Soft Computing (2022) 26:4673–4683
https://doi.org/10.1007/s00500-022-06995-z

FUZZY SYSTEMS AND THEIR MATHEMATICS

Generalized hesitant fuzzy numbers and their application in solving MADM problems based on TOPSIS method

Abazar Keikha[1]

Accepted: 4 March 2022 / Published online: 9 April 2022

Abstract

Generalized hesitant fuzzy numbers (GHFNs) are able to directly manage situations in which we may encounter a finite set of known values with a finite set of degrees of doubt as quantitative approximations of an uncertain situation/quantification of a linguistic expression. They are new extensions of hesitant fuzzy sets, which have been considered in this paper. In fact, in this paper, GHFNs will be utilized to model the uncertainty of the assessment values of options against criteria in multi-attribute decision making (MADM) problems. It means that all of the elements of decision matrix are GHFNs. Then, the technique for order of preference by similarity to ideal solution (TOPSIS) method, as a very successful method in solving MADM problems, will be updated to be used with GHFNs. To this end, the distance between GHFNs must be defined to obtain the distances between given alternatives from each of two subjective alternatives (positive/negative ideal solutions). Thus, three existing famous distance measures, i.e., general distance (d_g), Hamming distance (d_h), and Euclidean distance (d_e) measures, have been updated for GHFNs firstly. Then, the new TOPSIS method will be proposed based on GHFNs. Finally, the numerical examples have been appointed to illustrate the proposed method, analyze comparatively and validate it.

Keywords Generalized hesitant fuzzy numbers · Distance measures · Hesitant fuzzy numbers · Hesitant fuzzy sets · Multi-attribute decision making problems

1 Introduction

Uncertainty and decision making (Denoeux 2014; Bellman and Zadeh 1970), as two widely used concepts of human daily life, get more complicated with the progression of communication and technology. Quantification of qualitative data is only one source of uncertainty that can be modeled using fuzzy sets theory and its generalizations (Zadeh 1965; Atanassov 1983; Karnik and Mendel 2001; Faizi et al. 2018; Muhammad et al. 2020; Ibrahim et al. 2021; Al-shami et al. 2022). Hesitant fuzzy sets (HFSs) (Torra 2010), as the latest extension of fuzzy sets, are suitable for situations in which decision makers (DMs) are hesitated between a finite set of some values from [0,1], that is called hesitant fuzzy elements (HFEs). What is used in practical applications of HFSs are HFEs. Therefore, the development of mathematical principles of HFEs, such as comparison methods (Liao et al. 2014; Liao and Xu 2017), arithmetic and aggregation operations (Torra 2010; Liao and Xu 2014; Verma and Sharma 2013; Verma 2015; Xia and Xu 2011; Xu et al. 2013; Wei 2012; Zhang 2013; Liao and Xu 2014, 2015, 2017; Zhang 2016), determining the distance between two HFEs (Xu and Xia 2011a, b; Tong and Yu 2016), correlation coefficient and entropy measure (Xu and Xia 2011b; Tong and Yu 2016; Liao et al. 2015; Xu and Xia 2012), etc., was considered by researchers. Recently, various definitions of hesitant fuzzy numbers (HFNs), as the newly extension of HFSs, have been proposed. Deli (2020) and Deli and Karaaslan (2021) defined each element of a HFE as a trapezoidal fuzzy number which are common in the real parameters defining the trapezoid, but each has different heights, and called it generalized trapezoidal hesitant fuzzy (GTHF) numbers. Ranjbar et al. (2020) applied a finite set of fuzzy numbers from [0, 1] than crisp values from this interval as the elements of a HFE and called it hesitant fuzzy number. Keikha (2021a) combined a crisp positive real value with a finite set of some values between 0 and 1 and called it hesitant fuzzy number. In this regard, a HFN is displayed by $\langle a; \{\gamma_1, \gamma_2, \ldots, \gamma_n\}\rangle$, where a is a positive real value and $\gamma_i \in [0, 1],\ i = 1, 2, \ldots, n$.

