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METHODOLOGIES AND APPLICATION

On Alexandrov L-fuzzy nearness (II)

Enas H. Elkordy[1] · **Ahmed A. Ramadan**[1] · **Reham M. Ahmed**[1]

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Abstract

The Alexandrov L-fuzzy nearness is a new addition to the L-fuzzy systems that is the base for intelligent systems and its wide applications in various fields. This paper represents the connections among L-fuzzy nearness and various L-fuzzy systems such as: L-fuzzy rough sets, L-fuzzy semi-topogenous orders and L-fuzzy uniformities in complete residuated lattices. Moreover, we show that there is a Galois correspondence between the categories of those mentioned systems.

Keywords Complete residuated lattice · L-fuzzy rough set · L-fuzzy uniformity · L-fuzzy semi-topogenous order · Alexandrov L-fuzzy nearness · Galois correspondence

1 Introduction

Intelligent systems with fuzzy, uncertain and incomplete information are more easy to handle after Zadeh (1965) proposed the fuzzy set theory with the structure of membership value $[0, 1]$ in 1965. After that, Goguen (1967) replaced that structure with an arbitrary set and introduced the fuzzy topology which gave more varieties to study more fuzzy structures as Bělohlávek (2002b); Bělohlávek and Krupka (2015); Chang (1968); Fang (2007); Höhle and Klement (1995); Ramadan (1992); Ramadan and Kim (2018); Rodabaugh and Klement (2003); Šostak (1989).

Ward and Dilworth (1939) introduced the complete residuated lattice which is an algebraic structure for many valued logic. Through it, Bělohlávek (2002a) could give us the L-fuzzy sets which was capable of modeling vague and uncertain information systems.

Radzikowska and Kerre (2004) replaced the equivalence relation in Pawlak's rough set Pawlak (1982) by arbitrary relation to handle more uncertainty. Yao and Lin (1996) showed that upper and lower approximations of a set are nothing but closure and interior of it. Hence, they could propose several models of rough sets.

Quasi-uniformities in fuzzy sets have different approaches as follows: the entourage approach of Lowen (1981) and Höhle (1982) based on powersets of the form $L^{X\times X}$, the quasi-uniform covering approach of Kotzé (1999), the unification approach of Hutton (1977) based on the powersets of the form $L^{X^{L^X}}$ and the Rodabaugh (1988) as a generalization of unification of Gutiérrez (2003). Gutiérrez introduced L-valued Hutton quasi-uniformity where a quadruple $(L, \leq, \odot, *)$ is defined by a GL-monoid $(L, *)$ dominated by $\odot$, a cl-quasi-monoid $(L, \leq, \odot)$. They obtained the relation between Hutton, Lowen and Höhle categories. Lattice-valued fuzzy quasi-uniformity in entourage approach is studied by Ramadan et al. (2006).

The concept of topogenous structures was introduced in 1963 by Hungarian mathematician Császár (1963) that allowed to develop a unified approach of topologies, proximities and uniformities. In the same monograph, Császár developed the basics of the theory of topogenous structures and investigated spaces: uniform spaces and proximity spaces as a particular display of topogenous space. In the period of 1963–1991, Katsaras has published a series of papers (some of them in collaboration with Petalas) in which fuzzy topogenous spaces were defined and studied. In these papers, fuzziness was interpreted in Chang's sense (1968), that is fuzzy topogenous structures were realized as a crisp on the family $[0, 1]^X$ of fuzzy subsets of a set X.

Lately and as a unified structure and extension of Pawlak's rough set (1982; 1991), specifically in 2019, Ramadan et al. (2019) introduced the concept of Alexandrov L-fuzzy nearness and arose a great deal of relationships among it, L-fuzzy topological spaces and the L-fuzzy preproximities. That took the applications to multi-attribute decision making to a whole new level.

✉ Enas H. Elkordy
enas.elkordi@science.bsu.edu.eg

[1] Department of Mathematics and Computer Science, Faculty of Science, Beni-Suef University, Beni Suef 62511, Egypt

Šostak in (2004) proposed the concept of fuzzy category which is an ordinary category modified in such a way that "potential" objects and "potential" morphisms are such only to a certain degree and this degree can be any element of the corresponding lattice.

The degree approach has been developed extensively in the theory of fuzzy topology, fuzzy convergence and fuzzy convex structure (e.g., Yue and Fang (2006), Shi (2011), Li and Shi (2010), Pang and Shi (2014); Pang (2017)). Actually, special mappings between structured spaces and the structured space itself can also be endowed with some degrees. In (2014), Pang defined degrees of continuous mappings and open mappings between L-fuzzifying topological spaces. Liang and Shi (2014) further defined the degrees of continuous mappings and open mappings between L-fuzzy topological spaces and investigated their relationship. Li et al. (2019) defined the degrees of special mappings in the theory of L-convex spaces and investigated their properties. Xiu and Li in (2019) defined a degree approach to L-continuity, L-closedness and L-openness for mappings between L-co-topological spaces, and their connections were studied.

This paper's content is organized as follows. In section 2, we recall some fundamental concepts and related definitions. In section 3, we introduce Alexandrov L-fuzzy nearness, L-fuzzy approximation operators and Alexandrov L-fuzzy uniformities and give relations among them. In section 4, we show interesting relations also between Alexandrov L-fuzzy nearness and L-fuzzy semi-topogenous structure. Finally, in section 5, we define the degree to which a mapping is LF-continuous and equip it to the previous spaces. Galois correspondence between their categories is proved besides the adjunctions between the considered categories.

2 Preliminaries

Definition 1 Bělohlávek (2002b); Blount and Tsinakis (2003); Turunen (1999) An algebra $(L, \wedge, \vee, \odot, \to, \perp, \top)$ is called a complete residuated lattice if it satisfies the following conditions:

(C1) $(L, \leq, \vee, \wedge, \perp, \top)$ is a complete lattice with the greatest element $\top$ and the least element $\perp$,
(C2) $(L, \odot, \top)$ is a commutative monoid,
(C3) $x \odot y \leq z$ iff $x \leq y \to z$ for all $x, y, z \in L$.

In this paper, we assume that $(L, \leq, \odot, ^*)$ is a complete residuated lattice with an order reversing involution * which is defined by

$$x \oplus y = (x^* \odot y^*)^*, \quad x^* = x \to \perp.$$

For each $\alpha \in L$ and $f \in L^X$, we denote $(\alpha \to f), (\alpha \odot f), \alpha_X \in L^X$ as $(\alpha \to f)(x) = \alpha \to f(x), (\alpha \odot f)(x) = \alpha \odot f(x), \alpha_X(x) = \alpha$,

$$\top_x(y) = \begin{cases} \top, & \text{if } y = x, \\ \perp, & \text{otherwise,} \end{cases} \quad \top_x^*(y) = \begin{cases} \perp, & \text{if } y = x, \\ \top, & \text{otherwise} \end{cases}$$

Some basic properties of the binary operation $\odot$ and residuated operation $\to$ are collected in the following lemma that can be found in Bělohlávek (2002b); Blount and Tsinakis (2003); Šostak (2004).

Lemma 1 *Bělohlávek (2002b); Blount and Tsinakis (2003); Hájek (1998); Rodabaugh and Klement (2003); Turunen (1999) For each $x, y, z, x_i, y_i, w \in L$, we have the following properties:*

(1) $\top \to x = x$, $\perp \odot x = \perp$,
(2) *if* $y \leq z$, *then* $x \odot y \leq x \odot z$, $x \oplus y \leq x \oplus z$, $x \to y \leq x \to z$ *and* $z \to x \leq y \to x$,
(3) $x \leq y$ *iff* $x \to y = \top$,
(4) $(\bigwedge_i y_i)^* = \bigvee_i y_i^*$, $(\bigvee_i y_i)^* = \bigwedge_i y_i^*$,
(5) $x \to (\bigwedge_i y_i) = \bigwedge_i (x \to y_i)$,
(6) $(\bigvee_i x_i) \to y = \bigwedge_i (x_i \to y)$,
(7) $x \odot (\bigvee_i y_i) = \bigvee_i (x \odot y_i)$,
(8) $(\bigwedge_i x_i) \oplus y = \bigwedge_i (x_i \oplus y)$,
(9) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$,
(10) $x \odot y = (x \to y^*)^*$, $x \oplus y = x^* \to y$ *and* $x \to y = y^* \to x^*$,
(11) $(x \to y) \odot (z \to w) \leq (x \odot z) \to (y \odot w)$,
(12) $x \to y \leq (x \odot z) \to (y \odot z)$ *and* $(x \to y) \odot (y \to z) \leq x \to z$,
(13) $(x \to y) \odot (z \to w) \leq (x \oplus z) \to (y \oplus w)$,
(14) $x \odot (x \to y) \leq y$ *and* $y \leq x \to (x \odot y)$ *and* $x \to (y \oplus z) \leq (x \to y)^* \to z$,
(15) $\bigvee_{i\in\Gamma} x_i \to \bigvee_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$, $\bigwedge_{i\in\Gamma} x_i \to \bigwedge_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$,
(16) $(x \odot y) \odot (z \oplus w) \leq (x \odot z) \oplus (y \odot w)$,
(17) $z \to x \leq (x \to y) \to (z \to y)$ *and* $y \to z \leq (x \to y) \to (x \to z)$.

Definition 2 Bělohlávek (2002b) Let X be a nonempty set, then the mapping $R : X \times X \to L$ is called an L-fuzzy relation on X and for all $x, y, z \in X$ the relation R is said to be

(1) reflexive if $R(x, x) = \top$,
(2) symmetric if $R(x, y) = R(y, x)$,
(3) transitive if $R(x, y) \odot R(y, z) \leq R(x, z)$.

An L-fuzzy relation on X is called an L-fuzzy pre-order if it is reflexive and transitive and called an L-fuzzy equivalence relation if it is reflexive, symmetric and transitive.

Definition 3 Bělohlávek (2002b); Fang (2010) Let X be a nonempty set, define a binary mapping $\mathcal{S} : L^X \times L^X \to L$ by

$$\mathcal{S}(f, g) = \bigwedge_{x \in X}(f(x) \to g(x)).$$

Then, for each $f, g, h, k \in L^X$ and $\alpha \in L$, the following properties hold:

(1) $\mathcal{S}$ is an L-partial order on L^X,
(2) $f \leq g$ iff $\mathcal{S}(f, g) \geq \top$,
(3) if $f \leq g$, then $\mathcal{S}(h, f) \leq \mathcal{S}(h, g)$ and $\mathcal{S}(f, h) \leq \mathcal{S}(g, h)$ $\forall\, h \in L^X$,
(4) $\mathcal{S}(f, g) \odot \mathcal{S}(k, h) \leq \mathcal{S}(f \oplus k, g \oplus h)$ and $\mathcal{S}(f, g) \odot \mathcal{S}(k, h) \leq \mathcal{S}(f \odot k, g \odot h)$,
(5) $\bigwedge_{i \in \Gamma} \mathcal{S}(f_i, g_i) \leq \mathcal{S}(\bigwedge_{i \in \Gamma} f_i, \bigwedge_{i \in \Gamma} g_i)$,
(6) $\mathcal{S}(f, h) = \bigvee_{g \in L^X}(\mathcal{S}(f, g) \odot \mathcal{S}(g, h))$.

If $\phi : X \to Y$ is a mapping, then for $f, g \in L^X$ and $h, k \in L^Y$, we have

$$\mathcal{S}(f, h) \leq \mathcal{S}(\phi^{\leftarrow}(f), \phi^{\leftarrow}(h)),$$
$$\mathcal{S}(h, k) \leq \mathcal{S}(\phi^{\leftarrow}(h), \phi^{\leftarrow}(k)),$$

and the equalities hold if ϕ is bijective.

Definition 4 Pawlak (1982); Zhu (2009) A mapping $\overline{R} : L^X \to L^X$ is called an L-fuzzy upper approximation operator on X if the following holds:

(UO1) $\overline{R}(\perp_X) = \perp_X$,
(UO2) $\overline{R}(f) \geq f$ for all $f \in L^X$,
(UO3) $\overline{R}(\bigvee_{i \in \Gamma} f_i) = \bigvee_{i \in \Gamma} \overline{R}(f_i)$ for all $f_i \in L^X$,
(UO4) $\overline{R}(\alpha \odot f) = \alpha \odot \overline{R}(f)$.

An L-fuzzy upper approximation operator is called
(T) topological if $\overline{R}(\overline{R}(f)) \leq \overline{R}(f)$ for all $f \in L^X$,
(UAS) $(L, \oplus)$-fuzzy upper approximation operator on X if $\overline{R}(f \oplus g) \leq \overline{R}(f) \oplus \overline{R}(g)$.

Lemma 2 *An L-fuzzy upper approximation operator $\overline{R}$ on X is topological if and only if* $\bigvee_{z \in X} \overline{R}(\top_y)(z) \odot \overline{R}(\top_z)(x) \leq \overline{R}(\top_y)(x)$.

