathcal{I}_e}$. Then $H = \cup_{t\in T}(G_t \setminus I_t)$ where $G_t \in \tau_e$ and $I_t \in \mathcal{I}_e$. So $G_t = (S_t)_e$, $I_t = (R_t)_e$ for some $S_t \in \tau$ and $R_t \in \mathcal{I}$. Than means $\cup_{t\in T}(S_t \setminus R_t) \in \tau_{\mathcal{I}}$, so $(\cup_{t\in T}(S_t \setminus R_t))_e \in (\tau_{\mathcal{I}})_e$. Since $(\cup_{t\in T}(S_t \setminus R_t))_e = \cup_{t\in T}((S_t)_e \setminus (R_t)_e) = \cup_{t\in T}(G_t \setminus I_t) = H$, $H \in (\tau_{\mathcal{I}})_e$. So $(\tau_e)_{\mathcal{I}_e} \subset (\tau_{\mathcal{I}})_e$. □

Theorem 2 *Let $\{(U, \sigma_e) : e \in E\}$ be an indexed family of topological spaces and $S \subset E \times U$ and $\mathcal{I}$ be an ideal on $E \times U$. Then*

$$D_{\oplus_{e\in E}\sigma_e,\mathcal{I}}(S) = \cup_{e\in E} D_{\sigma_e[e],\mathcal{I}_{U[e]}}(S \cap U[e]).$$

Proof Let $(e, u) \in D_{\oplus_{e\in E}\sigma_e,\mathcal{I}}(S)$ and $H \in \sigma_e[e]$, $(e, u) \in H$. Since $H \in \oplus_{e\in E}\sigma_e$ and $H \cap S = H \cap S \cap U[e] \notin \mathcal{I}$, $H \cap S \cap U[e] \notin \mathcal{I}_{U[e]}$. That means $(e, u) \in D_{\sigma_e[e],\mathcal{I}_{U[e]}}(S \cap U[e])$, so $(e, u) \in \cup_{e\in E} D_{\sigma_e[e],\mathcal{I}_{U[e]}}(S \cap U[e])$.

Let $(e, u) \in D_{\sigma_e[e],\mathcal{I}_{U[e]}}(S \cap U[e])$ and $(e, u) \in H \in \oplus_{e\in E}\sigma_e$. Since $(e, u) \in H \cap U[e] \in \sigma_e[e]$ (by Lemma 5 item (2)), $H \cap U[e] \cap S \notin \mathcal{I}_{U[e]}$. That means $H \cap S \notin \mathcal{I}$, so $(e, u) \in D_{\oplus_{e\in E}\sigma_e,\mathcal{I}}(S)$. □

Theorem 3 *Let $\{(U, \sigma_e) : e \in E\}$ be an indexed family of topological spaces and $\mathcal{I}$ be an ideal on $E \times U$. Then*

$$(\oplus_{e\in E}\sigma_e)_{\mathcal{I}} = \oplus_{e\in E}(\sigma_e)_{\mathcal{I}_e}.$$

Proof $G \in (\oplus_{e\in E}\sigma_e)_{\mathcal{I}}$ if and only if $G = \cup_{t\in T}(G_t \setminus I_t)$ and $G_t \in \oplus_{e\in E}\sigma_e$, $I_t \in \mathcal{I}$ if and only if (by Lemma 5 item (2)) $G = \cup_{t\in T}(G_t \setminus I_t)$ and $(G_t)_e \in \sigma_e$, $(I_t)_e \in \mathcal{I}_e$ for any $e \in E$ if and only if $G = \cup_{t\in T}(G_t \setminus I_t)$ and $\cup_{t\in T}((G_t)_e \setminus (I_t)_e) \in (\sigma_e)_{\mathcal{I}_e}$ for any $e \in E$ if and only if $G_e = \cup_{t\in T}((G_t)_e \setminus (I_t)_e) \in (\sigma_e)_{\mathcal{I}_e}$ for any $e \in E$ if and only if $G \in \oplus_{e\in E}(\sigma_e)_{\mathcal{I}_e}$, by Lemma 5 item (2). □

Corollary 2 *A subset S of $E \times U$ is closed (open) in $(E \times U, (\oplus_{e\in E}\sigma_e)_{\mathcal{I}})$ if and only if $S \cap U[e]$ is closed (open) in $(U[e], (\sigma_e[e])_{\mathcal{I}_{U[e]}})$ for any $e \in E$ if and only if S_e is closed (open) in $(U, (\sigma_e)_{\mathcal{I}_e})$ for any $e \in E$.*

Proof A proof follows from Theorem 3, Lemma 5 item (2) and from Remark 3 item (1), i.e., from a homeomorphism between $(U[e], (\sigma_e[e])_{\mathcal{I}_{U[e]}})$ and $(U, (\sigma_e)_{\mathcal{I}_e})$. □

4 Relations and set-valued mappings

Any subset S of the Cartesian product $E \times U$ is a binary relation from a set E to a set U. By $\mathbf{R}(E, U)$, we denote the set of all binary relations from E to U. Two relations A, B are equal if and only if $A_e = B_e$ for any $e \in E$. The operations of the sum $S \cup T$, $\cup_{t\in T} S_t$, intersection $S \cap T$, $\cap_{t\in T} S_t$, complement S^c and the difference $S \setminus T$ of relations are defined in the obvious way as in the set theory.

By $F : E \to 2^U$ we denote a set-valued mapping (multifunction) from E to power set 2^U of U. The set of all set-valued mappings from E to 2^U is denoted by $\mathbf{F}(E, U)$. A set-valued mapping F for which $F(e) = \{u\}$ and it is empty-valued otherwise is denoted by F_e^u.

If F, G are two set-valued mappings, then $F \subset G$ $(F = G)$ means $F(e) \subset G(e)$ $(F(e) = G(e))$ for any $e \in E$. So if $G \in \mathbf{F}(E, U)$, then $F_e^u \subset G \Leftrightarrow u \in G(e)$. The difference $F \setminus G$ of F and G is defined as a set-valued mapping given by $(F \setminus G)(e) = F(e) \setminus G(e)$ for any $e \in E$. The intersection (union) of family $\{G_t : t \in T\}$ of set-valued mappings is defined as a set-valued mapping $H : E \to 2^U$ for which $H(e) = \cap_{t\in T} G_t(e)$ $(H(e) = \cup_{t\in T} G_t(e))$ for any $e \in E$, and it is denoted by $\cap_{t\in T} G_t$ $(\cup_{t\in T} G_t)$. For the intersection (union) of two set-valued mappings F and G, we use notation $F \cap G$ $(F \cup G)$. The complement F^c of F is defined as a set-valued mapping for which $F^c(e) = U \setminus F(e)$ for all $e \in E$.

A graph of $G \in \mathbf{F}(E, U)$ is a set $Gr(G) = \{(e, u) \in E \times U : u \in G(e)\}$ and it is a subset of $E \times U$, hence $Gr(G) \in \mathbf{R}(E, U)$. So, any set-valued mapping G determines a relation from $\mathbf{R}(E, U)$ denoted by $\mathbf{R}_G$ where

$$\mathbf{R}_G = Gr(G) = \cup_{e\in E}\varphi_e(G(e)),$$
$$(\mathbf{R}_G)_e = (Gr(G))_e = \varphi_e^{-1}(Gr(G)) = G(e).$$

On the other hand, any relation $S \in \mathbf{R}(E, U)$ determines a set-valued mapping $\mathbf{F}_S$ from E to 2^U where

$$\mathbf{F}_S(e) = \varphi_e^{-1}(S) = S_e.$$

From the definitions of $\mathbf{R}_G$ and $\mathbf{F}_S$ and from the equality of two relations and the equality of two multifunctions, we have $\mathbf{F}_{\mathbf{R}_G}(e) = (\mathbf{R}_G)_e = G(e)$ and $(\mathbf{R}_{\mathbf{F}_S})_e = \mathbf{F}_S(e) = S_e$ for any $e \in E$, so

$$\mathbf{F}_{\mathbf{R}_G} = G, \quad \mathbf{R}_{\mathbf{F}_S} = S.$$

It is useful to note the next conditions are equivalent:

(1) $\mathbf{F}_S = G$,
(2) $\mathbf{F}_S(e) = G(e)$ for any $e \in E$,
(3) $S_e = G(e)$ for any $e \in E$,
(4) $S_e = (\mathbf{R}_G)_e$ for any $e \in E$,
(5) $S = \mathbf{R}_G$.

5 Soft ideal topological space and ideal topological space

Definition 3 (Maji et al. 2003; Shabir and Naz 2011) Let E, U be two nonempty sets.