✉ Abazar Keikha
abazar_keikha@yahoo.com

[1] Department of Mathematics, Velayat University, Iranshahr, Iran

These are some of the introduced tools to model uncertainty. Meta-heuristic optimization algorithms (Abualigah et al. 2021c) such as reptile search algorithm (RSA) (Abualigah et al. 2022), the arithmetic optimization algorithm (Abualigah et al. 2021a), applications, deployments, and integration of internet of drones (IoD) (Abualigah et al. 2021b) are another research fields for solving uncertain practical problems. Perhaps the most fundamental question be,"Why is the need for so much diversity in uncertainty modeling tools?' The answer is presented in the mixed language with delicacy of the eminent scholar (Pollack 2003). In referring to the diversity of sources of uncertainty, he refers to the "garden of uncertainty". From this, it can be understood that as we explore more the around environment, newer sources of uncertainty may be discovered, that in the past were either not accepted as new sources, and were wrongly modeled using existing tools, or were not discussed at all. Another similar answer has been given by Klir (2006), when he classified the real-world problems into organized simplicity problems, organized complexity problems, and disorganized complexity problems. In this scientific classification, he refers to organized complexity problems as a group that, despite covering a huge volume of experimental issues, has received less attention. This, of course, also confirms the previous claim of the wrong modeling of some problems through existing mathematical tools.

Another question is, do we really need generalized hesitant fuzzy numbers? What are the disadvantages of using the available tools in this field? The answers to these questions and the advantages of the proposed method of the article are described by examining an example below.

Consider solving the simple problem of determining the number of homeless people in a state. In fact, the reference set X would be defined as $X = \bigcup_i \{x_i\}$, in which x_i is the number of homeless people in ith city. Looking at the reports in this field does not indicate a definite number, and we will encounter a finite set of numbers as $x_i = \{a_{i1}, a_{i2}, \ldots, a_{im}\}$. The DMs are not able to sift through these values to remove all but one value, and therefore, accept all values with degrees of doubt that expressed as a HFE $\{\gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{in}\}$, interpreted as different degrees of acceptance x_i. It is showed that the given problem is an organized complexity problem, because it has finite factors. Variety of math tools from simple to complex are offered for modeling such values. Averaging is one of the simplest and of course the oldest methods in such situations. In this case, if we consider only the mean of the values, the problem is transferred from the inaccurate space to the exact one. If we simultaneously consider the mean of the values and the mean of the degrees of doubt, the result is a discrete fuzzy number, and the given organized complexity problem is transferred to the category of organized simplicity problems, too. Also, we can consider x_i as an interval $x_i = [a_{i(1)}, a_{i(m)}]$, in which $a_{i(1)} = \min\{a_{i1}, a_{i2}, \ldots, a_{im}\}$, and $a_{i(m)} = \max\{a_{i1}, a_{i2}, \ldots, a_{im}\}$. Then, we will have an infinite set of values, but with the same membership grades as 1. Furthermore, modeling can be done using type-1/interval type-2/intuitionistic fuzzy numbers. In all of these modeling methods, the two finite sets of values and degrees of membership are transformed into two infinitely corresponding sets. In the case of using interval values, and various types of fuzzy numbers, the problem space is changed from discrete to continuous, and a new problem is created and solved instead of the original. This transition, although it provides many tools for mathematical analysis, but in addition to increasing uncertainty, practically leaves the main problem unresolved. In other words, by changing the nature of the real-world' problems from organized complexity problems to disorganized complexity problems, a large volume of them have been remained intact (Klir 2006).

To model such situations, where the DMs are hesitated between a finite set of positive real values with a finite set of hesitation degrees, the HFNs have been extended to generalized HFNs (GHFNs) (Keikha 2021b). Let $h(A) = \{a_1, a_2, \ldots, a_m\}$ be the finite set of possible values which are hesitated by $mh(A) = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$. By merging these two parts, a GHFN $\tilde{\tilde{A}} = \langle h(A); mh(A) \rangle = \langle \{a_1, a_2, \ldots, a_m\}; \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ is obtained, in which $h(A)$ called the real part of GHFN $\tilde{\tilde{A}}$, and $mh(A)$ is its membership part. For example, in homeless problem, the number of homeless people in ith city can be expressed by GHFN $\tilde{\tilde{x}}_i = \langle \{a_{i1}, a_{i2}, \ldots, a_{im}\}; \{\gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{in}\} \rangle$. It can be seen that in this modeling method, i.e., applying GHFNs, the problem space did not change and practically the data in the same original format are used without the slightest change. On the other hand, another important feature of GHFNs is the conversion of the opinions of a group of DMs on a single issue in the form of a GHFN, which may themselves have some degrees of uncertainty, too. It is hoped that with the development of the mathematical methodology of generalized hesitant fuzzy numbers and their practical application, a large number of problems that in Klir' belief have been remained untouched (Klir 2006) will be solved. GHFNs can be used in various fields, such as mechanism design, performance evaluation of employees/organizations/companies, future studies, stock markets, future markets, supply chain, society networks, cosmology, medical diagnostic and telemedicine (Glaz et al. 2021), decision making, and planning.

