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FUZZY SYSTEMS AND THEIR MATHEMATICS

# Complex q-rung orthopair fuzzy Frank aggregation operators and their application to multi-attribute decision making

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## Abstract
The complex q-rung orthopair fuzzy sets (Cq-ROFSs) can serve as a generalization of q-rung orthopair fuzzy sets (q-ROFSs) and complex fuzzy sets FS (CFSs). Cq-ROFSs provide more freedom for people handling uncertainty and vagueness by the truth and falsity grades on the condition that the sum of the q-powers of the real part and imaginary part is within the unit interval. Further, Frank operational laws are an extended form of Archimedes' T mode and Archimedes' S mode and Frank aggregation operators have a certain parameter which makes them more flexible and more generalized than many other aggregation operators in the process of information fusion. The objectives of this paper are to extend the Frank operations to the complex q-rung orthopair fuzzy environment and to introduce their score function and accuracy function. Meanwhile, some complex q-rung fuzzy Frank aggregation operators are developed, such as the complex q-rung orthopair fuzzy Frank weighted averaging (Cq-ROFFWA) operator, the complex q-rung orthopair fuzzy Frank weighted geometric (Cq-ROFFWG) operator, and the complex q-rung orthopair fuzzy Frank ordered weighted averaging (Cq-ROFFOWA) operator, and their special cases are discussed. In addition, an innovative MADM method is introduced according to the propounded operators to deal with multi-attribute decision-making problems under the complex q-rung orthopair fuzzy environment. Consequently, the practicability and effectiveness of the created methods are proposed by parameter exploration and comparative analysis.

**Keywords** Complex q-rung orthopair fuzzy set · Aggregation operators · Frank operator · Multi-attribute decision making

## 1 Introduction

Multi-attribute decision-making (MADM) problems have gradually become an important area in decision science. Since the objects are usually uncertain, fuzzy numbers (Chou et al. 2006; Fan and Feng 2009) are more fit for expressing some attribute values involved in MADM problems. The fuzzy sets (FSs) (Zadeh 1965) which were defined in 1965 are suitable to express fuzzy multi-criteria decision-making problems with vagueness. Atanassov and Gargov (1989) introduced intuitionistic fuzzy sets (IFSs), which contain a membership function and a non-membership function. In the last few decades, scholars (Xu 2007; Chen et al. 2011; Zhang 2013) have introduced many relevant intuitionistic fuzzy aggregation operators. However, in some special conditions, the membership degree plus the non-membership degree may be greater than one. IFSs could not describe this situation, so Yager (2013) introduced Pythagorean fuzzy sets (PFSs). A PFS is

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characterized by the sum of the squares of the membership function and the non-membership function being less than or equal to 1. The application of PFS is wider than that of IFSs in expressing the uncertainty in MADM problems. Ever since the first appearance of PFS, there are many studies (Yager 2013, 2014; Zhang and Xu 2014; Peng and Yang 2015, 2016; Zhang 2016a, 2016b; Gou et al. 2016; Mahanta and Panda 2021; Ma et al. 2021;Du et al. 2017; Akram et al.2020; Liu et al. 2021a, 2021b, 2021c, 2021d, 2021e; Xian and Cheng 2021; Zhang and Ma 2020; Sarkar and Biswas 2020; Shakeel et al. 2020) on MADM problems under Pythagorean fuzzy circumstances.

However, in some special conditions, if a decision-maker gives 0.8 for membership degree and 0.7 for non-membership degree, then the sum of the squares of both is greater than 1. For addressing such types of difficulties, Yager (2016) explored the q-rung orthopair fuzzy set (QROFS) with a condition that the sum of the q-power of membership degree and the q-power of non-membership degree is restricted to [0, 1]. The QROFS is usually able to cope with much higher degrees of uncertainty. Since it was initiated, numerous researchers have exploited and utilized it in various areas (Wang et al. 2019; Du et al. 2021; Xing et al. 2019; Ju et al. 2019a, 2019b, 2020; Gao et al. 2019; Garg 2021a; Mahmood and Ali 2021a, 2021b; Aydemir and Yilmaz Gündüz 2020; Rawat and Komal 2022; Akram et al. 2021; Liu et al.2021a, 2021b, 2021c, 2021d, 2021e).

According to the above prevailing studies, similar approaches are limited and are unable to represent the partial ignorance of the data and its fluctuations at a given period of time. To handle this, Ramot et al. (2002) introduced complex FS (CFS). Additionally, Alkouri and Salleh (2012) presented the theory of complex IFS (CIFS), which extended the range of the supporting grade and the supporting against grade from real numbers to complex numbers with a unit disc, and the CIFS can describe two-dimensional information.

Based on the CIFS, Ullah et al. (2020) proposed the complex PFS (CPFS), which is useful for efficiently describing uncertain and unreliable information. A characteristic of CPFS is that the sum of the square of the real part (and imaginary part) of complex membership grade and the square of the real part (and imaginary part) of complex non-membership grade is limited to [0, 1]. However, the CIFS and CPFS could not cope with information whose sum of the squares of the real part (also for the imaginary part) of both grades exceeds the unit interval. For example, some experts are asked to provide their preference about the degree of an alternative $D_i$ on a criterion $C_j$, and the complex-valued membership grade may be $0.93e^{i2\pi(0.84)}$, and the complex-valued non-membership grade may be $0.86e^{i2\pi(0.75)}$. To describe this type of information, Liu et al. (2020b, 2020c, 2020a) proposed the theory of complex QROFS (Cq-ROFS) with the condition the sum of the q-power of the real part (also for an imaginary part) of membership degree and real part (imaginary part) of the non-membership degree is belonging to [0, 1]. Since the Cq-ROFS is an extension of CIFS and CPFS to deal with real-life problems, the complex q-ROFS becomes a powerful tool for solving uncertain problems. Garg et al. (2021a) explored the aggregation operators under Cq-ROFSs. Afterward, the Cq-ROFS is usually able to deal with much higher degrees of uncertainty. The application of the Cq-ROFS is more exquisitely than that of CIFS and CPFS to describe the ambiguity in real-decision activities.

Since the Cq-ROFS' appearance, scholars and various researchers have paid great attention to multiple attribute group decision-making problems under complex q-rung orthopair fuzzy circumstances. For example, Liu et al. (2020b, 2020c, 2021a, 2021b, 2021c2020a) propounded distance measures and cosine similarity measures for Cq-ROFS and proposed the Schweizer–Sklar operator, Muirhead mean operator, BM operator, and Einstein operators to Cq-ROFS. Mahmood and Ali (2021a, 2021b) defined complex q-rung orthopair fuzzy Hamacher aggregation operators for Cq-ROFS. Yuan et al. (2020) propounded some fuzzy 2-tuple linguistic Maclaurin symmetric mean operators based on the Cq-ROFS. Garg et al. (2021c) developed complex interval-valued q-rung orthopair fuzzy set. Zeeshan and Mahmood (2020) extended Maclaurin symmetric mean operators in complex q-rung orthopair fuzzy environment. Garg et al. (2021b) defined complex q-rung orthopair Hamy Mean Operators.

Frank t-norm and t-conorm (1979) are generalizations of probabilistic and Lukasiewicz t-norm and t-conorm. The Frank t-norm and t-conorm are more flexible and adequate to deal with practical decision making since they have a parameter. In recent years, the Frank operator has received increased attention from the scientific community, and it has achieved a lot of researcher results on different fuzzy sets. Some operations on various fuzzy sets have been introduced based on the Frank T-norm and S-norm, such as intuitionistic Frank operations (Xia et al. 2012; Zhang et al. 2015), single-valued neutrosophic Frank operations (Garg 2016), hesitant Frank operations (Qin et al. 2016), dual-hesitant fuzzy Frank operations (Wang et al. 2016), interval intuitionistic linguistic Frank aggregation operators (Du and Hou 2018), Frank prioritized Bonferroni mean operations (Ji et al. 2018), linguistic intuitionistic fuzzy Frank weighted Heronian mean operator (Peng et al. 2018), interval-valued probabilistic hesitant fuzzy aggregation operators (Yahya et al. 2021), triangular interval type-2 fuzzy Frank operations (Qin and Liu 2014), interval-valued

Pythagorean Frank power operations (Yang et al. 2018), picture fuzzy Frank weighted averaging operator (Seikh and Mandal 2021a). Some researchers also studied various mathematical properties of the Frank t-norms (Liu et al. 2018; Xing et al. 2019; Zhou et al. 2019; Seikh and Mandal 2021b).

To the best of our knowledge, Frank t-norm and t-conorm are a generalization of algebraic triangular t-norm and t-conorm and Lukasiewicz t-norm and t-conorm. Furthermore, compared with algebraic operators, Einstein operators and many other operators, the application of Frank operator rule is more flexible and robust. The Frank t-norm and t-conorm involve a parameter, which make them more robust and resilient than other triangular norms in the procedure of fusion of information. Frank operator has less calculation quantity than many operators such as Heronian Mean (HM) operator and Bonferroni mean (BM) operator, and Frank operator is only one kind of t-norm satisfying the compatibility rule.

The primary motivations and contributions of this article are shown as follows:

1. Since the Cq-ROFS is an extension of CIFS and CPFS, it has more proficient, extensive and reliable than existing notions like CIFS and CPFS to deal with unpredictable information in the decision-making process. Further, up to now, there is not any research on the combination of complex q-rung orthopair fuzzy circumstance with Frank aggregation operator. So, it is very significant to extend the Frank operator to solve MCDM problems under complex q-rung orthopair fuzzy circumstance.
2. The Frank operator based on CQROFS is a meaningful concept to cope with two-dimensional information in a single set. So the goals of this article are to present complex q-rung orthopair fuzzy Frank weighted averaging (Cq-ROFFWA) operator, complex q-rung orthopair fuzzy Frank weighted geometric (Cq-ROFFWG) operator and complex q-rung orthopair fuzzy Frank ordered weighted averaging (Cq-ROFFOWA) operator.
3. To discuss some properties of these operators, such as the monotonicity, idempotency and boundedness, the relationships between these operators are put forward.
4. To design two different innovative approaches based on Cq-ROFFWA operator and Cq-ROFFWG operator.
5. To propound two example of its application, which validate the feasibility and reliability of the proposed methods. Furthermore, comparing the propounded methods with the existing methods, we will conclude that the proposed methods are superior to the existing methods, and we will show that the Frank t-conorm and t-norm aggregation operators make the aggregation process more flexible.

The article is organized as below. We review some basic concepts and definitions of the existing sets in Sect. 2. In Sect. 3, we develop the operational laws of Frank t-norm and t-conorm under complex q-rung orthopair fuzzy circumstances. In Sect. 4, we propose some complex q-rung orthopair fuzzy Frank operators. In Sect. 6, we define two approaches for MADM with complex q-rung orthopair fuzzy information based on the developed operators. In Sect. 7, a numerical example is introduced to show the efficiency and a contrastive study is given to certificate the merits of the proposed method. Some conclusion remarks are listed in Sect. 8.

## 2 Preliminaries

**Definition 1** (Liu et al. 2020a, b) Cq-ROFS is defined as follows:

$C = \langle (x, (u_C(x), v_C(x)))/x \in X \rangle,$

where $u_C(x){:}X \to \{w_1 : w_1 \in C, |w_1| \le 1\}$, $v_C(x):\ X \to \{w_2 : w_2 \in C, |w_2| \le 1\}$, such that $u_C(x) = a_1 + ib_1, v_C(x) = a_2 + ib_2$, provided that $0 \le |w_1|^q + |w_2|^q \le 1$, or $u_C(x) = G_C(x)e^{i.2\pi W_{G_C}(x)}$ and $v_C(x) = H_C(x)e^{i.2\pi W_{H_C}(x)}$, where $0 \le G_C^q(x) + H_C^q(x) \le 1$and $0 \le W_{G_C}^q(x) + W_{H_C}^q(x) \le 1$. The complex Cq-RFN is given as $C = \left(G_C(x)e^{i.2\pi W_{G_C}(x)}, H_C(x)e^{i.2\pi W_{H_C}(x)}\right)$.

**Definition 2** (Liu et al. 2020a, b) For any Cq-ROFN $C_1 = \left(G_1 e^{i2\pi W_{G_1}}, H_1 e^{i2\pi W_{H_1}}\right)$, the score function and accuracy function are proposed as below

$$S(C_1) = \frac{1}{2}\left(G_1^q + W_{G_1}^q - H_1^q - W_{H_1}^q\right), \tag{1}$$

$$H(C_1) = \frac{1}{2}\left(G_1^q + W_{G_1}^q + H_1^q + W_{H_1}^q\right), \tag{2}$$

where $S(C_1) \in [-1, 1]$ and $H(C_1) \in [0, 1]$.

**Definition 3** (Liu et al. 2020a, b) For any two Cq-ROFSs,$C_1$ and $C_2$, the following rules are used:

1. If $S(C_1) > S(C_2)$, then $C_1 > C_2$;
2. If $S(C_1) = S(C_2)$, then

(1) If $H(C_1) > H(C_2)$, then $C_1 > C_2$;
(2) If $H(C_1) = H(C_2)$, then $C_1 = C_2$.

**Definition 4** (Liu et al. 2020a, b) For any two Cq-ROFNs $C_1 = \left(G_1 e^{i2\pi W_{G_1}}, H_1 e^{i2\pi W_{H_1}}\right)$ and $C_2 = \left(G_2 e^{i2\pi W_{G_2}}, H_2 e^{i2\pi W_{H_2}}\right)$, with $\lambda \ge 1$, then

(1) $C_1 \oplus C_2 = \left( \left(G_1^q + G_2^q - G_1^q G_2^q\right)^{\frac{1}{q}} e^{i2\pi \left(W_{G_1}^q + W_{G_2}^q - W_{G_1}^q W_{G_2}^q\right)^{\frac{1}{q}}}, H_1 H_2 e^{i2\pi W_{H_1} W_{H_2}} \right);$

(2) $C_1 \otimes C_2 = \left( G_1 G_2 e^{i2\pi W_{G_1} W_{G_2}}, \left(H_1^q + H_2^q - H_1^q H_2^q\right)^{\frac{1}{q}} e^{i2\pi \left(W_{H_1}^q + W_{H_2}^q - W_{H_1}^q W_{H_2}^q\right)^{\frac{1}{q}}} \right);$

(3) $\lambda C_1 = \left( \left(1 - \left(1 - G_1^q\right)^{\lambda}\right)^{\frac{1}{q}} e^{i2\pi \left(1 - \left(1 - W_{G_1}^q\right)^{\lambda}\right)^{\frac{1}{q}}}, H_1^{\lambda} e^{i2\pi W_{H_1}^{\lambda}} \right);$

(4) $C_1^{\lambda} = \left( G_1^{\lambda} e^{i2\pi W_{G_1}^{\lambda}}, \left(1 - \left(1 - H_1^q\right)^{\lambda}\right)^{\frac{1}{q}} e^{i2\pi \left(1 - \left(1 - W_{H_1}^q\right)^{\lambda}\right)^{\frac{1}{q}}} \right).$

# 3 Complex q-rung orthopair fuzzy Frank aggregation operators

Under this part, we investigate Frank's operational laws for complex q-rung orthopair fuzzy numbers, which are developed as follows.

**Theorem 1** *For any three Cq-ROFNs* $C_1 = \left(G_1 e^{i2\pi W_{G_1}}, H_1 e^{i2\pi W_{H_1}}\right)$, $C_2 = \left(G_2 e^{i2\pi W_{G_2}}, H_2 e^{i2\pi W_{H_2}}\right)$ *and* $C = \left(G e^{i2\pi W_G}, H e^{i2\pi W_H}\right)$, *with any* $k, k_1, k_2 > 0$, *then*

(1) $C_1 \oplus C_2 = C_2 \oplus C_1;$

(2) $C_1 \otimes C_2 = C_2 \otimes C_1;$

(3) $k \cdot (C_1 \oplus C_2) = kC_1 \oplus kC_2;$

(4) $(k_1 k_2) \cdot C = k_1 \cdot (k_2 \cdot C);$

(5) $k_1 C \oplus k_2 C = (k_1 + k_2) \oplus C;$

(6) $(C_1 \otimes C_2)^k = C_1^k \otimes C_2^k;$

(7) $C_1^{k_1} \otimes C_1^{k_2} = C_1^{k_1 + k_2}.$

*The proof of Theorem 1 is given in "Appendix."*

**Theorem 2** *For any three Cq-ROFNs* $C_1 = \left(G_1 e^{i2\pi W_{G_1}}, H_1 e^{i2\pi W_{H_1}}\right)$ *and* $C_2 = \left(G_2 e^{i2\pi W_{G_2}}, H_2 e^{i2\pi W_{H_2}}\right)$, *then*

(1) $C_1^c \vee C_2^c = (C_1 \wedge C_2)^c;$

(2) $C_1^c \oplus C_2^c = (C_1 \otimes C_2)^c;$

(3) $C_1^c \otimes C_2^c = (C_1 \oplus C_2)^c;$

(4) $(C_1 \vee C_2) \oplus (C_1 \wedge C_2) = C_1 \oplus C_2;$

(5) $(C_1 \vee C_2) \otimes (C_1 \wedge C_2) = C_1 \otimes C_2;$

(6) $(C_1 \vee C_2) \wedge C_3 = (C_1 \wedge C_3) \vee (C_1 \wedge C_3);$

(7) $(C_1 \vee C_2) \oplus C_3 = (C_1 \oplus C_3) \vee (C_1 \oplus C_3);$

(8) $(C_1 \wedge C_2) \oplus C_3 = (C_1 \oplus C_3) \wedge (C_1 \oplus C_3);$

(9) $(C_1 \wedge C_2) \otimes C_3 = (C_1 \otimes C_3) \wedge (C_1 \otimes C_3).$

*The proof of Theorem 2 can be proven from the Frank operational laws of Cq-ROFNs, so it is omitted here.*

## 3.1 Complex q-rung orthopair fuzzy Frank aggregation operators

In theme of this subsection, the complex q-rung orthopair fuzzy Frank aggregation operators are introduced, and the relevant properties are given

**Definition 5** Let $C_j = \left(G_j e^{i2\pi W_{G_j}}, H_j e^{i2\pi W_{H_j}}\right) (j = 1, 2, \ldots, n)$ be a family of Cq-ROFNs.$w_j$ be the weight of $C_j$, meeting $w_j \in [0, 1]$, and $\sum_{j=1}^{n} w_j = 1$. The complex q-rung orthopair fuzzy Frank weighted averaging (Cq-ROFFWA) operator is initiated by

$$\mathrm{Cq\text{-}ROFFWA}(C_1, C_2, \ldots C_n) = \mathop{\oplus}_{j=1}^{n} w_j C_j = \left( \begin{array}{l} \left(1 - \log_\tau\left(1 + \prod_{j=1}^{n}\left(\tau^{1-(G_j)^q} - 1\right)^{w_j}\right)\right)^{\frac{1}{q}} e^{i2\pi \left(1 - \log_\tau\left(1 + \prod_{j=1}^{n}\left(\tau^{1-(W_{G_j})^q} - 1\right)^{w_j}\right)\right)^{\frac{1}{q}}}, \\ \left(\log_\tau\left(1 + \prod_{j=1}^{n}\left(\tau^{(H_j)^q} - 1\right)^{w_j}\right)\right)^{\frac{1}{q}} e^{i2\pi \left(\log_\tau\left(1 + \prod_{j=1}^{n}\left(\tau^{(W_{H_j})^q} - 1\right)^{w_j}\right)\right)^{\frac{1}{q}}} \end{array} \right) \quad (3)$$

The aggregated value by using Definition 5 is still a Cq-ROFN.