(1) If $F : E \to 2^U$ is a set-valued mapping, then F is called a soft set over U with respect to E. A soft set F for which $F(e) = \emptyset$ ($F(e) = U$) for any $e \in E$ is called the null soft set (the full soft set) and F_e^u is called a soft point.
(2) A soft set F is a soft subset of G (F is contained in G or G contains F), if $F(e) \subset G(e)$ for any $e \in E$. The complement of soft set F is defined as a soft set F^c where $F^c(e) = U \setminus F(e)$ for all $e \in E$. The intersection (union) of a family of soft sets $\{G_t : t \in T\}$ is defined as a soft set G where $G(e) = \cap_{t \in T} G_t(e)$ ($G(e) = \cup_{t \in T} G_t(e)$) for all $e \in E$.
(3) The family of all soft sets over U with respect to E is denoted by $SS(E, U)$. It is clear $SS(E, U) = \mathbf{F}(E, U)$. The family of all soft points is denoted by $SP(E, U)$.

Definition 4 (Maji et al. 2003; Shabir and Naz 2011) Let E, U be two nonempty sets. A soft topological space over U with respect to E is a triplet (E, U, τ) where $\tau \subset SS(E, U)$ is closed under finite intersection, arbitrary union of soft sets and contains the null soft set and the full soft set. A soft set from τ is called a soft open set, and its complement is called a soft closed set. If H is a soft set, then a soft closure (a soft interior) of H denoted by $scl_\tau(H)$ ($sint_\tau(H)$) is defined as the intersection (union) of all soft closed (soft open) sets containing H (contained in H).

Definition 5 (Al Ghour and Hamed Worood (2020); Gharib and Abd El-latif (2019); Kandil et al. (2014)) A nonempty family $\mathcal{I} \subset SS(E, U)$ of soft sets is called a soft ideal on U with respect to E if

(1) $A \cup B \in \mathcal{I}$ for any $A, B \in \mathcal{I}$,
(2) if $B \in \mathcal{I}$ and $A \subset B$, then $A \in \mathcal{I}$.

If τ is a soft topology over U with respect to E, then $(E, U, \tau, \mathcal{I})$ is called a soft ideal topological space over U with respect to E.

Definition 6 Let $S \in \mathbf{R}(E, U)$ and $G \in \mathbf{F}(E, U)$. We say S, G corresponds to G, S if $S = \mathbf{R}_G$, $G = \mathbf{F}_S$, respectively. Moreover, S and G are mutually corresponding if $S = \mathbf{R}_G$ and $G = \mathbf{F}_S$. A family $\mathcal{C} \subset \mathbf{R}(E, U)$, $\mathcal{B} \subset \mathbf{F}(E, U)$ corresponds to a family $\mathcal{B} \subset \mathbf{F}(E, U)$, $\mathcal{C} \subset \mathbf{R}(E, U)$ if $\mathcal{C} = \mathbf{R}_\mathcal{B} := \{\mathbf{R}_G : G \in \mathcal{B}\}$, $\mathcal{B} = \mathbf{F}_\mathcal{C} := \{\mathbf{F}_S : S \in \mathcal{C}\}$, respectively. Finally $\mathcal{C}$ and $\mathcal{B}$ are mutually corresponding if $\mathcal{C}$ corresponds to $\mathcal{B}$ and $\mathcal{B}$ corresponds to $\mathcal{C}$.

The following theorem deals with the mutual correspondence between ideal topological spaces and soft ideal topological spaces, and plays an important role in the transformation of soft topological problems into topological ones. For the correspondence between topological spaces and soft topological spaces, see Matejdes (2016) ; Matejdes (2021a), Matejdes (2021c).

Theorem 4 *There is a one-to-one correspondence between the family of all soft ideal topological spaces over U with respect to E and the family of all ideal topological spaces on $E \times U$ as follows:*

(1) If $(E, U, \tau, \mathcal{I})$ is a soft ideal topological space, then $(E \times U, \mathbf{R}_\tau, \mathbf{R}_\mathcal{I})$ is an ideal topological space where $\mathbf{R}_\tau = \{\mathbf{R}_G : G \in \tau\}$, $\mathbf{R}_\mathcal{I} = \{\mathbf{R}_I : I \in \mathcal{I}\}$, i.e., $G \in \tau \Leftrightarrow \mathbf{R}_G \in \mathbf{R}_\tau$ and $A \in \mathcal{I} \Leftrightarrow \mathbf{R}_A \in \mathbf{R}_\mathcal{I}$. We say $(E \times U, \mathbf{R}_\tau, \mathbf{R}_\mathcal{I})$ is corresponding to $(E, U, \tau, \mathcal{I})$.
(2) If $(E \times U, \tau, \mathcal{I})$ is a ideal topological space, then $(E, U, \mathbf{F}_\tau, \mathbf{F}_\mathcal{I})$ is a soft ideal topological space where $\mathbf{F}_\tau = \{\mathbf{F}_G : G \in \tau\}$, $\mathbf{F}_\mathcal{I} = \{\mathbf{F}_I : I \in \mathcal{I}\}$, i.e., $G \in \tau \Leftrightarrow \mathbf{F}_G \in \mathbf{F}_\tau$ and $A \in \mathcal{I} \Leftrightarrow \mathbf{F}_A \in \mathbf{F}_\mathcal{I}$. We say $(E, U, \mathbf{F}_\tau, \mathbf{F}_\mathcal{I})$ is corresponding to $(E \times U, \tau, \mathcal{I})$.
(3) Similar correspondence holds between (E, U, τ) and $(E \times U, \mathbf{R}_\tau)$, $(E \times U, \tau)$ and $(E, U, \mathbf{F}_\tau)$, respectively, see Matejdes (2016)*;* Matejdes (2021a)*,* Matejdes (2021c)*.*

Remark 4 By the above theorem, $(E \times U, \mathbf{R}_{\mathbf{F}_\tau}, \mathbf{R}_{\mathbf{F}_\mathcal{I}}) = (E \times U, \tau, \mathcal{I})$ is corresponding to $(E, U, \mathbf{F}_\tau, \mathbf{F}_\mathcal{I})$ and vice versa. So $(E \times U, \tau, \mathcal{I})$ and $(E, U, \mathbf{F}_\tau, \mathbf{F}_\mathcal{I})$ ($(E, U, \tau, \mathcal{I})$ and $(E \times U, \mathbf{R}_\tau, \mathbf{R}_\mathcal{I})$) are mutually corresponding. Similarly, we say a topology τ (a soft topology τ) and a soft topology $\mathbf{F}_\tau$ (a topology $\mathbf{R}_\tau$) (an ideal $\mathcal{I}$ (a soft ideal $\mathcal{I}$) and a soft ideal $\mathbf{F}_\mathcal{I}$ (an ideal $\mathbf{R}_\mathcal{I}$)) are mutually corresponding. If $(E \times U, \tau_1, \mathcal{I}_1)$ is an ideal topological space and $(E, U, \tau_2, \mathcal{I}_2)$ is a soft topological space, then they are mutually corresponding if $\mathbf{F}_{\tau_1} = \tau_2$ and $\mathbf{F}_{\mathcal{I}_1} = \mathcal{I}_2$ if and only if $\mathbf{R}_{\tau_2} = \tau_1$ and $\mathbf{R}_{\mathcal{I}_2} = \mathcal{I}_1$.

Any subset of $E \times U$ uniquely corresponds to a soft set. The set $E \times U$ ($\emptyset$) corresponds to the full soft set $\mathbf{F}_{E \times U}$ (the null soft set $\mathbf{F}_\emptyset$). Any set from a soft topology τ (a topology τ) corresponds to an open set (a soft open set) from $\mathbf{R}_\tau$ ($\mathbf{F}_\tau$), and its complement corresponds to a closed set (a soft closed set).

The next theorem summarizes the properties of the operators $\mathbf{F} : \mathbf{R}(E, U) \to \mathbf{F}(E, U)$ and $\mathbf{R} : \mathbf{F}(E, U) \to \mathbf{R}(E, U)$. For item (1), see the conditions at the end of the previous section and item (2) is trivial. For items (3)-(9), see Matejdes (2021c).

Theorem 5 *Let $(E \times U, \tau_1)$ and (E, U, τ_2) be mutually corresponding. If $H, G, G_t \in SS(E, U)$ and $A, B, S_t \in \mathbf{R}(E, U)$, $t \in T$, then*

(1) the next conditions are equivalent

(a) H and B are mutually corresponding,

(b) $H = \mathbf{F}_B$,
(c) $H(e) = \mathbf{F}_B(e)$ for any $e \in E$,
(d) $H(e) = B_e$ for any $e \in E$,
(e) $(\mathbf{R}_H)_e = B_e$ for any $e \in E$,
(f) $\mathbf{R}_H = B$.