MADM problems are constructed of a finite set of alternatives/options, and a finite set of attributes/criteria. In many of the proposed methods for solving MADM problems, alternatives evaluation against to all criteria is the first step (Tzeng and Huang 2011; Zhang et al. 2020). Due to the fact that subjective judgments and evaluations always have some degrees of uncertainty, the development of these methods in inaccurate environments was considered by researchers, which led to the introduction of fuzzy-based methods (for different types of fuzzy sets) (Palczewski and Sałabun 2019; Garg et al. 2020; Nan and Zhang 2014; Xu and Zhang 2012; Sun and Ouyang 2015; Aggrawal 2021; Atef et al. 2021; Al-shami 2021). Therefore, the analytical hierarchy process (AHP) method, the analytical network process (ANP) method, TOPSIS method, Choquet integral (CI) method, viekriterijumsko kompromisno rangiranje (VIKOR) method, simple additive weighting (SAW) method, elimination and choice expressing reality (ELECTERE) method, etc. (Tzeng and Huang 2011) have been updated to be applied with the various uncertainty theories. As the hesitant extension of such methods we can refer to, hesitant fuzzy TOPSIS (HFTOPSIS) method (Xu and Zhang 2012), hesitant fuzzy VIKOR (HFVIKOR) method (Liao and Xu 2013), hesitant fuzzy power average-based method (Liao et al. 2018), hesitant fuzzy COMET (Faizi et al. 2018), hesitant fuzzy preference relation (Liao et al. 2014), hesitant fuzzy aggregation operators (Wei 2012; Zhang 2013; Liao and Xu 2014, 2015), some approaches to hesitant fuzzy MADM problems with incomplete weight information (Wei et al. 2014).

Also, utilizing HFNs, a hybrid technique TOPSIS-CI based on combining Choquet integral (CI) and TOPSIS methods (Garg et al. 2020), and mean-based averaging methods (Keikha 2021a), is proposed to solve multi-attribute group decision making (MAGDM) problems. As we know, the TOPSIS method is completely dependent to distance measures. So, in each expansion of it to inaccurate environment, it is necessary to develop the distance measures, firstly (Xu and Xia 2011a, b; Tong and Yu 2016; Li et al. 2015).

GHFNs may be used in some practical MADM problems to construct decision matrix or model the human' assessments. We can replace them by one of the popular types of fuzzy numbers/intuitionistic fuzzy numbers/ interval numbers, etc., and then use an appropriate method for solving the given problem. As discussed earlier, converting the given data to one of the previous types will cause us to solve another problem instead of the main problem and practically leave the real problem unresolved. Therefore, the best way is to use the data directly in the solution process, which of course requires the development of existing methods and updating some of the required concepts. In this paper, TOPSIS method, as a widely used and popular method in other fields, will be extended to be used with GHFNs. To do this, we need to find a way to determine the distance between the GHFNs. Generalized distance measure, Hamming distance measure, and Euclidean distance measure are three successful distance measure which are used before along with other types of fuzzy numbers. These important distance measures will be generalized to determine the distances of GHNs, firstly. We will then update this method for use with generalized hesitant fuzzy numbers by redefining each step of the common TOPSIS method.

Table 1 List of some used abbreviation symbols in this article

Abbreviation symbols	Description
HFNs	Hesitant fuzzy numbers
HFSs	Hesitant fuzzy sets
HFEs	Hesitant fuzzy elements
GHFNs	Generalized hesitant fuzzy numbers
TOPSIS	Technique for order of preference by similarity to ideal solution
MADM	Multi attribute decision making
AGHFNs	Adjusted generalized hesitant fuzzy numbers
d_e	Euclidean distance
d_h	Hamming distance
d_g	Generalized distance

The main advantage of this method is that it does not change the nature of the data and solves exactly the current problem. But other previous methods, by being in such situations, first change the nature of the data through some common simplifications such as averaging or generalization such as interval/fuzzy numbers. Therefore, their obtained answer is not suitable for the current problem because it was obtained in a space other than the real space.
Also, converting and solving a MAGDM problem into a MADM problem will be done easily, by considering the group evaluations of experts from an option against a criterion as a GHFN.