***Proof*** By using mathematical induction to prove Eq. (3).

Case 1: if $n = 2$, then

$$w_1C_1 = \left( \begin{array}{l} \left(1-\log_\tau\left(1+\frac{\left(\tau^{1-G_1^q}-1\right)^{w_1}}{(\tau-1)^{w_1-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-W_{G_1}^q}-1\right)^{w_1}}{(\tau-1)^{w_1-1}}\right)\right)^{\frac{1}{q}}}, \\ \left(\log_\tau\left(1+\frac{\left(\tau^{H_1^q}-1\right)^{w_1}}{(\tau-1)^{w_1-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\frac{\left(\tau^{W_{H_1}^q}-1\right)^{w_1}}{(\tau-1)^{w_1-1}}\right)\right)^{\frac{1}{q}}} \end{array} \right)$$

$$w_2C_2 = \left( \begin{array}{l} \left(1-\log_\tau\left(1+\frac{\left(\tau^{1-G_2^q}-1\right)^{w_2}}{(\tau-1)^{w_2-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-W_{G_2}^q}-1\right)^{w_2}}{(\tau-1)^{w_2-1}}\right)\right)^{\frac{1}{q}}}, \\ \left(\log_\tau\left(1+\frac{\left(\tau^{H_2^q}-1\right)^{w_2}}{(\tau-1)^{w_2-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\frac{\left(\tau^{W_{H_2}^q}-1\right)^{w_2}}{(\tau-1)^{w_2-1}}\right)\right)^{\frac{1}{q}}} \end{array} \right)$$

$$\text{Cq - ROFFWA}(C_1, C_2, \ldots C_n) = w_1C_1 + w_2C_2$$

$$= \left( \left(1-\log_\tau\left(1+\frac{\left(\tau^{1-G_1^q}-1\right)^{w_1}}{(\tau-1)^{w_1-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-W_{G_1}^q}-1\right)^{w_1}}{(\tau-1)^{w_1-1}}\right)\right)^{\frac{1}{q}}}, \left(\log_\tau\left(1+\frac{\left(\tau^{H_1^q}-1\right)^{w_1}}{(\tau-1)^{w_1-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\frac{\left(\tau^{W_{H_1}^q}-1\right)^{w_1}}{(\tau-1)^{w_1-1}}\right)\right)^{\frac{1}{q}}} \right) \oplus$$

$$\left( \left(1-\log_\tau\left(1+\frac{\left(\tau^{1-G_2^q}-1\right)^{w_2}}{(\tau-1)^{w_2-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-W_{G_2}^q}-1\right)^{w_2}}{(\tau-1)^{w_2-1}}\right)\right)^{\frac{1}{q}}}, \left(\log_\tau\left(1+\frac{\left(\tau^{H_2^q}-1\right)^{w_2}}{(\tau-1)^{w_2-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\frac{\left(\tau^{W_{H_2}^q}-1\right)^{w_2}}{(\tau-1)^{w_2-1}}\right)\right)^{\frac{1}{q}}} \right);$$

$$= \left( \begin{array}{l} \left(1-\log_\tau\left(1+\frac{\left(\frac{\left(\tau^{1-G_1^q}-1\right)^{w_1}}{(\tau-1)^{w_1-1}}\right)\left(\frac{\left(\tau^{1-G_2^q}-1\right)^{w_2}}{(\tau-1)^{w_2-1}}\right)}{(\tau-1)}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\frac{\left(\frac{\left(\tau^{1-W_{G_1}^q}-1\right)^{w_1}}{(\tau-1)^{w_1-1}}\right)\left(\frac{\left(\tau^{1-W_{G_2}^q}-1\right)^{w_2}}{(\tau-1)^{w_2-1}}\right)}{(\tau-1)}\right)\right)^{\frac{1}{q}}}, \\ \left(\log_\tau\left(1+\frac{\left(\frac{\left(\tau^{H_1^q}-1\right)^{w_1}}{(\tau-1)^{w_1-1}}\right)\left(\frac{\left(\tau^{H_2^q}-1\right)^{w_2}}{(\tau-1)^{w_2-1}}\right)}{\tau-1}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\frac{\frac{\left(\tau^{W_{H_1}^q}-1\right)^{w_1}}{(\tau-1)^{w_1-1}}\frac{\left(\tau^{W_{H_2}^q}-1\right)^{w_2}}{(\tau-1)^{w_2-1}}}{\tau-1}\right)\right)^{\frac{1}{q}}} \end{array} \right)$$

$$= \left( \begin{array}{l} \left(1-\log_\tau\left(1+\frac{\left(\tau^{1-G_1^q}-1\right)^{w_1}\left(\tau^{1-G_2^q}-1\right)^{w_2}}{(\tau-1)^{\sum_{j=1}^{2}w_j-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-W_{G_1}^q}-1\right)^{w_1}\left(\tau^{1-W_{G_2}^q}-1\right)^{w_2}}{(\tau-1)^{\sum_{j=1}^{2}w_j-1}}\right)\right)^{\frac{1}{q}}}, \\ \left(\log_\tau\left(1+\frac{\left(\tau^{H_1^q}-1\right)^{w_1}\left(\tau^{H_2^q}-1\right)^{w_2}}{(\tau-1)^{\sum_{j=1}^{2}w_j-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\frac{\left(\tau^{W_{H_1}^q}-1\right)^{w_1}\left(\tau^{W_{H_2}^q}-1\right)^{w_2}}{(\tau-1)^{\sum_{j=1}^{2}w_j-1}}\right)\right)^{\frac{1}{q}}} \end{array} \right)$$

$$= \left( \begin{array}{l} \left(1-\log_\tau\left(1+\prod_{j=1}^{2}\left(\tau^{1-(G_j)^q}-1\right)^{w_j}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\prod_{j=1}^{2}\left(\tau^{1-\left(W_{G_j}\right)^q}-1\right)^{w_j}\right)\right)^{\frac{1}{q}}}, \\ \left(\log_\tau\left(1+\prod_{j=1}^{2}\left(\tau^{(H_j)^q}-1\right)^{w_j}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\prod_{j=1}^{2}\left(\tau^{\left(W_{H_j}\right)^q}-1\right)^{w_j}\right)\right)^{\frac{1}{q}}} \end{array} \right)$$

By Theorem 1, we prove the result of $w_1C_1 \oplus w_2C_2$ is also Cq-ROFN. For $n = 2$, the result is kept.

Case 2 Next, check for $k = m$, then Eq. (3) is hold obviously.

$$\text{Cq - ROFFWA}(C_1, C_2, \ldots C_m)$$

$$= \left( \begin{array}{l} \left(1-\log_\tau\left(1+\prod_{j=1}^{m}\frac{\left(\tau^{1-G_j^q}-1\right)^{w_1}}{(\tau-1)^{\sum_{j=1}^{m}w_j-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\prod_{j=1}^{m}\left(\frac{\tau^{1-\left(W_{G_j}\right)^q}-1}{(\tau-1)^{\sum_{j=1}^{m}w_j-1}}\right)^{w_j}\right)\right)^{\frac{1}{q}}}, \\ \left(\log_\tau\left(1+\prod_{j=1}^{m}\frac{\left(\tau^{(H_j)^q}-1\right)^{w_j}}{(\tau-1)^{\sum_{j=1}^{m}w_j-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\prod_{j=1}^{m}\left(\frac{\tau^{\left(W_{H_j}\right)^q}-1}{(\tau-1)^{\sum_{j=1}^{m}w_j-1}}\right)^{w_j}\right)\right)^{\frac{1}{q}}} \end{array} \right)$$

Case 3 lastly, we check for $k = m + 1$, such that

$$\begin{aligned}
&\text{Cq - ROFFWA}(C_1, C_2, \ldots C_m, C_{m+1}) \\
&= \left( \begin{array}{c} \left(1-\log_\tau\left(1+\prod_{j=1}^{m}\frac{\left(\tau^{1-G_j^q}-1\right)^{w_1}}{(\tau-1)^{\sum_{j=1}^{m}w_j-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\prod_{j=1}^{m}\left(\frac{\tau^{1-\left(W_{G_j}\right)^q}-1}{(\tau-1)^{\sum_{j=1}^{m}w_j-1}}\right)^{w_j}\right)\right)^{\frac{1}{q}}}, \\ \left(\log_\tau\left(1+\prod_{j=1}^{m}\frac{\left(\tau^{(H_j)^q}-1\right)^{w_j}}{(\tau-1)^{\sum_{j=1}^{m}w_j-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\prod_{j=1}^{m}\left(\frac{\tau^{\left(W_{H_j}\right)^q}-1}{(\tau-1)^{\sum_{j=1}^{m}w_j-1}}\right)^{w_j}\right)\right)^{\frac{1}{q}}} \end{array} \right) \oplus w_{m+1}C_{m+1} \\
&= \left( \begin{array}{c} \left(1-\log_\tau\left(1+\prod_{j=1}^{m}\frac{\left(\tau^{1-G_j^q}-1\right)^{w_1}}{(\tau-1)^{\sum_{j=1}^{m}w_j-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\prod_{j=1}^{m}\left(\frac{\tau^{1-\left(W_{G_j}\right)^q}-1}{(\tau-1)^{\sum_{j=1}^{m}w_j-1}}\right)^{w_j}\right)\right)^{\frac{1}{q}}}, \\ \left(\log_\tau\left(1+\prod_{j=1}^{m}\frac{\left(\tau^{(H_j)^q}-1\right)^{w_j}}{(\tau-1)^{\sum_{j=1}^{m}w_j-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\prod_{j=1}^{m}\left(\frac{\tau^{\left(W_{H_j}\right)^q}-1}{(\tau-1)^{\sum_{j=1}^{m}w_j-1}}\right)^{w_j}\right)\right)^{\frac{1}{q}}} \end{array} \right) \\
&\oplus \left( \begin{array}{c} \left(1-\log_\tau\left(1+\frac{\left(\tau^{1-G_{m+1}^q}-1\right)^{w_{m+1}}}{(\tau-1)^{w_{m+1}-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-W_{G_{m+1}}^q}-1\right)^{w_{m+1}}}{(\tau-1)^{w_{m+1}-1}}\right)\right)^{\frac{1}{q}}}, \\ \left(\log_\tau\left(1+\frac{\left(\tau^{H_{m+1}^q}-1\right)^{w_{m+1}}}{(\tau-1)^{w_{m+1}-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\frac{\left(\tau^{W_{H_{m+1}}^q}-1\right)^{w_{m+1}}}{(\tau-1)^{w_{m+1}-1}}\right)\right)^{\frac{1}{q}}} \end{array} \right)
\end{aligned}$$

$$
= \left( \begin{array}{l}
\left(\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-\left(1-\log_\tau\left(1+\prod_{j=1}^{m}\left(\tau^{1-(G_j)^q}-1\right)^{w_j}\right)\right)}-1\right)\left(\tau^{1-\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-G^q_{m+1}}-1\right)^{w_{m+1}}}{(\tau-1)^{w_{m+1}-1}}\right)\right)}-1\right)}{(\tau-1)}\right)\right)\right)^{\frac{1}{q}} \\
e^{i2\pi\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-\left(\left(1-\log_\tau\left(1+\prod_{j=1}^{m}\left(\tau^{1-\left(W_{G_j}\right)^q}-1\right)^{w_j}\right)\right)\right)}-1\right)\left(\tau^{1-\left(\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-W^q_{G_{m+1}}}-1\right)^{w_{m+1}}}{(\tau-1)^{w_{m+1}-1}}\right)\right)\right)}-1\right)}{(\tau-1)}\right)\right)^{\frac{1}{q}}}, \\
\left(\log_\tau\left(1+\frac{\left(\tau^{\left(\log_\tau\left(1+\prod_{j=1}^{m}\left(\tau^{(H_j)^q}-1\right)^{w_j}\right)\right)}-1\right)\left(\tau^{\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-G^q_{m+1}}-1\right)^{w_{m+1}}}{(\tau-1)^{w_{m+1}-1}}\right)\right)}-1\right)}{\tau-1}\right)\right)^{\frac{1}{q}} \\
e^{i2\pi\left(\log_\tau\left(1+\frac{\left(\tau^{\left(\log_\tau\left(1+\prod_{j=1}^{m}\left(\tau^{\left(W_{H_j}\right)^q}-1\right)^{w_j}\right)\right)}-1\right)\left(\tau^{\log_\tau\left(1+\frac{\left(\tau^{W^q_{H_{m+1}}}-1\right)^{w_{m+1}}}{(\tau-1)^{w_{m+1}-1}}\right)}-1\right)}{\tau-1}\right)\right)^{\frac{1}{q}}}
\end{array} \right);
$$

$$
= \left( \begin{array}{l}
\left(1-\log_\tau\left(1+\frac{\left(1+\prod_{j=1}^{m}\left(\tau^{1-(G_j)^q}-1\right)^{w_j}\right)\left(\tau^{1-G^q_{m+1}}-1\right)^{w_{m+1}}}{(\tau-1)^{\sum_{j=1} k+1 w_j-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\frac{\left(\prod_{j=1}^{m}\left(\tau^{1-\left(W_{G_j}\right)^q}-1\right)^{w_j}\right)\left(\tau^{1-W^q_{G_{m+1}}}-1\right)^{w_{m+1}}}{(\tau-1)^{\sum_{j=1} k+1 w_j-1}}\right)\right)^{\frac{1}{q}}}, \\
\left(\log_\tau\left(1+\frac{\left(\prod_{j=1}^{m}\left(\tau^{(H_j)^q}-1\right)^{w_j}\right)\left(\frac{\left(\tau^{1-G^q_{m+1}}-1\right)^{w_{m+1}}}{(\tau-1)^{w_{m+1}-1}}\right)}{(\tau-1)^{\sum_{j=1} k+1 w_j-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\frac{\prod_{j=1}^{m}\left(\tau^{\left(W_{H_j}\right)^q}-1\right)^{w_j}\left(\tau^{W^q_{H_{m+1}}}-1\right)^{w_{m+1}}}{(\tau-1)^{\sum_{j=1} k+1 w_j-1}}\right)\right)^{\frac{1}{q}}}
\end{array} \right)
$$

$$
= \left( \begin{array}{l}
\left(1-\log_\tau\left(1+\prod_{j=1}^{n}\left(\tau^{1-(G_j)^q}-1\right)^{w_j}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\prod_{j=1}^{n}\left(\tau^{1-\left(W_{G_j}\right)^q}-1\right)^{w_j}\right)\right)^{\frac{1}{q}}}, \\
\left(\log_\tau\left(1+\prod_{j=1}^{n}\left(\tau^{(H_j)^q}-1\right)^{w_j}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\prod_{j=1}^{n}\left(\tau^{\left(W_{H_j}\right)^q}-1\right)^{w_j}\right)\right)^{\frac{1}{q}}}
\end{array} \right)
$$

For $n = k + 1$, the result is kept. Hence Eq. (3). It is kept for all $n$.

**Theorem 3** (Idempotency) *Let* $C_j (j = 1, 2, ..., n)$ *be a set of be a family of Cq-ROFNs, if all* $a_j{=}a_0 (j = 1, 2, ..., n)$, *then,*

$$\text{Cq - ROFFWA}(C_1, C_2, \ldots, C_n) = C_0$$

***Proof*** We know that $C_0 = \left(G_0 e^{i2\pi W_{G_0}}, H_0 e^{i2\pi W_{H_0}}\right)$

$$G_{-} = \left(1 - \log_\tau\left(1 + \prod_{j=1}^{n}\left(\tau^{1-G_{-}^{q}} - 1\right)^{w_j}\right)\right)^{1/q}$$
$$\leq \left(1 - \log_\tau\left(1 + \prod_{j=1}^{n}\left(\tau^{1-G_{j}^{q}} - 1\right)^{w_j}\right)\right)^{1/q}$$
$$\leq \left(1 - \log_\tau\left(1 + \prod_{j=1}^{n}\left(\tau^{1-G_{+}^{q}} - 1\right)^{w_j}\right)\right)^{1/q} = G_{+}$$

$$\text{Cq - ROFFWA}(C_1, C_2, \ldots C_n)$$
$$= \begin{pmatrix} \left(1 - \log_\tau\left(1 + \prod_{j=1}^{n}\left(\tau^{1-(G_0)^q} - 1\right)^{w_j}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1 - \log_\tau\left(1 + \prod_{j=1}^{n}\left(\tau^{1-\left(W_{G_0}\right)^q} - 1\right)^{w_j}\right)\right)^{\frac{1}{q}}}, \\ \left(\log_\tau\left(1 + \prod_{j=1}^{n}\left(\tau^{(H_0)^q} - 1\right)^{w_j}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1 + \prod_{j=1}^{n}\left(\tau^{\left(W_{H_0}\right)^q} - 1\right)^{w_j}\right)\right)^{\frac{1}{q}}} \end{pmatrix}$$
$$= \left(G_0 e^{i2\pi W_{G_0}}, H_0 e^{i2\pi W_{H_0}}\right) = C_0.$$

**Theorem 4** (boundedness) *Let* $C_j (j = 1, 2, ..., n)$ *be a set of be a family of Cq-ROFNs,* $C_{-} = \min_{1 \leq j \leq n}\{C_j\}$, $C_{+} = \max_{1 \leq j \leq n}\{C_j\}$, $H_{-} = \min_{1 \leq j \leq n}\{H_j\}$, $H_{+} = \max_{1 \leq j \leq n}\{H_j\}$, $W_{G_{-}} = \min_{1 \leq j \leq n}\{W_{G_j}\}$, $W_{G_{+}} = \max_{1 \leq j \leq n}\{W_{G_j}\}$, $W_{H_{-}} = \min_{1 \leq j \leq n}\{W_{H_j}\}$, $W_{H_{+}} = \max_{1 \leq j \leq n}\{W_{H_j}\}$, *then*

$$C_{-} \leq \text{Cq - ROFFWA}(C_1, C_2, \ldots C_n) \leq C_{+}$$

***Proof*** Since $C_{-} = \min_{1 \leq j \leq n}\{C_j\}, C_{+} = \max_{1 \leq j \leq n}\{C_j\}, H_{-} = \min_{1 \leq j \leq n}\{H_j\}, H_{+} = \max_{1 \leq j \leq n}\{H_j\}$,

$W_{G_{-}} = \min_{1 \leq j \leq n}\{W_{G_j}\}, W_{G_{+}} = \max_{1 \leq j \leq n}\{W_{G_j}\}$,

Therefore, we can obtain

$$W_{G_{-}} = \left(1 - \log_\tau\left(1 + \prod_{j=1}^{n}\left(\tau^{1-W_{G_{-}}^{q}} - 1\right)^{w_j}\right)\right)^{1/q}$$
$$\leq \left(1 - \log_\tau\left(1 + \prod_{j=1}^{n}\left(\tau^{1-W_{G_j}^{q}} - 1\right)^{w_j}\right)\right)^{1/q}$$
$$\leq \left(1 - \log_\tau\left(1 + \prod_{j=1}^{n}\left(\tau^{1-W_{G+}^{q}} - 1\right)^{w_j}\right)\right)^{1/q} = W_{G+}$$

$$H_{-} = \left(\log_\tau\left(1 + \prod_{j=1}^{n}\left(\tau^{H_{-}^{q}} - 1\right)^{w_j}\right)\right)^{1/q}$$
$$\leq \left(\log_\tau\left(1 + \prod_{j=1}^{n}\left(\tau^{H_{j}^{q}} - 1\right)^{w_j}\right)\right)^{1/q}$$
$$\leq \left(\log_\tau\left(1 + \prod_{j=1}^{n}\left(\tau^{H_{+}^{q}} - 1\right)^{w_j}\right)\right)^{1/q} = H_{+}$$

$$W_{H_{-}} = \left(\log_\tau\left(1 + \prod_{j=1}^{n}\left(\tau^{W_{H_{-}}^{q}} - 1\right)^{w_j}\right)\right)^{1/q}$$
$$\leq \left(\log_\tau\left(1 + \prod_{j=1}^{n}\left(\tau^{W_{H_j}^{q}} - 1\right)^{w_j}\right)\right)^{1/q}$$
$$\leq \left(\log_\tau\left(1 + \prod_{j=1}^{n}\left(\tau^{W_{H_{+}}^{q}} - 1\right)^{w_j}\right)\right)^{1/q} = W_{H_{+}}$$

So, $S(C_j) = \frac{1}{2}\left(G_j^q + W_{G_j}^q - H_j^q - W_{H_j}^q\right) \leq \frac{1}{2}\left(G_+^q + W_{G_+}^q - H_-^q - W_{H_-}^q\right) = S(C_+)$

$$S(C_j) = \frac{1}{2}\left(G_j^q + W_{G_j}^q - H_j^q - W_{H_j}^q\right) \geq \frac{1}{2}\left(G_-^q + W_{G_-}^q - H_+^q - W_{H_+}^q\right) = S(C_-),$$

then we have

$$C_- \leq \text{Cq - ROFFWA}(C_1, C_2, ...C_n) \leq C_+$$

**Theorem 5** (Monotonicity) *If $C_j(j = 1, 2, ..., n)$ and $C'_j(j = 1, 2, ..., n)$ are two collections of Cq-ROFNs, and $C'_j \leq C_j(j = 1, 2, ..., n)$, then.*

Cq - ROFFWA$(C'_1, C'_2, ...C'_n) \leq$ Cq - ROFFWA$(C_1, C_2, \ldots C_n)$.