(2) $\mathbf{F}_{\{(e,u)\}} = F_e^u$, $\mathbf{R}_{F_e^u} = \{(u, e)\}$, $(e, u) \in A$ if and only if $F_e^u \subset \mathbf{F}_A$, $F_e^u \subset H$ if and only if $(e, u) \in \mathbf{R}_H$.
(3) H is soft open (soft closed) in (E, U, τ_2) if and only if $\mathbf{R}_H$ is open (closed) in $(E \times U, \tau_1)$ and A is open (closed) in $(E \times U, \tau_1)$ if and only if $\mathbf{F}_A$ is soft open (soft closed) in (E, U, τ_2).
(4) $\mathbf{F}_{A\cap B} = \mathbf{F}_A \cap \mathbf{F}_B$, $\mathbf{F}_{A\cup B} = \mathbf{F}_A \cup \mathbf{F}_B$, $\mathbf{F}_{\cap_{t\in T} S_t} = \cap_{t\in T}\mathbf{F}_{S_t}$, $\mathbf{F}_{\cup_{t\in T} S_t} = \cup_{t\in T}\mathbf{F}_{S_t}$.
(5) $\mathbf{R}_{H\cap G} = \mathbf{R}_H \cap \mathbf{R}_G$, $\mathbf{R}_{H\cup G} = \mathbf{R}_H \cup \mathbf{R}_G$, $\mathbf{R}_{\cap_{t\in T} G_t} = \cap_{t\in T}\mathbf{R}_{G_t}$, $\mathbf{R}_{\cup_{t\in T} G_t} = \cup_{t\in T}\mathbf{R}_{G_t}$.
(6) $\mathbf{R}_{\mathbf{F}_A} = A$, $\mathbf{F}_{\mathbf{R}_H} = H$, $\mathbf{F}_{A\setminus B} = \mathbf{F}_A \setminus \mathbf{F}_B$, $\mathbf{R}_{H\setminus G} = \mathbf{R}_H \setminus \mathbf{R}_G$.
(7) $scl_{\tau_2}(H) = \mathbf{F}_{cl_{\tau_1}(\mathbf{R}_H)}$, $sint_{\tau_2}(H) = \mathbf{F}_{int_{\tau_1}(\mathbf{R}_H)}$, $scl_{\tau_2}(\mathbf{F}_A) = \mathbf{F}_{cl_{\tau_1}(A)}$, $sint_{\tau_2}(\mathbf{F}_A) = \mathbf{F}_{int_{\tau_1}(A)}$.
(8) $cl_{\tau_1}(A) = \mathbf{R}_{scl_{\tau_2}(\mathbf{F}_A)}$, $int_{\tau_1}(A) = \mathbf{R}_{sint_{\tau_2}(\mathbf{F}_A)}$, $cl_{\tau_1}(\mathbf{R}_H) = \mathbf{R}_{scl_{\tau_2}(H)}$, $int_{\tau_1}(\mathbf{R}_H) = \mathbf{R}_{sint_{\tau_2}(H)}$.
(9) $scl_{\tau_2}(H \cup G) = scl_{\tau_2}(H) \cup scl_{\tau_2}(G)$, $sint_{\tau_2}(H \cap G) = sint_{\tau_2}(H) \cap sint_{\tau_2}(G)$.

The methods of constructing new topological spaces from old ones and the one-to-one correspondence between the family of topological spaces and soft topological spaces allow the introduction of soft topological spaces. Some of them are introduced in the following definition.

Definition 7 In this definition, we introduce a soft topological sum, a soft topological subspace, and a soft topology corresponding to $\tau_{\mathcal{I}}$.

(1) Let $\{(U, \sigma_e) : e \in E\}$ be an indexed family of topological spaces. A soft topological sum of $\{(U, \sigma_e) : e \in E\}$ is defined as a soft topology $\mathbf{F}_{\oplus_{e\in E}\sigma_e}$ and it is denoted by $\oplus^s_{e\in E}\sigma_e$. Note $\mathbf{F}_{\oplus_{e\in E}\sigma_e}$ is equal to $\{H : E \to 2^U : H(e) \in \sigma_e$ for all $e \in E\}$ which is a soft topology and $\mathbf{F}_{\sigma_e[e]} = \{H : E \to 2^U : H(e) \in \sigma_e$ and $H(f) = \emptyset$ *for* $f \neq e\}$ is its soft subbase. So $\mathbf{F}_{\oplus_{e\in E}\sigma_e} = \oplus^s_{e\in E}\sigma_e$ (see notation $\oplus_{e\in E}\sigma_e$ in Al Ghour and Hamed Worood (2020)). Specially if $\sigma_e = \mathfrak{J}$ for any $e \in E$, then $\mathbf{F}_{\oplus_{e\in E}\mathfrak{J}} = \oplus^s_{e\in E}\mathfrak{J} = \tau(\mathfrak{J})$ where $\tau(\mathfrak{J}) = \{F \in SS(E, U) : F(e) \in \mathfrak{J}$ for any $e \in E\}$ is a soft topology from Al Ghour and Hamed Worood (2020).
(2) If $Y \subset U$, then a soft topological subspace of (E, U, τ) on Y is defined as the corresponding soft topological space to a topological subspace $(E \times Y, (\mathbf{R}_\tau)_{E\times Y})$ where $(\mathbf{R}_\tau)_{E\times Y}$ is a subspace topology on $E \times Y$ derived from $\mathbf{R}_\tau$. A soft topological subspace of (E, U, τ) on Y is denoted by Al Ghour and Hamed Worood (2020) by (E, Y, τ_Y), see Lemma 9.
(3) If $(E \times U, \tau, \mathcal{I})$ is an ideal topological space, then we can define a soft ideal topological space by $(E, U, \mathbf{F}_{\tau_{\mathcal{I}}})$, see Lemma 6 and Lemma 8 (1).

Lemma 6 *Let $(E \times U, \tau, \mathcal{I})$ be an ideal topological space. Then $\mathbf{F}_{\tau_{\mathcal{I}}} = (\mathbf{F}_\tau)_{\mathbf{F}_{\mathcal{I}}}$ where $(\mathbf{F}_\tau)_{\mathbf{F}_{\mathcal{I}}}$ denotes a soft topology generated by a soft base $\{G \setminus I : G \in \mathbf{F}_\tau, I \in \mathbf{F}_{\mathcal{I}}\} = \mathbf{F}_{\mathcal{B}_{\tau,\mathcal{I}}}$.*

Proof The equation $\{G \setminus I : G \in \mathbf{F}_\tau, I \in \mathbf{F}_{\mathcal{I}}\} = \mathbf{F}_{\mathcal{B}_{\tau,\mathcal{I}}}$ and the fact that $\{G \setminus I : G \in \mathbf{F}_\tau, I \in \mathbf{F}_{\mathcal{I}}\}$ is a soft base is trivial.

$H \in \mathbf{F}_{\tau_{\mathcal{I}}}$ if and only if $\mathbf{R}_H \in \tau_{\mathcal{I}}$ if and only if $\mathbf{R}_H = \cup_{t\in T}(G_t \setminus I_t)$ where $G_t \in \tau$ and $I_t \in \mathcal{I}$ if and only if $H = \cup_{t\in T}(\mathbf{F}_{G_t} \setminus \mathbf{F}_{I_t})$ (by Theorem 5 (4), (6)) where $\mathbf{F}_{G_t} \in \mathbf{F}_\tau$ and $\mathbf{F}_{I_t} \in \mathbf{F}_{\mathcal{I}}$ if and only if $H \in (\mathbf{F}_\tau)_{\mathbf{F}_{\mathcal{I}}}$. That means $\mathbf{F}_{\tau_{\mathcal{I}}} = (\mathbf{F}_\tau)_{\mathbf{F}_{\mathcal{I}}}$. □

In the following, $\tau, \hat{\tau}$ $(\mathcal{I}, \hat{\mathcal{I}})$ denotes a topology on $E \times X$, a soft topology over U with respect to E (an ideal on $E \times X$, a soft ideal on U with respect to E), respectively.

Lemma 7 *Let $(E \times U, \tau, \mathcal{I})$, $(E, U, \hat{\tau}, \hat{\mathcal{I}})$ be an ideal topological space, soft ideal topological space, respectively. Then*

(1) For any $e \in E$ the families $\hat{\tau}_e := \{F(e) : F \in \hat{\tau}\}$ and $\tau_e := \{G_e : G \in \tau\}$ are the topologies on U. If $(E \times U, \tau)$ and $(E, U, \hat{\tau})$ are mutually corresponding, then $(\mathbf{R}_{\hat{\tau}})_e = \tau_e = \hat{\tau}_e = (\mathbf{F}_\tau)_e$ for any $e \in E$.
(2) $\hat{\tau} = \mathbf{F}_\tau$ and $\hat{\mathcal{I}} = \mathbf{F}_{\mathcal{I}}$ $(\hat{\tau} = \mathbf{F}_\tau)$ if and only if $\tau = \mathbf{R}_{\hat{\tau}}$ and $\mathcal{I} = \mathbf{R}_{\hat{\mathcal{I}}}$ $(\tau = \mathbf{R}_{\hat{\tau}})$ if and only if $(E \times U, \tau, \mathcal{I})$ and $(E, U, \hat{\tau}, \hat{\mathcal{I}})$ $((E \times U, \tau)$ and $(E, U, \hat{\tau}))$ are mutually corresponding.
(3) The next conditions are equivalent.