Due to the many used abbreviation symbol in this article, Table 1 is provided with the necessary explanations for easier use.

So, the remainder of this article is organized as follows. HFSs, HFNs and GHFNs and some related concepts , such as distance measures of HFEs and HFNs, arithmetic operations of GHFNs, that are required later in this article, will be discussed in Sect. 2. Section 3 presents methods to determine the Euclidean distance (d_e), Hamming distance (d_h), generalized distance (d_g) between two given GHFNs. Section 4 consists the general structure of MADM problems, the traditional TOPSIS method, and proposed a novel TOPSIS method to be used with GHFNs. The proposed method will be illustrated by numerical examples, validated and analyzed comparatively in Sect. 5. The concluding remarks consisting of describing academic implications, major findings, and directions for future research as the conclusion section of this article are given in Sect. 6.

2 Hesitant fuzzy sets and hesitant fuzzy numbers

HFSs, HFNs and GHFNs are reviewed in this section.

HFSs theory has been proposed to model uncertainty, where decision maker is hesitated between a finite set of some values from [0, 1]. Let $h(x) = \{\gamma_i | i = 1, 2, \ldots, n; \gamma_i \in [0, 1]\}$, and X be a reference set. Then, $E = \{< x, h(x) > | x \in X\}$ is called a HFS, in which $h(x)$ is called a hesitant fuzzy element (HFE) (Xia and Xu 2011). HFEs are not necessarily the same in length, and there exist some methods to adjusted them. Consider two HFEs h_1 and h_2, where $|h_1| = m$ and $|h_2| = n$ with $m < n$. To adjust them, it is real to add $n-m$ elements to the HFE h_1. In the optimistic mode, the maximum element of h_1, in the pessimistic approach its minimum element, in the indifference approach the value of 0.5 (Liao and Xu 2017), and otherwise the power average of the available elements in h_1 (Liao et al. 2018), proposed to be iterated $n - m$ times in the set h_1.

As one of the most important and fundamental concepts in HFSs theory, we can refer to distance measure. Li et al. (2015) and Li et al. (2015) addressed some axiomatic definition and distance measures.

Definition 1 Let h_1, h_2 and h_3 be three arbitrary HFEs. A distance measure d for HFSs must be satisfied the following properties:

(i) $0 \le d(h_1, h_2) \le 1$;
(ii) $d(h_1, h_2) = 0 \quad \Longleftrightarrow \quad h_1 = h_2$;
(iii) $d(h_1, h_2) = d(h_2, h_1)$;
(iv) $d(h_1, h_2) \le d(h_1, h_3) + d(h_3, h_2)$;
(v) $h_1 \preccurlyeq h_2 \preccurlyeq h_3 \;\Rightarrow d(h_1, h_2) \le d(h_1, h_3)$ & $d(h_2, h_3) \le d(h_1, h_3)$.

Computing the distance between given HFSs has also discussed by many researchers, and there are many methods in this regard (Xu and Xia 2011a, b; Tong and Yu 2016; Li et al. 2015).

Definition 2 (Xu and Xia 2011a) Consider two adjusted HFEs h_1 and h_2. Let $h_{j(i)}$ be the ith smallest value of h_j and l be the maximum length of the given HFEs, then

(i) $d_{hnh}(h_1, h_2) = \frac{1}{l}\sum_{i=1}^{l} |h_{1(i)} - h_{2(i)}|$;
(ii) $d_{hne}(h_1, h_2) = \sqrt{\frac{1}{l}\sum_{i=1}^{l} |h_{1(i)} - h_{2(i)}|^2}$;
(iii) $d_{hng}(h_1, h_2) = \left[\frac{1}{l}\sum_{i=1}^{l} |h_{1(i)} - h_{2(i)}|^{\lambda}\right]^{\frac{1}{\lambda}}, \quad \lambda > 0$;
(iv) $d_{ghnh}(h_1, h_2) = \left[\max_i |h_{1(i)} - h_{2(i)}|^{\lambda}\right]^{\frac{1}{\lambda}}, \quad \lambda > 0$;

are called hesitant normalized Hamming, Euclidean, generalized, and Hausdorff distances, respectively.