In the following, we will discuss some characteristics of Cq - ROFFWA operator.

**Theorem 6** *Let $C_j(j = 1, 2, ..., n)$ be a set of be a family of Cq-ROFNs, $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $C_j(j = 1, 2, ..., n)$, satisfying $0 \leq w_j \leq 1$ and $\sum_{j=1}^n w_j = 1$.*

1. If $\tau \to 1$, the Cq - ROFFWA operator is reduced to a complex q-rung orthopair fuzzy averaging operator (Cq-ROFWA), which is initiated by

$$\lim_{\tau\to 1} \text{Cq - ROFFWA}(C_1, C_2, ...C_n) = \left(\left(1 - \prod_{j=1}^n \left(\left(1 - G_j^q\right)^{w_j}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1 - \prod_{j=1}^n \left(\left(1 - W_{G_j}^q\right)^{w_j}\right)\right)^{\frac{1}{q}}}, \prod_{j=1}^n (H_j)^{w_j} e^{i2\pi \prod_{j=1}^n W_{H_j}^{w_j}}\right)$$

***Proof*** Because $\tau \to 1$, $\prod_{j=1}^n \left(\tau^{1-G_j^q} - 1\right)^{w_j} \to 0$,

Therefore, we can obtain that

$$\left(1 - \log_\tau\left(1 + \prod_{j=1}^n \left(\tau^{1-G_j^q} - 1\right)^{w_j}\right)\right)^{\frac{1}{q}} = \left(1 - \frac{\ln\left(1 + \prod_{j=1}^n \left(\tau^{1-G_j^q} - 1\right)^{w_j}\right)}{\ln \tau}\right)^{\frac{1}{q}} = \left(1 - \frac{\prod_{j=1}^n \left(\tau^{1-G_j^q} - 1\right)^{w_j}}{\ln \tau}\right)^{\frac{1}{q}}$$

According to Taylor's expansion, we can get

$$\tau^{1-G_j^q} = 1 + \left(1 - G_j^q\right) \ln \tau + \frac{\left(1 - G_j^q\right)}{2} (\ln \tau)^2 + \ldots$$

Then we have

$$\tau^{1-G_j^q} - 1 = \left(1 - G_j^q\right) \ln \tau + \mathrm{O}(\ln \tau)$$

$$\left(1 - \frac{\prod_{j=1}^n \left(\tau^{1-G_j^q} - 1\right)^{w_j}}{\ln \tau}\right)^{\frac{1}{q}} = \left(1 - \frac{\prod_{j=1}^n \left(\left(1 - G_j^q\right)^{w_j}\right) \ln \tau}{\ln \tau}\right)^{\frac{1}{q}} = \left(1 - \prod_{j=1}^n \left(\left(1 - G_j^q\right)^{w_j}\right)\right)^{\frac{1}{q}}$$

Similarly, we have $\tau \to 1$, $e^{i2\pi\left(1 - \log_\tau\left(1 + \prod_{j=1}^n \left(\tau^{1-\left(W_{G_j}\right)^q} - 1\right)^{w_j}\right)\right)^{\frac{1}{q}}} \to e^{i2\pi\left(1 - \prod_{j=1}^n \left(1 - W_{G_j}^q\right)^{w_j}\right)^{\frac{1}{q}}}$.

If $\tau \to 1$, $\prod_{j=1}^n \left(\tau^{H_j^q} - 1\right)^{w_j} \to 0$,.

Then we get

$$\left(\log_\tau\left(1 + \prod_{j=1}^n \left(\tau^{H_j^q} - 1\right)^{w_j}\right)\right)^{\frac{1}{q}} = \left(\frac{\ln\left(1 + \prod_{j=1}^n \left(\tau^{H_j^q} - 1\right)^{w_j}\right)}{\ln \tau}\right)^{\frac{1}{q}} = \left(\frac{\prod_{j=1}^n \left(\tau^{H_j^q} - 1\right)^{w_j}}{\ln \tau}\right)^{\frac{1}{q}}$$

According to Taylor's expansion, we can get

$$\tau^{H_j^q} = 1 + H_j^q \ln \tau + \frac{H_j^{2q}}{2} (\ln \tau)^2 + \ldots$$

since $\tau > 1$, we have

$$\tau^{H_j^q}-1=H_j^q\ln\tau+\mathrm{O}(\ln\tau)$$

$$\left(\frac{\prod_{j=1}^{n}\left(\tau^{H_j^q}-1\right)^{w_j}}{\ln\tau}\right)^{\frac{1}{q}}=\left(\frac{\prod_{j=1}^{n}\left(H_j^q\ln\tau\right)^{w_j}}{\ln\tau}\right)^{\frac{1}{q}}=\prod_{j=1}^{n}\left(H_j\right)^{w_j}$$

Similarly, we have $\tau\rightarrow 1$, $e^{i2\pi\left(\log_\tau\left(1+\prod_{j=1}^{n}\left(\tau^{\left(w_{H_j}\right)^q}-1\right)^{w_j}\right)\right)^{\frac{1}{q}}}\rightarrow e^{i2\pi\prod_{j=1}^{n}\left(w_{H_j}\right)^{w_j}}$.

Therefore, we can obtain

$$\lim_{\tau\rightarrow 1}\mathrm{Cq\text{-}ROFFWA}(C_1,C_2,\ldots C_n)$$

$$=\lim_{\tau\rightarrow 1}\left(\begin{array}{c}\left(1-\log_\tau\left(1+\prod_{j=1}^{n}\left(\tau^{1-\left(G_j\right)^q}-1\right)^{w_j}\right)\right)^{\frac{1}{q}}e^{i2\pi\left(1-\log_\tau\left(1+\prod_{j=1}^{n}\left(\tau^{1-\left(w_{G_j}\right)^q}-1\right)^{w_j}\right)\right)^{\frac{1}{q}}},\\ \left(\log_\tau\left(1+\prod_{j=1}^{n}\left(\tau^{\left(H_j\right)^q}-1\right)^{w_j}\right)\right)^{\frac{1}{q}}e^{i2\pi\left(\log_\tau\left(1+\prod_{j=1}^{n}\left(\tau^{\left(w_{H_j}\right)^q}-1\right)^{w_j}\right)\right)^{\frac{1}{q}}}\end{array}\right)$$

$$=\lim_{\tau\rightarrow 1}\left(\left(1-\frac{\prod_{j=1}^{n}\left(\tau^{1-G_j^q}-1\right)^{w_j}}{\ln\tau}\right)^{\frac{1}{q}}e^{i2\pi\left(1-\frac{\prod_{j=1}^{n}\left(\tau^{1-W_{G_j}^q}-1\right)^{w_j}}{\ln\tau}\right)^{\frac{1}{q}}},\left(\frac{\prod_{j=1}^{n}\left(\tau^{H_j^q}-1\right)^{w_j}}{\ln\tau}\right)^{\frac{1}{q}}e^{i2\pi\left(\frac{\prod_{j=1}^{n}\left(\tau^{\left(W_{H_j}\right)^q}-1\right)^{w_j}}{\ln\tau}\right)^{\frac{1}{q}}}\right)$$

$$=\left(\left(1-\prod_{j=1}^{n}\left(\left(1-G_j^q\right)^{w_j}\right)\right)^{\frac{1}{q}}e^{i2\pi\left(1-\prod_{j=1}^{n}\left(\left(1-W_{G_j}^q\right)^{w_j}\right)\right)^{\frac{1}{q}}},\prod_{j=1}^{n}\left(H_j\right)^{w_j}e^{i2\pi\prod_{j=1}^{n}W_{H_j}^{w_j}}\right)$$

2. When $\tau\rightarrow+\infty$, the Cq - ROFFWA operator is reduced to a tradition arithmetic weighted average operator, that is:

$$\lim_{\tau\rightarrow+\infty}\mathrm{Cq\text{-}ROFFWA}$$
$$=\left(\left(\sum_{j=1}^{n}w_j\left(G_j\right)^q\right)^{\frac{1}{q}}e^{i2\pi\left(\sum_{j=1}^{n}w_jW_{G_j}^q\right)^{\frac{1}{q}}},\left(\sum_{j=1}^{n}w_jH_j^q\right)^{\frac{1}{q}}e^{i2\pi\left(\sum_{j=1}^{n}w_jW_{H_j}^q\right)^{\frac{1}{q}}}\right)$$

***Proof***

$$\lim_{\tau\to 1} \mathrm{Cq\text{-}ROFFWA}(C_1, C_2, \ldots C_n)$$

$$= \lim_{\tau\to+\infty}\left(\begin{array}{c}\left(1-\log_\tau\left(1+\prod_{j=1}^{n}\left(\tau^{1-(G_j)^q}-1\right)^{w_j}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\prod_{j=1}^{n}\left(\tau^{1-\left(w_{G_j}\right)^q}-1\right)^{w_j}\right)\right)^{\frac{1}{q}}},\\ \left(\log_\tau\left(1+\prod_{j=1}^{n}\left(\tau^{(H_j)^q}-1\right)^{w_j}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\prod_{j=1}^{n}\left(\tau^{\left(w_{H_j}\right)^q}-1\right)^{w_j}\right)\right)^{\frac{1}{q}}}\end{array}\right)$$

$$= \left(\begin{array}{c}\lim_{\tau\to+\infty}\left(1-\lim_{\tau\to+\infty}\frac{\ln\left(1+\prod_{j=1}^{n}\left(\tau^{1-G_j^q}-1\right)^{w_j}\right)}{\ln\tau}\right)^{\frac{1}{q}} e^{i2\pi\lim_{\tau\to+\infty}\left(1-\lim_{\tau\to+\infty}\frac{\ln\left(1+\prod_{j=1}^{n}\left(\tau^{1-\left(w_{G_j}\right)^q}-1\right)^{w_j}\right)}{\ln\tau}\right)^{\frac{1}{q}}},\\ \left(\lim_{\tau\to+\infty}\frac{\ln\left(1+\prod_{j=1}^{n}\left(\tau^{H_j^q}-1\right)^{w_j}\right)}{\ln\tau}\right)^{\frac{1}{q}} e^{i2\pi\left(\lim_{\tau\to+\infty}\frac{\ln\left(1+\prod_{j=1}^{n}\left(\tau^{\left(w_{H_j}\right)^q}-1\right)^{w_j}\right)}{\ln\tau}\right)^{\frac{1}{q}}}\end{array}\right)$$

Based on the L'Hospital's rule, we have

$$= \left(\begin{array}{c}\left(\left(1-\lim_{\tau\to+\infty}\frac{\prod_{j=1}^{n}\left(\tau^{1-G_j^q}-1\right)^{w_j}}{1+\prod_{j=1}^{n}\left(\tau^{1-G_j^q}-1\right)^{w_j}}\left(\sum_{j=1}^{n}w_j\left(1-G_j^q\right)\frac{\tau^{1-G_j^q}}{\left(\tau^{1-G_j^q}-1\right)}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\lim_{\tau\to+\infty}\frac{\prod_{j=1}^{n}\left(\tau^{1-\left(w_{G_j}\right)^q}-1\right)^{w_j}}{1+\prod_{j=1}^{n}\left(\tau^{1-\left(w_{G_j}\right)^q}-1\right)^{w_j}}\left(\sum_{j=1}^{n}w_j\left(1-\left(w_{G_j}\right)^q\right)\frac{\tau^{1-\left(w_{G_j}\right)^q}}{\left(\tau^{1-\left(w_{G_j}\right)^q}-1\right)}\right)\right)^{\frac{1}{q}}},\\ \left(\lim_{\tau\to+\infty}\frac{\frac{\prod_{j=1}^{n}\left(\tau^{H_j^q}-1\right)^{w_j}}{\left(1+\prod_{j=1}^{n}\left(\tau^{H_j^q}-1\right)^{w_j}\right)}\left(\sum_{j=1}^{n}\frac{w_jH_j^q\tau^{H_j^q-1}}{\left(\tau^{H_j^q}-1\right)}\right)}{\frac{1}{\tau}}\right)^{\frac{1}{q}} e^{i2\pi\left(\lim_{\tau\to+\infty}\frac{\frac{\prod_{j=1}^{n}\left(\tau^{\left(w_{H_j}\right)^q}-1\right)^{w_j}}{\left(1+\prod_{j=1}^{n}\left(\tau^{\left(w_{H_j}\right)^q}-1\right)^{w_j}\right)}\left(\sum_{j=1}^{n}\frac{w_j\left(w_{H_j}\right)^q\tau^{\left(w_{H_j}\right)^q-1}}{\left(\tau^{\left(w_{H_j}\right)^q}-1\right)}\right)}{\frac{1}{\tau}}\right)^{\frac{1}{q}}}\end{array}\right)$$

$$= \left( \left(1 - \sum_{j=1}^{n} w_j \left(1 - G_j^q\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\sum_{j=1}^{n} w_j\left(1-\left(W_{G_j}\right)^q\right)\right)^{\frac{1}{q}}}, \right.$$

$$\left.\left(\lim_{\tau\to+\infty} \frac{\prod_{j=1}^{n} \left(\tau^{H_j^q}-1\right)^{w_j}}{\left(1+\prod_{j=1}^{n}\left(\tau^{H_j^q}-1\right)^{w_j}\right)} \left(\sum_{j=1}^{n} \frac{w_j H_j^q \tau^{H_j^q}}{\left(\tau^{H_j^q}-1\right)}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\lim_{\tau\to+\infty}\frac{\prod_{j=1}^{n}\left(\tau^{\left(W_{H_j}\right)^q}-1\right)^{w_j}}{\left(1+\prod_{j=1}^{n}\left(\tau^{\left(W_{H_j}\right)^q}-1\right)^{w_j}\right)}\left(\sum_{j=1}^{n}\frac{w_j\left(W_{H_j}\right)^q\tau^{\left(W_{H_j}\right)^q}}{\left(\tau^{\left(W_{H_j}\right)^q}-1\right)}\right)\right)^{\frac{1}{q}}}\right)$$

$$= \left( \left(\sum_{j=1}^{n} w_j\left(G_j\right)^q\right)^{\frac{1}{q}} e^{i2\pi\left(\sum_{j=1}^{n} w_j W_{G_j}^q\right)^{\frac{1}{q}}}, \left(\sum_{j=1}^{n} w_j H_j^q\right)^{\frac{1}{q}} e^{i2\pi\left(\sum_{j=1}^{n} w_j W_{H_j}^q\right)^{\frac{1}{q}}} \right)$$

This completes the proof.