(a) $(E \times U, \tau)$ and $(E, U, \hat{\tau})$ are mutually corresponding,
(b) $\mathbf{F}_{\mathcal{B}}$ is a soft base of $(E, U, \hat{\tau})$ for any base $\mathcal{B}$ of $(E \times U, \tau)$,
(c) $\mathbf{R}_{\hat{\mathcal{B}}}$ is a base of $(E \times U, \tau)$ for any soft base $\hat{\mathcal{B}}$ of $(E, U, \hat{\tau})$.

Proof (1) It is clear that $\hat{\tau}_e$ and τ_e are the topologies on U. Since $\mathbf{R}_{\hat{\tau}} = \tau$ and $\mathbf{F}_\tau = \hat{\tau}$, $(\mathbf{R}_{\hat{\tau}})_e = \tau_e$ and $(\mathbf{F}_\tau)_e = \hat{\tau}_e$.

Since $G \in \tau$ and $\mathbf{F}_G \in \hat{\tau}$ are mutually corresponding, by Theorem 5 (1) (c), (d), $G_e = \mathbf{F}_G(e)$ for any $e \in E$. That means $A \in \tau_e$ if and only if $A = G_e = \mathbf{F}_G(e)$ for some $G \in \tau$ if and only if $A = \mathbf{F}_G(e)$ for some $\mathbf{F}_G \in \hat{\tau}$ if and only if $A \in \hat{\tau}_e$.

(2) follows from Remark 4.

(3) In the following, we use the rules of Theorem 5. $(a) \Rightarrow (b)$ Let $(E \times U, \tau)$ and $(E, U, \hat{\tau})$ be mutually corresponding. Let $\mathcal{B} = \{G_t : t \in T\}$ be a base of $(E \times U, \tau)$. If $G \in \hat{\tau}$, then $\mathbf{R}_G \in \tau$ so $\mathbf{R}_G = \cup_{t\in T_0 \subset T} G_t$ and $G_t \in \mathcal{B}$. That means

$G = \mathbf{F}_{\mathbf{R}_G} = \mathbf{F}_{\cup_{t\in T_0\subset T} G_t} = \cup_{t\in T_0\subset T}\mathbf{F}_{G_t}$ and $\mathbf{F}_{G_t} \in \mathbf{F}_{\mathcal{B}}$. That means $\mathbf{F}_{\mathcal{B}}$ is a soft base of $(E, U, \hat{\tau})$.

$(b) \Rightarrow (c)$ Let $\hat{\mathcal{B}} = \{G_t : t \in T\}$ be a soft base of $(E, U, \hat{\tau})$. If $G \in \tau$, then $G = \cup_{s\in S} G_s$ where $G_s \in \mathcal{B}_0$ for some base $\mathcal{B}_0$ of τ. So $\mathbf{F}_G = \mathbf{F}_{\cup_{s\in S} G_s} = \cup_{s\in S}\mathbf{F}_{G_s}$ and $\mathbf{F}_{G_s} \in \mathbf{F}_{\mathcal{B}_0}$. Since $\mathbf{F}_{\mathcal{B}_0}$ is a base of $\hat{\tau}$, $\mathbf{F}_{G_s} \in \hat{\tau}$. Then for any $s \in S$, $\mathbf{F}_{G_s} = \cup_{i\in I} H_s^i$ where $H_s^i \in \hat{\mathcal{B}}$. That means $\mathbf{F}_G = \cup_{s\in S}\cup_{i\in I} H_s^i$, so $G = \mathbf{R}_{\cup_{s\in S}\cup_{i\in I} H_s^i} = \cup_{s\in S}\cup_{i\in I}\mathbf{R}_{H_s^i}$ and $\mathbf{R}_{H_s^i} \in \mathbf{R}_{\hat{\mathcal{B}}}$. That means $\mathbf{R}_{\hat{\mathcal{B}}}$ is a base of τ.

$(c) \Rightarrow (a)$ By item (2), it is sufficient to prove $\tau = \mathbf{R}_{\hat{\tau}}$. Let $\hat{\mathcal{B}} \subset \hat{\tau}$ be a base of $(E, U, \hat{\tau})$. Then $\mathbf{R}_{\hat{\mathcal{B}}} \subset \tau$ is a base of $(E \times U, \tau)$. If $A \in \tau$, then $A = \cup_{t\in T} G_t$ where $G_t \in \mathbf{R}_{\hat{\mathcal{B}}} \subset \mathbf{R}_{\hat{\tau}}$. Since $\mathbf{R}_{\hat{\tau}}$ is a topology (see Theorem 4), $A \in \mathbf{R}_{\hat{\tau}}$. On the other hand, if $A \in \mathbf{R}_{\hat{\tau}}$, then $A = \mathbf{R}_S$ where $S \in \hat{\tau}$. Since $\hat{\mathcal{B}}$ is a soft base of $(E, U, \hat{\tau})$, $S = \cup_{t\in T} G_t$ where $G_t \in \hat{\mathcal{B}}$. Then $A = \mathbf{R}_{\cup_{t\in T} G_t} = \cup_{t\in T}\mathbf{R}_{G_t}$ where $\mathbf{R}_{G_t} \in \mathbf{R}_{\hat{\mathcal{B}}} \subset \tau$. Since τ is a topology, $A \in \tau$. □

Lemma 8 *Let $(E \times U, \tau, \mathcal{I})$ and $(E, U, \hat{\tau}, \hat{\mathcal{I}})$ be mutually corresponding (i.e., $\hat{\tau} = \mathbf{F}_\tau$ and $\hat{\mathcal{I}} = \mathbf{F}_{\mathcal{I}}$) and $\{(U, \sigma_e) : e \in E\}$ be an indexed family of topological spaces. Then*

(1) $\mathbf{F}_{\tau_{\mathcal{I}}} = (\mathbf{F}_\tau)_{\mathbf{F}_{\mathcal{I}}} = \hat{\tau}_{\hat{\mathcal{I}}}$ and $\mathbf{R}_{\hat{\tau}_{\hat{\mathcal{I}}}} = (\mathbf{R}_{\hat{\tau}})_{\mathbf{R}_{\hat{\mathcal{I}}}} = \tau_{\mathcal{I}}$. That means $\tau_{\mathcal{I}}$ and $\hat{\tau}_{\hat{\mathcal{I}}}$ are mutually corresponding.

(2) $(\oplus^s_{e\in E}\sigma_e)_{\hat{\mathcal{I}}} = (\oplus^s_{e\in E}\sigma_e)_{\mathbf{F}_{\mathcal{I}}} = (\mathbf{F}_{\oplus_{e\in E}\sigma_e})_{\mathbf{F}_{\mathcal{I}}} = \mathbf{F}_{(\oplus_{e\in E}\sigma_e)_{\mathcal{I}}} = \mathbf{F}_{\oplus_{e\in E}(\sigma_e)_{\mathcal{I}_e}} = \oplus^s(\sigma_e)_{\mathcal{I}_e}$. So, $(\oplus^s_{e\in E}\sigma_e)_{\hat{\mathcal{I}}}$ and $(\oplus_{e\in E}\sigma_e)_{\mathcal{I}} = \oplus_{e\in E}(\sigma_e)_{\mathcal{I}_e}$ are mutually corresponding.

(3) $cl_\tau(G) = \mathbf{R}_{scl_{\hat{\tau}}(\mathbf{F}_G)}$, $scl_{\hat{\tau}}(H) = \mathbf{F}_{cl_\tau(\mathbf{R}_H)}$ for any subset G of $E \times U$, for any soft set H, respectively where $scl_{\hat{\tau}}$ is the soft closure operator with respect to $\hat{\tau}$, see Definition 4.

Proof (1) By Lemma 6, $\mathbf{F}_{\tau_{\mathcal{I}}} = (\mathbf{F}_\tau)_{\mathbf{F}_{\mathcal{I}}} = \hat{\tau}_{\hat{\mathcal{I}}}$, so $\tau_{\mathcal{I}} = \mathbf{R}_{\hat{\tau}_{\hat{\mathcal{I}}}}$. By Lemma 6, $\mathbf{F}_{(\mathbf{R}_{\hat{\tau}})_{(\mathbf{R}_{\hat{\mathcal{I}}})}} = (\mathbf{F}_{\mathbf{R}_{\hat{\tau}}})_{(\mathbf{F}_{\mathbf{R}_{\hat{\mathcal{I}}}})} = \hat{\tau}_{\hat{\mathcal{I}}}$, so $\mathbf{R}_{\hat{\tau}_{\hat{\mathcal{I}}}} = (\mathbf{R}_{\hat{\tau}})_{\mathbf{R}_{\hat{\mathcal{I}}}} = \tau_{\mathcal{I}}$.

(2) follows from Definition 7, Lemma 6, Theorem 3, Definition 7.