Proof Since $f = \bigvee_{y \in X} \top_y \odot f(y)$, then we have

$$\begin{aligned}
\overline{R}(f)(x) &= \overline{R}\Big(\bigvee_{y \in X} \top_y \odot f(y)\Big)(x) \\
&= \bigvee_{y \in X} \overline{R}(\top_y \odot f(y))(x) \\
&= \bigvee_{y \in X} \overline{R}(\top_y)(x) \odot f(y) \\
&\geq \bigvee_{y \in X} \left(\bigvee_{z \in X} \overline{R}(\top_y)(z) \odot \overline{R}(\top_z)(x)\right) \odot f(y) \\
&= \bigvee_{z \in X} \overline{R}(\top_z)(x) \odot \left(\bigvee_{y \in X} (\overline{R}(\top_y)(z) \odot f(y))\right) \\
&= \bigvee_{z \in X} \overline{R}(\top_z)(x) \odot \overline{R}(f)(z) = \overline{R}(\overline{R}(f))(x).
\end{aligned}$$

Conversely, we have $\overline{R}(\top_x)(y) \geq \overline{R}(\overline{R}(\top_x))(y) = \bigvee_{z \in X} \overline{R}(\top_z)(y) \odot \overline{R}(\top_x)(z)$. □

Definition 5 Pawlak (1982); Zhu (2009) A mapping $\underline{R} : L^X \to L^X$ is called an L-fuzzy lower approximation operator on X if the following holds:
(LO1) $\underline{R}(\top_X) = \top_X$,
(LO2) $\underline{R}(f) \leq f$ for all $f \in L^X$,
(LO3) $\underline{R}(\bigwedge_{i \in \Gamma} f_i) = \bigwedge_{i \in \Gamma} \underline{R}(f_i)$ for all $f_i \in L^X$,
(LO4) $\underline{R}(\alpha \to f) = \alpha \to \underline{R}(f)$.
An L-fuzzy lower approximation operator is called
(T) topological if $\underline{R}(\underline{R}(f)) \geq \underline{R}(f)$ for all $f \in L^X$,
(LAS) $(L, \odot)$-fuzzy lower approximation operator on X if $\underline{R}(f \odot g) \geq \underline{R}(f) \odot \underline{R}(g)$.

Lemma 3 *An L-fuzzy lower approximation operator $\underline{R}$ on X is topological if and only if* $\bigwedge_{z \in X} \underline{R}(\top_y^*)(z) \oplus \underline{R}(\top_z^*)(x) \geq \underline{R}(\top_y^*)(x)$.

Proof Since $f^*(x) = \left(\bigvee_{y \in X} \top_y \odot f(y)\right)^*(x) = \bigwedge_{y \in X} (f(y) \to \top_y^*)(x)$, then we have $f(x) = \bigwedge_{y \in X}(f^*(y) \to \top_y^*)(x)$. Hence, by Lemma 1(14), we have

$$\begin{aligned}
\underline{R}(f)(x) &= \bigwedge_{y \in X} (f^*(y) \to \underline{R}(\top_y^*)(x)) \\
&\leq \bigwedge_{y \in X} \left(f^*(y) \to \bigwedge_{z \in X} (\underline{R}(\top_y^*)(z) \oplus \underline{R}(\top_z^*)(x))\right) \\
&= \bigwedge_{y \in X} \bigwedge_{z \in X} (f^*(y) \to (\underline{R}(\top_y^*)(z) \oplus \\
&\qquad \underline{R}(\top_z^*)(x))) \\
&\leq \bigwedge_{y \in X} \bigwedge_{z \in X} \left((f^*(y) \to \underline{R}(\top_y^*)(z))^* \to \underline{R}(\top_z^*)(x)\right) \\
&= \bigwedge_{z \in X} \left(\bigwedge_{y \in X} (f^*(y) \to \underline{R}(\top_y^*)(z))^* \to \underline{R}(\top_z^*)(x)\right) \\
&= \bigwedge_{z \in X} (\underline{R}^*(f)(z) \to \underline{R}(\top_z^*)(x)) = \underline{R}(\underline{R}(f))(x).
\end{aligned}$$

Conversely, by Lemma 1(10), we have

$$\begin{aligned}
\underline{R}(\top_x^*)(y) &\leq \underline{R}(\underline{R}(\top_x^*)(y)) \\
&= \bigwedge_{z \in X} (\underline{R}^*(\top_x^*)(z) \to \underline{R}(\top_z^*)(y)) \\
&= \bigwedge_{z \in X} (\underline{R}(\top_x^*)(z) \oplus \underline{R}(\top_z^*)(y)).
\end{aligned}$$

□

Remark 1 Pawlak (1982) Let $\overline{R}$ and $\underline{R}$ be L-fuzzy upper and L-fuzzy lower approximation operators on X, respectively, then the pair $(\underline{R}(f), \overline{R}(f))$ is called an L-fuzzy rough set for f.

Lemma 4 *Oh and Kim (2017) For each $f, g \in L^X$, define two mappings $u_{f,g}, u^{-1}_{f,g} : X \times X \to L$ by*

$$u_{f,g}(x, y) = f(x) \to g(y),\ u^{-1}_{f,g}(x, y) = u_{f,g}(y, x).$$

Then, the following holds:

(1) $\top_{X\times X} = u_{\perp_X,\perp_X} = u_{\top_X,\top_X}$,
(2) if $f_2 \le f_1$ *and* $g_1 \le g_2$, *then* $u_{f_1,g_1} \le u_{f_2,g_2}$,
(3) for any $u_{f,g} \in L^{X\times X}$ *and* $h \in L^X$, *it holds that* $u_{f,h} \circ u_{h,g} \le u_{f,g}$ *where*

$$u_{f,h}(x, y) \circ u_{h,g}(y, z) = \bigvee_{y\in X} ((f(x) \to h(y)) \odot (h(y) \to g(z))),$$

(4) $u_{\bigvee_{i\in\Gamma} f_i,g} = \bigwedge_{i\in\Gamma} u_{f_i,g}, u_{f,\bigwedge_{i\in\Gamma} g_i} = \bigwedge_{i\in\Gamma} u_{f,g_i}$,
(5) $u_{\alpha\odot f,\, g} = \alpha \to u_{f,g}$ *and* $u_{f,\alpha\to g} = \alpha \to u_{f,g}$,
(6) $u_{\alpha\odot f,\alpha\odot g} \ge u_{f,g}$ *and* $u_{\alpha\to f,\alpha\to g} \ge u_{f,g}$,
(7) $u_{f,g} = \bigwedge_{z\in X}(f(z) \to u_{\top_z,g}), u_{f,g} = \bigwedge_{z\in X}(g^*(z) \to u_{f,\top^*_z})$,
(8) $u_{f,g} = \bigwedge_{y,z\in X}(f(y) \to (g^*(z) \to u_{\top_y,\top^*_z}))$,
(9) $u^{-1}_{f,g} = u_{g^*,f^*}$.

Definition 6 Ramadan et al. (2015a, b) A mapping $\mathcal{U} : L^{X\times X} \to L$ is called an Alexandrov L-fuzzy pre-uniformity on X iff the following conditions hold:

(AU1) there exists $u \in L^{X\times X}$ such that $\mathcal{U}(u) = \top$,
(AU2) if $v \le u$, then $\mathcal{U}(v) \le \mathcal{U}(u)$,
(AU3) for every $u_i \in L^{X\times X}$, $\mathcal{U}(\bigwedge_{i\in\Gamma} u_i) = \bigwedge_i \mathcal{U}(u_i)$,
(AU4) $\mathcal{U}(u) \le \bigwedge_{x\in X} u(x, x)$,
(AU5) $\mathcal{U}(\alpha \to u) = \alpha \to \mathcal{U}(u)$ for each $\alpha \in L$.

The pair $(X,\mathcal{U})$ is called an Alexandrov L-fuzzy pre-uniformity.

An Alexandrov L-fuzzy pre-uniformity is an Alexandrov L-fuzzy quasi-uniformity if

(AQ) $\mathcal{U}(u) \le \bigvee\{\mathcal{U}(v) \odot \mathcal{U}(w) \mid v \circ w \le u\}$, where

$$v \circ w(x, z) = \bigvee_{y\in X} v(y, z) \odot w(x, y).$$

An Alexandrov L-fuzzy quasi-uniformity is an Alexandrov L-fuzzy uniformity if

(U) $\mathcal{U}(u) = \mathcal{U}(u^{-1})$, where $u^{-1}(x, y) = u(y, x)$.

An Alexandrov L-fuzzy pre-uniformity is separated if

(SE) $\mathcal{U}(u_{\top_x,\top_x}) = \top$ or $\mathcal{U}^*(u_{\top^*_x,\top^*_x}) = \perp$ for each $x \in X$.

Remark 2 Oh and Kim (2017); Ramadan et al. (2015a, b) Let $(X,\mathcal{U})$ be an L-fuzzy pre-uniformity and by (U1),(U2), then we have $\mathcal{U}(\top_{X\times X}) = \top$ because $u \le \top_{X\times X}$ for all $u \in L^{X\times X}$.

Definition 7 El-Dardery et al. (2013); Ramadan et al. (2015a) A mapping $\xi : L^X \times L^X \to L$ is called an Alexandrov L-fuzzy semi-topogenous order on X if it satisfies the following axioms:

(ST1) $\xi(\top_X, \top_X) = \xi(\perp_X, \perp_X) = \top$,
(ST2) $\xi(f, g) \le S(f, g)$,
(ST3) if $f_1 \le f, g \le g_1$, then $\xi(f, g) \le \xi(f_1, g_1)$,
(ST4) $\xi(\bigvee_{i\in\Gamma} f_i, g) = \bigwedge_{i\in\Gamma} \xi(f_i, g),\ \xi(f, \bigwedge_{i\in\Gamma} g_i) = \bigwedge_{i\in\Gamma} \xi(f, g_i)$.

For every $f, f_1, f_2, g, g_1, g_2 \in L^X$, an Alexandrov L-fuzzy semi-topogenous order ξ on X is called

(1) $(L, \oplus)$-fuzzy topogenous order if $\xi(f_1\oplus f_2, g_1\oplus g_2) \ge \xi(f_1, g_1) \odot \xi(f_2, g_2)$,
(2) $(L, \odot)$-fuzzy co-topogenous order if

$$\xi(f_1 \odot f_2, g_1 \odot g_2) \ge \xi(f_1, g_1) \odot \xi(f_2, g_2),$$

(3) $(L, \odot)$-fuzzy topogenous (resp. co-topogenous) space if $\xi \le \xi \circ \xi$, where

$$(\xi_1 \circ \xi_2)(f, g) = \bigvee_{h\in L^X} \xi_1(f, h) \odot \xi_2(h, g),$$

(4) topological if $\bigwedge_{y\in X} \xi(\top_x, \top^*_y) \oplus \xi(\top_y, \top^*_z) \ge \xi(\top_x, \top^*_z)$,
(5) stratified if $\xi(\alpha \odot f, g) = \alpha \to \xi(f, g)$,
(6) co-stratified if $\xi(f, \alpha \to g) = \alpha \to \xi(f, g)$,
(7) strong if $\xi(\alpha \odot f, \alpha \odot g) \ge \xi(f, g),\ \xi(\alpha \to f, \alpha \to g) \ge \xi(f, g)$,
(8) separated if $\xi(\top_x, \top_x) = \top,\ \xi(\top^*_x, \top^*_x) = \top$.

Proposition 1 *El-Dardery et al. (2013); Ramadan et al. (2015a) Let ξ be an Alexandrov L-fuzzy semi-topogenous order on X, define a mapping $\mathcal{I}_\xi : L^X \to L^X$ by*

$$\mathcal{I}_\xi(f) = \bigvee_{g\in L^X} \xi(g^*, g^*) \odot \mathcal{S}(g, f) \odot g \quad \forall\ f, g \in L^X.$$

Then, the pair $(X, \mathcal{I}_\xi)$ is an Alexandrov L-fuzzy interior space.

Proposition 2 *El-Dardery et al. (2013); Ramadan et al. (2015a) Let $(X, \mathcal{C})$ be an Alexandrov L-fuzzy closure space, define a mapping $\xi_\mathcal{C} : L^X \times L^X \to L$ by $\xi_\mathcal{C}(f, g) = \bigwedge_{x\in X} (\mathcal{C}(f)(x) \to g(x))$. Then, $\xi_\mathcal{C}$ is an Alexandrov L-fuzzy semi-topogenous order on X.*

Proposition 3 *El-Dardery et al. (2013); Ramadan et al. (2015a) Let $(X, \mathcal{I})$ be an Alexandrov L-fuzzy interior space, define a mapping $\xi_{\mathcal{I}} : L^X \times L^X \to L$ by $\xi_{\mathcal{I}}(f, g) = \bigwedge_{x \in X} (g^*(x) \to \mathcal{I}(f^*)(x))$. Then, $\xi_{\mathcal{I}}$ is an Alexandrov L-fuzzy semi-topogenous order on X.*

3 Alexandrov *L*-fuzzy nearness, *L*-fuzzy approximation operators and Alexandrov *L*-fuzzy uniformities

This section is divided into two parts; in the first part, we investigate the relationship between the L-fuzzy nearness and L-fuzzy upper (lower) approximation operators, while the second part addresses the relation between the L-fuzzy nearness and L-fuzzy uniformities.