**Theorem 7** *Let* $C_j (j = 1, 2, ..., n)$ *be a set of be a family of Cq-ROFNs,* $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ *is the weight vector of* $C_j (j = 1, 2, ..., n)$, *and* $0 \le w_j \le 1$. *At the same time,* $\sum_{j=1}^{n} w_j = 1$, *then*

$$\text{Cq - ROFFWA}(C_1 \oplus C, C_2 \oplus C, \ldots, C_n \oplus C) = \text{Cq - ROFFWA}(C_1, C_2, \ldots, C_n) \oplus C$$

*The proof of Theorem 7 is given in "Appendix."*

**Theorem 8** *Let* $C_j (j = 1, 2, ..., n)$ *be a set of be a family of Cq-ROFNs,* $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ *is the weight vector of* $C_j (j = 1, 2, ..., n)$, *and* $0 \le w_j \le 1$. *At the same time,* $\sum_{j=1}^{n} w_j = 1$, *then*

$$\text{Cq - ROFFWA}(k \cdot C_1 \oplus C, k \cdot C_2 \oplus C, \ldots, k \cdot C_n \oplus C) = k\text{Cq - ROFFWA}(C_1, C_2, \ldots, C_n) \oplus C$$

*The proof of Theorem 8 is given in "Appendix."*

**Theorem 9** *Let* $C_j (j = 1, 2, ..., n)$ *be a set of be a family of Cq-ROFNs,* $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ *is the weight vector of* $C_j (j = 1, 2, ..., n)$, *and* $0 \le w_j \le 1$. *At the same time,* $\sum_{j=1}^{n} w_j = 1$ *and* $k > 0$, *then*

$$\text{Cq - ROFFWA}(kC_1, kC_2, ..., kC_n) = k\text{Cq - ROFFWA}(C_1, C_2, ..., C_n)$$

*The proof of Theorem 9 is given in "Appendix."*

**Theorem 10** *Let* $C_j (j = 1, 2, ..., n)$ *be a set of be a family of Cq-ROFNs,* $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ *is the weight vector of* $C_j (j = 1, 2, ..., n)$, *and* $0 \le w_j \le 1$. *At the same time,* $\sum_{j=1}^{n} w_j = 1$, *then*

$$\text{Cq - ROFFWA}(C_1 \oplus C'_1, C_2 \oplus C'_2, \ldots, C_n \oplus C'_n) = \text{Cq - ROFFWA}(C_1, C_2, ..., C_n) \oplus \text{Cq - ROFFWA}\left(C'_1, C'_2, ..., C'_n\right)$$

**Definition 6** Let $C_j = \left(G_j e^{i2\pi W_{G_j}}, H_j e^{i2\pi W_{H_j}}\right) (j = 1, 2, ..., n)$ be a family of Cq-ROFNs.$w_j$ be the weight of $C_j$, meeting $w_j \in [0, 1]$, and $\sum_{j=1}^{n} w_j = 1$. The complex q-rung orthopair fuzzy Frank weighted geometric (Cq-ROFFWG) operator is initiated by

$$\mathrm{Cq\text{ - }ROFFWG}(C_1, C_2, \ldots C_n) = \mathop{\otimes}_{j=1}^{n} C_j^{w_j}$$

$$= \begin{pmatrix} \left(\log_\tau\left(1+\prod_{j=1}^{n}\left(\tau^{(G_j)^q}-1\right)^{w_j}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\prod_{j=1}^{n}\left(\tau^{(w_{G_j})^q}-1\right)^{w_j}\right)\right)^{\frac{1}{q}}} \\ \left(1-\log_\tau\left(1+\prod_{j=1}^{n}\left(\tau^{1-(H_j)^q}-1\right)^{w_j}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\prod_{j=1}^{n}\left(\tau^{1-(w_{H_j})^q}-1\right)^{w_j}\right)\right)^{\frac{1}{q}}} \end{pmatrix} \tag{4}$$

In particular, if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, the Cq - ROFFWG operator will be simplified to Cq - ROFFG operator, that is

$$\mathrm{Cq\text{ - }ROFFG}(C_1, C_2, \ldots C_n) = \mathop{\otimes}_{j=1}^{n} C_j^{\frac{1}{n}}$$

$$= \begin{pmatrix} \left(\log_\tau\left(1+\prod_{j=1}^{n}\left(\tau^{(G_j)^q}-1\right)^{\frac{1}{n}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\prod_{j=1}^{n}\left(\tau^{(w_{G_j})^q}-1\right)^{\frac{1}{n}}\right)\right)^{\frac{1}{q}}}, \\ \left(1-\log_\tau\left(1+\prod_{j=1}^{n}\left(\tau^{1-(H_j)^q}-1\right)^{\frac{1}{n}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\prod_{j=1}^{n}\left(\tau^{1-(w_{H_j})^q}-1\right)^{\frac{1}{n}}\right)\right)^{\frac{1}{q}}} \end{pmatrix}$$

**Theorem 11** *Let* $a_j (j = 1, 2, \ldots, n), a'_j (j = 1, 2, \ldots, n)$ *be two set of Cq-ROFNs,* $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ *is the weight vector of* $\mathrm{C}_j (j = 1, 2, \ldots, n)$, *and* $0 \le w_j \le 1$. *At the same time,* $\sum_{j=1}^{n} w_j = 1$, *then*

1. $\mathrm{Cq\text{ - }ROFFWA}(a_1^c, a_2^c, \ldots, a_n^c) = \mathrm{Cq\text{ - }ROFFWG}(a_1, a_2, \ldots, a_n)^c$
2. $\mathrm{Cq\text{ - }ROFFWG}(a_1^c, a_2^c, \ldots, a_n^c) = \mathrm{Cq\text{ - }ROFFWA}(a_1, a_2, \ldots, a_n)^c$

Similar to the Cq - ROFFWA operator, the Cq - ROFFWG operator also has bounded, monotonic and idempotent properties, and we can obtain the following properties of Cq - ROFFWG operator as follows.

**Lemma 1** [73]. *Let* $x_j > 0, \lambda_j > 0$, $j = 1,2,\ldots,n$, *and* $\sum_{j=1}^{n} \lambda_j = 1$, *then*

$$\prod_{j=1}^{n} x_j^{\lambda_j} \le \sum_{j=1}^{n} \lambda_j x_j$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

**Theorem 12** *Let* $C_j (j = 1, 2, \ldots, n)$ *be a collection of Cq-ROFNs,* $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ *is the weight vector of* $C_j (j = 1, 2, \ldots, n)$, *and* $0 \le w_j \le 1$. *At the same time,* $\sum_{j=1}^{n} w_j = 1$, *then*

$$\mathrm{Cq\text{ - }ROFFWG}(C_1, C_2, \ldots C_n) \le \mathrm{Cq\text{ - }ROFFWA}(C_1, C_2, \ldots C_n)$$

*The proof can be shown in appendix.*

**Theorem 13** *Let* $C_j (j = 1, 2, \ldots, n)$ *be a collection of Cq-ROFNs, let C is a Cq-ROFN,* $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ *is the weight vector of* $C_j (j = 1, 2, \ldots, n)$, *and* $0 \le w_j \le 1$. *At the same time,* $\sum_{j=1}^{n} w_j = 1$, *then*

$$\mathrm{Cq\text{ - }ROFFWG}(\mathrm{C}_1^\lambda \otimes \mathrm{C}, \mathrm{C}_2^\lambda \otimes \mathrm{C}, \ldots, \mathrm{C}_n^\lambda \otimes \mathrm{C}) = (\mathrm{Cq\text{ - }ROFFWG}(C_1, C_2, \ldots C_n))^\lambda \otimes \mathrm{C}$$

**Theorem 14** *Let* $C_j (j = 1, 2, \ldots, n)$ *be a collection of Cq-ROFNs,* $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ *is the weight vector of* $C_j (j = 1, 2, \ldots, n)$, *and* $0 \le w_j \le 1$. *At the same time,* $\sum_{j=1}^{n} w_j = 1, \lambda > 0$, *then*

$$\mathrm{Cq\text{ - }ROFFWG}(\mathrm{C}_1^\lambda, \mathrm{C}_2^\lambda, \ldots, \mathrm{C}_n^\lambda) = (\mathrm{Cq\text{ - }ROFFWG}(\mathrm{C}_1, \mathrm{C}_2, \ldots, \mathrm{C}_n))^\lambda$$

**Theorem 15** *Let* $C_j (j = 1, 2, \ldots, n)$ *be a collection of Cq-ROFNs, and C is a Cq-ROFN,* $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ *is the* *weight vector of* $C_j (j = 1, 2, ..., n)$, *and* $0 \leq w_j \leq 1$. *At the same time*, $\sum_{j=1}^{n} w_j = 1, \lambda > 0$,*then*

$$\begin{aligned}&\mathrm{Cq\text{ - }ROFFWG}(\mathrm{C}_1 \otimes \mathrm{C}, \mathrm{C}_2 \otimes \mathrm{C}, \ldots, \mathrm{C}_n \otimes \mathrm{C}) \\ &\quad = \mathrm{Cq\text{ - }ROFFWG}(\mathrm{C}_1, \mathrm{C}_2, \ldots, \mathrm{C}_n) \otimes \mathrm{C}\end{aligned}$$

**Theorem 16** *Let* $C_j (j = 1, 2, ..., n)$ *be a collection of Cq-ROFNs, and C is a Cq-ROFN*, $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ *is the weight vector of* $C_j (j = 1, 2, ..., n)$, *and* $0 \leq w_j \leq 1$. *At the same time*, $\sum_{j=1}^{n} w_j = 1, \lambda > 0$,*then*

$$\begin{aligned}&\mathrm{Cq\text{ - }ROFFWG}(\mathrm{C}_1 \otimes \mathrm{C}, \mathrm{C}_2 \otimes \mathrm{C}, \ldots, \mathrm{C}_n \otimes \mathrm{C}) \\ &\quad = \mathrm{Cq\text{ - }ROFFWG}(\mathrm{C}_1, \mathrm{C}_2, \ldots, \mathrm{C}_n) \otimes \mathrm{C}\end{aligned}$$

**Theorem 17** *Let* $C_j (j = 1, 2, ..., n)$ *and* $\mathrm{B}_j (j = 1, 2, ..., n)$ *be two collections of Cq-ROFNs*, *then*

$$\begin{aligned}&\mathrm{Cq\text{ - }ROFFWG}(\mathrm{C}_1 \otimes B_1, \mathrm{C}_2 \otimes B_2, \ldots, \mathrm{C}_n \otimes B_n) \\ &= \mathrm{Cq\text{ - }ROFFWG}(\mathrm{C}_1, \mathrm{C}_2, \ldots, \mathrm{C}_n) \\ &\quad \otimes \mathrm{Cq\text{ - }ROFFWG}(B_1, B_2, \ldots, B_n)\end{aligned}$$

By discussing different values of the parameters, the following special cases are given in the follows.

**Theorem 18** *Let* $C_j (j = 1, 2, ..., n)$ *be a collection of Cq-ROFNs*, $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ *is the weight vector of* $C_j (j = 1, 2, ..., n)$, *and* $0 \leq w_j \leq 1$, *then*

1. If $\tau \to 1$, the Cq - ROFFWG operator is reduced to a complex q-rung orthopair fuzzy averaging operator, which is given as:

$$\begin{aligned}&\lim_{\tau \to 1} \mathrm{Cq\text{ - }ROFFWG}\ (C_1, C_2, \ldots C_n) \\ &= \left( \prod_{j=1}^{n} (G_j)^{w_j} e^{i2\pi \prod_{j=1}^{n} W_{G_j}^{w_j}}, \left(1 - \prod_{j=1}^{n} \left( \left(1 - H_j^q\right)^{w_j} \right) \right)^{\frac{1}{q}} \right. \\ &\quad \left. e^{i2\pi \left(1 - \prod_{j=1}^{n} \left( \left(1 - W_{H_j}^q\right)^{w_j} \right) \right)^{\frac{1}{q}}} \right)\end{aligned}$$

2. If $\tau \to +\infty$, then Cq - ROFFWG operator is reduced to the traditional arithmetic weighted average operator, which is given as:

$$\begin{aligned}&\lim_{\tau \to +\infty} \mathrm{Cq\text{ - }ROFFWG} \\ &= \left( \left( \sum_{j=1}^{n} w_j G_j^q \right)^{\frac{1}{q}} e^{i2\pi \left( \sum_{j=1}^{n} w_j W_{G_j}^q \right)^{\frac{1}{q}}}, \left( \sum_{j=1}^{n} w_j (H_j)^q \right)^{\frac{1}{q}} e^{i2\pi \left( \sum_{j=1}^{n} w_j W_{H_j}^q \right)^{\frac{1}{q}}} \right)\end{aligned}$$

## 3.2 Complex q-rung orthopair fuzzy Frank ordered weighted aggregation operators

**Definition 7** Let $C_j (j = 1, 2, ..., n)$ be a collection of Cq-ROFNs, and $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ be the aggregation-associated vector of $C_j (j = 1, 2, ..., n)$, and $\omega_i \in [0, 1]$, at the same time $\sum_{i=1}^{n} \omega_i = 1$.Then, the complex q-rung orthopair fuzzy Frank ordered weighted average (Cq - ROFFOWA) operator is a mapping Cq - ROFFOWA:$\Omega^n \to \Omega$, if

$$\begin{aligned}&\mathrm{Cq\text{ - }ROFFOWA}(C_1, C_2, \ldots C_n) = \mathop{\oplus}_{j=1}^{n} \omega_j C_{\sigma(j)} \\ &= \left( \begin{array}{l} \left(1 - \log_\tau \left(1 + \prod_{j=1}^{n} \left(\tau^{1 - G_{\sigma(j)}^q} - 1\right)^{\omega_j} \right) \right)^{\frac{1}{q}} e^{i2\pi \left(1 - \log_\tau \left(1 + \prod_{j=1}^{n} \left(\tau^{1 - \left(w_{G_{\sigma(j)}}\right)^q} - 1\right)^{\omega_j} \right) \right)^{\frac{1}{q}}}, \\ \left(\log_\tau \left(1 + \prod_{j=1}^{n} \left(\tau^{H_{\sigma(j)}^q} - 1\right)^{\omega_j} \right) \right)^{\frac{1}{q}} e^{i2\pi \left(\log_\tau \left(1 + \prod_{j=1}^{n} \left(\tau^{\left(w_{H_{\sigma(j)}}\right)^q} - 1\right)^{\omega_j} \right) \right)^{\frac{1}{q}}} \end{array} \right)\end{aligned}$$

where $\Omega$ is the set of complex q-rung orthopair fuzzy variables, $C_{\sigma(j)}$ is the $j$th largest element in $C_j (j = 1, 2, \ldots, n)$, and $C_{\sigma(j-1)} \geq C_{\sigma(j)}$; then, the aggregated value by the Cq - ROFFOWA operator is still a Cq-ROFE.

Similar to the Cq - ROFFWA operator, the Cq - ROFFOWA operator also has the properties of boundedness, idempotency and monotonicity, and the Cq - ROFFOWA operator also has the commutativity property.

**Theorem 19** (Commutativity) *Let $C_j (j = 1, 2, ..., n)$ be a collection of Cq-ROFNs, and $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ be the aggregation-associated vector of $C_j (j = 1, 2, ..., n)$, and $\omega_i \in [0, 1]$, at the same time $\sum_{i=1}^{n} \omega_i = 1$. If $(C'_1, C'_2, ..., C'_n)$ is any permutation of $(C_1, C_2, ..., C_n)$, then*

$$\begin{aligned} &\text{Cq - ROFFOWA}(C_1, C_2, \ldots C_n) \\ &\quad = Cq - ROFFOWA\left(C'_1, C'_2, \ldots C'_n\right) \end{aligned}$$

## 4 MADM approach based on proposed operators

In this section, we propose the MAGDM technique based on complex q-rung orthopair fuzzy Franks operators. Let $A = \{A_1, A_2, ..., A_m\}$ is a set of $m$ alternatives, $C = \{C_1, C_2, ..., C_n\}$ is the set of $n$ attributes, with the attribute weight vector $w = (w_1, w_2, ..., w_n)^T$, satisfying $w_j \in [0, 1]$ with $\sum_{j=1}^{n} w_j = 1$. Suppose that $D = (d_{ij})_{m\times n}$ is a complex q-rung orthopair fuzzy decision matrix, where $C_{ij}$ takes the form of the complex q-rung orthopair fuzzy number given for the alternative $A_i (i = 1, 2, ..., m)$ with respect to attribute $C_j$.

*Step 1* complex q-rung orthopair fuzzy number-based decision matrix $D = (d_{ij})_{m\times n}$ is established.

*Step 2* Use Eqs. (3) and (4) to aggregate the decision matrix.

$$\mathrm{C}_j = \text{Cq - ROFFWA}\left(C_{j1}, C_{j2}, \ldots C_{jn}\right), j = 1, 2, 3, \ldots, m \tag{5}$$

or

$$\mathrm{C}_j = \text{Cq - ROFFWG}\left(C_{j1}, C_{j2}, \ldots C_{jn}\right), j = 1, 2, 3, \ldots, m \tag{6}$$

*Step 3* Use Eq. (1) to calculate the score value of $C_j (j = 1, 2, ..., m)$ obtained in Step (2).

*Step 4* Rank the feasible alternatives $A_i (i = 1, 2, ...m)$ based on their score value. If there is equal between two score values $S(a_i)$ and $S(a_k)$, then we need to compute the degree of accuracy values $H(a_i)$ and $H(a_k)$ of the alternatives $A_i$ and $A_k$ ($i$, $k$ = 1,2,…,$m$), respectively, by Eq. (2), and select the best one.

*Step 5* End.

## 5 Applications based on Frank aggregation operators for Cq-ROFNs

### 5.1 An application for emergency management system

**Numerical Example 1:** In order to minimize the impact of public health emergencies on the society and the country, strengthening emergency management capacity and effectively controlling the disaster effect of public health emergencies are important issues to all countries. How to evaluate the current public crisis emergency response capacity and optimize the emergency management system have always been the focus of various countries.

This example is based on Cq-ROFNs which is adapted from Ref Liu et al. (2021c), Du et al. (2017). There are five possible emerging technology companies in a panel; $\mathrm{A}_i (i = 1, 2, 3, 4, 5)$, the experts need to give an evaluation according to four attributes: (1)$C_1$ is the emergency forecasting capability; (2)$C_2$ is the emergency process capability; (3)$C_3$ is the after-disaster loss evaluation capability; (4)$C_4$ is the emergency support capability; (5)$C_5$ is the after-disaster reconstruction capability. The important degree of the attribute is $w = (0.28, 0.35, 0.16, 0.21)^T$. The assessment values of each alternative are given in the form of Cq-ROFNs. Then the proposed approach is utilized in the following matrix (Table 1):

**1.** The method of using the Cq-ROFFWA operator in Eq. (5) to aggregate all complex q-rung orthopair fuzzy assessment values of the alternative$A_i (i = 1, 2, 3, 4, 5)$**,** and we choose $q = 1.5$ and $\tau = 3$ to study the final ranking results, which is shown in Table 2.

Hence, the best alternative is $A_5$, which represents the emerging technology company (Table 3).

## 6 The method of using the Cq-ROFFWA or Cq-ROFFWA operator

Utilize the Cq-ROFFWA operator or Cq-ROFFWG operator in Eqs. (5) and (6) to aggregate all complex q-rung orthopair fuzzy assessment values of the alternative $A_i (i = 1, 2, 3, 4, 5)$ on the attributes $C_j (j = 1, 2, 3, 4)$ shown in Table 1 into the overall assessment values $h_i (i = 1, 2, 3, 4)$. In order to check the effect of the parameters $q$ and $\tau$ for this example, we choose different $q$

**Table 1** Complex q-rung orthopair fuzzy decision matrix $R = (a_{ij})_{m\times n}$

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ |
|---|---|---|---|---|
| $A_1$ | $(0.7e^{i2\pi(0.65)}, 0.2e^{i2\pi(0.3)})$ | $(0.6e^{i2\pi(0.6)}, 0.3e^{i2\pi(0.31)})$ | $(0.8e^{i2\pi(0.5)}, 0.2e^{i2\pi(0.11)})$ | $(0.73e^{i2\pi(0.6)}, 0.25e^{i2\pi(0.31)})$ |
| $A_2$ | $(0.6e^{i2\pi(0.66)}, 0.21e^{i2\pi(0.32)})$ | $(0.7e^{i2\pi(0.71)}, 0.2e^{i2\pi(0.33)})$ | $(0.83e^{i2\pi(0.7)}, 0.3e^{i2\pi(0.2)})$ | $(0.75e^{i2\pi(0.3)}, 0.27e^{i2\pi(0.4)})$ |
| $A_3$ | $(0.4e^{i2\pi(0.68)}, 0.22e^{i2\pi(0.35)})$ | $(0.41e^{i2\pi(0.68)}, 0.3e^{i2\pi(0.4)})$ | $(0.45e^{i2\pi(0.8)}, 0.4e^{i2\pi(0.5)})$ | $(0.44e^{i2\pi(0.8)}, 0.28e^{i2\pi(0.41)})$ |
| $A_4$ | $(0.3e^{i2\pi(0.64)}, 0.26e^{i2\pi(0.31)})$ | $(0.5e^{i2\pi(0.65)}, 0.31e^{i2\pi(0.2)})$ | $(0.6e^{i2\pi(0.6)}, 0.3e^{i2\pi(0.34)})$ | $(0.5e^{i2\pi(0.81)}, 0.32e^{i2\pi(0.4)})$ |
| $A_5$ | $(0.73e^{i2\pi(0.67)}, 0.3e^{i2\pi(0.35)})$ | $(0.8e^{i2\pi(0.65)}, 0.45e^{i2\pi(0.2)})$ | $(0.86e^{i2\pi(0.7)}, 0.2e^{i2\pi(0.41)})$ | $(0.76e^{i2\pi(0.7)}, 0.29e^{i2\pi(0.51)})$ |

**Table 2** When $\tau = 3, q = 1.5$, aggregation values of the Cq-ROFNs

| Method | Getting values |
|---|---|
| A1 = Cq—ROFFWA(C11, C12,..., C1n) | $(0.69529e^{i2\pi(0.60063)}, 0.24194e^{i2\pi(0.26170)})$ |
| A2 = Cq—ROFFWA(C21, C22,..., C2n) | $(0.71346e^{i2\pi(0.63352)}, 0.23070e^{i2\pi(0.31533)})$ |
| A3 = Cq—ROFFWA(C31, C32,..., C3n) | $\left(0.42029e^{i2\pi\left(0.73058\right)}, 0.28455e^{i2\pi(0.40195)}\right)$ |
| A4 = Cq—ROFFWA(C41, C42,..., C4n) | $\left(0.47127e^{i2\pi(0.68224)}, 0.29565e^{i2\pi\left(0.28599\right)}\right)$ |
| A5 = Cq—ROFFWA(C51, C52,..., C5n) | $(0.78165e^{i2\pi(0.67476)}, 0.32359e^{i2\pi(0.32251)})$ |

and $\tau$ a to study the final ranking results of this example, which is shown below.