(3) follows from Theorem 5 (7), (8). □

6 Soft ω-open sets and strongly soft ω-open sets

In the next remark, we specify the above results to those of Al Ghour and Hamed Worood (2020) concerning soft ω-open sets and strongly soft ω-open sets. Readers are referred to Al Ghour and Hamed Worood (2020) for the following notations: $coc(U, E)$, $scoc(U, E)$, $CSS(U, E)$, $SCSS(U, E)$, $SP(E, U)$, $\hat{\tau}_c$, $\hat{\tau}_{sc}$, $\hat{\tau}_\omega$, $\hat{\tau}_{s\omega}$ where $\hat{\tau}$ is a soft topology over U with respect to E.

Definition 8 By $\mathcal{I}^s$, $\mathcal{I}^0$, we denote an ideal of all countable subsets of $E \times U$, an ideal of all subsets I of $E \times U$ such that $I_e \subset U$ is countable for any $e \in E$, respectively. Let $\hat{\mathcal{I}}^0$, $\hat{\mathcal{I}}^s$ be the corresponding soft ideal to $\mathcal{I}^0$, $\mathcal{I}^s$, i.e, $\mathbf{F}_{\mathcal{I}^0} = \hat{\mathcal{I}}^0 \Leftrightarrow \mathbf{R}_{\hat{\mathcal{I}}^0} = \mathcal{I}^0$, $\mathbf{F}_{\mathcal{I}^s} = \hat{\mathcal{I}}^s \Leftrightarrow \mathbf{R}_{\hat{\mathcal{I}}^s} = \mathcal{I}^s$, respectively.

Remark 5 Let $(E \times U, \tau,)$ and $(E, U, \hat{\tau})$ be mutually corresponding, i.e., $\mathbf{F}_\tau = \hat{\tau} \Leftrightarrow \mathbf{R}_{\hat{\tau}} = \tau$. Then

(1) $\mathcal{I}^s$, $\mathcal{I}^0$ is a constant ideal on $E \times U$, respectively, i.e., $\mathcal{I}^s_e = \mathcal{I}^0_e$ (= an ideal of all countable subsets of U) and $\mathcal{I}^s_{U[e]} = \mathcal{I}^0_{U[e]}$ (= an ideal of all countable subsets of $U[e]$) for any $e \in E$. It is clear $\mathcal{I}^s \subset \mathcal{I}^0$, $\hat{\mathcal{I}}^s \subset \hat{\mathcal{I}}^0$.

(2) $\mathcal{I}^0$ corresponds to the collection of all countable soft sets $CSS(U, E)$ ($G \in CSS(U, E) \Leftrightarrow G(e)$ is countable for any $e \in E$), i.e., $CSS(U, E) = \mathbf{F}_{\mathcal{I}^0} = \hat{\mathcal{I}}^0 \Leftrightarrow \mathbf{R}_{CSS(U,E)} = \mathcal{I}^0 = \mathbf{R}_{\hat{\mathcal{I}}^0}$.

(3) $\mathcal{I}^s$ corresponds to the collection of all strongly countable soft sets $SCSS(U, E)$ ($G \in SCSS(U, E) \Leftrightarrow G(e)$ is countable for any $e \in E$ and $\{e : G(e) \neq \emptyset\}$ is countable), i.e., $SCSS(U, E) = \mathbf{F}_{\mathcal{I}^s} = \hat{\mathcal{I}}^s \Leftrightarrow \mathbf{R}_{SCSS(U,E)} = \mathcal{I}^s = \mathbf{R}_{\hat{\mathcal{I}}^s}$.
Since $\hat{\mathcal{I}}^s \subset \hat{\mathcal{I}}^0$, $SCSS(U, E) \subset CSS(U, E)$, see Al Ghour and Hamed Worood (2020), Proposition 11.
Moreover, $\mathcal{I}^s = \mathcal{I}^0$ if and only if E is countable, so $SCSS(U, E) = CSS(U, E)$ if and only if E is countable, see Al Ghour and Hamed Worood (2020) Theorem 16.

(4) The base $\mathcal{B}_{\tau,\mathcal{I}^0}$ corresponds to the soft base $\hat{\tau}_c = \{G \setminus I : G$ is soft open, $I \in CSS(U, E)\} = \mathbf{F}_{\mathcal{B}_{\tau,\mathcal{I}^0}} \Leftrightarrow \mathbf{R}_{\hat{\tau}_c} = \mathcal{B}_{\tau,\mathcal{I}^0}$. So, $\mathbf{F}_{\mathcal{B}_{\tau,\mathcal{I}^0}}$ is a soft base for $\hat{\tau}_\omega$, by Lemma 7, see Al Ghour and Hamed Worood (2020) Theorem 2.

(5) The base $\mathcal{B}_{\tau,\mathcal{I}^s}$ corresponds to the soft base $\hat{\tau}_{sc} = \{G \setminus I : G$ is soft open, $I \in SCSS(U, E)\} = \mathbf{F}_{\mathcal{B}_{\tau,\mathcal{I}^s}} \Leftrightarrow \mathbf{R}_{\hat{\tau}_{sc}} = \mathcal{B}_{\tau,\mathcal{I}^s}$. So, $\mathbf{F}_{\mathcal{B}_{\tau,\mathcal{I}^s}}$ is a soft base for $\hat{\tau}_{s\omega}$, by Lemma 7, see Al Ghour and Hamed Worood (2020) Theorem 18.

(6) The topology $co_{E\times U,\mathcal{I}^0}$ corresponds to the cocountable soft topology $coc(U, E)$, so
$coc(U, E) = \mathbf{F}_{co_{E\times U,\mathcal{I}^0}} \Leftrightarrow \mathbf{R}_{coc(U,E)} = co_{E\times U,\mathcal{I}^0} = (co_{E\times U,\mathcal{I}^0})_{\mathcal{I}^0}$, by Remark 1 (4).
The topology $co_{E\times U,\mathcal{I}^s}$ corresponds to the strongly cocountable soft topology $scoc(U, E)$, so
$scoc(U, E) = \mathbf{F}_{co_{E\times U,\mathcal{I}^s}} \Leftrightarrow \mathbf{R}_{scoc(U,E)} = co_{E\times U,\mathcal{I}^s} = (co_{E\times U,\mathcal{I}^s})_{\mathcal{I}^s}$, by Remark 1 (4).
Clearly $co_{E\times U,\mathcal{I}^s} \subset co_{E\times U,\mathcal{I}^0}$, see Remark 1 (5) and $co_{E\times U,\mathcal{I}^s} = co_{E\times U,\mathcal{I}^0}$ if and only if E is countable. So, $scoc(U, E) = coc(U, E)$ if and only if E is countable.

(7) $A \in \hat{\tau}_\omega$ if and only if $A = \cup_{t\in T} G_t$ where $G_t \in \hat{\tau}_c$ if and only if $A = \cup_{t\in T}(F_t \setminus I_t)$ where $F_t \in \hat{\tau}$ and $I_t \in CSS(U, E) = \mathbf{F}_{\mathcal{I}^0}$ if and only if $A \in (\hat{\tau})_{\mathbf{F}_{\mathcal{I}^0}}$. Similarly $A \in \hat{\tau}_{s\omega}$ if and only if $A \in (\hat{\tau})_{\mathbf{F}_{\mathcal{I}^s}}$. By Lemma 8 (1),

$\hat{\tau}_\omega = (\hat{\tau})_{\mathbf{F}_{\mathcal{I}^0}} = (\hat{\tau})_{\hat{\mathcal{I}}^0} = (\mathbf{F}_\tau)_{\mathbf{F}_{\mathcal{I}^0}} = \mathbf{F}_{\tau_{\mathcal{I}^0}}$,
$\hat{\tau}_{s\omega} = (\hat{\tau})_{\mathbf{F}_{\mathcal{I}^s}} = (\hat{\tau})_{\hat{\mathcal{I}}^s} = (\mathbf{F}_\tau)_{\mathbf{F}_{\mathcal{I}^s}} = \mathbf{F}_{\tau_{\mathcal{I}^s}}$.

(8) By item (7), $\tau_{\mathcal{I}^0}$ and $\hat{\tau}_\omega$, $\tau_{\mathcal{I}^s}$ and $\hat{\tau}_{s\omega}$ are mutually corresponding, respectively. Consequently
$(\oplus_{e\in E}\sigma_e)_{\mathcal{I}^0}$ and $(\oplus^s_{e\in E}\sigma_e)_\omega$,
$(\oplus_{e\in E}\sigma_e)_{\mathcal{I}^s}$ and $(\oplus^s_{e\in E}\sigma_e)_{s\omega}$,
$(co_{E\times U,\mathcal{I}^0})_{\mathcal{I}^0}$ and $(coc(U,E))_\omega$,
$(co_{E\times U,\mathcal{I}^s})_{\mathcal{I}^s}$ and $(scoc(U,E))_{s\omega}$
are mutually corresponding, respectively. So
$\mathbf{F}_{(\oplus_{e\in E}\sigma_e)_{\mathcal{I}^0}} = (\oplus^s_{e\in E}\sigma_e)_\omega$,
$\mathbf{F}_{(\oplus_{e\in E}\sigma_e)_{\mathcal{I}^s}} = (\oplus^s_{e\in E}\sigma_e)_{s\omega}$,
$\mathbf{F}_{(co_{E\times U,\mathcal{I}^0})_{\mathcal{I}^0}} = (coc(U,E))_\omega$,
$\mathbf{F}_{(co_{E\times U,\mathcal{I}^s})_{\mathcal{I}^s}} = (scoc(U,E))_{s\omega}$.