In some other uncertain problems, such as self-assessments, the DMs may be encounter with some crisp predetermined/recorded values which are not accepted completely and must be considered during evaluations. Hesitant fuzzy numbers, as the generalization of HFSs, are suitable to model such problems (Keikha 2021a; Garg et al. 2020). A HFN consists of two parts: real part, and HFE part, i.e., $\tilde{E} = \langle a; \{\gamma_1, \gamma_2, \ldots, \gamma_n\}\rangle$ is called a HFN, in which a is a positive real number, and $\gamma_i \in [0, 1]$ are hesitation/satisfaction degrees.

Definition 3 (Garg et al. 2020; Keikha 2021c) Let $\tilde{a}_H = \langle a, h(a)\rangle$ and $\tilde{b}_H = \langle b, h(b)\rangle$ be two arbitrary HFNs where, $a, b \in \mathbb{R}$, $h(a)$ with cardinality $|h(a)| = k$ and $h(b)$ with cardinality $|h(b)| = l$ are two sets of some values in [0, 1]. Their Euclidean distance (d_e) and Hamming distance (d_h) were be defined as follows:

$$d_e(\tilde{a}_H, \tilde{b}_H) = \sqrt{\frac{1}{1 + k \times l}\left(|a-b|^2 + \sum_{\substack{\gamma_i \in h(a),\\ \gamma_j \in h(b)}} |\gamma_i - \gamma_j|^2\right)},$$

$$d_h(\tilde{a}_H, \tilde{b}_H) = \frac{1}{1 + k \times l}\left(|a-b| + \sum_{\substack{\gamma_i \in h(a),\\ \gamma_j \in h(b)}} |\gamma_i - \gamma_j|\right).$$

In some other problems, it may be that the real part of a HFN is not a crisp value and expressed by a finite set of real values. In such cases, HFNs must be extended to what called GHFNs. In fact, $\tilde{\tilde{A}}^H = \langle\{a_1, a_2, \ldots, a_n\}; \{\gamma_1, \gamma_2, \ldots, \gamma_m\}\rangle = \langle h(A), mh(A)\rangle$, in which $a_i,\ i = 1, 2, \ldots, n$ are positive real values, and $\gamma_j \in [0, 1],\ j = 1, 2, \ldots, m$ are membership/satisfaction degrees, is nominated as a GHFN. It is easy to see that a GHFN is called a HFN if $|h(A)| = 1$, or a discrete fuzzy number if $|h(A)| = |mh(A)| = 1$.

GHFNs $\tilde{\tilde{A}}^H = \langle\{a_1, a_2, \ldots, a_m\}; \{\gamma_1, \gamma_2, \ldots \gamma_n\}\rangle$ and $\tilde{\tilde{B}}^H = \langle\{b_1, b_2, \ldots, b_k\}; \{\lambda_1, \lambda_2, \ldots, \lambda_l\}\rangle$ are called adjusted GHFNs (AGHFNs), if $m = k$ and $n = l$. If not so, without lose of generality let $m < k$ or $n < l$, $a^{max} = \max\{a_1, a_2, \ldots, a_n\}$, $a^{min} = \min\{a_1, a_2, \ldots, a_n\}$, $\gamma^{max} = \max\{\gamma_1, \gamma_2, \ldots, \gamma_m\}$, and $\gamma^{min} = \min\{\gamma_1, \gamma_2, \ldots, \gamma_m\}$. Then, $\tilde{\tilde{A}}^H$ must be extended: for optimistic DM, a^{max} or γ^{max} will be repeated $k-m$ or $l-n$ times in $\{a_1, a_2, \ldots, a_n\}$ or $\{\gamma_1, \gamma_2, \ldots \gamma_m\}$, respectively. It is done by a^{min} or γ^{min} for pessimistic DM, and $\overline{a} = \frac{a_1+a_2+\cdots+a_m}{m}$ or 0.5 for indifference DM, similarly.

Definition 4 (Keikha 2021b) Consider two AGHFNs $\tilde{\tilde{A}}^H = \langle\{a_1, a_2, \ldots, a_m\}; \{\gamma_1, \gamma_2, \ldots, \gamma_n\}\rangle$ and $\tilde{\tilde{B}}^H = \langle\{b_1, b_2, \ldots, b_m\}; \{\lambda_1, \lambda_2, \ldots, \lambda_n\}\rangle$. Then, for any positive real value w we have:

(i) $w\tilde{\tilde{A}}^H = \langle \{wa_1, wa_2, \ldots, wa_m\}; \{\gamma_1, \gamma_2, \ldots, \gamma_n\} \rangle$, $w > 0$;
(ii) $(\tilde{\tilde{A}}^H)^w = \langle \{(a_1)^w, (a_2)^w, \ldots, (a_m)^w\}; \{\gamma_1, \gamma_2, \ldots, \gamma_n\} \rangle$, $w > 0$;
(iii) $\tilde{\tilde{A}}^H \oplus \tilde{\tilde{B}}^H = \langle \cup_i \{a_{(i)} + b_{(i)}\}; mh(A) \cup mh(B) \rangle$;
(iv) $\tilde{\tilde{A}}^H \otimes \tilde{\tilde{B}}^H = \langle \cup_i \{a_{(i)} b_{(i)}\}; \cup_i \min\{\gamma_i, \lambda_i\} \rangle$;

where $a_{(1)}, a_{(2)}, \ldots, a_{(m)}$ is a permutation of $a_1, a_2, \ldots, a_m$ such that $a_{(1)} \le a_{(2)} \le \cdots \le a_{(m)}$, and $b_{(1)}, b_{(2)}, \ldots, b_{(m)}$ is a permutation of $b_1, b_2, \ldots, b_m$ such that $b_{(1)} \le b_{(2)} \le \cdots \le b_{(m)}$.

GHFNs are at the starting point of a long and dark way, and like any new scientific subject, they need much attempts in both theoretical and application aspects. In order to propose a TOPSIS-based method for solving uncertain MADM problems that are classified in organized complexity problems, we need to define the distance measure of GHFNs. These process will be done in the next sections.

3 Distance functions of GHFNs

In some practical applications of GHFNs, such as decision-making problems, it is needed to determine the distances between them. Generalized, Hamming and Euclidean distances are the most famous which have been updated for different types of fuzzy numbers and are used (Guha and Chakraborty 2010b, a; Sang et al. 2014). These distance functions will be updated to be used with GHFNs, in this Section.

Definition 5 Let $\tilde{\tilde{A}}^H = \langle \{a_1, a_2, \ldots, a_n\}; \{\gamma_1, \gamma_2, \ldots, \gamma_m\} \rangle$, and $\tilde{\tilde{B}}^H = \langle \{b_1, b_2, \ldots, b_n\}; \{\lambda_1, \lambda_2, \ldots, \lambda_m\} \rangle$ be two arbitrary given AGHFNs. Then, their Generalized distance $d_g\left(\tilde{\tilde{A}}^H, \tilde{\tilde{B}}^H\right)$, Hamming distance $d_h\left(\tilde{\tilde{A}}^H, \tilde{\tilde{B}}^H\right)$, and Euclidean distance $d_e\left(\tilde{\tilde{A}}^H, \tilde{\tilde{B}}^H\right)$ can be defined as follows:

$$d_g\left(\tilde{\tilde{A}}^H, \tilde{\tilde{B}}^H\right) = \Big[\frac{1}{n}\sum_{i=1}^{n} |a_{(i)} - b_{(i)}|^{\lambda} + \frac{1}{m}\sum_{j=1}^{m} |\gamma_{(j)} - \lambda_{(j)}|^{\lambda}\Big]^{\frac{1}{\lambda}},$$
$$d_h\left(\tilde{\tilde{A}}^H, \tilde{\tilde{B}}^H\right) = \frac{1}{n}\sum_{i=1}^{n} |a_{(i)} - b_{(i)}| + \frac{1}{m}\sum_{j=1}^{m} |\gamma_{(j)} - \lambda_{(j)}|,$$
$$d_e\left(\tilde{\tilde{A}}^H, \tilde{\tilde{B}}^H\right) = \sqrt{\frac{1}{n}\sum_{i=1}^{n} |a_{(i)} - b_{(i)}|^2 + \frac{1}{m}\sum_{j=1}^{m} |\gamma_{(j)} - \lambda_{(j)}|^2},$$

where $._{(t)}$ is the tth largest value in its corresponding set.