(1) **Effect of parameter** $q$.

In this subsection, we examine the effect of the parameter δ on the ranking results. We obtained the ranking results for different values of the parameter δ, as shown in Table 4 ($\tau$=3). As shown in Tables 4 and 5, the parameter $q$ affected the ranking results. However, the best alternative did not change.

**(2) Effect of parameter** $\tau$.

In this subsection, we examine the effect of the parameter $q$ on the ranking results. We obtained the ranking results for different values of $q$, as shown in Tables 6 and 7 ($q = 4$). As indicated in Tables 6 and 7, the parameter $q$ affected the ranking results. However, the best alternative did not change.

Further, when$q = 4$, we select different $\tau$ to study the ranking of enterprises, as shown in Tables 6 and 7.

From Tables 4, 5, 6 and 7, we can find that the given approaches like Cq - ROFFWA operator,Cq - ROFFWG operator is giving the same ranking values, and $A_5$ is found to be the best one.

## 7 Comparison analyses

To elaborate the effectiveness and practicability of the created method, we give a comparative analysis of the developed operators with several previous decision methods, including Mahmood' approach (2021) based on Hamacher aggregation operators for Cq - ROFS. Liu' approach (2021c) is based on Einstein operator for Cq - ROFS, Liu' approach (2020b) for Cq - ROFS and Garg' approach (2021b) based on Hamy Mean Operators for Cq - ROFS, and we will discuss this example in the following three parts.

(1) Cq-ROFHWA operator and Cq-ROFHWG operator.

To facilitate the comparative analysis, we give the complex q-rung orthopair fuzzy Hamacher weighted average (Cq-ROFHWA) (Mahmood and Ali 2021a, 2021b) and Cq-ROFHWG operators to deal with this example. From Tables 8 and 9, it is clear that for the different values of parameter q, the same ranking results are obtained, and $A_5$ is found to be the best one.

(2) Cq-ROFEWA operator and Cq-ROFEWG operator.

In a comparison analysis with existing complex q-rung orthopair fuzzy Einstein weighted average (Cq-ROFEWA) operator (Liu et al. 2021c) and Cq-ROFEWG operator, we can obtain the ranking result as follows:

From Tables 10 and 11, it is obvious that the ranking results are the same for diverse parameter $q$ by utilizing Cq - ROFEWG operator or Cq - ROFEWG operator, and $A_5$ is found to be the best one.

(3) Cq-ROFWHM operator.

Compared with Cq-ROFWHM (Garg et al. 2021b) operator, the results are shown below.

(4) Cq - ROFWA operator and Cq - ROFWG operator.

**Table 3** Score values of the alternatives

| Method | Score values |
|---|---|
| A1 = Cq—ROFFWA(C11, C12,..., C1n) | $S(A_1) = 0.39618$ |
| A2 = Cq—ROFFWA(C21, C22,..., C2n) | $S(A_2) = 0.4095$ |
| A3 = Cq—ROFFWA(C31, C32,..., C3n) | $S(A_3) = 0.24515$ |
| A4 = Cq—ROFFWA(C41, C42,..., C4n) | $S(A_4) = 0.28667$ |
| A5 = Cq—ROFFWA(C51, C52,..., C5n) | $S(A_5) = 0.439054$ |

Compared with Cq - ROFWA (Liu et al.2020b) operator and Cq - ROFWG operator the results are shown in Tables 13 and 14.

From Tables 13 and 14, the ranking results are similar to our proposed method, which proves the authenticity and the rationality of the given method.

In this article, firstly, we compared the proposed methods with each other based on different parameters. Secondly, we make a comparison between the proposed and existing methods. As can be seen from Tables 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 and 14, by comparing the proposed method with the Cq-ROFHWA, Cq-ROFHWG, Cq-ROFEWA, Cq-ROFHWG, Cq-ROFWA, Cq-ROFWG and Cq-ROFWHM operators, the conclusion is the same and is the best desirable alternative, which shows the rationality of the defined method in this article. Thirdly, the proposed method is more flexible and universal than the methods based on the Cq-ROFWA operator and the Cq-ROFWG operator in dealing with the MADM problems according to different parameters, and the calculation amount of Mahmood' method (Mahmood and Ali2021a, 2021b), Liu' methods (Liu et al. 2020b, 2021c) and Garg H' approach (2021b) is very higher than that of our created method, and our methods can reflect the decision makers preferences more flexible based on the parameter. According to the analysis above, the proposed methods are better than the other approaches.

## 7.1 An application choosing veil for COVID-19

**Numerical Example 2:** The MCDM issue is cited from Ref Yang et al. (2021). In the current extreme situation of the spread of COVID-19, six types of veils are commonly available in the market, including clinical careful covers, particulate respirators (N95/KN95 or more), clinical defensive covers, clinical defensive covers, dispensable clinical covers, conventional non-clinical covers and gas covers. Individuals need to buy the antivirus veil from six rising star antivirus products. In addition, individuals

**Table 4** When $\tau = 3$, ranking results based on Cq - ROFFWA operator by using the different $q$

| $q$ | $S(a_1)$ | $S(a_2)$ | $S(a_3)$ | $S(a_4)$ | $S(a_5)$ | Ranking |
|---|---|---|---|---|---|---|
| 1.1 | 0.400405 | 0.404738 | 0.237438 | 0.287663 | 0.416741 | $A_5 \succ A_2 \succ A_1 \succ A_4 \succ A_3$ |
| 1.5 | 0.396189 | 0.4095 | 0.245152 | 0.28667 | 0.439054 | $A_5 \succ A_2 \succ A_1 \succ A_4 \succ A_3$ |
| 2 | 0.359788 | 0.383096 | 0.234496 | 0.262104 | 0.429349 | $A_5 \succ A_2 \succ A_1 \succ A_4 \succ A_3$ |
| 10 | 0.021947 | 0.034676 | 0.026826 | 0.018283 | 0.056009 | $A_5 \succ A_2 \succ A_3 \succ A_1 \succ A_4$ |
| 30 | 0.000111 | 0.00033 | 0.000232 | 0.000189 | 0.00069 | $A_5 \succ A_2 \succ A_3 \succ A_4 \succ A_1$ |
| 50 | 0.000209 | 0.000636 | 0.000341 | 0.000352 | 0.001001 | $A_5 \succ A_2 \succ A_4 \succ A_3 \succ A_1$ |
| 100 | 0.00000000001 | 0.00000000064 | 0.00000000003 | 0.00000000007 | 0.00000000217 | $A_5 \succ A_2 \succ A_4 \succ A_3 \succ A_1$ |

**Table 5** When $\tau = 3$, ranking results based on Cq - ROFFWG operator by using the different $q$

| $q$ | $S(a_1)$ | $S(a_2)$ | $S(a_3)$ | $S(a_4)$ | $S(a_5)$ | Ranking |
|---|---|---|---|---|---|---|
| 1.1 | 0.383397 | 0.37087 | 0.228081 | 0.26568 | 0.393402 | $A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$ |
| 1.5 | 0.37759 | 0.369112 | 0.234322 | 0.261309 | 0.412969 | $A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$ |
| 2 | 0.341045 | 0.33702 | 0.222729 | 0.234767 | 0.402898 | $A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$ |
| 5 | 0.111429 | 0.115601 | 0.097802 | 0.073698 | 0.202971 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 10 | 0.014045 | 0.016209 | 0.019398 | 0.009223 | 0.050005 | $A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$ |
| 30 | 0.013957 | 0.016017 | 0.019076 | 0.009124 | 0.049503 | $A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$ |
| 50 | 0.013952 | 0.013925 | 0.018914 | 0.009002 | 0.049526 | $A_5 \succ A_3 \succ A_1 \succ A_2 \succ A_4$ |
| 60 | 0.00000000006 | 0.00000000023 | 0.00000000165 | 0.00000000002 | 0.00000013 | $A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$ |

**Table 6** If $q = 4$, ranking results based on the Cq - ROFFWA operator by using the different $\tau$

| $\tau$ | $S(a_1)$ | $S(a_2)$ | $S(a_3)$ | $S(a_4)$ | $S(a_5)$ | Ranking |
|---|---|---|---|---|---|---|
| 1.1 | 0.184625 | 0.223135 | 0.145845 | 0.136866 | 0.282018 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 2 | 0.183882 | 0.221653 | 0.145196 | 0.135849 | 0.281551 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 6 | 0.182834 | 0.219548 | 0.144253 | 0.134434 | 0.280845 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 10 | 0.182471 | 0.218817 | 0.143915 | 0.13395 | 0.280579 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 30 | 0.181907 | 0.217679 | 0.143366 | 0.133198 | 0.280124 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 60 | 0.181671 | 0.2172 | 0.143121 | 0.132882 | 0.279905 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 80 | 0.181594 | 0.217044 | 0.143038 | 0.132779 | 0.279826 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 100 | 0.181541 | 0.216937 | 0.14298 | 0.132708 | 0.27977 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 1000 | 0.18124 | 0.216323 | 0.142612 | 0.132296 | 0.279364 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 10,000 | 0.181152 | 0.216138 | 0.142461 | 0.132167 | 0.279134 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 1,000,000 | 0.181108 | 0.216047 | 0.14232 | 0.132095 | 0.278821 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |

**Table 7** If $q = 4$, ranking results based on Cq - ROFFWG operator by using the different $\tau$

| $\tau$ | $S(a_1)$ | $S(a_2)$ | $S(a_3)$ | $S(a_4)$ | $S(a_5)$ | Ranking |
|---|---|---|---|---|---|---|
| 1.1 | 0.166435 | 0.167829 | 0.132488 | 0.110833 | 0.264624 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 2 | 0.167176 | 0.169836 | 0.133075 | 0.111543 | 0.265112 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 6 | 0.168469 | 0.173413 | 0.134075 | 0.112814 | 0.265899 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 10 | 0.169039 | 0.175018 | 0.134504 | 0.113386 | 0.266219 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 30 | 0.170192 | 0.178329 | 0.135347 | 0.114566 | 0.266819 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 60 | 0.170868 | 0.18031 | 0.135824 | 0.115273 | 0.26714 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 80 | 0.171137 | 0.181107 | 0.13601 | 0.115557 | 0.267261 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 100 | 0.171341 | 0.181714 | 0.13615 | 0.115773 | 0.267351 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 1000 | 0.173203 | 0.187428 | 0.137356 | 0.117796 | 0.268082 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 10,000 | 0.17466 | 0.192152 | 0.138196 | 0.119443 | 0.268537 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 1,000,000 | 0.176611 | 0.199047 | 0.139116 | 0.121791 | 0.26898 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |

**Table 8** When $k = 3$, ranking results based on the Cq-ROFHWA operator by using the different q

| $q$ | $S(a_1)$ | $S(a_2)$ | $S(a_3)$ | $S(a_4)$ | $S(a_5)$ | Ranking |
|---|---|---|---|---|---|---|
| 1.1 | 0.437597 | 0.434858 | 0.129117 | 0.210158 | 0.462234 | $A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$ |
| 1.5 | 0.435419 | 0.433218 | 0.142457 | 0.220977 | 0.46033 | $A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$ |
| 5 | 0.433038 | 0.440938 | 0.202252 | 0.278003 | 0.451915 | $A_5 \succ A_2 \succ A_1 \succ A_4 \succ A_3$ |
| 10 | 0.026639 | 0.038981 | 0.016673 | 0.011672 | 0.06849 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 20 | 0.001668 | 0.003386 | 0.001502 | 0.001061 | 0.007026 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 80 | 0.0000000018 | 0.000000035847 | 0.000000002179 | 0.000000003341 | 0.000000097571 | $A_5 \succ A_2 \succ A_4 \succ A_3 \succ A_1$ |

consider four criteria to decide their choice: (1)$C_1$ is the spillage rate that is the adhesiveness of the veil structure configuration to cover the human face; (2)$C_2$ is the reusability; (3)$C_3$ is the nature of crude materials; (4)$C_4$ is the filtration productivity. All criteria values are benefit type. The weight vector of the criteria is $w = (0.15, 0.15, 0.2, 0.5)^{\mathrm{T}}$. The five potential options are assessed regarding the four criteria by CQROFSs, and complex q-rung orthopair fuzzy decision matrix R is listed in Table 15.

Because the proposed MADM based on CQROFSs with its Cq - ROFFWA operator has different q values, it generates some usefully special cases. To examine the usefulness and proficiency of the proposed approach, we next

**Table 9** When $k = 3$, ranking results based on the Cq-ROFHWG operator by using the different $q$

| $q$ | $S(a_1)$ | $S(a_2)$ | $S(a_3)$ | $S(a_4)$ | $S(a_5)$ | Ranking |
|---|---|---|---|---|---|---|
| 1.1 | 0.38628 | 0.37738 | 0.22993 | 0.26927 | 0.39799 | $A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$ |
| 1.5 | 0.38012 | 0.37535 | 0.23632 | 0.26455 | 0.41757 | $A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$ |
| 5 | 0.11182 | 0.11648 | 0.09826 | 0.07409 | 0.20348 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 10 | 0.01407 | 0.01626 | 0.01944 | 0.00925 | 0.05008 | $A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$ |
| 20 | 0.00026 | 0.00040 | 0.00075 | 0.00016 | 0.00345 | $A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$ |
| 50 | 0.0000000027 | 0.0000000084 | 0.0000000427 | 0.0000000009 | 0.0000017088 | $A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$ |

**Table 10** When $k = 2$, ranking results based on the Cq - ROFEWA operator by using the different q

| $q$ | $S(a_1)$ | $S(a_2)$ | $S(a_3)$ | $S(a_4)$ | $S(a_5)$ | Ranking |
|---|---|---|---|---|---|---|
| 1.1 | 0.441705 | 0.439769 | 0.118027 | 0.203928 | 0.466271 | $A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$ |
| 1.5 | 0.440447 | 0.439108 | 0.129267 | 0.213466 | 0.464914 | $A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$ |
| 5 | 0.442137 | 0.447826 | 0.187336 | 0.267599 | 0.460177 | $A_5 \succ A_2 \succ A_1 \succ A_4 \succ A_3$ |
| 10 | 0.029347 | 0.041919 | 0.012811 | 0.00937 | 0.074812 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 20 | 0.001872 | 0.00376 | 0.001131 | 0.000802 | 0.007875 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 80 | 0.0000000021 | 0.0000000403 | 0.0000000016 | 0.0000000025 | 0.0000001098 | $A_5 \succ A_2 \succ A_4 \succ A_1 \succ A_3$ |
| 100 | 0.000000000024 | 0.000000000971 | 0.000000000019 | 0.000000000037 | 0.0000000032 | $A_5 \succ A_2 \succ A_4 \succ A_1 \succ A_3$ |

**Table 11** When $k = 2$, ranking results based on the Cq - ROFEWG operator by using the different $q$

| $q$ | $S(a_1)$ | $S(a_2)$ | $S(a_3)$ | $S(a_4)$ | $S(a_5)$ | Ranking |
|---|---|---|---|---|---|---|
| 1.1 | 0.38446 | 0.37314 | 0.22877 | 0.26691 | 0.39511 | $A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$ |
| 1.5 | 0.37848 | 0.37108 | 0.23506 | 0.26235 | 0.41470 | $A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$ |
| 5 | 0.11145 | 0.11561 | 0.09787 | 0.07373 | 0.20309 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 10 | 0.01404 | 0.01620 | 0.01939 | 0.00922 | 0.05000 | $A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$ |
| 20 | 0.00026 | 0.00040 | 0.00074 | 0.00016 | 0.00345 | $A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$ |
| 50 | 0.000000002 | 0.00000000838 | 0.00000004267 | 0.00000000095 | 0.00000170878 | $A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$ |

present ranking results based on Cq - ROFFWA operator when $\tau = 3$, six cases of $q = 1$, $q = 1.1$, $q = 2$, $q = 3$, $q = 5$, and $q = 10$. These decision matrix items are shown in Table 16.

If $q = 1$, then the Cq - ROFFWA operator is reduced to the complex intuitionistic fuzzy Frank aggregation (CIFWA) operator.

If $q = 2$, then the Cq - ROFFWA operator is reduced to the complex Pythagorean fuzzy Frank aggregation (CPFWA) operator.

If $q = 3$, then the Cq - ROFFWA operator is reduced to the complex Fermatean fuzzy Frank aggregation (CPFFA) operator.

From Table 16, we can get that $A_6$ is the best one. The prominent characteristic of the Cq - ROFFWA operator is that the decision makers can choose the appropriate parameter value $q$ and $\tau$ by their preferences. The defined method has an ideal property about the parameter value $q$ and $\tau$ which provides people to choose appropriate values on the basis of risk preferences. If the decision maker is risk averse, we can select the parameter $\tau$ as large as possible, and if the decision maker is risk loving, we can choose the parameter $\tau$ as small as possible. Since Frank t-norm and t-conorm are a generalization of algorithms such as Algebra, Einstein and Hamacher's t-norm and t-conorm, it is more general in dealing with multi-attribute decision-making problems. Further, the complex q-rung orthopair fuzzy Frank aggregation operators consider the rejection level when they are compared with complex intuition fuzzy Frank aggregation operator, complex Pythagorean fuzzy Frank aggregation operator and complex q-rung orthopair fuzzy Frank aggregation operators.