(9) Since $(coc(U,E))_\omega = \mathbf{F}_{(co_{E\times U,\mathcal{I}^0})_{\mathcal{I}^0}} = \mathbf{F}_{co_{E\times U,\mathcal{I}^0}} = coc(U,E)$ and $(scoc(U,E))_{s\omega} = \mathbf{F}_{(co_{E\times U,\mathcal{I}^s})_{\mathcal{I}^s}} = \mathbf{F}_{co_{E\times U,\mathcal{I}^s}} = scoc(U,E)$ (see item (8), Remark 1 (4) and item (6)),
$(coc(U,E))_\omega = coc(U,E)$,
$(scoc(U,E))_{s\omega} = scoc(U,E)$,
see Al Ghour and Hamed Worood (2020) Corollary 1 and 7.
If E is countable, then $coc(U,E) = scoc(U,E) = (coc(U,E))_\omega = (coc(U,E))_{s\omega}$, see item (6).

(10) A set $G \in \tau_{\mathcal{I}^s}$ ($G \in \tau_{\mathcal{I}^0}$) is called a strongly ω-open set (ω-open set) and it corresponds to a strongly soft ω-open set (soft ω-open set) from $\hat{\tau}_{s\omega}$ ($\hat{\tau}_\omega$). According Al Ghour and Hamed Worood (2020), $\hat{\tau}_{s\omega}$ ($\hat{\tau}_\omega$) is called the soft topology of all strongly soft ω-open sets (soft ω-open sets).

(11) Since $\mathcal{I}^s_e = \mathcal{I}^0_e$ (= an ideal of all countable subsets of U, see Remark 5 (1)), for any topology σ on U,
$$(\sigma)_{\mathcal{I}^s_e} = (\sigma)_{\mathcal{I}^0_e} = \sigma_\omega$$
where σ_ω is a topology on U generated by a base $\{G\backslash A : G \in \sigma$ and A is countable$\}$. Consequently
$$(\tau_e)_{\mathcal{I}^s_e} = (\tau_e)_{\mathcal{I}^0_e} = (\tau_e)_\omega = (\hat{\tau}_e)_\omega,$$
by Lemma 7 (1).

(12) $\{(e,u)\}$ corresponds to a soft point F^u_e (e_x, see Al Ghour and Hamed Worood (2020)).

(13) $\{\{(e,u)\} : e \in E, u \in U\}$ corresponds to $SP(E,U)$.

(14) Since $\tau_{\mathcal{I}^0}$ and $\hat{\tau}_\omega$, $\tau_{\mathcal{I}^s}$ and $\hat{\tau}_{s\omega}$ are mutually corresponding, respectively (see Remark 5 (8)), by Theorem 5 (7), (8)
$cl_{\tau_{\mathcal{I}^0}}(G) = \mathbf{R}_{scl_{\hat{\tau}_\omega}(\mathbf{F}_G)}$, $scl_{\hat{\tau}_\omega}(H) = \mathbf{F}_{cl_{\tau_{\mathcal{I}^0}}(\mathbf{R}_H)}$,
$cl_{\tau_{\mathcal{I}^s}}(G) = \mathbf{R}_{scl_{\hat{\tau}_{s\omega}}(\mathbf{F}_G)}$, $scl_{\hat{\tau}_{s\omega}}(H) = \mathbf{F}_{cl_{\tau_{\mathcal{I}^s}}(\mathbf{R}_H)}$
where $scl_{\hat{\tau}_\omega}$, $scl_{\hat{\tau}_{s\omega}}$ is the soft closure operator with respect to $\hat{\tau}_\omega$, $\hat{\tau}_{s\omega}$, respectively, and G is a subset of $E \times X$ and H is a soft set.

7 Application of results for an ideal of countable sets

Corollary 3 *Let $\{(U,\sigma_e)) : e \in E\}$ be an indexed family of topological spaces and $(U,\mathfrak{J})$ be a topological space. Then*

(1) $(\oplus^s_{e\in E}\sigma_e)_\omega = (\oplus^s_{e\in E}\sigma_e)_{s\omega} = \oplus^s_{e\in E}(\sigma_e)_\omega$,
(2) $(\tau(\mathfrak{J}))_\omega = (\tau(\mathfrak{J}))_{s\omega} = \tau(\mathfrak{J}_\omega)$,

see Al Ghour and Hamed Worood (2020) *Theorem* 8, 26, *Corollary* 1, 4, 11, 12, 13.

Proof (1) By Remark 5 (11)) and Theorem 3,
$(\oplus_{e\in E}\sigma_e)_{\mathcal{I}^0} = \oplus_{e\in E}(\sigma_e)_{\mathcal{I}^0_e} = \oplus_{e\in E}(\sigma_e)_{\mathcal{I}^s_e} = (\oplus_{e\in E}\sigma_e)_{\mathcal{I}^s}$.
That means
$\mathbf{F}_{(\oplus_{e\in E}\sigma_e)_{\mathcal{I}^0}} = \mathbf{F}_{\oplus_{e\in E}(\sigma_e)_{\mathcal{I}^0_e}} = \mathbf{F}_{\oplus_{e\in E}(\sigma_e)_{\mathcal{I}^s_e}} = \mathbf{F}_{(\oplus_{e\in E}\sigma_e)_{\mathcal{I}^s}}$.
By Remark 5 (8),
$\mathbf{F}_{(\oplus_{e\in E}\sigma_e)_{\mathcal{I}^0}} = (\oplus^s_{e\in E}\sigma_e)_\omega$,
$\mathbf{F}_{(\oplus_{e\in E}\sigma_e)_{\mathcal{I}^s}} = (\oplus^s_{e\in E}\sigma_e)_{s\omega}$.
By Remark 5 (11) and Definition 7,
$\mathbf{F}_{\oplus_{e\in E}(\sigma_e)_{\mathcal{I}^s_e}} = \mathbf{F}_{\oplus_{e\in E}(\sigma_e)_{\mathcal{I}^0_e}} = \mathbf{F}_{\oplus_{e\in E}(\sigma_e)_\omega} = \oplus^s_{e\in E}(\sigma_e)_\omega$.
That means
$(\oplus^s_{e\in E}\sigma_e)_\omega = \oplus^s_{e\in E}(\sigma_e)_\omega = (\oplus^s_{e\in E}\sigma_e)_{s\omega}$.
(2) If $\sigma_e = \mathfrak{J}$ for any $e \in E$, then by (1)
$(\oplus^s_{e\in E}\mathfrak{J})_\omega = (\oplus^s_{e\in E}\mathfrak{J})_{s\omega} = \oplus^s_{e\in E}(\mathfrak{J})_\omega$.
Using notation from Al Ghour and Hamed Worood (2020) (see Definition 7),
$(\tau(\mathfrak{J}))_\omega = (\tau(\mathfrak{J}))_{s\omega} = \tau(\mathfrak{J}_\omega)$. □

Let $(E,U,\hat{\tau})$ be a soft topological space and $Y \subset U$. If $F \in SS(E,U)$, then a soft set F_Y is defined as $F_Y(e) = F(e) \cap Y$ for any $e \in E$. A family $\hat{\tau}_Y = \{F_Y : F \in \hat{\tau}\}$ is called a relative soft topology on Y, see Al Ghour and Hamed Worood (2020). Similarly, we define a soft ideal $\hat{\mathcal{I}}_Y = \{I_Y : I \in \hat{\mathcal{I}}\}$ where $\hat{\mathcal{I}}$ is a soft ideal.

Lemma 9 *Let $(E,U,\hat{\tau},\hat{\mathcal{I}})$ and $(E\times U,\tau,\mathcal{I})$ be mutually corresponding, $Y \subset U$. Then*

(1) $\mathbf{F}_{\tau_{E\times Y}} = \hat{\tau}_Y = (\mathbf{F}_\tau)_Y$,
(2) $\mathbf{F}_{\mathcal{I}_{E\times Y}} = \hat{\mathcal{I}}_Y = (\mathbf{F}_\mathcal{I})_Y$,
(3) $(\hat{\tau}_Y)_{\hat{\mathcal{I}}_Y} = (\hat{\tau}_{\hat{\mathcal{I}}})_Y$.