Example 1 Let $\tilde{\tilde{A}}^H = \langle \{2, 3, 4, 5\}; \{.4, .6, .7, .9\} \rangle$ and $\tilde{\tilde{B}}^H = \langle \{1, 3, 6, 7\}; \{.5, .6, .8, 1\} \rangle$. Then,

$$d_h(\tilde{\tilde{A}}^H, \tilde{\tilde{B}}^H) = \frac{1}{4}(1+0+2+2) + \frac{1}{4}(.1+0+.1+.1) = 1.325,$$
$$d_e(\tilde{\tilde{A}}^H, \tilde{\tilde{B}}^H) = \sqrt{\frac{1}{4}(1+0+4+4) + \frac{1}{4}(.01+0+.01+.01)} = 1.503,$$
Get $\lambda = 3$, $d_g(\tilde{\tilde{A}}^H, \tilde{\tilde{B}}^H) = [\frac{1}{4}(1^3+0^3+2^3+2^3) + \frac{1}{4}(.001+0+.001+.001)]^{\frac{1}{3}} = 1.62.$

Remark The generalized distance measure $d_g\big(\tilde{\tilde{A}}^H, \tilde{\tilde{B}}^H\big)$ is reduced to Hamming distance measure $d_h\big(\tilde{\tilde{A}}^H, \tilde{\tilde{B}}^H\big)$ if $\lambda = 1$, and for $\lambda = 2$ it is called Euclidean distance measure $d_e\big(\tilde{\tilde{A}}^H, \tilde{\tilde{B}}^H\big)$, in which $\tilde{\tilde{A}}^H$ and $\tilde{\tilde{B}}^H$ are arbitrary GHFNs.

Theorem *Let $\tilde{\tilde{A}}^H = \langle \{a_1, a_2, \ldots, a_n\}; \{\gamma_1, \gamma_2, \ldots, \gamma_m\} \rangle$, $\tilde{\tilde{B}}^H = \langle \{b_1, b_2, \ldots, b_n\}; \{\lambda_1, \lambda_2, \ldots, \lambda_m\} \rangle$, and $\tilde{\tilde{C}}^H = \langle \{c_1, c_2, \ldots, c_n\}; \{\nu_1, \nu_2, \ldots, \nu_m\} \rangle$, be arbitrary given AGHFNs. Then,*

(i) $d(\tilde{\tilde{A}}^H, \tilde{\tilde{A}}^H) = 0$;
(ii) $d(\tilde{\tilde{A}}^H, \tilde{\tilde{B}}^H) = d(\tilde{\tilde{B}}^H, \tilde{\tilde{A}}^H)$;
(iii) $d(\tilde{\tilde{A}}^H, \tilde{\tilde{C}}^H) \le d(\tilde{\tilde{A}}^H, \tilde{\tilde{B}}^H) + d(\tilde{\tilde{B}}^H, \tilde{\tilde{C}}^H)$;

where the distance function d can be interpreted as d_g, d_h, and d_e.

Proof For any $x, y, z \in \mathbb{R}$, it is easy to see that $|x-y| = |y-x|$, and $|x-y| \le |x-z| + |z-y|$. Using these properties, and regarding to the definitions of distance functions d_g, d_h, and d_e, the proof is obvious. □

4 MADM problems and GHFNs

In this section, as a practical application of GHFNs, they will be used to model the uncertainty contained in a MADM problem. Then, the TOPSIS method, as a widely used method in solving MADM problem, will be generalized to the new type of hesitant fuzzy numbers, i.e., GHFNs.

Let there exist r candidates $C_1, C_2, \ldots, C_r$ to be ranked based on s attributes $\mathsf{a}_1, \mathsf{a}_2, \ldots, \mathsf{a}_s$, in ascending order. Usually each candidate is evaluated against to all attribute, and arranged in a matrix, called decision matrix, i.e., $D = [d_{ij}]_{r \times s}$, in which d_{ij} means the evaluation value of ith candidate against to jth attribute. Because of existing uncertainty in any subjective judgment or other qualitative evaluations, they must be modeled to quantitative values for doing extra mathematical processes. In this paper, GHFNs have been used in the modeling process. Then, in the solving process, the TOPSIS method will be extended to do it.

4.1 The traditional TOPSIS method

The TOPSIS method (Tzeng and Huang 2011) is based on the following steps:

Step 1 Constructing the decision matrix $D = [d_{ij}]_{r\times s}$ via the assessment of all candidates against to all attributes, are expressed by crisp values.

Step 2 Normalizing decision matrix to $N = [n_{ij}]_{r\times s}$, called normalized decision matrix:

$$n_{ij} = \begin{cases} \frac{d_{ij}-d_j^{min}}{d_j^{max}-d_j^{min}}, \ j \in B; \\ \frac{d_j^{max}-d_{ij}}{d_j^{max}-d_j^{min}}, \ j \in C. \end{cases}$$

where, B is the set of attributes with positive aspect or benefit attributes (the larger is better), and C is the set of attributes with negative aspect or cost attributes (the smaller is better).