**Table 12** When $q = 3$, ranking results based on the Cq-ROFWHM operator by using the different $x$

| $x$ | $S(a_1)$ | $S(a_2)$ | $S(a_3)$ | $S(a_4)$ | $S(a_5)$ | Ranking |
|---|---|---|---|---|---|---|
| 2 | 0.718563 | 0.721737 | 0.649679 | 0.648689 | 0.779716 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 3 | 0.712878 | 0.71244 | 0.643013 | 0.639795 | 0.775243 | $A_5 \succ A_1 \succ A_2 \succ A_3 \succ A_4$ |
| 4 | 0.708174 | 0.705694 | 0.635857 | 0.633196 | 0.768629 | $A_5 \succ A_1 \succ A_2 \succ A_3 \succ A_4$ |

**Table 13** Ranking results based on the Cq - ROFWA operator by using the different $q$

| $q$ | $S(a_1)$ | $S(a_2)$ | $S(a_3)$ | $S(a_4)$ | $S(a_5)$ | Ranking |
|---|---|---|---|---|---|---|
| 1.1 | 0.402609 | 0.408894 | 0.238689 | 0.290738 | 0.419788 | $A_5 \succ A_2 \succ A_1 \succ A_4 \succ A_3$ |
| 1.5 | 0.398267 | 0.414047 | 0.246492 | 0.289777 | 0.441607 | $A_5 \succ A_2 \succ A_1 \succ A_4 \succ A_3$ |
| 4 | 0.184755 | 0.223394 | 0.145958 | 0.137046 | 0.282098 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 10 | 0.022084 | 0.034989 | 0.027026 | 0.018523 | 0.05625 | $A_5 \succ A_2 \succ A_3 \succ A_1 \succ A_4$ |
| 20 | 0.001274 | 0.002706 | 0.002281 | 0.001617 | 0.005407 | $A_5 \succ A_2 \succ A_3 \succ A_4 \succ A_1$ |
| 50 | 0.0000011 | 0.00000727 | 0.00000264 | 0.00000279 | 0.00001573 | $A_5 \succ A_2 \succ A_4 \succ A_3 \succ A_1$ |

**Table 14** Ranking results based on the Cq - ROFWG operator by using the different $q$

| $q$ | $S(a_1)$ | $S(a_2)$ | $S(a_3)$ | $S(a_4)$ | $S(a_5)$ | Ranking |
|---|---|---|---|---|---|---|
| 1.1 | 0.381670 | 0.365080 | 0.226894 | 0.262694 | 0.390520 | $A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$ |
| 1.5 | 0.375824 | 0.362390 | 0.233072 | 0.258312 | 0.410543 | $A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$ |
| 4 | 0.166315 | 0.167506 | 0.132392 | 0.110719 | 0.264542 | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| 10 | 0.013957 | 0.016033 | 0.019245 | 0.009141 | 0.049770 | $A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$ |
| 20 | 0.000264 | 0.000397 | 0.000744 | 0.000162 | 0.003439 | $A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$ |
| 50 | 0.00000000274 | 0.000000008377 | 0.000000042674 | 0.000000000946 | 0.000001708757 | $A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$ |

**Table 15** Complex q-rung orthopair fuzzy decision matrix $R = (a_{ij})_{m \times n}$

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ |
|---|---|---|---|---|
| $A_1$ | $\left(0.3e^{i2\pi(0.4)}, 0.2e^{i2\pi(0.1)}\right)$ | $\left(0.31e^{i2\pi(0.41)}, 0.21e^{i2\pi(0.11)}\right)$ | $\left(0.32e^{i2\pi(0.41)}, 0.22e^{i2\pi(0.12)}\right)$ | $\left(0.33e^{i2\pi(0.43)}, 0.23e^{i2\pi(0.13)}\right)$ |
| $A_2$ | $\left(0.5e^{i2\pi(0.6)}, 0.4e^{i2\pi(0.3)}\right)$ | $\left(0.51e^{i2\pi(0.61)}, 0.41e^{i2\pi(0.31)}\right)$ | $\left(0.52e^{i2\pi(0.62)}, 0.42e^{i2\pi(0.32)}\right)$ | $\left(0.53e^{i2\pi(0.63)}, 0.43e^{i2\pi(0.33)}\right)$ |
| $A_3$ | $\left(0.4e^{i2\pi(0.68)}, 0.22e^{i2\pi(0.35)}\right)$ | $\left(0.41e^{i2\pi(0.68)}, 0.3e^{i2\pi(0.4)}\right)$ | $\left(0.45e^{i2\pi(0.8)}, 0.4e^{i2\pi(0.5)}\right)$ | $\left(0.44e^{i2\pi(0.8)}, 0.28e^{i2\pi(0.41)}\right)$ |
| $A_4$ | $\left(0.1e^{i2\pi(0.3)}, 0.4e^{i2\pi(0.2)}\right)$ | $\left(0.11e^{i2\pi(0.31)}, 0.41e^{i2\pi(0.21)}\right)$ | $\left(0.12^{i2\pi(0.32)}, 0.42e^{i2\pi(0.22)}\right)$ | $\left(0.13e^{i2\pi(0.33)}, 0.43e^{i2\pi(0.23)}\right)$ |
| $A_5$ | $\left(0.7e^{i2\pi(0.5)}, 0.1e^{i2\pi(0.2)}\right)$ | $\left(0.71e^{i2\pi(0.51)}, 0.11e^{i2\pi(0.21)}\right)$ | $\left(0.72e^{i2\pi(0.52)}, 0.12e^{i2\pi(0.22)}\right)$ | $\left(0.73e^{i2\pi(0.53)}, 0.13e^{i2\pi(0.23)}\right)$ |
| $A_6$ | $\left(0.8e^{i2\pi(0.7)}, 0.1e^{i2\pi(0.1)}\right)$ | $\left(0.81e^{i2\pi(0.71)}, 0.11e^{i2\pi(0.11)}\right)$ | $\left(0.82e^{i2\pi(0.72)}, 0.12e^{i2\pi(0.12)}\right)$ | $\left(0.83e^{i2\pi(0.73)}, 0.13e^{i2\pi(0.13)}\right)$ |

To sum up, two practical examples are given in the paper to illustrate the application of the defined method and to demonstrate its effectiveness and practicality.

The key advantages of the method defined in the paper are as follows:

1. The complex q-rung orthopair fuzzy Frank aggregation operators generalize the existing Frank aggregation operators, the complex q-rung orthopair fuzzy Einstein aggregation operators, the complex q-rung orthopair fuzzy weighted averaging operators, the complex intuitionistic fuzzy Frank aggregation operator, the complex Pythagorean fuzzy Frank aggregation operator, the complex Fermatean fuzzy Frank aggregation operator. It is clear that the Cq - ROFFWA operator in this paper has more general

**Table 16** When $\tau = 3$, ranking results based on Cq - ROFFWA operator by using the different $q$

| $q$ | $S(a_1)$ | $S(a_2)$ | $S(a_3)$ | $S(a_4)$ | $S(a_5)$ | $S(a_6)$ | Ranking |
|---|---|---|---|---|---|---|---|
| 1 | 0.1994 | 0.2003 | 0.2457 | − 0.099 | 0.4505 | 0.6508 | $A_6 \succ A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$ |
| 1.1 | 0.1917 | 0.2040 | 0.2528 | − 0.095 | 0.4493 | 0.6818 | $A_6 \succ A_5 \succ A_2 \succ A_3 \succ A_1 \succ A_4$ |
| 2 | 0.1076 | 0.1885 | 0.2601 | − 0.053 | 0.3638 | 0.5822 | $A_6 \succ A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$ |
| 3 | 0.0471 | 0.1366 | 0.2208 | − 0.0250 | 0.2516 | 0.4621 | $A_6 \succ A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$ |
| 5 | 0.0079 | 0.0571 | 0.1383 | − 0.0051 | 0.1163 | 0.2837 | $A_6 \succ A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$ |
| 10 | 0.00009 | 0.0049 | 0.0410 | − 0.00008 | 0.01979 | 0.0888 | $A_6 \succ A_3 \succ A_5 \succ A_2 \succ A_1 \succ A_4$ |

and more information in terms of both membership and non-membership degrees than other above operators. Therefore, the complex q-rung orthopair fuzzy Frank aggregation operators can be applied to deal with more uncertain and complicated information of real-life decision-making problems.

2. Complex q-rung orthopair fuzzy term which can solve the uncertainty more precisely than q-rung orthopair fuzzy set, complex fuzzy set and q-rung orthopair uncertain linguistic fuzzy set in qualitative, and many fuzzy set cannot deal with MADM under complex q-rung environment such as the rough set and neutrosophic set, but the complex q-rung orthopair fuzzy number can be used to deal with such problems effectively.

3. The prominent characteristic of the proposed method is that the parameter value $q$ and $\tau$ can be changed by their preferences.

4. Membership and non-membership degree are complements of the complex fuzzy terms, which can show us how much degree that an attribute value belongs to or not belongs to a complex q-rung term.

## 8 Conclusion

In this paper, we extended the Frank operations to the complex q-rung orthopair fuzzy environment according to the definition of the complex q-rung orthopair fuzzy set and the Frank aggregation operator. To begin with, certain Frank operational laws of complex q-rung orthopair fuzzy set (Cq-ROFS) were introduced. Meanwhile, we developed a family of complex q-rung orthopair fuzzy Frank averaging operators, such as the Cq-ROFFWA operator, the Cq-ROFFWG operator and the Cq-ROFFOWA operator. Some desirable properties of the introduced operators were given. In addition, based on the defined operators, a novel approach for MADM was proposed under the complex q-rung orthopair fuzzy environment. Numerical examples were given to demonstrate the effectiveness and usefulness of the defined method by comparison with other existing methods.

The Frank aggregation operators based on the complex q-rung orthopair fuzzy set are an important complement to related research, and the method defined in this paper may add a new direction for solving MCDM problems. We have created a precedent to combine Frank aggregation operators with complex q-rung orthopair fuzzy set and make use of the complex q-rung orthopair fuzzy Frank aggregation operators to deal with decision-making problems. Since the method proposed in this paper cannot handle multi-attribute decision making in complex q-rung linguistic fuzzy environments or complex q-rung fuzzy N-soft environments, in future work, we will extend Frank aggregation operators under complex q-rung linguistic orthopair fuzzy environments, complex q-rung fuzzy N-soft environments and T-spherical fuzzy environments, or develop Frank aggregation operators to 3, 4-quasirung fuzzy sets (Seikh and Mandal, 2022). In addition, we shall further generalize these defined operators to deal with Biogas plant implementation problem (Karmakar et al.2021), plastic ban problem (Seikh et al. 2021a), market share problem (Seikh et al.2021b), social network analysis (Liu et al. 2022a), social trust propagation mechanism (Liu et al.2022b) and incomplete probabilistic linguistic preference relations (Wang et al. 2021; Liu et al.2020a, 2020b, 2020c), or extend the aggregation operators to other domains, such as pattern recognition, cluster analysis and investment decisions.

## Appendix

***Proof of Theorem 1*** We prove the parts (1, 5, 6), and the others are similar with them.

1. Let us prove part (1), we get

$$C_1 \oplus C_2 = \begin{pmatrix} \left(1-\log_\tau\left(1+\frac{\left(\tau^{1-G_1^q}-1\right)\left(\tau^{1-G_2^q}-1\right)}{(\tau-1)}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-W_{G_1}^q}-1\right)\left(\tau^{1-W_{G_2}^q}-1\right)}{(\tau-1)}\right)\right)^{\frac{1}{q}}}, \\ \left(\log_\tau\left(1+\frac{\left(\tau^{H_1^q}-1\right)\left(\tau^{H_2^q}-1\right)}{\tau-1}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\frac{\left(\tau^{W_{H_1}^q}-1\right)\left(\tau^{W_{H_2}^q}-1\right)}{\tau-1}\right)\right)^{\frac{1}{q}}} \end{pmatrix}$$

$$= \begin{pmatrix} \left(1-\log_\tau\left(1+\frac{\left(\tau^{1-G_2^q}-1\right)\left(\tau^{1-G_1^q}-1\right)}{(\tau-1)}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-W_{G_2}^q}-1\right)\left(\tau^{1-W_{G_1}^q}-1\right)}{(\tau-1)}\right)\right)^{\frac{1}{q}}}, \\ \left(\log_\tau\left(1+\frac{\left(\tau^{H_2^q}-1\right)\left(\tau^{H_1^q}-1\right)}{\tau-1}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\frac{\left(\tau^{W_{H_2}^q}-1\right)\left(\tau^{W_{H_1}^q}-1\right)}{\tau-1}\right)\right)^{\frac{1}{q}}} \end{pmatrix}$$

$$= C_2 \oplus C_1.$$

2. Let us prove part (5), we get

$k \cdot (C_1 \oplus C_2) = kC_1 \oplus kC_2;$

$$C_1 \oplus C_2 = \begin{pmatrix} \left(1-\log_\tau\left(1+\frac{\left(\tau^{1-G_1^q}-1\right)\left(\tau^{1-G_2^q}-1\right)}{(\tau-1)}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-W_{G_1}^q}-1\right)\left(\tau^{1-W_{G_2}^q}-1\right)}{(\tau-1)}\right)\right)^{\frac{1}{q}}}, \\ \left(\log_\tau\left(1+\frac{\left(\tau^{H_1^q}-1\right)\left(\tau^{H_2^q}-1\right)}{\tau-1}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\frac{\left(\tau^{W_{H_1}^q}-1\right)\left(\tau^{W_{H_2}^q}-1\right)}{\tau-1}\right)\right)^{\frac{1}{q}}} \end{pmatrix}$$

$$kC_1 = \left( \left(1-\log_\tau\left(1+\frac{\left(\tau^{1-G_1^q}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-W_{G_1}^q}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}}}, \left(\log_\tau\left(1+\frac{\left(\tau^{H_1^q}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\frac{\left(\tau^{W_{H_1}^q}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}}} \right),$$

$$kC_2 = \left( \left(1-\log_\tau\left(1+\frac{\left(\tau^{1-G_2^q}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-W_{G_2}^q}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}}}, \left(\log_\tau\left(1+\frac{\left(\tau^{H_2^q}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\frac{\left(\tau^{W_{H_2}^q}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}}} \right)$$

Then, we have

$$kC_1 \oplus kC_2 = \left( \begin{array}{l} \left(1 - \log_\tau\left(1 + \frac{\left(\frac{\left(\tau^{1-G_1^q} - 1\right)^k}{(\tau-1)^{k-1}}\right)\left(\frac{\left(\tau^{1-G_2^q} - 1\right)^k}{(\tau-1)^{k-1}}\right)}{(\tau-1)}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1 - \log_\tau\left(1 + \frac{\left(\frac{\left(\tau^{1-W_{G_1}^q} - 1\right)^k}{(\tau-1)^{k-1}}\right)\left(\frac{\left(\tau^{1-W_{G_2}^q} - 1\right)^k}{(\tau-1)^{k-1}}\right)}{(\tau-1)}\right)\right)^{\frac{1}{q}}}, \\ \left(\log_\tau\left(1 + \frac{\left(\frac{\left(\tau^{H_2^q} - 1\right)^k}{(\tau-1)^{k-1}}\right)\left(\frac{\left(\tau^{H_2^q} - 1\right)^k}{(\tau-1)^{k-1}}\right)}{\tau-1}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1 + \frac{\left(\frac{\left(\tau^{W_{H_1}^q} - 1\right)^k}{(\tau-1)^{k-1}}\right)\left(\frac{\left(\tau^{W_{H_2}^q} - 1\right)^k}{(\tau-1)^{k-1}}\right)}{\tau-1}\right)\right)^{\frac{1}{q}}} \end{array} \right)$$

$$= \left( \begin{array}{l} \left(1 - \log_\tau\left(1 + \frac{\left(\tau^{1-G_1^q} - 1\right)^k\left(\tau^{1-G_2^q} - 1\right)^k}{(\tau-1)^{2k-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1 - \log_\tau\left(1 + \frac{\left(\left(\tau^{1-W_{G_1}^q} - 1\right)^k\right)\left(\left(\tau^{1-W_{G_2}^q} - 1\right)^k\right)}{(\tau-1)^{2k-1}}\right)\right)^{\frac{1}{q}}}, \\ \left(\log_\tau\left(1 + \frac{\left(\tau^{H_2^q} - 1\right)^k\left(\tau^{H_2^q} - 1\right)^k}{(\tau-1)^{2k-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1 + \frac{\left(\tau^{W_{H_1}^q} - 1\right)^k\left(\tau^{W_{H_2}^q} - 1\right)^k}{(\tau-1)^{2k-1}}\right)\right)^{\frac{1}{q}}} \end{array} \right)$$

$$k(\mathrm{C}_1 \oplus \mathrm{C}_2) = \left( \begin{array}{l} \left( \left( 1 - \log_\tau \left( 1 + \frac{\left( \tau^{1-\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-\mathrm{G}_1^q}-1\right)\left(\tau^{1-\mathrm{G}_2^q}-1\right)}{(\tau-1)}\right)\right)} - 1 \right)^k}{(\tau-1)^{k-1}} \right) \right) \right)^{\frac{1}{q}} e^{i2\pi \left( 1-\log_\tau \left( 1+\frac{\left( \tau^{1-\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-W_{G_1}^q}-1\right)\left(\tau^{1-W_{G_2}^q}-1\right)}{(\tau-1)}\right)\right)} -1 \right)^k}{(\tau-1)^{k-1}} \right) \right)^{\frac{1}{q}}}, \\ \left( \left( \log_\tau \left( 1 + \frac{\left( \tau^{\left(\log_\tau\left(1+\frac{\left(\tau^{\mathrm{H}_1^q}-1\right)\left(\tau^{\mathrm{H}_2^q}-1\right)}{\tau-1}\right)\right)} - 1 \right)^k}{(\tau-1)^{k-1}} \right) \right) \right)^{\frac{1}{q}} e^{i2\pi \left( \log_\tau \left( 1+\frac{\left( \tau^{\left(\log_\tau\left(1+\frac{\left(\tau^{W_{H_1}^q}-1\right)\left(\tau^{W_{H_2}^q}-1\right)}{\tau-1}\right)\right)} -1 \right)^k}{(\tau-1)^{k-1}} \right) \right)^{\frac{1}{q}}} \end{array} \right)$$

$$= \left( \begin{array}{l} \left( 1 - \log_\tau \left( 1 + \frac{\left(\tau^{1-G_1^q}-1\right)^k \left(\tau^{1-G_2^q}-1\right)^k}{(\tau-1)^{2k-1}} \right) \right)^{\frac{1}{q}} e^{i2\pi \left( 1-\log_\tau \left( 1+\frac{\left(\left(\tau^{1-W_{G_1}^q}-1\right)^k\right)\left(\left(\tau^{1-W_{G_2}^q}-1\right)^k\right)}{(\tau-1)^{2k-1}} \right) \right)^{\frac{1}{q}}}, \\ \left( \log_\tau \left( 1 + \frac{\left(\tau^{H_2^q}-1\right)^k \left(\tau^{H_2^q}-1\right)^k}{(\tau-1)^{2k-1}} \right) \right)^{\frac{1}{q}} e^{i2\pi \left( \log_\tau \left( 1+\frac{\left(\tau^{W_{H_1}^q}-1\right)^k \left(\tau^{W_{H_2}^q}-1\right)^k}{(\tau-1)^{2k-1}} \right) \right)^{\frac{1}{q}}} \end{array} \right)$$

So, $k(\mathrm{C}_1 \oplus \mathrm{C}_2) = k\mathrm{C}_1 \oplus k\mathrm{C}_2$.