Proof (1) A set $A \in \tau_{E\times Y}$ if and only if $A = G \cap (E \times Y)$ where $G \in \tau$ if and only if $\mathbf{F}_A(e) = \varphi_e^{-1}(G \cap (E\times Y)) =$

$\varphi_e^{-1}(G) \cap \varphi_e^{-1}(E \times Y) = \mathbf{F}_G(e) \cap Y = (\mathbf{F}_G)_Y(e)$ where $\mathbf{F}_G \in \hat{\tau}$ if and only if $\mathbf{F}_A \in \hat{\tau}_Y$. So $\mathbf{F}_{\tau_{E\times Y}} = \hat{\tau}_Y = (\mathbf{F}_\tau)_Y$.

(2) is similar.

(3) By Lemma 2, $(\tau_{E\times Y})_{\mathcal{I}_{E\times Y}} = (\tau_{\mathcal{I}})_{E\times Y}$, so $\mathbf{F}_{(\tau_{E\times Y})_{\mathcal{I}_{E\times Y}}} = \mathbf{F}_{(\tau_{\mathcal{I}})_{E\times Y}}$. By (1) and Lemma 6, $\mathbf{F}_{(\tau_{\mathcal{I}})_{E\times Y}} = (\mathbf{F}_{\tau_{\mathcal{I}}})_Y = ((\mathbf{F}_\tau)_{\mathbf{F}_{\mathcal{I}}})_Y = (\hat{\tau}_{\hat{\mathcal{I}}})_Y$.

By Lemma 6 and (1), (2), $\mathbf{F}_{(\tau_{E\times Y})_{\mathcal{I}_{E\times Y}}} = (\mathbf{F}_{\tau_{E\times Y}})_{\mathbf{F}_{\mathcal{I}_{E\times Y}}} = (\hat{\tau}_Y)_{\mathbf{F}_{\mathcal{I}_{E\times Y}}} = (\hat{\tau}_Y)_{\hat{\mathcal{I}}_Y}$. That means $(\hat{\tau}_Y)_{\hat{\mathcal{I}}_Y} = (\hat{\tau}_{\hat{\mathcal{I}}})_Y$. □

Corollary 4 *Let $(E, U, \hat{\tau})$ be a soft topological space. Then*

$$(\hat{\tau}_Y)_\omega = (\hat{\tau}_\omega)_Y,$$
$$(\hat{\tau}_Y)_{s\omega} = (\hat{\tau}_{s\omega})_Y,$$

see Al Ghour and Hamed Worood (2020) *Theorem* 15, 34.

Proof Since $\hat{\tau}_{\hat{\mathcal{I}}^0} = \hat{\tau}_\omega$, $\hat{\tau}_{\hat{\mathcal{I}}^s} = \hat{\tau}_{s\omega}$ (see Remark 5 (7)), $(\hat{\tau}_{\hat{\mathcal{I}}^0})_Y = (\hat{\tau}_\omega)_Y$, $(\hat{\tau}_{\hat{\mathcal{I}}^s})_Y = (\hat{\tau}_{s\omega})_Y$. Moreover, $\hat{\mathcal{I}}^0_Y$, $\hat{\mathcal{I}}^s_Y$ is a soft ideal of all countable-valued mappings, a soft ideal of all countable-valued mapping and nonempty-valued mappings on a countable set, so $(\hat{\tau}_Y)_{\hat{\mathcal{I}}^0_Y} = (\hat{\tau}_Y)_\omega$, $(\hat{\tau}_Y)_{\hat{\mathcal{I}}^s_Y} = (\hat{\tau}_Y)_{s\omega}$ (see Remark 5 (7)), respectively. Both equations follow from the equation $(\hat{\tau}_Y)_{\hat{\mathcal{I}}_Y} = (\hat{\tau}_{\hat{\mathcal{I}}})_Y$, see Lemma 9 (3). □

Corollary 5 *Let $(E, U, \hat{\tau})$ be a soft topological space. Then*

$$(\hat{\tau}_\omega)_e = (\hat{\tau}_e)_\omega = (\hat{\tau}_{s\omega})_e,$$

see Al Ghour and Hamed Worood (2020) Theorem 25, Corollary 10.

Proof Let $(E \times U, \tau)$ be the corresponding topological space to $(E, U, \hat{\tau})$. By Theorem 1 item (2)

$$(\tau_{\mathcal{I}^0})_e = (\tau_e)_{\mathcal{I}^0_e}, \quad (\tau_{\mathcal{I}^s})_e = (\tau_e)_{\mathcal{I}^s_e}.$$

Since $(\tau_e)_{\mathcal{I}^s_e} = (\tau_e)_{\mathcal{I}^0_e}$ (see Remark 5 item (11)),

$$(\tau_{\mathcal{I}^0})_e = (\tau_e)_{\mathcal{I}^0_e} = (\tau_e)_{\mathcal{I}^s_e} = (\tau_{\mathcal{I}^s})_e.$$

Since $\tau_{\mathcal{I}_0}$ and $\hat{\tau}_{\hat{\mathcal{I}}_0}$ are mutually corresponding (see Lemma 8), by Lemma 7 (1) and Remark 5 (11)

$$(\mathbf{F}_{\tau_{\mathcal{I}^0}})_e = (\hat{\tau}_e)_\omega = (\hat{\tau}_e)_\omega = (\mathbf{F}_{\tau_{\mathcal{I}^s}})_e.$$

By Remark 5 (7),
$(\hat{\tau}_\omega)_e = (\hat{\tau}_e)_\omega = (\hat{\tau}_{s\omega})_e$. □

Corollary 6 *Let $(E, U, \hat{\tau})$ be a soft topological space. If $G \in \hat{\tau}_\omega$, $G \in \hat{\tau}_{s\omega}$, then $G(e) \in (\hat{\tau}_e)_\omega$, $G(e) \in (\hat{\tau}_e)_\omega$, respectively, see* Al Ghour and Hamed Worood (2020) *Corollary* 3, 9.

Proof Since $\tau_{\mathcal{I}^0}$ and $\hat{\tau}_\omega$, $\tau_{\mathcal{I}^s}$ and $\hat{\tau}_{s\omega}$ are mutually corresponding, respectively (see Remark 5 (8)), $\mathbf{R}_G \in \tau_{\mathcal{I}^0}$, $\mathbf{R}_G \in \tau_{\mathcal{I}^s}$, respectively. By Theorem 5 (1) (d) (e), Theorem 1 and Remark 5 (11), $G(e) = (\mathbf{R}_G)_e \in (\tau_e)_{\mathcal{I}^0_e} = (\hat{\tau}_e)_\omega$, $G(e) = (\mathbf{R}_G)_e \in (\tau_e)_{\mathcal{I}^s_e} = (\tau_e)_{\mathcal{I}^0_e} = (\hat{\tau}_e)_\omega$, respectively. □

Recall an ideal $\mathcal{I}$ on $E \times U$ is τ-codense where τ is a topology on $E \times U$ if $\mathcal{I} \cap \tau = \{\emptyset\}$, see Kaniewski et al. (1998). So the corresponding soft variant can be defined as follows: A soft ideal $\hat{\mathcal{I}}$ is $\hat{\tau}$-soft codense if $\hat{\mathcal{I}} \cap \hat{\tau} = \{\mathbf{F}_\emptyset\}$ where $\hat{\tau}$ is a soft topology. That means, see Al Ghour and Hamed Worood (2020), $(E, U, \hat{\tau})$ is soft anti-locally countable (strongly soft anti-locally countable) if and only if $\hat{\mathcal{I}}^0 \cap \hat{\tau} = \{\mathbf{F}_\emptyset\}$ ($\hat{\mathcal{I}}^s \cap \hat{\tau} = \{\mathbf{F}_\emptyset\}$) if and only if $\hat{\mathcal{I}}^0$ is $\hat{\tau}$-soft codense ($\hat{\mathcal{I}}^s$ is $\hat{\tau}$-soft codense).

Corollary 7 *Let $(E, U, \hat{\tau})$ be a soft topological space. Then*

(1) $(E,U,\hat{\tau})$ is soft anti-locally countable, i.e., $\hat{\mathcal{I}}^0 \cap \hat{\tau} = \{\mathbf{F}_\emptyset\}$ (strongly soft anti-locally countable, i.e., $\hat{\mathcal{I}}^s \cap \hat{\tau} = \{\mathbf{F}_\emptyset\}$) if and only if $(E, U, \hat{\tau}_\omega)$ is soft anti-locally countable, i.e., $\hat{\mathcal{I}}^0 \cap \hat{\tau}_{\hat{\mathcal{I}}_0} = \{\mathbf{F}_\emptyset\}$ ($(E, U, \hat{\tau}_{s\omega})$ is strongly soft anti-locally countable, i.e., $\hat{\mathcal{I}}^s \cap \hat{\tau}_{\hat{\mathcal{I}}_s} = \{\mathbf{F}_\emptyset\}$), see Al Ghour and Hamed Worood (2020) *Theorem* 13, 32.