Definition 8 Ramadan et al. (2019) A mapping $\mathcal{N} : L^X \times L^X \to L$ is called an Alexandrov L-fuzzy nearness on X if it satisfies the following axioms:

(N1) $\mathcal{N}(\bot_X, \top_x) = \bot$,
(N2) $\mathcal{N}(f, \top_x) \geq f(x)$,
(N3) if $f \leq g$, then $\mathcal{N}(f, \top_x) \leq \mathcal{N}(g, \top_x)\ \ \forall f, g \in L^X$,
(N4) $\mathcal{N}(\bigvee_{i \in \Gamma} f_i, \top_x) = \bigvee_{i \in \Gamma} \mathcal{N}(f_i, \top_x)$, $\mathcal{N}(\top_x, \bigvee_{i \in \Gamma} f_i) = \bigvee_{i \in \Gamma} \mathcal{N}(\top_x, f_i)$,
(N5) for all $\alpha \in L$, $f \in L^X$ we have

$$\mathcal{N}(\alpha \odot f, \top_x) = \alpha \odot \mathcal{N}(f, \top_x) = \mathcal{N}(f, \alpha \odot \top_x).$$

The pair $(X, \mathcal{N})$ is called an Alexandrov L-fuzzy nearness. An Alexandrov L-fuzzy nearness $(X, \mathcal{N})$ is called

(1) topological if $\bigvee_{y \in X} \mathcal{N}(\top_x, \top_y) \odot \mathcal{N}(\top_y, \top_z) \leq \mathcal{N}(\top_x, \top_z)$,
(2) $(L, \oplus)$-fuzzy nearness if for every $f_1, f_2 \in L^X$, we have

$$\mathcal{N}(f_1 \oplus f_2, \top_x) \leq \mathcal{N}(f_1, \top_x) \oplus \mathcal{N}(f_2, \top_x),$$

(3) $(L, \oplus)$-fuzzy co-nearness if for every $f_1, f_2 \in L^X$, we have

$$\mathcal{N}(\top_x, f_1 \oplus f_2) \leq \mathcal{N}(\top_x, f_1) \oplus \mathcal{N}(\top_x, f_2),$$

(4) symmetric if $\mathcal{N}^s = \mathcal{N}$, where $\mathcal{N}^s(f, \top_x) = \mathcal{N}(\top_x, f)$,
(5) separated if $\mathcal{N}(\top_x^*, \top_x) = \mathcal{N}(\top_x, \top_x^*) = \bot$ for every $x \in X$.

Theorem 1 *Let $\overline{R}$ be an Alexandrov L-fuzzy upper approximation operator on X, define a mapping $\mathcal{N}_{\overline{R}} : L^X \times L^X \to L$ by*

$$\mathcal{N}_{\overline{R}}(f, \top_x) = \bigvee_{x \in X} \overline{R}(f)(x) \ \ \forall x \in X.$$

Then, the following holds:

(1) $\mathcal{N}_{\overline{R}}$ is an Alexandrov L-fuzzy nearness on X,
(2) if $\overline{R}$ is topological, then $\mathcal{N}_{\overline{R}}$ is topological,
(3) if $(X, \overline{R})$ is an $(L, \oplus)$-fuzzy upper approximation space, then $(X, \mathcal{N})$ is an $(L, \oplus)$-fuzzy nearness.

Proof (2)

$$\begin{aligned}
&\bigvee_{y \in X} \mathcal{N}_{\overline{R}}(\top_x, \top_y) \odot \mathcal{N}_{\overline{R}}(\top_y, \top_z) \\
&\quad = \bigvee_{y \in X} (\bigvee_{y \in X} \overline{R}(\top_x)(y) \odot \bigvee_{z \in X} \overline{R}(\top_y)(z)) \\
&\quad = \bigvee_{y,z \in X} \big(\overline{R}(\top_x)(y) \odot \overline{R}(\top_y)(z)\big) \\
&\quad \leq \bigvee_{z \in X} \overline{R}(\top_x)(z) = \mathcal{N}_{\overline{R}}(\top_x, \top_z).
\end{aligned}$$

□

Theorem 2 *Let $\underline{R}$ be an Alexandrov L-fuzzy lower approximation operator on X, define a mapping $\mathcal{N}_{\underline{R}} : L^X \times L^X \to L$ by*

$$\mathcal{N}_{\underline{R}}(f, \top_x) = \bigvee_{x \in X} \underline{R}^*(f^*)(x) \ \ \forall x \in X.$$

Then, the following holds:

(1) $\mathcal{N}_{\underline{R}}$ is an Alexandrov L-fuzzy nearness on X,
(2) if $\underline{R}$ is topological, then $\mathcal{N}_{\underline{R}}$ is topological,
(3) if $(X, \underline{R})$ is an $(L, \odot)$-fuzzy lower approximation space, then $(X, \mathcal{N})$ is an $(L, \oplus)$-fuzzy nearness.

Proof (2)

$$\begin{aligned}
&\bigvee_{y \in X} \mathcal{N}_{\underline{R}}(\top_x, \top_y) \odot \mathcal{N}_{\underline{R}}(\top_y, \top_z) \\
&\quad = \bigvee_{y \in X} \left(\bigvee_{y \in X} \underline{R}^*(\top_x^*)(y) \odot \bigvee_{z \in X} \underline{R}^*(\top_y^*)(z) \right) \\
&\quad \leq \bigvee_{z \in X} \underline{R}^*(\top_x^*)(z) = \mathcal{N}_{\underline{R}}(\top_x, \top_z).
\end{aligned}$$

□

Theorem 3 *Let $\mathcal{N}$ be an Alexandrov L-fuzzy nearness on X, define two mappings $\overline{R}_{\mathcal{N}}, \underline{R}_{\mathcal{N}} : X \times X \to L$ as*

$$\overline{R}_{\mathcal{N}}(f)(x) = \bigvee_{y \in X} \mathcal{N}(\top_x, \top_y) \odot f(y),\ \ \underline{R}_{\mathcal{N}}(f)(x) = \bigwedge_{y \in X} (\mathcal{N}(\top_x, \top_y) \to f(y))$$

for all $x, y \in X$, $f \in L^X$. Then, the following holds:

(1) $\overline{R}_{\mathcal{N}}$ *is an L-fuzzy upper approximation operator such that*

$$\overline{R}_{\mathcal{N}}(\top_y)(x) = \mathcal{N}(\top_x, \top_y),$$

(2) $\underline{R}_{\mathcal{N}}$ *is an L-fuzzy lower approximation operator such that*

$$\underline{R}_{\mathcal{N}}(\top_y^*)(x) = \mathcal{N}^*(\top_x, \top_y),$$

(3) if $\mathcal{N}$ *is topological, then* $\overline{R}_{\mathcal{N}}$ *and* $\underline{R}_{\mathcal{N}}$ *both are topological,*

(4) if L is idempotent, then $\overline{R}_{\mathcal{N}}$ *is an* $(L, \oplus)$*-fuzzy upper approximation operator on X and* $\underline{R}_{\mathcal{N}}$ *is an* $(L, \odot)$*-fuzzy lower approximation operator on X,*

(5) $\overline{R}_{\mathcal{N}}(f) = \underline{R}^*_{\mathcal{N}}(f^*),\ \underline{R}_{\mathcal{N}}(f) = \overline{R}^*_{\mathcal{N}}(f^*)$.

Proof (3)(T)

$$\begin{aligned}\overline{R}_{\mathcal{N}}(\overline{R}_{\mathcal{N}}(f))(x) &= \bigvee_{y\in X} \mathcal{N}(\top_x, \top_y) \odot \overline{R}_{\mathcal{N}}(f)(y)\\ &= \bigvee_{y\in X} \mathcal{N}(\top_x, \top_y) \odot \left(\bigvee_{z\in X} \mathcal{N}(\top_y, \top_z) \odot f(z)\right)\\ &= \bigvee_{z\in X} \left(\bigvee_{y\in X} \mathcal{N}(\top_x, \top_y) \odot \mathcal{N}(\top_y, \top_z)\right) \odot f(z)\\ &\leq \bigvee_{z\in X} \mathcal{N}(\top_x, \top_z) \odot f(z) = \overline{R}_{\mathcal{N}}(f)(x),\end{aligned}$$

$$\begin{aligned}\underline{R}_{\mathcal{N}}(\underline{R}_{\mathcal{N}}(f))(x) &= \bigwedge_{y\in X} (\mathcal{N}(\top_x, \top_y) \to \underline{R}_{\mathcal{N}}(f)(y))\\ &= \bigwedge_{y\in X} (\mathcal{N}(\top_x, \top_y) \to \left(\bigwedge_{z\in X} (\mathcal{N}(\top_y, \top_z) \to f(z))\right)\\ &= \bigwedge_{z\in X} \left((\bigvee_{y\in X} \mathcal{N}(\top_x, \top_y) \odot \mathcal{N}(\top_y, \top_z)) \to f(z)\right)\\ &\geq \bigwedge_{z\in X} (\mathcal{N}(\top_x, \top_z) \to f(z)) = \underline{R}_{\mathcal{N}}(f)(x).\end{aligned}$$

(5)

$$\begin{aligned}\overline{R}^*_{\mathcal{N}}(f^*)(x) &= \left(\bigvee_{y\in X} \mathcal{N}(\top_x, \top_y) \odot f^*(y)\right)^*\\ &= \bigwedge_{y\in X} (\mathcal{N}(\top_x, \top_y) \to f(y)) = \underline{R}_{\mathcal{N}}(f)(x),\end{aligned}$$

$$\begin{aligned}\underline{R}^*_{\mathcal{N}}(f^*)(x) &= \left(\bigwedge_{y\in X} (\mathcal{N}(\top_x, \top_y) \to f^*(y))\right)^*\\ &= \bigvee_{y\in X} \mathcal{N}(\top_x, \top_y) \odot f(y) = \overline{R}_{\mathcal{N}}(f)(x).\end{aligned}$$

□

Theorem 4 *Let* $\mathcal{U}$ *be an Alexandrov L-fuzzy pre-uniformity on X, define a mapping* $\mathcal{N}_{\mathcal{U}} : L^X \times L^X \to L$ *by*

$$\mathcal{N}_{\mathcal{U}}(f, \top_x) = \mathcal{U}^*(u_{f,\top_x^*}) \quad \forall\, x \in X.$$

Then, the following holds:

(1) $\mathcal{N}_{\mathcal{U}}$ *is an Alexandrov L-fuzzy nearness on X,*
(2) if $\mathcal{U}$ *is an Alexandrov L-fuzzy uniformity on X, then* $\mathcal{N}_{\mathcal{U}}$ *is symmetric,*
(3) if $\mathcal{U}$ *is separated, then* $\mathcal{N}_{\mathcal{U}}$ *is separated,*
(4) $\mathcal{N}_{\mathcal{U}}(f, \top_x) = \bigvee_{y\in X} \mathcal{U}^*(u_{\top_y,\top_x^*})$,
(5) if $\mathcal{U}^*(u_{\top_x,\top_z^*}) \geq \bigvee_{y\in X} \mathcal{U}^*(u_{\top_x,\top_y^*}) \odot \mathcal{U}^*(u_{\top_y,\top_z^*})$*, then* $\mathcal{N}_{\mathcal{U}}$ *is topological.*

Proof (5)(T)

$$\begin{aligned}&\bigvee_{y\in X} \mathcal{N}(\top_x, \top_y) \odot \mathcal{N}(\top_y, \top_z)\\ &\quad= \bigvee_{y\in X} \mathcal{U}^*(u_{\top_x,\top_y^*}) \odot \mathcal{U}^*(u_{\top_y,\top_z^*})\\ &\quad\leq \mathcal{U}^*(u_{\top_x,\top_z^*}) = \mathcal{N}(\top_x, \top_z).\end{aligned}$$

□

Theorem 5 *Let* $\mathcal{N}$ *be an Alexandrov L-fuzzy nearness on X, define a mapping* $\mathcal{U}_{\mathcal{N}} : L^{X\times X} \to L$ *by*

$$\mathcal{U}_{\mathcal{N}}(u) = \bigwedge_{x,y\in X} (\mathcal{N}(\top_x, \top_y) \to u(x, y)).$$

Then, the following holds:

(1) $\mathcal{U}_{\mathcal{N}}$ *is an Alexandrov L-fuzzy pre-uniformity on X,*
(2) if $\mathcal{N}$ *is symmetric, then* $\mathcal{U}_{\mathcal{N}}$ *is an Alexandrov L-fuzzy uniformity on X,*
(3) if $\mathcal{N}$ *is separated, then* $\mathcal{U}_{\mathcal{N}}$ *is separated,*
(4) if $\mathcal{N}$ *is topological, then*
$\mathcal{U}^*_{\mathcal{N}}(u_{\top_x,\top_z^*}) \geq \bigvee_{y\in X} \mathcal{U}^*_{\mathcal{N}}(u_{\top_x,\top_y^*}) \odot \mathcal{U}^*_{\mathcal{N}}(u_{\top_y,\top_z^*})$.