3. The proof of part (2,4,5) is straightforward. So, it has been omitted here.

4. Let us consider part (6), we have

$$C_1 \otimes C_2 = \left( \begin{array}{c} \left(\log_\tau\left(1+\dfrac{\left(\tau^{G_1^q}-1\right)\left(\tau^{G_2^q}-1\right)}{\tau-1}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\frac{\left(\tau^{W^q_{G_1}}-1\right)\left(\tau^{W^q_{G_2}}-1\right)}{\tau-1}\right)\right)^{\frac{1}{q}}}, \\ \left(1-\log_\tau\left(1+\dfrac{\left(\tau^{1-H_1^q}-1\right)\left(\tau^{1-H_2^q}-1\right)}{(\tau-1)}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-W^q_{H_1}}-1\right)\left(\tau^{1-W^q_{H_2}}-1\right)}{(\tau-1)}\right)\right)^{\frac{1}{q}}} \end{array} \right)$$

$$(C_1 \otimes C_2)^k = \left( \begin{array}{c} \left(\log_\tau\left(1+\dfrac{\left(\tau^{\left(\log_\tau\left(1+\frac{\left(\tau^{G_1^q}-1\right)\left(\tau^{G_2^q}-1\right)}{\tau-1}\right)\right)}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\frac{\left(\tau^{\left(\log_\tau\left(1+\frac{\left(\tau^{W^q_{G_1}}-1\right)\left(\tau^{W^q_{G_2}}-1\right)}{\tau-1}\right)\right)}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}}} \\ \left(1-\log_\tau\left(1+\dfrac{\left(\tau^{1-\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-H_1^q}-1\right)\left(\tau^{1-H_2^q}-1\right)}{(\tau-1)}\right)\right)}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}} \\ e^{i2\pi\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-W^q_{H_1}}-1\right)\left(\tau^{1-W^q_{H_2}}-1\right)}{(\tau-1)}\right)\right)}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}}} \end{array} \right).$$

$$= \left( \begin{array}{c} \left(\log_\tau\left(1+\dfrac{\left(\left(\tau^{G_1^q}-1\right)\left(\tau^{G_2^q}-1\right)\right)^k}{(\tau-1)^{2k-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\frac{\left(\left(\tau^{W^q_{G_1}}-1\right)\left(\tau^{W^q_{G_2}}-1\right)\right)^k}{(\tau-1)^{2k-1}}\right)\right)^{\frac{1}{q}}} \\ \left(1-\log_\tau\left(1+\dfrac{\left(\left(\tau^{1-H_1^q}-1\right)\left(\tau^{1-H_2^q}-1\right)\right)^k}{(\tau-1)^{2k-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\frac{\left(\left(\tau^{1-W^q_{H_1}}-1\right)\left(\tau^{1-W^q_{H_2}}-1\right)\right)^k}{(\tau-1)^{2k-1}}\right)\right)^{\frac{1}{q}}} \end{array} \right).$$

According to the definition, we have

$$C_1^k = \left( \begin{array}{c} \left(\log_\tau\left(1+\frac{\left(\tau^{G_1^q}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\frac{\left(\tau^{W^q_{G_1}}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}}} \\ \left(1-\log_\tau\left(1+\frac{\left(\tau^{1-H_1^q}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-W^q_{H_1}}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}}} \end{array} \right),$$

$$C_2^k = \left( \begin{array}{c} \left(\log_\tau\left(1+\frac{\left(\tau^{G_2^q}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\frac{\left(\tau^{W^q_{G_2}}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}}} \\ \left(1-\log_\tau\left(1+\frac{\left(\tau^{1-H_2^q}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-W^q_{H_2}}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}}} \end{array} \right)$$

Then, we have

$$C_1^k \otimes C_2^k = \left( \begin{array}{c} \left(\log_\tau\left(1+\frac{\left(\tau^{\left(\log_\tau\left(1+\frac{\left(\tau^{G_1^q}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)}-1\right)\left(\tau^{\left(\log_\tau\left(1+\frac{\left(\tau^{G_2^q}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)}-1\right)}{\tau-1}\right)\right)^{\frac{1}{q}} \\ e^{i2\pi\log_\tau\left(1+\frac{\left(\tau^{\left(\log_\tau\left(1+\frac{\left(\tau^{W^q_{G_1}}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)}-1\right)\left(\tau^{\left(\log_\tau\left(1+\frac{\left(\tau^{W^q_{G_2}}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)}-1\right)}{\tau-1}\right)^{\frac{1}{q}}}, \\ \left(1-\log_\tau\left(1+\frac{\left(\tau^{1-\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-H_1^q}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)}-1\right)\left(\tau^{1-\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-H_2^q}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)}-1\right)}{(\tau-1)}\right)\right)^{\frac{1}{q}} \\ e^{i2\pi\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-W^q_{H_1}}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)}-1\right)\left(\tau^{1-\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-W^q_{H_2}}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)}-1\right)}{(\tau-1)}\right)\right)^{\frac{1}{q}}} \end{array} \right);$$

$$
= \left( \begin{array}{c}
\left( \log_\tau \left( 1 + \frac{\left( \frac{\left(\tau^{G_1^q} - 1\right)^k}{(\tau-1)^{k-1}} \right) \left( \frac{\left(\tau^{G_2^q} - 1\right)^k}{(\tau-1)^{k-1}} \right)}{\tau - 1} \right) \right)^{\frac{1}{q}} e^{i2\pi \log_\tau \left( 1 + \frac{\left( \frac{\left(\tau^{W^q_{G_1}} - 1\right)^k}{(\tau-1)^{k-1}} \right) \left( \frac{\left(\tau^{W^q_{G_1}} - 1\right)^k}{(\tau-1)^{k-1}} \right)}{\tau - 1} \right)^{\frac{1}{q}}}, \\
\left( 1 - \log_\tau \left( 1 + \frac{\left( \frac{\left(\tau^{1-H_1^q} - 1\right)^k}{(\tau-1)^{k-1}} \right) \left( \frac{\left(\tau^{1-H_2^q} - 1\right)^k}{(\tau-1)^{k-1}} \right)}{(\tau - 1)} \right) \right)^{\frac{1}{q}} e^{i2\pi \left( 1 - \log_\tau \left( 1 + \frac{\left( \frac{\left(\tau^{1-W^q_{H_1}} - 1\right)^k}{(\tau-1)^{k-1}} \right) \left( \frac{\left(\tau^{1-W^q_{H_2}} - 1\right)^k}{(\tau-1)^{k-1}} \right)}{(\tau - 1)} \right) \right)^{\frac{1}{q}}}
\end{array} \right)
$$

$$
= \left( \begin{array}{c}
\log_\tau \left( 1 + \frac{\left( \left(\tau^{G_1^q} - 1\right)^k \right) \left(\tau^{G_2^q} - 1\right)^k}{(\tau-1)^{2k-1}} \right)^{\frac{1}{q}} e^{i2\pi \log_\tau \left( 1 + \frac{\left( \left(\tau^{W^q_{G_1}} - 1\right)^k \right) \left(\tau^{W^q_{G_2}} - 1\right)^k}{(\tau-1)^{2k-1}} \right)^{\frac{1}{q}}}, \\
\left( 1 - \log_\tau \left( 1 + \frac{\left( \left(\tau^{1-H_1^q} - 1\right)^k \right) \left( \left(\tau^{1-H_2^q} - 1\right)^k \right)}{(\tau-1)^{2k-1}} \right) \right)^{\frac{1}{q}} e^{i2\pi \left( 1 - \log_\tau \left( 1 + \frac{\left(\tau^{1-W^q_{H_1}} - 1\right)^k \left(\tau^{1-W^q_{H_2}} - 1\right)^k}{(\tau-1)^{2k-1}} \right) \right)^{\frac{1}{q}}}
\end{array} \right)
$$

$(C_1 \otimes C_2)^k$.

***Proof of Theorem 8*** Since

$$
C_j \oplus C = \left( \begin{array}{c}
\left( 1 - \log_\tau \left( 1 + \frac{\left(\tau^{1-G_j^q} - 1\right)\left(\tau^{1-G^q} - 1\right)}{(\tau - 1)} \right) \right)^{\frac{1}{q}} e^{i2\pi \left( 1 - \log_\tau \left( 1 + \frac{\left(\tau^{1-W^q_{G_j}} - 1\right)\left(\tau^{1-W^q_{G}} - 1\right)}{(\tau-1)} \right) \right)^{\frac{1}{q}}}, \\
\left( \log_\tau \left( 1 + \frac{\left(\tau^{H_j^q} - 1\right)\left(\tau^{H^q} - 1\right)}{\tau - 1} \right) \right)^{\frac{1}{q}} e^{i2\pi \left( \log_\tau \left( 1 + \frac{\left(\tau^{W^q_{H_j}} - 1\right)\left(\tau^{W^q_{H}} - 1\right)}{\tau-1} \right) \right)^{\frac{1}{q}}}
\end{array} \right)
$$

So, we have

$$
\begin{aligned}
&\mathrm{Cq\text{-}ROFFWA}(C_1 \oplus C, C_2 \oplus C, \ldots, C_n \oplus C)\\
&= \left( \begin{array}{l}
\left(1-\log_\tau\left(1+\prod_{j=1}^{n}\left(\tau^{\log_\tau\left(1+\frac{\left(\tau^{1-G_j^q}-1\right)\left(\tau^{1-G^q}-1\right)}{(\tau-1)}\right)}-1\right)^{w_j}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\prod_{j=1}^{n}\left(\tau^{\log_\tau\left(1+\frac{\left(\tau^{1-W_{G_j}^q}-1\right)\left(\tau^{1-W_G^q}-1\right)}{(\tau-1)}\right)}-1\right)^{w_j}\right)\right)^{\frac{1}{q}}},\\
\left(\log_\tau\left(1+\prod_{j=1}^{n}\left(1+\frac{\left(\tau^{H_j^q}-1\right)\left(\tau^{H^q}-1\right)}{\tau-1}-1\right)^{w_j}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\prod_{j=1}^{n}\left(1+\frac{\left(\tau^{W_{H_j}^q}-1\right)\left(\tau^{W_H^q}-1\right)}{\tau-1}-1\right)^{w_j}\right)\right)^{\frac{1}{q}}}
\end{array} \right)\\
&= \left( \begin{array}{l}
\left(1-\log_\tau\left(1+\prod_{j=1}^{n}\left(\frac{\left(\tau^{1-G_j^q}-1\right)\left(\tau^{1-G^q}-1\right)}{(\tau-1)}\right)^{w_j}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\prod_{j=1}^{n}\left(\frac{\left(\tau^{1-W_{G_j}^q}-1\right)\left(\tau^{1-W_G^q}-1\right)}{(\tau-1)}\right)^{w_j}\right)\right)^{\frac{1}{q}}},\\
\left(\log_\tau\left(1+\prod_{j=1}^{n}\left(\frac{\left(\tau^{H_j^q}-1\right)\left(\tau^{H^q}-1\right)}{\tau-1}\right)^{w_j}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\prod_{j=1}^{n}\left(\frac{\left(\tau^{W_{H_j}^q}-1\right)\left(\tau^{W_H^q}-1\right)}{\tau-1}\right)^{w_j}\right)\right)^{\frac{1}{q}}}
\end{array} \right)
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&\mathrm{Cq\text{-}ROFFWA}(C_1, C_2, \ldots, C_n) \oplus C\\
&= \left( \begin{array}{l}
\left(1-\log_\tau\left(1+\frac{\left(\tau^{\left(\log_\tau\left(1+\prod_{j=1}^{n}\left(\tau^{1-G_j^q}-1\right)^{w_j}\right)\right)}-1\right)\left(\tau^{1-G^q}-1\right)}{(\tau-1)}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\frac{\left(\tau^{\left(\log_\tau\left(1+\prod_{j=1}^{n}\left(\tau^{1-W_{G_j}^q}-1\right)^{w_j}\right)\right)}-1\right)\left(\tau^{1-W_G^q}-1\right)}{(\tau-1)}\right)\right)^{\frac{1}{q}}},\\
\left(\log_\tau\left(1+\frac{\left(\prod_{j=1}^{n}\left(\tau^{H_j^q}-1\right)^{w_j}\right)\left(\tau^{H^q}-1\right)}{\tau-1}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\frac{\left(\prod_{j=1}^{n}\left(\tau^{W_{H_j}^q}-1\right)^{w_j}\right)\left(\tau^{W_H^q}-1\right)}{\tau-1}\right)\right)^{\frac{1}{q}}}
\end{array} \right)
\end{aligned}
$$

$$=\left(\begin{array}{c}\left(1-\log_\tau\left(1+\frac{\left(\prod_{j=1}^{n}\left(\tau^{1-G_j^q}-1\right)^{w_j}\right)\left(\tau^{1-G^q}-1\right)}{(\tau-1)}\right)\right)^{\frac{1}{q}}e^{i2\pi\left(1-\log_\tau\left(1+\frac{\left(\prod_{j=1}^{n}\left(\tau^{1-W_{G_j}^q}-1\right)^{w_j}\right)\left(\tau^{1-W_G^q}-1\right)}{(\tau-1)}\right)\right)^{\frac{1}{q}}},\\ \left(\log_\tau\left(1+\frac{\left(\prod_{j=1}^{n}\left(\tau^{H_j^q}-1\right)^{w_j}\right)\left(\tau^{H^q}-1\right)}{\tau-1}\right)\right)^{\frac{1}{q}}e^{i2\pi\left(\log_\tau\left(1+\frac{\left(\prod_{j=1}^{n}\left(\tau^{W_{H_j}^q}-1\right)^{w_j}\right)\left(\tau^{W_H^q}-1\right)}{\tau-1}\right)\right)^{\frac{1}{q}}}\end{array}\right)$$

So, we have
Cq - ROFFWA$(C_1\oplus C, C_2\oplus C,\ldots,C_n\oplus C)=$
Cq - ROFFWA$(C_1,C_2,\ldots,C_n)\oplus C$.

***Proof of Theorem 9*** Since

$$kC_j=\left(\begin{array}{c}\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-G_j^q}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}}e^{i2\pi\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-W_{G_j}^q}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}}},\\ \left(\log_\tau\left(1+\frac{\left(\tau^{H_j^q}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}}e^{i2\pi\left(\log_\tau\left(1+\frac{\left(\tau^{W_{H_j}^q}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}}}\end{array}\right)$$

Furthermore,

$$kC_j\oplus C=\left(\begin{array}{c}\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-G_j^q}-1\right)^k\left(\tau^{1-G^q}-1\right)}{(\tau-1)^k}\right)\right)^{\frac{1}{q}}e^{i2\pi\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-W_{G_j}^q}-1\right)^k\left(\tau^{1-W_G^q}-1\right)}{(\tau-1)^k}\right)\right)^{\frac{1}{q}}},\\ \left(\log_\tau\left(1+\frac{\left(\tau^{H_j^q}-1\right)^k\left(\tau^{H^q}-1\right)}{(\tau-1)^k}\right)\right)^{\frac{1}{q}}e^{i2\pi\left(\log_\tau\left(1+\frac{\left(\tau^{W_{H_j}^q}-1\right)^k\left(\tau^{W_H^q}-1\right)}{(\tau-1)^k}\right)\right)^{\frac{1}{q}}}\end{array}\right)$$

Therefore, we have

Cq - ROFFWA$(k\cdot C_1\oplus C, k\cdot C_2\oplus C,\ldots,k\cdot C_n\oplus C)$

$$= \left( \begin{array}{l} \left( \left( 1 - \log_\tau \left( 1 + \prod_{j=1}^{n} \left( \tau^{1-\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-G_j^q}-1\right)^k\left(\tau^{1-G^q}-1\right)}{(\tau-1)^k}\right)\right)} - 1 \right)^{w_j} \right) \right) \right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\prod_{j=1}^{n}\left(\tau^{1-\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-W_{G_j}^q}-1\right)^k\left(\tau^{1-W_G^q}-1\right)}{(\tau-1)^k}\right)\right)}-1\right)^{w_j}\right)\right)^{\frac{1}{q}}}, \\ \left( \log_\tau \left( 1+\prod_{j=1}^{n} \left( \tau^{\log_\tau\left(1+\frac{\left(\tau^{H_j^q}-1\right)^k\left(\tau^{H^q}-1\right)}{(\tau-1)^k}\right)} - 1 \right)^{w_j} \right) \right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\prod_{j=1}^{n}\left(\tau^{\log_\tau\left(1+\frac{\left(\tau^{W_{H_j}^q}-1\right)^k\left(\tau^{W_H^q}-1\right)}{(\tau-1)^k}\right)}-1\right)^{w_j}\right)\right)^{\frac{1}{q}}} \end{array} \right)$$

$$= \left( \begin{array}{l} \left( 1 - \log_\tau \left( 1 + \prod_{j=1}^{n} \frac{\left(\tau^{1-G_j^q} - 1\right)^{kw_j} \left(\tau^{1-G^q} - 1\right)}{(\tau - 1)^k} \right) \right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\prod_{j=1}^{n}\frac{\left(\tau^{1-W_{G_j}^q}-1\right)^{kw_j}\left(\tau^{1-W_G^q}-1\right)}{(\tau-1)^k}\right)\right)^{\frac{1}{q}}}, \\ \left( \log_\tau \left( 1+\prod_{j=1}^{n} \frac{\left(\tau^{H_j^q} - 1\right)^{kw_j} \left(\tau^{H^q} - 1\right)}{(\tau - 1)^k} \right) \right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\prod_{j=1}^{n}\frac{\left(\tau^{W_{H_j}^q}-1\right)^{kw_j}\left(\tau^{W_H^q}-1\right)}{(\tau-1)^k}\right)\right)^{\frac{1}{q}}} \end{array} \right)$$

Furthermore,

$$k \cdot C\text{q - ROFFWA}(C_1, C_2, \ldots, C_n)$$

$$= \left( \begin{array}{l} \left( 1 - \log_\tau \left( 1 + \frac{\prod_{j=1}^{n} \left(\tau^{1-G_j^q} - 1\right)^{kw_j}}{(\tau - 1)^{k-1}} \right) \right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\frac{\prod_{j=1}^{n}\left(\tau^{1-W_{G_j}^q}-1\right)^{kw_j}}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}}}, \\ \left( \log_\tau \left( 1 + \frac{\prod_{j=1}^{n} \left(\tau^{H_j^q} - 1\right)^{kw_j}}{(\tau - 1)^{k-1}} \right) \right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\frac{\prod_{j=1}^{n}\left(\tau^{W_{H_j}^q}-1\right)^{kw_j}}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}}} \end{array} \right)$$

Then,

$$\mathrm{kCq\text{-}ROFFWA}(C_1, C_2, \ldots, C_n) \oplus C$$

$$= \left( \left(1 - \log_\tau\left(1 + \frac{\left(\frac{\prod_{j=1}^{n}\left(\tau^{1-G_j^q} - 1\right)^{kw_j}}{(\tau-1)^{k-1}}\right)\left(\tau^{1-G^q} - 1\right)}{(\tau-1)}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\frac{\left(\frac{\prod_{j=1}^{n}\left(\tau^{1-W_{G_j}^q}-1\right)^{kw_j}}{(\tau-1)^{k-1}}\right)\left(\tau^{1-W_G^q}-1\right)}{(\tau-1)}\right)\right)^{\frac{1}{q}}}, \left(\log_\tau\left(1+\frac{\left(\frac{\prod_{j=1}^{n}\left(\tau^{H_j^q} - 1\right)^{kw_j}}{(\tau-1)^{k-1}}\right)\left(\tau^{H^q} - 1\right)}{\tau - 1}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\frac{\left(\frac{\prod_{j=1}^{n}\left(\tau^{W_{H_j}^q}-1\right)^{kw_j}}{(\tau-1)^{k-1}}\right)\left(\tau^{W_H^q}-1\right)}{\tau-1}\right)\right)^{\frac{1}{q}}} \right)$$

$$= \left( \left(1 - \log_\tau\left(1 + \frac{\prod_{j=1}^{n}\left(\tau^{1-G_j^q} - 1\right)^{kw_j}\left(\tau^{1-G^q} - 1\right)}{(\tau-1)^k}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\frac{\prod_{j=1}^{n}\left(\tau^{1-W_{G_j}^q}-1\right)^{kw_j}\left(\tau^{1-W_G^q}-1\right)}{(\tau-1)^k}\right)\right)^{\frac{1}{q}}}, \left(\log_\tau\left(1+\frac{\prod_{j=1}^{n}\left(\tau^{H_j^q} - 1\right)^{kw_j}\left(\tau^{H^q} - 1\right)}{(\tau-1)^k}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\frac{\prod_{j=1}^{n}\left(\tau^{W_{H_j}^q}-1\right)^{kw_j}\left(\tau^{W_H^q}-1\right)}{(\tau-1)^k}\right)\right)^{\frac{1}{q}}} \right)$$

Then, we can get
$\mathrm{Cq\text{-}ROFFWA}(k \cdot C_1 \oplus C, k \cdot C_2 \oplus C, \ldots, k \cdot C_n \oplus C) = \mathrm{kCq\text{-}ROFFWA}(C_1, C_2, \ldots, C_n) \oplus C$.