(2) If $(E, U, \hat{\tau})$ is soft anti-locally countable, strongly soft anti-locally countable, then

$$scl_{\hat{\tau}}(H) = scl_{\hat{\tau}_\omega}(H),$$
$$scl_{\hat{\tau}}(H) = scl_{\hat{\tau}_{s\omega}}(H),$$

for any $H \in \hat{\tau}_\omega$, for any $H \in \hat{\tau}_{s\omega}$, respectively, see Al Ghour and Hamed Worood (2020) *Theorem* 14, 33.

(3) If $(E, U, \hat{\tau})$ is soft anti-locally countable, then

$$scl_{\hat{\tau}}(H) = scl_{\hat{\tau}_\omega}(H) = scl_{\hat{\tau}_{s\omega}}(H)$$

for any $H \in \hat{\tau}_{s\omega}$.

Proof (1)$(E,U,\hat{\tau})$ is soft anti-locally countable (strongly soft anti-locally countable) if and only if $\hat{\mathcal{I}}^0 \cap \hat{\tau} = \{\mathbf{F}_\emptyset\}$ ($\hat{\mathcal{I}}^s \cap \hat{\tau} = \{\mathbf{F}_\emptyset\}$) if and only if $\mathcal{I}^0 \cap \tau = \{\emptyset\}$ ($\mathcal{I}^s \cap \tau = \{\emptyset\}$) if and only if $\mathcal{I}^0 \cap \tau_{\mathcal{I}^0} = \{\emptyset\}$ ($\mathcal{I}^s \cap \tau_{\mathcal{I}^s} = \{\emptyset\}$) (see Lemma 4 (1)) if and only if $\hat{\mathcal{I}}^0 \cap \hat{\tau}_{\hat{\mathcal{I}}_0} = \{\mathbf{F}_\emptyset\}$ ($\hat{\mathcal{I}}^s \cap \hat{\tau}_{\hat{\mathcal{I}}_s} = \{\mathbf{F}_\emptyset\}$) if and only if $(E, U, \hat{\tau}_\omega)$ is soft anti-locally countable ($(E, U, \hat{\tau}_{s\omega})$ is strongly soft anti-locally countable).

(2) Suppose $\mathcal{I}^0 \cap \tau = \{\emptyset\}$, i.e., $(E, U, \hat{\tau})$ is soft anti-locally countable. By Lemma 3

$$cl_\tau(G) = cl_{\tau_{\mathcal{I}^0}}(G) = D_{\tau,\mathcal{I}^0}(G),$$

for any $G \in \tau_{\mathcal{I}^0}$. By Theorem 5 (7) and Remark 5 (14),

$$scl_{\hat{\tau}}(\mathbf{F}_G) = \mathbf{F}_{cl_\tau(G)} = \mathbf{F}_{cl_{\tau_{\mathcal{I}^0}}(G)} = scl_{\hat{\tau}_\omega}(\mathbf{F}_G).$$

Then for any $H \in \hat{\tau}_\omega = \mathbf{F}_{\tau^0_{\mathcal{I}}}$ (see Remark 5 (7)), $H = \mathbf{F}_S$ for some $S \in \tau^0_{\mathcal{I}}$ and

$$scl_{\hat{\tau}}(\mathbf{F}_S) = scl_{\hat{\tau}_\omega}(\mathbf{F}_S),$$
$$scl_{\hat{\tau}}(H) = scl_{\hat{\tau}_\omega}(H).$$

Similarly, if $\mathcal{I}^s \cap \tau = \{\emptyset\}$, i.e., $(E, U, \hat{\tau})$ is strongly soft anti-locally countable, then for any $H \in \hat{\tau}_{s\omega}$

$$scl_{\hat{\tau}}(H) = scl_{\hat{\tau}_{s\omega}}(H).$$

(3) Suppose $\mathcal{I}^0 \cap \tau = \{\emptyset\}$, i.e., $(E, U, \hat{\tau})$ is soft anti-locally countable. Since $\mathcal{I}^s \cap \tau \subset \mathcal{I}^0 \cap \tau = \{\emptyset\}$ ($\tau_{\mathcal{I}^s} \subset \tau_{\mathcal{I}^0}$, see Remark 1 (5)), $(E, U, \hat{\tau})$ is strongly soft anti-locally countable. Then

$$scl_{\hat{\tau}}(H) = scl_{\hat{\tau}_\omega}(H) = scl_{\hat{\tau}_{s\omega}}(H)$$

for any $H \in \hat{\tau}_{s\omega}$. □

Corollary 8 *Recall a subset A of $(X, \tau, \mathcal{I})$ locally belongs to $\mathcal{I}$, if $A \cap D_{\tau,\mathcal{I}}(A) = \emptyset$, i.e., for any $x \in A$ there is $G \in \tau$ containing x such that $A \cap U \in \mathcal{I}$, see* Kaniewski et al. (1998). *So, X locally belongs to $\mathcal{I}$ if and only if for any $x \in X$ there is an open set G containing x such that $G \in \mathcal{I}$. That means $(E, U, \hat{\tau})$ is soft locally countable (strongly soft locally countable), see* Al Ghour and Hamed Worood (2020) *if and only if for any $F^u_e \in SP(E, U)$ there is a set $G \in \hat{\tau}$ containing F^u_e such that $G \in \hat{\mathcal{I}}^0$ ($\hat{\mathcal{I}}^s$).*

Since $F^u_e \in \hat{\mathcal{I}}^0$ ($F^u_e \in \hat{\mathcal{I}}^s$) for any $(e, u) \in E \times U$, then Theorem 10, 29, Corollary 5, 14 of Al Ghour and Hamed Worood (2020) *follow from Lemma* 4 (2a).

Corollary 9 *By the correspondence between the family of soft topological spaces and the family of topological spaces (see Theorem 4), a soft topological space $(E, U, \hat{\tau})$ is soft Lindelöf (soft weakly Lindelöf), see* Al Ghour and Hamed Worood (2020) *if and only if the corresponding topological space $(E \times U, \tau)$ is Lindelöf (weakly Lindelöf). So, the next assertions of* Al Ghour and Hamed Worood (2020) *follow directly from above results: Theorem 35, see Lemma* 4 (5), *Theorem 36, see Lemma* 4 (3), *Theorem* 37, *see Remark* 5 (6), *Lemma* 4 (5), *Theorem* 38, *see Lemma* 5 (3), *Corollary* 16, *see Lemma* 4 (5), *Remark* 1 (1), *Theorem* 39, *see Corollary* 3 (1), *Lemma* 4 (5), *Theorem* 40, *see Lemma* 4 (8), *Theorem* 41, *see Lemma* 4 (7), *Corollary* 17, *see Lemma* 5 (3), *Theorem* 45, *see Corollary* 7, *Lemma* 4 (4).

Recall that many results of Al Ghour and Hamed Worood (2020) hold for arbitrary soft ideal. In addition to $\hat{\mathcal{I}}^0$ and $\hat{\mathcal{I}}^s$, we can consider a soft ideal $\hat{\mathcal{I}}_0$ where $\mathcal{I}_0 = \{B \subset E \times U : B^u$ is countable for any $u \in U\}$ and $B^u = \{e \in E : (e, u) \in B\}$.

The next assertions from Al Ghour and Hamed Worood (2020) follow directly from the above-obtained results. Namely

Theorem 2, 3, see Remark 1 (1),
Proposition 9, see Remark 1 (1),
Theorem 4, 21, see Remark 1 (3),
Theorem 5, 22, see Remark 1 (2),
Theorem 7, see Theorem 1 (2),
Theorem 18, see Remark 1 (1), (5),
Theorem 19 (b), see Remark 1 (5f),
Theorem 20, see Remark 1 (1),
Proposition 12, see Remark 1 (1),
Theorem 21, see Remark 1 (3),
Theorem 23, see Remark 1 (5e),
Theorem 11, 30, see Lemma 4 (2b),
Theorem 42, see Lemma 4 (6) (where $E \times U = \cup_{e \in E} U[e]$ and by Remark 2 (1), $(U[e], \sigma_e[e])$ is weakly Lindelöf if and only if (U, σ_e) is weakly Lindelöf).

Note that the examples from Al Ghour and Hamed Worood (2020) also have their topological variants.

8 Conclusion

This paper contributes to the expanding literature on soft topology. We prove that soft topologies can be characterized by crisp topologies. This is based on bilateral transition that produces soft topologies from crisp topologies and vice versa. Both constructions are explicit and amenable to mathematical manipulations. Various consequences demonstrate that this transition has far reaching implications for the development of soft topology and its extensions.

We have clearly documented the advantage of this bilateral transition in which all notions and results of soft ideal topological spaces have crisp counterparts in ideal topological spaces. This means that the concepts and results that relate to soft ideal topological spaces are fully covered and derivable from standard methods of general topology. From this point of view, we can also evaluate the results from Al Ghour and Hamed Worood (2020) as a copy of known results. Therefore, in further research of soft topological spaces, we propose avoiding the methods and results that are counterparts (consequences) of topological concepts and rather to focus on applications of soft topological spaces.

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Declarations

Conflict of interest The author declares that he has no conflict of interest.

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