Proof (4) Since

$$\begin{aligned}\mathcal{U}^*_{\mathcal{N}}(u_{\top_x,\top_y^*}) &= \bigvee_{x,y\in X} \mathcal{N}(\top_x, \top_y) \odot u^*_{\top_x,\top_y^*}(x, y)\\ &= \bigvee_{x,y\in X} \mathcal{N}(\top_x, \top_y) \odot (\top_x(x) \to \top_y^*(y))^*\\ &= \bigvee_{x,y\in X} \mathcal{N}(\top_x, \top_y),\end{aligned}$$

then

$$\begin{aligned}&\bigvee_{y\in X} \mathcal{U}^*(u_{\top_x,\top_y^*}) \odot \mathcal{U}^*(u_{\top_y,\top_z^*})\\ &\quad= \bigvee_{y\in X} \left(\bigvee_{x,y\in X} \mathcal{N}(\top_x, \top_y) \odot \bigvee_{y,z\in X} \mathcal{N}(\top_y, \top_z)\right)\\ &\quad\leq \bigvee_{x,z\in X} \mathcal{N}(\top_x, \top_z) = \mathcal{U}^*(u_{\top_x,\top_z^*}).\end{aligned}$$

□

4 Alexandrov L-fuzzy nearness and Alexandrov L-fuzzy topogenous space

In this section, we introduce the relationship between Alexandrov L-fuzzy nearness and Alexandrov L-fuzzy topogenous space and raise some characteristics among other previously given systems.

Theorem 6 *Let ξ be a strong Alexandrov L-fuzzy topogenous order on X, define a mapping $\mathcal{N}_\xi : L^X \times L^X \to L$ by*

$$\mathcal{N}_\xi(f, \top_x) = \xi^*(f, \top_x^*) \ \ \forall f \in L^X.$$

Then,

(1) $(X, \mathcal{N}_\xi)$ is an Alexandrov L-fuzzy nearness,
(2) if ξ is an $(L, \oplus)$-fuzzy topogenous order on X, then $(X, \mathcal{N}_\xi)$ is an $(L, \oplus)$-fuzzy nearness,
(3) if ξ is topological, then $\mathcal{N}_\xi$ is topological,
(4) if ξ is separated, then $\mathcal{N}_\xi$ is separated.

Proof (2)

$$\begin{aligned}&\mathcal{N}_\xi(f_1 \oplus f_2, \top_x)\\ &= \xi^*(f_1 \oplus f_2, \top_x^*) = \big(\xi(f_1 \oplus f_2, \top_x^* \oplus \top_x^*)\big)^*\\ &\le \big(\xi(f_1, \top_x^*) \odot \xi(f_2, \top_x^*)\big)^* = \xi^*(f_1, \top_x^*) \oplus \xi^*(f_2, \top_x^*)\\ &= \mathcal{N}_\xi(f_1, \top_x) \oplus \mathcal{N}_\xi(f_2, \top_x).\end{aligned}$$

(3)

$$\begin{aligned}\mathcal{N}(\top_x, \top_z) = \xi^*(\top_x, \top_z^*) &= \big(\xi(\top_x, \top_z^*)\big)^*\\ &\ge \Big(\bigwedge_{y\in X} \xi(\top_x, \top_y^*) \oplus \xi(\top_y, \top_z^*)\Big)^*\\ &= \bigvee_{y\in X} \xi^*(\top_x, \top_y^*) \odot \xi^*(\top_y, \top_z^*)\\ &= \bigvee_{y\in X} \mathcal{N}_\xi(\top_x, \top_y) \odot \mathcal{N}_\xi(\top_y, \top_z).\end{aligned}$$

□

Theorem 7 *Let $(X, \mathcal{N})$ be an Alexandrov L-fuzzy nearness, define a mapping $\xi_\mathcal{N} : L^X \times L^X \to L$ by*

$$\xi_\mathcal{N}(f, g) = \bigwedge_{x\in X}(\mathcal{N}(f, \top_x) \to g(x)) \ \ \forall f, g \in L^X.$$

Then,

(1) $\xi_\mathcal{N}$ is a strong Alexandrov L-fuzzy semi-topogenous order on X,
(2) if $(X, \mathcal{N})$ is an $(L, \oplus)$-fuzzy nearness, then $\xi_\mathcal{N}$ is an $(L, \oplus)$-fuzzy topogenous order on X,
(3) if $\mathcal{N}$ is topological, then $\xi_\mathcal{N}$ is topological,
(4) if $\mathcal{N}$ is separated, then $\xi_\mathcal{N}$ is separated.

Proof (3)

$$\begin{aligned}&\bigwedge_{y\in X} \xi_\mathcal{N}(\top_x, \top_y^*) \oplus \xi_\mathcal{N}(\top_y, \top_z^*)\\ &= \bigwedge_{y\in X} \Big(\bigwedge_{y\in X}(\mathcal{N}(\top_x, \top_y) \to \top_y^*(y))\Big)\\ &\quad \oplus \big(\bigwedge_{z\in X}(\mathcal{N}(\top_y, \top_z) \to \top_z^*(z))\big)\\ &= \bigwedge_{y,z\in X} \mathcal{N}^*(\top_x, \top_y) \oplus \mathcal{N}^*(\top_y, \top_z)\\ &= \bigwedge_{z\in X} \Big(\bigvee_{y\in X} \mathcal{N}(\top_x, \top_y) \odot \mathcal{N}(\top_y, \top_z)\Big)^*\\ &\ge \bigwedge_{z\in X} \mathcal{N}^*(\top_x, \top_z) = \bigwedge_{z\in X}(\mathcal{N}(\top_x, \top_z) \to \top_z^*(z))\\ &= \xi(\top_x, \top_z^*).\end{aligned}$$

□

Corollary 1 *Let $(X, \mathcal{N})$ be an Alexandrov L-fuzzy nearness, define a mapping $\xi_\mathcal{N} : L^X \times L^X \to L$ by*

$$\xi_\mathcal{N}(f, g) = \bigwedge_{x\in X}(g^*(x) \to \mathcal{N}^*(\top_x, f)) \ \ \forall f, g \in L^X.$$

Then,

(1) $\xi_\mathcal{N}$ is a strong Alexandrov L-fuzzy semi-topogenous order on X,
(2) if $(X, \mathcal{N})$ is an $(L, \oplus)$-fuzzy nearness, then $\xi_\mathcal{N}$ is an $(L, \oplus)$-fuzzy topogenous order on X,
(3) if $\mathcal{N}$ is topological, then $\xi_\mathcal{N}$ is topological,
(4) if $\mathcal{N}$ is separated, then $\xi_\mathcal{N}$ is separated.

Theorem 8 *Let $(X, \mathcal{N})$ be an Alexandrov L-fuzzy nearness and ξ be an Alexandrov L-fuzzy co-topogenous order on X. Define a mapping $\mathcal{I}_{\xi_\mathcal{N}} : L^X \to L^X$ by*

$$\mathcal{I}_{\xi_\mathcal{N}}(f) = \bigvee_{g\in L^X} \xi(g^*, g^*) \odot \mathcal{S}(g, f) \odot g \quad \forall f, g \in L^X.$$

Then, $(X, \mathcal{I}_{\xi_\mathcal{N}})$ is an Alexandrov L-fuzzy interior space. Moreover, if $\mathcal{N}$ is symmetric, we have $\mathcal{I}_{\xi_\mathcal{N}}(f) \le \mathcal{I}_\mathcal{N}(f^*)$.

Proof

$$\begin{aligned}\mathcal{I}_{\xi_\mathcal{N}}(f)(x) &= \bigvee_{g\in L^X} \xi_\mathcal{N}(g^*, g^*) \odot \mathcal{S}(g, f) \odot g(x)\\ &= \bigvee_{g\in L^X} \big(\bigwedge_{x\in X}(\mathcal{N}(g^*, \top_x) \to g^*(x))\big)\\ &\quad \odot \mathcal{S}(g, f) \odot g(x)\\ &\le \bigwedge_{x\in X} (\mathcal{N}(f^*, \top_x) \to f^*(x))\\ &\quad \odot \mathcal{S}(f, f) \odot f(x)\\ &= \bigwedge_{x\in X} \big(\mathcal{N}(f^*, \top_x) \to f^*(x)\big) \odot f(x)\\ &= \bigwedge_{x\in X} \big(f(x) \to \mathcal{N}^*(f^*, \top_x)\big) \odot f(x)\\ &= \bigwedge_{x\in X} f(x) \odot \big(f(x) \to \mathcal{N}^*(f^*, \top_x)\big)\\ &\le \mathcal{N}^*(f^*, \top_x) = \mathcal{I}_\mathcal{N}(f^*)(x).\end{aligned}$$

Corollary 2 *Let $(X, \mathcal{N})$ be an Alexandrov L-fuzzy nearness, define a mapping $\xi_{\mathcal{C}_\mathcal{N}} : L^X \times L^X \to L$ by $\xi_{\mathcal{C}_\mathcal{N}}(f, g) = \bigwedge_{x\in X} (\mathcal{C}_\mathcal{N}(f)(x) \to g(x)) \ \forall f, g \in L^X$, where $\mathcal{C}_\mathcal{N}(f)(x) = \mathcal{N}(f, \top_x)$. Then, $(X, \xi_{\mathcal{C}_\mathcal{N}})$ is an Alexandrov L-fuzzy co-topogenous space.*

Theorem 9 *Let $(X, \mathcal{N})$ be an Alexandrov L-fuzzy nearness, define a mapping $\xi_{\mathcal{I}_{\mathcal{N}}} : L^X \times L^X \to L$ by*

$$\xi_{\mathcal{I}_{\mathcal{N}}}(f, g) = \bigwedge_{x\in X} (g^*(x) \to \mathcal{I}_{\mathcal{N}}(f^*)(x)) \quad \forall\, f, g \in L^X,$$

where $\mathcal{I}_{\mathcal{N}}(f)(x) = \mathcal{N}^*(\top_x, f^*)$. Then, we have the following properties:

(1) if $(X, \mathcal{N})$ is an Alexandrov L-fuzzy co-nearness, then $(X, \xi_{\mathcal{I}_{\mathcal{N}}})$ is an Alexandrov L-fuzzy co-topogenous space,
(2) if $\mathcal{N}$ is symmetric, then $\xi_{\mathcal{I}_{\mathcal{N}}} = \xi_{\mathcal{N}}$,
(3) if $\mathcal{N}$ is separated, then $\xi_{\mathcal{I}_{\mathcal{N}}}$ is separated.

Proof (1) If $\mathcal{N}$ is an Alexandrov L-fuzzy co-nearness on X, then

$$\begin{aligned}
&\xi_{\mathcal{I}_{\mathcal{N}}}(f_1 \oplus f_2, g_1 \oplus g_2)\\
&\quad = \bigwedge_{x\in X}((g_1 \oplus g_2)^*(x) \to \mathcal{I}_{\mathcal{N}}(f_1 \oplus f_2)^*(x))\\
&\quad = \bigwedge_{x\in X} \left((g_1^*(x) \odot g_2^*(x)) \to \mathcal{N}^*(\top_x, f_1 \oplus f_2)\right)\\
&\quad \geq \bigwedge_{x\in X} \left((g_1^*(x) \odot g_2^*(x)) \to \left(\mathcal{N}(\top_x, f_1) \oplus \mathcal{N}(\top_x, f_2)\right)^*\right)\\
&\quad = \bigwedge_{x\in X} \left((g_1^*(x) \odot g_2^*(x)) \to (\mathcal{N}^*(\top_x, f_1) \odot \mathcal{N}^*(\top_x, f_2))\right)\\
&\quad \geq \bigwedge_{x\in X} \Big(\left(g_1^*(x) \to \mathcal{N}^*(\top_x, f_1)\right) \odot \big(g_2^*(x)\\
&\qquad \to \mathcal{N}^*(\top_x, f_2)\big)\Big)\\
&\quad = \bigwedge_{x\in X} \left(g_1^*(x) \to \mathcal{N}^*(\top_x, f_1)\right) \odot \bigwedge_{x\in X} \big(g_2^*(x)\\
&\qquad \to \mathcal{N}^*(\top_x, f_2)\big)\\
&\quad = \xi_{\mathcal{I}_{\mathcal{N}}}(f_1, g_1) \odot \xi_{\mathcal{I}_{\mathcal{N}}}(f_2, g_2).
\end{aligned}$$

(2) If $\mathcal{N}$ is symmetric, then

$$\begin{aligned}
\xi_{\mathcal{I}_{\mathcal{N}}}(f, g) &= \bigwedge_{x\in X} \left(g^*(x) \to \mathcal{I}_{\mathcal{N}}(f^*)(x)\right)\\
&= \bigwedge_{x\in X} \left(g^*(x) \to \mathcal{N}^*(\top_x, f)\right)\\
&= \bigwedge_{x\in X} (\mathcal{N}(f, \top_x) \to g(x)) = \xi_{\mathcal{N}}.
\end{aligned}$$

(3) Easily proved.