***Proof of Theorem 10*** For the left-hand side of Theorem 10, we can have

$$kC_1 = \left( \left(1-\log_\tau\left(1+\frac{\left(\tau^{1-G_1^q}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-W^q_{G_1}}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}}}, \left(\log_\tau\left(1+\frac{\left(\tau^{H_1^q}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\frac{\left(\tau^{W^q_{H_1}}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}}} \right),$$

$$\mathrm{Cq\text{-}ROFFWA}(C_1, C_2, \ldots C_n) = \left( \left(1-\log_\tau\left(1+\prod_{j=1}^{n}\left(\frac{\left(\tau^{1-G_1^q}-1\right)^k}{(\tau-1)^{k-1}}\right)^{w_j}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\prod_{j=1}^{n}\left(\frac{\left(\tau^{1-\left(W_{G_j}\right)^q}-1\right)^k}{(\tau-1)^{k-1}}\right)^{w_j}\right)\right)^{\frac{1}{q}}}, \left(\log_\tau\left(1+\prod_{j=1}^{n}\left(\frac{\left(\tau^{H_j^q}-1\right)^k}{(\tau-1)^{k-1}}\right)^{w_j}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\prod_{j=1}^{n}\left(\frac{\left(\tau^{\left(W^q_{H_j}\right)}-1\right)^k}{(\tau-1)^{k-1}}\right)^{w_j}\right)\right)^{\frac{1}{q}}} \right)$$

Therefore, we can get

$$k\mathrm{Cq\text{-}ROFFWA}(C_1, C_2, \ldots, C_n) = \left( \left(1-\log_\tau\left(1+\frac{\left(\tau^{1-\left(1-\log_\tau\left(1+\prod_{j=1}^{n}\left(\tau^{1-G_j^q}-1\right)^{w_j}\right)\right)}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-\left(1-\log_\tau\left(1+\prod_{j=1}^{n}\left(\tau^{1-W^q_{G_j}}-1\right)^{w_j}\right)\right)}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}}}, \left(\log_\tau\left(1+\frac{\left(\tau^{\log_\tau\left(1+\prod_{j=1}^{n}\left(\tau^{H_j^q}-1\right)^{w_j}\right)}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\frac{\left(\tau^{\log_\tau\left(1+\prod_{j=1}^{n}\left(\tau^{W^q_{H_j}}-1\right)^{w_j}\right)}-1\right)^k}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}}} \right)$$

$$= \begin{pmatrix} \left(1-\log_\tau\left(1+\frac{\prod_{j=1}^{n}\left(\tau^{1-G_j^q}-1\right)^{kw_j}}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(1-\log_\tau\left(1+\frac{\prod_{j=1}^{n}\left(\tau^{1-W_{G_j}^q}-1\right)^{kw_j}}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}}}, \\ \left(\log_\tau\left(1+\frac{\prod_{j=1}^{n}\left(\tau^{H_j^q}-1\right)^{kw_j}}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}} e^{i2\pi\left(\log_\tau\left(1+\frac{\prod_{j=1}^{n}\left(\tau^{W_{H_j}^q}-1\right)^{kw_j}}{(\tau-1)^{k-1}}\right)\right)^{\frac{1}{q}}} \end{pmatrix}$$

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## Declarations

**Conflict of interest** The authors declare that there is no conflict of interest regarding the publication of this paper.

## References

Akram M, Ilyas F, Garg H (2020) Multi-criteria group decision making based on ELECTRE I method in Pythagorean fuzzy information. Soft Comput 24(5):3425–3453

Aydemir SB, Yilmaz Gündüz S (2020) Extension of multi-Moora method with some q-rung orthopair fuzzy Dombi prioritized weighted aggregation operators for multi-attribute decision making. Soft Comput 24(24):18545–18563

Akram M, Naz S, Edalatpanah SA (2021) Group decision-making framework under linguistic q-rung orthopair fuzzy Einstein models. Soft Comput 25(15):10309–10334

Alkouri AMJS, Salleh AR (2012) Complex intuitionistic fuzzy sets// AIP conference proceedings. Am Inst Phys 1482(1):464–470

Atanassov K, Gargov G (1989) Interval-valued intuitionistic fuzzy sets. Fuzzy Sets Syst 31(3):343–349

Chen TY, Wang HP, Lu YY (2011) A multi-criteria group decision-making approach based on interval-valued intuitionistic fuzzy sets: a comparative perspective. Expert Syst Appl 38(6):7647–7658

Chou TY, Chou S, Tzeng GH (2006) Evaluating IT/IS investments: a fuzzy multi-criteria decision model approach. Eur J Oper Res 173(3):1026–1046

Du YQ, Hou FJ, Zafar W (2017) A novel method for multiattribute decision making with interval-valued Pythagorean fuzzy linguistic information. Int J Intell Syst 32(10):1085–1112

Du YQ, Hou FJ (2018) Interval intuitionistic linguistic Frank aggregation operators and their application in decision making. Control Decis 33(01):119–125

Du YQ, Ren WJ, Du YH (2021) Some q-rung orthopair trapezoidal fuzzy linguistic hamacher aggregation operators and their applications. J Intell Fuzzy Syst 41(6):6285–6302

Fan ZP, Feng B (2009) A multiple attributes decision making method using individual and collaborative attribute data in a fuzzy environment. Inf Sci 179(20):3603–3618

Frank MJ (1979) On the simultaneous associativity of *F (x, y)* and *x+y− F (x, y)*. Aequ Math 19(1):194–226

Garg H, Ali Z, Mahmood T (2021a) Algorithms for complex interval-valued q-rung orthopair fuzzy sets in decision making based on aggregation operators, AHP, and TOPSIS. Expert Syst. https://doi.org/10.1111/exsy.12609

Garg H, Sirbiladze G, Ali Z (2021b) Hamy mean operators based on complex q-Rung Orthopair fuzzy setting and their application in multi-attribute decision making. Mathematics 9(18):1–29

Garg H (2021) New exponential operation laws and operators for interval-valued q-rung orthopair fuzzy sets in group decision making process. Neural Comput Appl 1:1–27. https://doi.org/10.1007/s00521-021-06036-0

Garg H, Naz S, Ziaa F (2021c) A ranking method based on Muirhead mean operator for group decision making with complex interval-valued q-rung orthopair fuzzy numbers. Soft Comput 25(22):14001–14027

Garg H (2016) Novel single-valued neutrosophic aggregated operators under Frank norm operation and its application to decision-making process. Int J Uncertain Quantif 6(4):361–375

Gou X, Xu Z, Ren P (2016) The properties of continuous Pythagorean fuzzy information. Int J Intell Syst 31(5):401–424

Gao J, Liang Z, Xu ZS (2019) Additive integrals of q -rung orthopair fuzzy functions. IEEE Trans Cybern 50(10):4406–4419

Ju YB, Luo C, Ma J (2019a) A novel multiple-attribute group decision-making method based on q-rung orthopair fuzzy generalized power weighted aggregation operators. Int J Intell Syst 34(9):2077–2103

Ju YB, Wang A, Ma J (2020) Some q-rung orthopair fuzzy 2-tuple linguistic Muirhead mean aggregation operators and their applications to multiple-attribute group decision making. Int J Intell Syst 35(1):184–213

Ju YB, Luo C, Ma J (2019b) Some interval-valued q-rung orthopair weighted averaging operators and their applications to multiple-attribute decision making. Int J Intell Syst 34(10):2584–2606

Ji P, Wang J, Zhang H (2018) Frank prioritized Bonferroni mean operator with single-valued neutrosophic sets and its application in selecting third-party logistics providers. Neural Comput Appl 30(3):799–823

Karmakar S, Seikh MR, Castillo O (2021) ype-2 intuitionistic fuzzy matrix games based on a new distance measure: Application to biogas-plant implementation problem. Appl Soft Comput 106:1–14. https://doi.org/10.1016/j.asoc.2021.107357

Liu PD, Ali Z, Mahmood T (2021a) Some cosine similarity measures and distance measures between complex q-rung orthopair fuzzy sets and their applications. Int J Comput Intell Syst 14(1):1653–1671

Liu PD, Mahmood T, Ali Z (2021b) Complex q-rung orthopair fuzzy Schweizer-Sklar Muirhead mean aggregation operators and their application in multi-criteria decision-making. J Intell Fuzzy Syst 3:1–23. https://doi.org/10.3233/JIFS-202440

Liu PD, Ali Z, Mahmood T (2021c) Generalized complex q-rung orthopair fuzzy Einstein averaging aggregation operators and their application in multi-attribute decision making. Complex Intell Syst 7(1):511–538

Liu PD, Ali Z, Mahmood T (2020a) Group decision-making using complex q-Rung orthopair fuzzy bonferroni mean. Int J Comput Intell Syst 13(1):822–851

Liu PD, Liu J, Chu Y (2018) Multi-attribute group decision making method based on neutrosophic trapezoidal fuzzy linguistic frank aggregation operators. J Intell Fuzzy Syst 35(1):779–791

Liu PD, Mahmood T, Ali Z (2020b) Complex q-rung orthopair fuzzy aggregation operators and their applications in multi-attribute group decision making. Information 11(1):5. https://doi.org/10.3390/INFO11010005

Liu Y, Wei G, Abdullah S et al (2021d) Banzhaf–Choquet-copula-based aggregation operators for managing q-rung orthopair fuzzy information. Soft Comput 25(10):6891–6914

Liu J, Fang M, Jin F (2021e) Pythagorean fuzzy linguistic decision support model based on consistency-adjustment strategy and consensus reaching process. Soft Comput 25(13):8205–8221

Liu PD, Wang P, Pedrycz W (2020c) Consistency-and consensus-based group decision-making method with incomplete probabilistic linguistic preference relations. IEEE Trans Fuzzy Syst 9(9):2565–2579

Liu PD, Li Y, Wang P (2022a) Opinion dynamics and minimum adjustment-driven consensus model for multi-criteria large-scale group decision making under a novel social trust propagation mechanism. IEEE Trans Fuzzy Syst. https://doi.org/10.1109/TFUZZ.2022.3186172

Liu PD, Zhang K, Wang P et al (2022b) A clustering-and maximum consensus-based model for social network large-scale group decision making with linguistic distribution. Inf Sci 602:269–297. https://doi.org/10.1016/j.ins.2022.04.038

Zadeh LA (1965) Fuzzy sets. Inf Control 8(3):338–353

Mahmood T, Ali Z (2021a) A novel approach of complex q-rung orthopair fuzzy hamacher aggregation operators and their application for cleaner production assessment in gold mines. J Ambient Intell Humaniz Comput 12(9):8933–8959

Mahanta J, Panda S (2021) Distance measure for Pythagorean fuzzy sets with varied applications. Neural Comput Appl 33(24):17161–17171

Ma X, Akram M, Zahid K (2021) Group decision-making framework using complex Pythagorean fuzzy information. Neural Comput Appl 33(6):2085–2105

Mahmood T, Ali Z (2021b) Entropy measure and TOPSIS method based on correlation coefficient using complex q-rung orthopair fuzzy information and its application to multi-attribute decision making. Soft Comput 25(2):1249–1275

Peng X, Yang Y (2015) Some results for Pythagorean fuzzy sets. Int J Intell Syst 30(11):1133–1160

Peng X, Yang Y (2016) Fundamental properties of interval-valued Pythagorean fuzzy aggregation operators. Int J Intell Syst 31(5):444–487

Peng H, Wang J, Cheng P (2018) A linguistic intuitionistic multi-criteria decision-making method based on the Frank Heronian mean operator and its application in evaluating coal mine safety. Int J Mach Learn Cybern 9(6):1053–1068

Qin J, Liu X (2014) Frank aggregation operators for triangular interval type-2 fuzzy set and its application in multiple attribute group decision making. J Appl Math 3:1–24. https://doi.org/10.1155/2014/923213

Rawat SS, Komal (2022) Multiple attribute decision making based on q-rung orthopair fuzzy Hamacher Muirhead mean operators. Soft Comput 26(5):2465–2487

Ramot D, Milo R, Friedman M, Kandel A (2002) Complex fuzzy sets. IEEE Trans Fuzzy Syst 10(2):171–186

Sarkar B, Biswas A (2020) A unified method for Pythagorean fuzzy multi-criteria group decision-making using entropy measure, linear programming and extended technique for ordering preference by similarity to ideal solution. Soft Comput 24(7):5333–5344

Shakeel M, Abdullah S, Aslam M (2020) Ranking methodology of induced Pythagorean trapezoidal fuzzy aggregation operators based on Einstein operations in group decision making. Soft Comput 24(10):7319–7334

Seikh MR, Mandal U (2021a) Q-rung orthopair fuzzy Frank aggregation operators and its application in multiple attribute decision-making with unknown attribute weights. Granular Comput. https://doi.org/10.1007/s41066-021-00290-2

Seikh MR, Mandal U (2021b) Some Picture Fuzzy Aggregation Operators based on Frank t-norm and t-conorm: Application to MADM Process. Informatica 45(3):447–461

Seikh MR, Mandal U (2022) Multiple attribute decision-making based on 3, 4-quasirung fuzzy sets. Granular Comput. https://doi.org/10.31449/inf.v45i3.3025

Seikh MR, Dutta S, Li DF (2021a) Solution of matrix games with rough interval pay-offs and its application in the telecom market share problem. Int J Intell Syst 36(10):6066–6100

Seikh MR, Karmakar S, Castillo O (2021b) A novel defuzzification approach of type-2 fuzzy variable to solving matrix games: An application to plastic ban problem. Iranian J Fuzzy Syst 18(5):155–172

Ullah K, Mahmood T, Ali Z (2020) On some distance measures of complex Pythagorean fuzzy sets and their applications in pattern recognition. Complex Intell Syst 6(1):15–27

Wang J, Wei GW, Lu J (2019) Some q-rung orthopair fuzzy Hamy mean operators in multiple attribute decision-making and their application to enterprise resource planning systems selection. Int J Intell Syst 34(10):2429–2458

Wang L, Shen Q, Zhu L (2016) Dual hesitant fuzzy power aggregation operators based on Archimedean t-conorm and t-norm and their application to multiple attribute group decision making. Appl Soft Comput 38:23–50. https://doi.org/10.1016/j.asoc.2015.09.012

Wang P, Liu PD, Chiclana F (2021) Multi-stage consistency optimization algorithm for decision making with incomplete probabilistic linguistic preference relation. Inf Sci 556:361–388. https://doi.org/10.1016/j.ins.2020.10.004

Xian S, Cheng Y (2021) Pythagorean fuzzy time series model based on Pythagorean fuzzy c-means and improved Markov weighted in the prediction of the new COVID-19 cases. Soft Comput 5(22):13881–13896

Xing Y, Zhang R, Zhou Z (2019) Some q-rung orthopair fuzzy point weighted aggregation operators for multi-attribute decision making. Soft Comput 23(22):11627–11649

Xia MM, Xu ZS, Zhu B (2012) Some issues on intuitionistic fuzzy aggregation operators based on Archimedean t-conorm and t-norm. Knowl-Based Syst 31:78–88. https://doi.org/10.1007/s00500-018-03712-7

Xu ZS (2007) Intuitionistic fuzzy aggregation operators. IEEE Trans Fuzzy Syst 15(6):1179–1187

Yahya M, Abdullah S, Chinram R (2021) Frank aggregation operators and their application to probabilistic hesitant fuzzy multiple attribute decision-Making. Int J Fuzzy Syst 23(1):194–215

Yager RR (2016) Generalized orthopair fuzzy sets. IEEE Trans Fuzzy Syst 25(5):1222–1230

Yager RR (2013) Pythagorean fuzzy subsets. In: Proceedings of the Joint IFSA World Congress and NAFIPS Annual Meeting, Edmonton, Canada, vol 24, pp 57–61. https://doi.org/10.1109/IFSA-NAFIPS.2013.6608375

Yager RR (2014) Pythagorean membership grades in multi-criteria decision making. IEEE Trans Fuzzy Syst 22(4):958–965

Yang MS, Ali Z, Mahmood T (2021) Complex q-rung orthopair uncertain linguistic partitioned Bonferroni mean operators with application in antivirus mask selection. Symmetry 13(2):249. https://doi.org/10.3390/sym13020249

Zhang ZM (2013) Interval-valued intuitionistic hesitant fuzzy aggregation operators and their application in group decision-making. J Appl Math 2013:377–384. https://doi.org/10.1155/2013/670285

Zhang XL, Xu ZS (2014) Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets. Int J Intell Syst 29(12):1061–1078

Zhou LP, Dong JY, Wan SP (2019) Two new approaches for multi-attribute group decision-making with interval-valued neutrosophic Frank aggregation operators and incomplete weights. IEEE Access 7:102727–102750. https://doi.org/10.1109/ACCESS.2019.2927133

Zhang X, Liu P, Wang Y (2015) Multiple attribute group decision making methods based on intuitionistic fuzzy frank power aggregation operators. J Intell Fuzzy Syst 29(5):2235–2246

Zhang H, Ma Q (2020) Three-way decisions with decision-theoretic rough sets based on Pythagorean fuzzy covering. Soft Comput 24(24):18671–18688

Zhang XL (2016a) Multi-criteria Pythagorean fuzzy decision analysis: a hierarchical QUALIFLEX approach with the closeness index-based ranking methods. Inf Sci 330:104–124. https://doi.org/10.1016/j.ins.2015.10.012

Zhang XL (2016b) A novel approach based on similarity measure for Pythagorean fuzzy multiple criteria group decision making. Int J Intell Syst 31(6):593–611. https://doi.org/10.1002/int.21796