Example 1 Let $X = \{h_i \mid i = \{1, ..., 3\}\}$ with h_i=house and $Y = \{e, b, w, c, i\}$ with e=expensive, b=beautiful, w=wooden, c=creative, i=in the green surroundings. Let $([0, 1], \odot, \to, ^*, 0, 1)$ be a complete residuated lattice as

$$\begin{aligned}
x \odot y &= \max\{0, x + y - 1\}, \quad x \to y\\
&= \min\{1 - x + y, 1\}, \quad x^* = 1 - x.
\end{aligned}$$

Let $R \in [0, 1]^{X\times Y}$ be a fuzzy information system as follows:

R	e	b	w	c	i
h_1	0.7	0.6	0.5	0.9	0.2
h_2	0.6	0.8	0.4	0.3	0.5
h_3	0.4	0.9	0.8	0.6	0.6

Define $[0, 1]$-fuzzy pre-orders $R_X^Y, R_X^{\{b,w\}} \in [0, 1]^{X\times Y}$ by

$$R_X^Y(h_i, h_j) = \bigwedge_{y\in Y}(R(h_i, y) \to R(h_j, y)),$$

$$R_X^{\{b,w\}}(h_i, h_j) = \bigwedge_{y\in\{b,w\}}(R(h_i, y) \to R(h_j, y)),$$

$$R_X^Y = \begin{pmatrix} 1 & 0.4 & 0.7\\ 0.7 & 1 & 0.8\\ 0.6 & 0.6 & 1 \end{pmatrix}, \quad R_X^{\{b,w\}} = \begin{pmatrix} 1 & 0.9 & 1\\ 0.8 & 1 & 1\\ 0.7 & 0.6 & 1 \end{pmatrix}.$$

For each $R \in \{R_X^Y, R_X^{\{b,w\}}\}$, we obtain $\mathcal{N}_R : L^X \times L^X \to L$ as

$$\mathcal{N}_R(f, \top_x) = \bigvee_{x,y\in X} R(x, y) \odot f(y) \quad \forall\, f \in L^X.$$

Hence, and by Ramadan et al. (2019), $\mathcal{N}_R$ is an Alexandrov L-fuzzy nearness on X. Moreover, $\mathcal{N}_R$ is topological.

(1) By Theorem 3, we obtain an L-fuzzy upper approximation operator $\overline{R}_{\mathcal{N}} : L^X \to L^X$ on X as $\overline{R}_{\mathcal{N}_R}(f)(x) = \bigvee_{x,y\in X} R(y, x) \odot f(y) \quad \forall\, f \in L^X$.

(T) Moreover, $\overline{R}_{\mathcal{N}_R}$ is topological since

$$\begin{aligned}
\overline{R}_{\mathcal{N}_R}(\overline{R}_{\mathcal{N}_R}(f))(x) &= \bigvee_{x,y\in X} R(y, x) \odot \overline{R}_{\mathcal{N}_R}(f)(y)\\
&= \bigvee_{x,y\in X} R(y, x) \odot \bigvee_{y,z\in X} R(z, y) \odot (f)(z)\\
&= \bigvee_{x,y,z\in X} R(z, y) \odot R(y, x) \odot (f)(z)\\
&\leq \bigvee_{x,z\in X} R(z, x) \odot (f)(z) = \overline{R}_{\mathcal{N}_R}(f).
\end{aligned}$$

(UAS) Finally, $\overline{R}_{\mathcal{N}_R}$ is an $(L, \oplus)$-fuzzy upper approximation operator on X since

$$\begin{aligned}
\overline{R}_{\mathcal{N}_R}(f \oplus g)(x) &= \bigvee_{x,y\in X} R(x, y) \odot (f \oplus g)(y)\\
&= \bigvee_{x,y\in X} R(x, y) \odot (f(y) \oplus g(y))\\
&\leq \left(\bigvee_{x,y\in X} R(x, y) \odot f(y)\right) \oplus\\
&\quad \left(\bigvee_{x,y\in X} R(x, y) \odot g(y)\right)\\
&= \overline{R}_{\mathcal{N}_R}(f) \oplus \overline{R}_{\mathcal{N}_R}(g).
\end{aligned}$$

(2) By Theorem 3, we obtain an L-fuzzy lower approximation operator $\underline{R}_{\mathcal{N}} : L^X \to L^X$ on X as $\underline{R}_{\mathcal{N}_R}(f)(x) = \bigwedge_{x,y\in X}(R(y, x) \to f(y)) \quad \forall\, f \in L^X$.

(T) Moreover, and by Lemma 1(9), $\underline{R}_{\mathcal{N}_R}$ is topological since

$$\begin{aligned}\underline{R}_{\mathcal{N}_R}(\underline{R}_{\mathcal{N}_R}(f))(x) &= \bigwedge_{x,y\in X}(R(y,x)\to \underline{R}_{\mathcal{N}_R}(f)(y))\\ &= \bigwedge_{x,y\in X}\left(R(y,x)\to\left(\bigwedge_{y,z\in X}R(z,y)\to(f)(z)\right)\right)\\ &= \bigwedge_{x,y,z\in X}((R(z,y)\odot R(y,x))\to(f)(z))\\ &\geq \bigwedge_{x,z\in X}(R(z,x)\to(f)(z)) = \underline{R}_{\mathcal{N}_R}(f).\end{aligned}$$

(LAS) Finally, $\underline{R}_{\mathcal{N}_R}$ is an $(L,\odot)$-fuzzy lower approximation operator on X since

$$\begin{aligned}\overline{R}_{\mathcal{N}_R}(f\odot g)(x) &= \bigwedge_{x,y\in X}\big(R(x,y)\to(f\odot g)(y)\big)\\ &= \bigwedge_{x,y\in X}\big(R(x,y)\to(f(y)\odot g(y))\big)\\ &\geq \left(\bigwedge_{x,y\in X}(R(x,y)\to f(y))\right)\\ &\quad\odot\left(\bigvee_{x,y\in X}(R(x,y)\to g(y))\right)\\ &= \underline{R}_{\mathcal{N}_R}(f)\odot\underline{R}_{\mathcal{N}_R}(g).\end{aligned}$$

(3) By Theorem 5, we obtain a mapping $\mathcal{U}_{\mathcal{N}} : L^{X\times X}\to L$ as

$\mathcal{U}_{\mathcal{N}_R}(u) = \bigwedge_{x,y\in X}(R(y,x)\to u(x,y))\ \forall\, u\in L^{X\times X}$, which is easily proved to be an Alexandrov L-fuzzy pre-uniformity on X.

(4) By Theorem 7, we obtain a strong Alexandrov L-fuzzy semi-topogenous order $\xi_{\mathcal{N}} : L^X\times L^X\to L$ on X by

$\xi_{\mathcal{N}_R}(f,g) = \bigwedge_{x,y\in X}(R(y,x)\to(f(y)\to g(x)))$ for all $f,g\in L^X$. But not separated.

5 Degrees of *LF*-mappings and Galois correspondences

Now, we will study the degree of continuity Xiu and Li (2019) for LF-near map and some special maps of spaces discussed in this paper, like: L-fuzzy lower (upper) approximation spaces, L-fuzzy uniform space, L-fuzzy topogenous space and vise versa. we will show the Galois correspondences between their categories.

Definition 9 Let $(X,\overline{R}_X)$ and $(Y,\overline{R}_Y)$ be two L-fuzzy upper approximation spaces and $\phi : X\to Y$ be a mapping. Then, $D_{\overline{R}}(\phi)$ defined by

$$D_{\overline{R}}(\phi) = \bigwedge\nolimits_{f\in Y} S\big(\overline{R}_X(\phi^{\leftarrow}(f)),\phi^{\leftarrow}(\overline{R}_Y(f))\big)$$

is the degree to which the mapping ϕ is an LF-upper approximation map.

If $D_{\overline{R}}(\phi)=\top$, then $\overline{R}_X(\phi^{\leftarrow}(f))\leq\phi^{\leftarrow}(\overline{R}_Y(f))$ for all $f\in L^Y$ which is exactly the definition of LF-upper approximation map.

Definition 10 Let $(X,\underline{R}_X)$ and $(Y,\underline{R}_Y)$ be two L-fuzzy lower approximation spaces and $\phi : X\to Y$ be a mapping. Then, $D_{\underline{R}}(\phi)$ defined by

$$D_{\underline{R}}(\phi) = \bigwedge\nolimits_{f\in L^Y} S\big(\phi^{\leftarrow}(\underline{R}_Y(f)),\underline{R}_X(\phi^{\leftarrow}(f))\big)$$

is the degree to which the mapping ϕ is an LF-lower approximation map.

If $D_{\underline{R}}(\phi)=\top$, then $\phi^{\leftarrow}(\underline{R}_Y(f))\leq\underline{R}_X(\phi^{\leftarrow}(f))$ for all $f\in L^Y$ which is exactly the definition of LF-lower approximation map.

Definition 11 Let $(X,\mathcal{U}_X)$ and $(Y,\mathcal{U}_Y)$ be two Alexandrov L-fuzzy pre-uniformities and $\phi : X\to Y$ be a mapping. Then, $D_{\mathcal{U}}(\phi)$ defined by

$$D_{\mathcal{U}}(\phi) = \bigwedge\nolimits_{v\in L^{Y\times Y}}\big(\mathcal{U}_Y(v)\to\mathcal{U}_X((\phi\times\phi)^{\leftarrow}(v))\big)$$

is the degree to which the mapping ϕ is an LF-uniformly continuous map.

If $D_{\mathcal{U}}(\phi)=\top$, then for every $v\in L^{Y\times Y}, \mathcal{U}_Y(v)\leq\mathcal{U}_X((\phi\times\phi)^{\leftarrow}(v))$ which is exactly the definition of LF-uniformly continuous map.

Definition 12 Let (X,ξ_X) and (Y,ξ_Y) two Alexandrov L-fuzzy topogenous spaces and $\phi : X\to Y$ be a mapping. Then, $D_{\xi}(\phi)$ defined by

$$D_{\xi}(\phi) = \bigwedge\nolimits_{f,g\in L^Y}\big(\xi_Y(f,g)\to\xi_X(\phi^{\leftarrow}(f),\phi^{\leftarrow}(g))\big)$$

is the degree to which the mapping ϕ is an LF-topogenous map.

If $D_{\xi}(\phi)=\top$, then $\xi_Y(f,g)\leq\xi_X(\phi^{\leftarrow}(f),\phi^{\leftarrow}(g))$ for all $f,g\in L^Y$ which is exactly the definition of LF-topogenous map.

Definition 13 Let $(X,\mathcal{N}_X)$ and $(Y,\mathcal{N}_Y)$ be two Alexandrov L-fuzzy nearness and $\phi : X\to Y$ be a mapping. Then, $D_{\mathcal{N}}(\phi)$ defined by

$$D_{\mathcal{N}} = \bigwedge\nolimits_{f\in L^Y}\bigwedge\nolimits_{x\in X}\big(\mathcal{N}_X(\phi^{\leftarrow}(f),\phi^{\leftarrow}(\top_{\phi(x)}))\to\mathcal{N}_Y(f,\top_{\phi(x)})\big),$$

or equivalently,

$$D_{\mathcal{N}} = \bigwedge_{f\in L^Y} \bigwedge_{x\in X} \big(\mathcal{N}_X(\phi^{\leftarrow}(\top_{\phi(x)}), \phi^{\leftarrow}(f)) \to \mathcal{N}_Y(\top_{\phi(x)}, f)\big)$$

is the degree to which a mapping ϕ is an LF-near map.

If $D_{\mathcal{N}}(\phi) = \top$, then $\mathcal{N}_X(\phi^{\leftarrow}(f), \phi^{\leftarrow}(\top_{\phi(x)})) \leq \mathcal{N}_Y(f, \top_{\phi(x)})$, or equivalently, $\mathcal{N}_X(\phi^{\leftarrow}(\top_{\phi(x)}), \phi^{\leftarrow}(f)) \leq \mathcal{N}_Y(\top_{\phi(x)}, f)$ for all $x \in X$, $f \in L^Y$, which is exactly the definition of LF-near map.

Definition 14 Adámek et al. (1990) A concrete category is a pair $(\mathcal{C}, U)$, where $\mathcal{C}$ is a category and $U : \mathcal{C} \to Set$ is a faithful functor (or a forgetful functor). For each $\mathcal{C}$-object X, $U(X)$ is called the underlying set of X. Thus, every object in a concrete category can be regarded as a structured set.

We write $\mathcal{C}$ for $(\mathcal{C}, U)$, if the concrete functor is obvious. All of the categories considered in this paper are concrete categories.

A concrete functor between two concrete categories $(\mathcal{C}, U)$ and $(\mathcal{D}, V)$ is a functor $G : \mathcal{C} \to \mathcal{D}$ with $U = V \circ G$, which means that G only changes the structures on the underlying sets. Hence, in order to define a concrete functor $G : \mathcal{C} \to \mathcal{D}$, we only consider the following two requirements.

First, we assign to each $\mathcal{C}$-object X, a $\mathcal{D}$-object $G(X)$ such that $V(G(X)) = U(X)$.

Second, we verify that if a function $f : U(X) \to U(Y)$ is a $\mathcal{C}$-morphism $X \to Y$, then it is also $\mathcal{D}$-morphism $G(X) \to G(Y)$.

Theorem 10 *Adámek et al. (1990) Suppose that $F : \mathcal{D} \to \mathcal{C}$, $G : \mathcal{C} \to \mathcal{D}$ are concrete functors. Then, the following conditions are equivalent*

(1) $\{id_Y : (F \circ G)(Y) \to Y \mid Y \in \mathcal{C}\}$ is a natural transformation from the functor $(F \circ G)$ to the identity functor on $\mathcal{C}$, and $\{id_X : X \to (G \circ F)(X) \mid X \in \mathcal{D}\}$ is a natural transformation from the identity functor on $\mathcal{D}$ to the functor $(G \circ F)$,

(2) for each $Y \in \mathcal{C}$, $id_Y : (F \circ G)(Y) \to Y \mid Y \in \mathcal{C}$ is a $\mathcal{C}$-morphism, and for each $X \in \mathcal{D}$, $id_X : X \to (G \circ F)(X) \mid X \in \mathcal{D}$ is a $\mathcal{D}$-morphism.

In this case, (F, G) is called a Galois correspondence between $\mathcal{C}$ and $\mathcal{D}$. If (F, G) is a Galois correspondence, then it is easy to check that F is left adjoint to G, or equivalently that G is a right adjoint to F.

The category of L-fuzzy lower approximation spaces with LF-lower approximation maps as morphisms is denoted by **LF-LAS**.

The category of L-fuzzy upper approximation spaces with LF-upper approximation maps as morphisms is denoted by **LF-UAS**.

The category of Alexandrov L-fuzzy pre-uniformities with LF-uniformly continuous maps as morphisms is denoted by **ALF-UNS**.

The category of Alexandrov L-fuzzy semi-topogenous spaces with LF-topogenous maps as morphisms is denoted by **ALF-TGS**.

The category of Alexandrov L-fuzzy near spaces with LF-near maps as morphisms is denoted by **ALF-NRS**.

Theorem 11 *Let $(X, \overline{R}_X)$ and $(Y, \overline{R}_Y)$ be two L-fuzzy upper approximation spaces and $\phi : X \to Y$ be a mapping, then $D_{\overline{R}}(\phi) \leq D_{\mathcal{N}_{\overline{R}}}(\phi)$.*

Proof

$$\begin{aligned}
D_{\mathcal{N}_{\overline{R}}}(\phi) &= \bigwedge_{f\in L^Y} \bigwedge_{x\in X} \big(\mathcal{N}_{\overline{R}_X}(\phi^{\leftarrow}(f), \phi^{\leftarrow}(\top_{\phi(x)})) \\
&\quad \to \mathcal{N}_{\overline{R}_Y}(f, \top_{\phi(x)})\big) \\
&= \bigwedge_{f\in L^Y} \bigwedge_{x\in X} \Big(\bigvee_{x\in X} \overline{R}_X(\phi^{\leftarrow}(f))(x) \\
&\quad \to \bigvee_{\phi(x)\in Y} \overline{R}_Y(f)(\phi(x))\Big) \\
&= \bigwedge_{f\in L^Y} \bigwedge_{x\in X} \Big(\bigvee_{x\in X} \\
&\quad \overline{R}_X(\phi^{\leftarrow}(f))(x) \to \bigvee_{x\in X} \phi^{\leftarrow}(\overline{R}_Y(f))(x)\Big) \\
&\geq \bigwedge_{f\in L^Y} S\big(\overline{R}_X(\phi^{\leftarrow}(f)), \phi^{\leftarrow}(\overline{R}_Y(f))\big) = D_{\overline{R}}(\phi).
\end{aligned}$$

The above theorem shows the correspondence $(X, \overline{R}_X) \vdash (X, \mathcal{N}_{\overline{R}_X})$ induced a concrete functor $\varPi$: **LF-UAS** $\to$ **ALF-NRS** with $\varPi(X, \overline{R}_X) = (X, \mathcal{N}_{\overline{R}_X})$, $\varPi(\phi) = \phi$. □

Theorem 12 *Let $(X, \underline{R}_X)$ and $(Y, \underline{R}_Y)$ be two L-fuzzy lower approximation spaces and $\phi : X \to Y$ be a mapping, then $D_{\underline{R}}(\phi) \leq D_{\mathcal{N}_{\underline{R}}}(\phi)$.*

Proof

$$\begin{aligned}
D_{\mathcal{N}_{\underline{R}}}(\phi) &= \bigwedge_{f\in L^Y} \bigwedge_{x\in X} \\
&\big(\mathcal{N}_{\underline{R}_X}(\phi^{\leftarrow}(f), \phi^{\leftarrow}(\top_{\phi(x)})) \to \mathcal{N}_{\underline{R}_Y}(f, \top_{\phi(x)})\big) \\
&= \bigwedge_{f\in L^Y} \bigwedge_{x\in X} (\bigvee_{x\in X} \\
&\underline{R}^*_X((\phi^{\leftarrow}(f))^*)(x) \to \bigvee_{\phi(x)\in Y} \underline{R}^*_Y(f^*)(\phi(x))) \\
&= \bigwedge_{f\in L^Y} \bigwedge_{x\in X} (\bigvee_{x\in X}
\end{aligned}$$

$$\underline{R}_X^*(\phi^{\leftarrow}(f^*))(x) \to \bigvee_{x\in X} \phi^{\leftarrow}(\underline{R}_Y^*(f^*))(x)\Big)$$
$$\geq \bigwedge_{f\in L^Y}\bigwedge_{x\in X}\big(\underline{R}_X^*(\phi^{\leftarrow}(f^*))(x)$$
$$\to (\phi^{\leftarrow}(\underline{R}_Y(f^*)))^*(x)\big)$$
$$= \bigwedge_{f\in L^Y} S\big(\phi^{\leftarrow}(\underline{R}_Y(f^*)), \underline{R}_X(\phi^{\leftarrow}(f^*))\big) = D_{\underline{R}}(\phi).$$

The above theorem shows the correspondence $(X, \underline{R}_X) \vdash (X, \mathcal{N}_{\underline{R}_X})$ induced a concrete functor Ω : **LF-LAS** $\to$ **ALF-NRS** with $\Omega(X, \underline{R}_X) = (X, \mathcal{N}_{\underline{R}_X})$, $\Omega(\phi) = \phi$. □

Theorem 13 *Let $(X, \mathcal{U}_X)$ and $(Y, \mathcal{U}_Y)$ be two L-fuzzy pre-uniformities and $\phi : X \to Y$ be a mapping, then $D_{\mathcal{U}}(\phi) \leq D_{\mathcal{N}_{\mathcal{U}}}(\phi)$.*

Proof For $x, y \in X$, $f, g \in L^{Y\times Y}$, we have

$$\begin{aligned}(\phi\times\phi)^{\leftarrow}(u_{f,g})(x, y) &= u_{f,g}(\phi(x), \phi(y))\\ &= f(\phi(x)) \to g(\phi(y))\\ &= \phi^{\leftarrow}(f)(x) \to \phi^{\leftarrow}(g)(y)\\ &= u_{\phi^{\leftarrow}(f),\phi^{\leftarrow}(g)}(x, y).\end{aligned}$$

Thus, for all $v \in L^{Y\times Y}$, $v \leq u_{f,\top^*_{\phi(x)}}$ we have

$$D_{\mathcal{N}_{\mathcal{U}}}(\phi) = \bigwedge_{f\in L^Y}\bigwedge_{x\in X}$$
$$\big(\mathcal{N}_{\mathcal{U}_X}(\phi^{\leftarrow}(f), \phi^{\leftarrow}(\top_{\phi(x)})) \to \mathcal{N}_{\mathcal{U}_Y}(f, \top_{\phi(x)})\big)$$
$$= \bigwedge_{f\in L^Y}\bigwedge_{x\in X}\big(\mathcal{U}_X^*(u_{\phi^{\leftarrow}(f),(\phi^{\leftarrow}(\top_{\phi(x)}))^*}) \to \mathcal{U}_Y^*(u_{f,\top^*_{\phi(x)}})\big)$$
$$= \bigwedge_{f\in L^Y}\bigwedge_{x\in X}\big(\mathcal{U}_X^*((\phi\times\phi)^{\leftarrow}(u_{f,\top^*_{\phi(x)}})) \to \mathcal{U}_Y^*(u_{f,\top^*_{\phi(x)}})\big)$$
$$= \bigwedge_{f\in L^Y}\bigwedge_{x\in X}\big(\mathcal{U}_Y(u_{f,\top^*_{\phi(x)}}) \to \mathcal{U}_X((\phi\times\phi)^{\leftarrow}(u_{f,\top^*_{\phi(x)}}))\big)$$
$$\geq \bigwedge_{v\in L^{Y\times Y}}(\mathcal{U}_Y(v)$$
$$\to \mathcal{U}_X((\phi\times\phi)^{\leftarrow}(v))) = D_{\mathcal{U}}(\phi).$$

The above theorem shows the correspondence $(X, \mathcal{U}_X) \vdash (X, \mathcal{N}_{\mathcal{U}_X})$ induced a concrete functor Φ : **ALF-UNS** $\to$ **ALF-NRS** with $\Phi(X, \mathcal{U}_X) = (X, \mathcal{N}_{\mathcal{U}_X})$, $\Phi(\phi) = \phi$. □

Theorem 14 *Let (X, ξ_X) and (Y, ξ_Y) be two Alexandrov L-fuzzy semi-topogenous spaces and $\phi : X \to Y$ be a mapping. Then, $D_{\xi}(\phi) \leq D_{\mathcal{N}_{\xi}}(\phi)$.*

Proof

$$D_{\mathcal{N}_{\xi}}(\phi) = \bigwedge_{f\in L^Y}\bigwedge_{x\in X}\Big(\mathcal{N}_{\xi_X}\big(\phi^{\leftarrow}(f), \phi^{\leftarrow}(\top_{\phi(x)})\big)$$
$$\to \mathcal{N}_{\xi_Y}(f, \top_{\phi(x)})\Big)$$
$$= \bigwedge_{f\in L^Y}\bigwedge_{x\in X}\Big(\xi_X^*\big(\phi^{\leftarrow}(f), (\phi^{\leftarrow}(\top_{\phi(x)}))^*\big)$$
$$\to \xi_Y^*(f, \top^*_{\phi(x)})\Big)$$
$$= \bigwedge_{f\in L^Y}\bigwedge_{x\in X}\Big(\xi_Y(f, \top^*_{\phi(x)})$$
$$\to \xi_X\big(\phi^{\leftarrow}(f), \phi^{\leftarrow}(\top^*_{\phi(x)})\big)\Big) = D_{\xi}(\phi).$$

The above theorem shows the correspondence $(X, \xi_X) \vdash (X, \mathcal{N}_{\xi_X})$ induced a concrete functor Θ : **ALF-TGS** $\to$ **ALF-NRS** with $\Theta(X, \xi_X) = (X, \mathcal{N}_{\xi_X})$, $\Theta(\phi) = \phi$.

Theorem 15 *Let $(X, \mathcal{N}_X)$ and $(Y, \mathcal{N}_Y)$ be two L-fuzzy near spaces and $\phi : X \to Y$ be a mapping, then*

(1) $D_{\mathcal{N}}(\phi) \leq D_{\overline{R}_{\mathcal{N}}}(\phi)$,
(2) $D_{\mathcal{N}}(\phi) \leq D_{\underline{R}_{\mathcal{N}}}(\phi)$,
(3) $D_{\mathcal{N}}(\phi) \leq D_{\mathcal{U}_{\mathcal{N}}}(\phi)$,
(4) $D_{\mathcal{N}}(\phi) \leq D_{\xi_{\mathcal{N}}}(\phi)$.

Proof (1)

$$D_{\overline{R}_{\mathcal{N}}}(\phi) = \bigwedge_{f\in L^Y} S\big(\overline{R}_{\mathcal{N}_X}(\phi^{\leftarrow}(f)), \phi^{\leftarrow}(\overline{R}_{\mathcal{N}_Y}(f))\big)$$
$$= \bigwedge_{f\in L^Y}\bigwedge_{x\in X}\big(\overline{R}_{\mathcal{N}_X}(\phi^{\leftarrow}(f))(x) \to \phi^{\leftarrow}(\overline{R}_{\mathcal{N}_Y}(f))(x)\big)$$
$$= \bigwedge_{f\in L^Y}\bigwedge_{x\in X}\big((\bigvee_{y\in X}\mathcal{N}_X(\top_x, \top_y) \odot \phi^{\leftarrow}(f)(y))$$
$$\to \left(\bigvee_{\phi(y)\in Y}\mathcal{N}_Y(\top_{\phi(x)}, \top_{\phi(y)}) \odot f(\phi(y)))\right)$$
$$\geq \bigwedge_{f\in L^Y}\bigwedge_{x,y\in X}\big((\mathcal{N}_X(\top_x, \top_y) \odot \phi^{\leftarrow}(f)(y))$$
$$\to (\mathcal{N}_Y(\top_{\phi(x)}, \top_{\phi(y)}) \odot \phi^{\leftarrow}(f)(y))\big)$$
$$\geq \bigwedge_{f\in L^Y}\bigwedge_{x,y\in X}\big(\mathcal{N}_X(\top_x, \top_y)$$
$$\to \mathcal{N}_Y(\top_{\phi(x)}, \top_{\phi(y)})\big) = D_{\mathcal{N}}(\phi).$$

This shows the correspondence $(X, \mathcal{N}_X) \vdash (X, \overline{R}_{\mathcal{N}_X})$ induced a concrete functor Γ : **ALF-NRS** $\to$ **LF-UAS** with $\Gamma(X, \mathcal{N}_X) = (X, \overline{R}_{\mathcal{N}_X})$, $\Gamma(\phi) = \phi$.

(2)

$$D_{\underline{R}_{\mathcal{N}}}(\phi) = \bigwedge_{f\in L^Y} S\big(\phi^{\leftarrow}(\underline{R}_{\mathcal{N}_Y}(f)), \underline{R}_{\mathcal{N}_X}(\phi^{\leftarrow}(f))\big)$$
$$= \bigwedge_{f\in L^Y}\bigwedge_{x\in X}\big(\phi^{\leftarrow}(\underline{R}_{\mathcal{N}_Y}(f))(x) \to \underline{R}_{\mathcal{N}_X}(\phi^{\leftarrow}(f))(x)\big)$$
$$= \bigwedge_{f\in L^Y}\bigwedge_{x\in X}\big((\bigwedge_{\phi(y)\in Y}\mathcal{N}_Y(\top_{\phi(x)}, \top_{\phi(y)}) \to f(\phi(y))$$

$$\rightarrow \left(\bigwedge_{y\in X} \mathcal{N}_X(\top_x, \top_y) \rightarrow \phi^{\leftarrow}(f)(y)) \right)$$
$$\geq \bigwedge_{f\in L^Y} \bigwedge_{x,y\in X} \big((\mathcal{N}_Y(\top_{\phi(x)}, \top_{\phi(y)}) \rightarrow f(\phi(y)))$$
$$\rightarrow (\mathcal{N}_X(\top_x, \top_y) \rightarrow f(\phi(y)))\big)$$
$$\geq \bigwedge_{x,y\in X} (\mathcal{N}_X(\top_x, \top_y)$$
$$\rightarrow \mathcal{N}_Y(\top_{\phi(x)}, \top_{\phi(y)})) = D_{\mathcal{N}}(\phi).$$

This shows the correspondence $(X, \mathcal{N}_X) \vdash (X, \underline{R}_{\mathcal{N}_X})$ induced a concrete functor Υ : **ALF-NRS** $\rightarrow$ **LF-LAS** with $\Upsilon(X, \mathcal{N}_X) = (X, \underline{R}_{\mathcal{N}_X}),\ \Upsilon(\phi) = \phi$.

(3) By Theorem 13, we have

$$D_{\mathcal{U}_{\mathcal{N}}}(\phi) = \bigwedge_{v\in L^{Y\times Y}} \big(\mathcal{U}_{\mathcal{N}_Y}(v) \rightarrow \mathcal{U}_{\mathcal{N}_X}((\phi\times\phi)^{\leftarrow}(v))\big)$$
$$= \bigwedge_{v\in L^{Y\times Y}} \Big(\bigwedge_{\phi(x),\phi(y)\in Y}$$
$$\big(\mathcal{N}_Y(\top_{\phi(x)}, \top_{\phi(y)}) \rightarrow v(\phi(x), \phi(y))\big)$$
$$\rightarrow \bigwedge_{x,y\in X} \big(\mathcal{N}_X(\top_x, \top_y)$$
$$\rightarrow (\phi\times\phi)^{\leftarrow}(v)(x, y)\big)\Big)$$
$$\geq \bigwedge_{x,y\in X} (\mathcal{N}_X(\top_{\phi^{\leftarrow}(x)}, \top_{\phi^{\leftarrow}(y)})$$
$$\rightarrow \mathcal{N}_Y(\top_x, \top_y)) = D_{\mathcal{N}}(\phi).$$

This shows the correspondence $(X, \mathcal{N}_X) \vdash (X, \mathcal{U}_{\mathcal{N}_X})$ induced a concrete functor Δ : **ALF-NRS** $\rightarrow$ **ALF-UNS** with $\Delta(X, \mathcal{N}_X) = (X, \mathcal{U}_{\mathcal{N}_X}), \Delta(\phi) = \phi$.

(4)

$$D_{\xi_{\mathcal{N}}}(\phi) = \bigwedge_{f,g\in L^Y} \big(\xi_{\mathcal{N}_Y}(f, g)$$
$$\rightarrow \xi_{\mathcal{N}_X}(\phi^{\leftarrow}(f), \phi^{\leftarrow}(g))\big)$$
$$= \bigwedge_{f,g\in L^Y} \left(\bigwedge_{\phi(x)\in Y} (\mathcal{N}_Y(f, \top_{\phi(x)}) \rightarrow g(\phi(x)) \right)$$
$$\rightarrow \left(\bigwedge_{x\in X} (\mathcal{N}_X(\phi^{\leftarrow}(f), \top_x) \rightarrow \phi^{\leftarrow}(g)(x)) \right)$$
$$\geq \bigwedge_{f,g\in L^Y} \bigwedge_{x\in X} \big((\mathcal{N}_Y(f, \top_{\phi(x)}) \rightarrow g(\phi(x)))$$
$$\rightarrow (\mathcal{N}_X(\phi^{\leftarrow}(f),$$
$$\phi^{\leftarrow}(\top_{\phi(x)})) \rightarrow g(\phi(x)))\big)$$
$$\geq \bigwedge_{f\in L^Y} \bigwedge_{x\in X} (\mathcal{N}_X(\phi^{\leftarrow}(f), \phi^{\leftarrow}(\top_{\phi(x)}))$$
$$\rightarrow \mathcal{N}_Y(f, \top_{\phi(x)})) = D_{\mathcal{N}}(\phi).$$

This shows the correspondence $(X, \mathcal{N}_X) \vdash (X, \xi_{\mathcal{N}_X})$ induced a concrete functor Ψ : **ALF-NRS** $\rightarrow$ **ALF-TGS** with $\Psi(X, \mathcal{N}_X) = (X, \xi_{\mathcal{N}_X}),\ \Psi(\phi) = \phi$. □

Proposition 4 *The pair* (Π, Γ) *forms a Galois correspondence between the category* ***LF-UAS*** *and the category* ***ALF-NRS***.

Proof $D_{\mathcal{N}}(id_X) = \top$, where $id_X : (X, (\Pi\circ\Gamma)(\mathcal{N}_X)) \rightarrow (X, \mathcal{N}_X)$. By Theorems 1 and 3, we have

$$D_{\mathcal{N}}(id_X) = D_{\mathcal{N}}(\Pi\circ\Gamma)$$
$$= \bigwedge_{f\in L^X} \bigwedge_{x\in X} \big(\mathcal{N}_{\overline{R}_{\mathcal{N}_X}}(f, \top_x) \rightarrow \mathcal{N}_X(f, \top_x)\big)$$
$$= \bigwedge_{f\in L^X} \bigwedge_{x\in X} \left(\bigvee_{x\in X} \overline{R}_{\mathcal{N}_X}(f)(x) \rightarrow \mathcal{N}_X(f, \top_x) \right)$$
$$= \bigwedge_{f\in L^X} \bigwedge_{x\in X} \big((\bigvee_{x,y\in X} \mathcal{N}_X(\top_x, \top_y)$$
$$\odot f(y)) \rightarrow \mathcal{N}_X(f, \top_x)\big)$$
$$= \bigwedge_{f\in L^X} \bigwedge_{x\in X} \big((\bigvee_{x\in X} \mathcal{N}_X(\top_x, \top_x)$$
$$\odot f(x)) \rightarrow \mathcal{N}_X(f, \top_x)\big)$$
$$= \bigwedge_{f\in L^X} \bigwedge_{x\in X} \big(\mathcal{N}_X(\bigvee_{x\in X} \top_x$$
$$\odot f(x), \top_x) \rightarrow \mathcal{N}_X(f, \top_x)\big)$$
$$= \bigwedge_{f\in L^X} \bigwedge_{x\in X} \big(\mathcal{N}_X(f, \top_x) \rightarrow \mathcal{N}_X(f, \top_x)\big) = \top.$$

Thus, Π is a left inverse of Γ for any $(X, \mathcal{N}_X) \in$ **ALF-NRS** and $id_X : (X, (\Pi\circ\Gamma)(\mathcal{N}_X)) \rightarrow (X, \mathcal{N}_X)$ is an LF-near map.

Secondly, we show that $D_{\overline{R}}(id_X) = \top$, where $id_X : (X, \overline{R}_X) \rightarrow (X, (\Gamma\circ\Pi)(\overline{R}_X))$,

$$D_{\overline{R}}(id_X) = D_{\overline{R}}(\Gamma\circ\Pi)$$
$$= \bigwedge_{f\in L^X} \bigwedge_{x\in X} \big(\overline{R}_X(f)(x) \rightarrow \overline{R}_{\mathcal{N}_{\overline{R}_X}}(f)(x)\big)$$
$$= \bigwedge_{f\in L^X} \bigwedge_{x\in X} \big(\overline{R}_X(f)(x)$$
$$\rightarrow \bigvee_{y\in X} \mathcal{N}_{\overline{R}_X}(\top_x, \top_y) \odot f(y)\big)$$
$$= \bigwedge_{f\in L^X} \bigwedge_{x\in X} \big(\overline{R}_X(f)(x)$$
$$\rightarrow \bigvee_{x,y\in X} \overline{R}_X(\top_x)(y) \odot f(y)\big)$$
$$\geq \bigwedge_{f\in L^X} \bigwedge_{x\in X} \big(\overline{R}_X(f)(x)$$

$$\to \bigvee_{x\in X} \overline{R}_X(\top_x)(x) \odot f(x)\big)$$
$$= \bigwedge_{f\in L^X}\bigwedge_{x\in X}\left(\overline{R}_X(f)(x) \to \overline{R}_X(\bigvee_{x\in X} \top_x \odot f(x))(x)\right)$$
$$= \bigwedge_{f\in L^X}\bigwedge_{x\in X}\big(\overline{R}_X(f)(x) \to \overline{R}_X(f)(x)\big) = \top.$$

Thus, Π is a right inverse of Γ for any $(X, \overline{R}_X) \in$ **LF-UAS** and $id_X : (X, \overline{R}_X) \to (X, (\Gamma\circ\Pi)(\overline{R}_X))$ is an LF-upper approximation map. The pair (Π, Γ) forms a Galois correspondence between the category **LF-UAS** and the category **ALF-NRS**. □

Proposition 5 *The pair (Ω, Υ) forms a Galois correspondence between the category **LF-LAS** and the category **ALF-NRS**.*

Proof Firstly, we show that $D_{\mathcal{N}}(id_X) = \top$, where $id_X : (X, (\Omega \circ \Upsilon)(\mathcal{N}_X)) \to (X, \mathcal{N}_X)$. By Theorems 2 and 3, we have

$$D_{\mathcal{N}}(id_X) = D_{\mathcal{N}}(\Omega \circ \Upsilon)$$
$$= \bigwedge_{f\in L^X}\bigwedge_{x\in X}\big(\mathcal{N}_{\underline{R}_{\mathcal{N}_X}}(f, \top_x) \to \mathcal{N}_X(f, \top_x)\big)$$
$$= \bigwedge_{f\in L^X}\bigwedge_{x\in X}\big(\bigvee_{x\in X} \underline{R}^*_{\mathcal{N}_X}(f^*)(x) \to \mathcal{N}_X(f, \top_x)\big)$$
$$= \bigwedge_{f\in L^X}\bigwedge_{x\in X}\big(\bigvee_{x\in X}(\bigwedge_{y\in X}(\mathcal{N}_X(\top_x, \top_y)$$
$$\to f^*(y)))^* \to \mathcal{N}_X(f, \top_x)\big)$$
$$= \bigwedge_{f\in L^X}\bigwedge_{x\in X}\big((\bigvee_{x,y\in X} \mathcal{N}_X(\top_x, \top_y)$$
$$\odot f(y)) \to \mathcal{N}_X(f, \top_x)\big)$$
$$= \bigwedge_{f\in L^X}\bigwedge_{x\in X}\big(\mathcal{N}_X(\bigvee_{x\in X}$$
$$\top_x \odot f(x), \top_x) \to \mathcal{N}_X(f, \top_x)\big)$$
$$= \bigwedge_{f\in L^X}\bigwedge_{x\in X}(\mathcal{N}_X(f, \top_x) \to \mathcal{N}_X(f, \top_x)) = \top.$$

Thus, Ω is a left inverse of Υ for any $(X, \mathcal{N}_X) \in$ **ALF-NRS** and $id_X : (X, (\Omega \circ \Upsilon)(\mathcal{N}_X)) \to (X, \mathcal{N}_X)$ is an LF-near map.

Secondly, we show that $D_{\underline{R}}(id_X) = \top$, where $id_X : (X, \underline{R}_X) \to (X, (\Upsilon \circ \Omega)(\underline{R}_X))$,

$$D_{\underline{R}}(id_X) = D_{\underline{R}}(\Upsilon \circ \Omega)$$
$$= \bigwedge_{f\in L^X} S\big(\underline{R}_{\mathcal{N}_{\underline{R}_X}}(f), \underline{R}_X(f)\big)$$
$$= \bigwedge_{f\in L^X}\bigwedge_{x\in X}\big(\underline{R}_{\mathcal{N}_{\underline{R}_X}}(f)(x) \to \underline{R}_X(f)(x)\big)$$
$$= \bigwedge_{f\in L^X}\bigwedge_{x\in X}\big((\bigwedge_{y\in X}(\mathcal{N}_{\underline{R}_X}$$
$$(\top_x, \top_y) \to f(y))) \to \underline{R}_X(f)(x)\big)$$
$$= \bigwedge_{f\in L^X}\bigwedge_{x,y\in X}\big((\bigvee_{y\in X}$$
$$\underline{R}^*_X(\top^*_x)(y) \to f(y)) \to \underline{R}_X(f)(x)\big)$$
$$\geq \bigwedge_{f\in L^X}\bigwedge_{x\in X}\big((f(x) \to \underline{R}_X(\bigwedge_{x\in X} \top^*_x)(x)) \to \underline{R}_X(f)(x)\big)$$
$$= \bigwedge_{f\in L^X}\bigwedge_{x\in X}\big(\underline{R}_X(f(x) \to \bigwedge_{x\in X} \top^*_x)(x) \to \underline{R}_X(f)(x)\big)$$
$$= \bigwedge_{f\in L^X}\bigwedge_{x\in X}\big(\underline{R}_X(\bigvee_{x\in X} f(x)$$
$$\odot \top_x)(x) \to \underline{R}_X(f)(x)\big)$$
$$= \bigwedge_{f\in L^X}\bigwedge_{x\in X}\big(\underline{R}_X(f)(x) \to \underline{R}_X(f)(x)\big) = \top.$$

Thus, Ω is a right inverse of Υ for any $(X, \underline{R}_X) \in$ **LF-LAS** and $id_X : (X, \underline{R}_X) \to (X, (\Upsilon \circ \Omega)(\underline{R}_X))$ is an LF-lower approximation map.

The pair (Ω, Υ) forms a Galois correspondence between the category **LF-LAS** and the category **ALF-NRS**. □

Proposition 6 *The pair (Φ, Δ) forms a Galois correspondence between the category **ALF-UNS** and the category **ALF-NRS**.*

Proof Firstly, we show that $D_{\mathcal{N}}(id_X) = \top$, where $id_X : (X, (\Phi \circ \Delta)(\mathcal{N}_X)) \to (X, \mathcal{N}_X)$. By Theorems 4 and 5, we have

$$D_{\mathcal{N}}(id_X) = D_{\mathcal{N}}(\Phi \circ \Delta)$$
$$= \bigwedge_{f\in L^X}\bigwedge_{x\in X}\big(\mathcal{N}_{\mathcal{U}_{\mathcal{N}_X}}(f, \top_x) \to \mathcal{N}_X(f, \top_x)\big)$$
$$= \bigwedge_{f\in L^X}\bigwedge_{x\in X}\big(\mathcal{U}^*_{\mathcal{N}_X}(u_{f,\top^*_x})$$
$$\to \mathcal{N}_X(f, \top_x)\big)$$
$$= \bigwedge_{f\in L^X}\bigwedge_{x\in X}\big(\bigvee_{x,y\in X}(\mathcal{N}_X(\top_x, \top_y)$$
$$\to u_{f,\top^*_x}(x, y))^* \to \mathcal{N}_X(f, \top_x)\big)$$
$$= \bigwedge_{f\in L^X}\bigwedge_{x\in X}\big(\bigvee_{x,y\in X}(\mathcal{N}_X(\top_x, \top_y)$$
$$\odot f(x) \odot \top_x(y)) \to \mathcal{N}_X(f, \top_x)\big)$$
$$= \bigwedge_{f\in L^X}\bigwedge_{x\in X}\big((\mathcal{N}_X(\bigvee_{x\in X}$$
$$\top_x \odot f(x), \top_x)) \to \mathcal{N}_X(f, \top_x)\big)$$
$$= \bigwedge_{f\in L^X}\bigwedge_{x\in X}\big(\mathcal{N}_X(f, \top_x) \to \mathcal{N}_X(f, \top_x)\big) = \top.$$

Thus, Φ is a left inverse of Δ for any $(X, \mathcal{N}_X) \in$ **ALF-NRS** and $id_X : (X, (\Phi \circ \Delta)(\mathcal{N}_X)) \to (X, \mathcal{N}_X)$ is an LF-near map.

Secondly, we show that $D_{\mathcal{U}}(id_X) = \top$, where $id_X : (X, \mathcal{U}_X) \to (X, (\Delta \circ \Phi)(\mathcal{U}_X))$. Since $u = \bigwedge_{x,y\in X}(u^*(x, y) \to u_{\top_x,\top_y^*})$, then we have

$$
\begin{aligned}
D_{\mathcal{U}}(id_X) &= D_{\mathcal{U}}(\Delta \circ \Phi)\\
&= \bigwedge_{u\in L^{X\times X}} \big(\mathcal{U}_{\mathcal{N}_{\mathcal{U}_X}}(u) \to \mathcal{U}_X(u)\big)\\
&= \bigwedge_{u\in L^{X\times X}} \big(\bigwedge_{x,y\in X} (\mathcal{N}_{\mathcal{U}_X}(\top_x, \top_y) \to u(x, y))\\
&\quad \to \mathcal{U}_X(u)\big)\\
&= \bigwedge_{u\in L^{X\times X}} \big(\bigwedge_{x,y\in X} (\mathcal{U}_X^*(u_{\top_x,\top_y^*})\\
&\quad \to u(x, y)) \to \mathcal{U}_X(u)\big)\\
&= \bigwedge_{u\in L^{X\times X}} \big(\mathcal{U}_X(\bigwedge_{x,y\in X} (u^*(x, y)\\
&\quad \to u_{\top_x,\top_y^*})) \to \mathcal{U}_X(u)\big)\\
&= \bigwedge_{u\in L^{X\times X}} \big(\mathcal{U}_X(u) \to \mathcal{U}_X(u)\big) = \top.
\end{aligned}
$$

Thus, Φ is a right inverse of Δ for any $(X, \mathcal{U}_X) \in$ **ALF-UNS** and $id_X : (X, \mathcal{U}_X) \to (X, (\Delta \circ \Phi)(\mathcal{U}_X))$ is an LF-uniformly continuous map.

The pair (Φ, Δ) forms a Galois correspondence between the category **ALF-UNS** and the category **ALF-NRS**. □

Proposition 7 *The pair (Θ, Ψ) forms a Galois correspondence between the category **ALF-TGS** and the category **ALF-NRS**.*

Proof Firstly, we show that $D_{\mathcal{N}}(id_X) = \top$, where $id_X : (X, (\Theta \circ \Psi)(\mathcal{N}_X)) \to (X, \mathcal{N}_X)$. By Theorems 6 and 7, we have

$$
\begin{aligned}
D_{\mathcal{N}}(id_X) &= D_{\mathcal{N}}(\Theta \circ \Psi)\\
&= \bigwedge_{f\in L^X} \bigwedge_{x\in X} \big(\mathcal{N}_{\xi_{\mathcal{N}_X}}(f, \top_x) \to \mathcal{N}_X(f, \top_x)\big)\\
&= \bigwedge_{f\in L^X} \bigwedge_{x\in X} \big(\xi^*_{\mathcal{N}_X}(f, \top_x^*) \to \mathcal{N}_X(f, \top_x)\big)\\
&= \bigwedge_{f\in L^X} \bigwedge_{x\in X} \big((\bigwedge_{x\in X} (\mathcal{N}_X(f, \top_x) \to \top_x^*(x)))^*\\
&\quad \to \mathcal{N}_X(f, \top_x)\big)\\
&= \bigwedge_{f\in L^X} \bigwedge_{x\in X} \big((\mathcal{N}_X(f, \top_x) \odot \top_x(x))\\
&\quad \to \mathcal{N}_X(f, \top_x)\big) = \top.
\end{aligned}
$$

Thus, Θ is a left inverse of Ψ for any $(X, \mathcal{N}_X) \in$ **ALF-NRS** and $id_X : (X, (\Theta \circ \Psi)(\mathcal{N}_X)) \to (X, \mathcal{N}_X)$ is an LF-near map.

Secondly, we show that $D_\xi(id_X) = \top$, where $id_X : (X, \xi_X) \to (X, (\Psi \circ \Theta)(\xi_X))$, we have

$$
\begin{aligned}
D_\xi(id_X) &= D_\xi(\Psi \circ \Theta)\\
&= \bigwedge_{f,g\in L^X} \big(\xi_{\mathcal{N}_{\xi_X}}(f, g) \to \xi_X(f, g)\big)\\
&= \bigwedge_{f,g\in L^X} \big(\bigwedge_{x\in X} (\mathcal{N}_{\xi_X}(f, \top_x)\\
&\quad \to g(x)) \to \xi_X(f, g)\big)\\
&= \bigwedge_{f,g\in L^X} \big(\bigwedge_{x\in X} (\xi_X^*(f, \top_x^*) \to g(x)) \to \xi_X(f, g)\big)\\
&= \bigwedge_{f,g\in L^X} \big(\bigwedge_{x\in X} (g^*(x) \to \xi_X(f, \top_x^*)) \to \xi_X(f, g)\big)\\
&= \bigwedge_{f,g\in L^X} \big(\xi_X(f, \bigwedge_{x\in X} (g^*(x) \to \top_x^*))) \to \xi_X(f, g)\big)\\
&= \bigwedge_{f,g\in L^X} \big(\xi_X(f, g) \to \xi_X(f, g)\big) = \top.
\end{aligned}
$$

Thus, Θ is a right inverse of Ψ for any $(X, \xi_X) \in$ **ALF-TGS** and $id_X : (X, \xi_X) \to (X, (\Psi \circ \Theta)(\xi_X))$ is an LF-topogenous map.

The pair (Θ, Ψ) forms a Galois correspondence between the category **ALF-TGS** and the category **ALF-NRS**. □

6 Conclusion

As a unified structure of extension of Pawlak's rough set Pawlak (1982, 1991), we have the following

(1) we reintroduce the Alexandrov L-fuzzy nearness and presented its relations with some L-fuzzy systems such as: L-fuzzy rough sets, L-fuzzy semi-topogenous orders and L-fuzzy uniformities in complete residuated lattice. Unlike the late paper Ramadan et al. (2019), we discussed its relations with different other systems: L-fuzzy topological spaces (interior, closure, co-topology) and L-fuzzy pre-proximities.
(2) in this paper, we present the degree of continuity concept and equip it to prove that property for the mentioned systems unlike the late paper Ramadan et al. (2019) in which we discussed the continuity property in the regular sense.
(3) in this paper, we extend the degree sense to demonstrate the Galois correspondence among the categories of Alexandrov L-fuzzy nearness **ALF-NRS**, L-fuzzy

lower approximation spaces **LF-LAS**, L-fuzzy upper approximation spaces **LF-UAS**, Alexandrov L-fuzzy pre-uniformities **ALF-UNS** and Alexandrov L-fuzzy semi-topogenous spaces **ALF-TGS** and prove its existence unlike the late paper Ramadan et al. (2019) in which we demonstrated it in the regular sense.

(4) like the late paper Ramadan et al. (2019), we present example 1 through fuzzy information system which confirm the feasibility of using the proposed approaches to solve daily problems.

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Declarations

Conflict of interest The authors declare that they have no conflict of interests

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