Step 3 Calculating weighted normalized decision matrix $NW = [n_{ij} \times w_i]_{r\times s} = [v_{ij}]$, if the attributes are weighted by weight vector $W = (w_1, w_2, \ldots, w_n)$.

Step 4 Determining positive ideal solution (PIS), and negative ideal solution (NIS) as two subjective alternatives:

$$\begin{aligned} A^+ &= \{p_1^+, p_2^+, \ldots, p_s^+\} \\ &= \{(max_i v_{ij} | j \in B)\&(min_i v_{ij} | j \in C)\}, \\ A^- &= \{p_1^-, p_2^-, \ldots, p_s^-\} \\ &= \{(min_i v_{ij} | j \in B)\&(max_i v_{ij} | j \in C)\}. \end{aligned}$$

Step 5 Calculating the distance of each alternative from subjective alternatives:

$$S_i^+ = \sqrt{\sum_{j=1}^{s}(v_{ij} - p_j^+)^2}, \qquad S_i^- = \sqrt{\sum_{j=1}^{s}(v_{ij} - p_j^-)^2}, \quad i = 1, 2, \ldots, r.$$

Step 6 Computing the relative distances $R_i = \frac{S_i^-}{S_i^+ + S_i^-}$, $i = 1, 2, \ldots, r$, and rank them.

Step 7 Reorder the candidates C_i according to order of R_i, $(i = 1, 2, \ldots, r)$.

4.2 Updating TOPSIS method with GHFNs

In this subsection, the traditional TOPSIS method will be extended to be applicable with GHFNs.

Step 1 Evaluation phase All options will be evaluated against to all criteria, and the assessment values of each option, which are allowed to be expressed by qualitative/linguistic terms, or uncertain values, are arranged in a decision matrix $D = [d_{ij}]_{r\times s}$.

Step 2 Modeling phase GHFNs have been applied to model uncertain elements of decision matrix or quantify the qualitative assessments. Then, the decision matrix $D = [d_{ij}]_{r\times s}$ converted to the hesitant decision matrix $H\tilde{\tilde{D}} = [\tilde{\tilde{d}}_{ij}]_{r\times s}$.

Step 3 Adjusting phase The GHFNs containing in the hesitant decision matrix $H\tilde{\tilde{D}}$ should be adjusted, as in $\tilde{\tilde{d}}_{ij} = \left\langle \{a_{ij}^{(1)}, a_{ij}^{(2)}, \ldots, a_{ij}^{(m)}\}; \{\gamma_{ij}^{(1)}, \gamma_{ij}^{(2)}, \ldots, \gamma_{ij}^{(n)}\} \right\rangle$, where $a_{ij}^{(l)}$, and$\gamma_{ij}^{(l)}$ are lth largest value in their corresponding set.

Step 4 Normalizing phase The elements of hesitant decision matrix $H\tilde{\tilde{D}}$ must be scale-less. It is done via the normalizing process, and normalized hesitant decision matrix $NH\tilde{\tilde{D}} = [\tilde{\tilde{n}}_{ij}]_{r\times s}$, in which $\tilde{\tilde{n}}_{ij} = \left\langle \{\alpha_{ij}^{(1)}, \alpha_{ij}^{(2)}, \ldots, \alpha_{ij}^{(m)}\}; \{\gamma_{ij}^{(1)}, \gamma_{ij}^{(2)}, \ldots, \gamma_{ij}^{(n)}\} \right\rangle$, will be resulted as:

$$\alpha_{ij}^{(l)} = \begin{cases} \frac{a_{ij}^{(l)}-a_j^{(l)min}}{a_j^{(l)max}-a_j^{(l)min}} \ j \in B; \\ \qquad\qquad\qquad\qquad l = 1, 2, \ldots, m \\ \frac{a_j^{(l)max}-a_{ij}^{(l)}}{a_j^{(l)max}-a_j^{(l)min}} \ j \in C. \end{cases}$$

where, $a_j^{(l)min} = \min_i\{a_{ij}^{(l)}\}$, $a_j^{(l)max} = \max_i\{a_{ij}^{(l)}\}$.

Step 5 Weighted phase If the attributes are weighted by weight vector $W = (w_1, w_2, \ldots, w_n)$, the weighted normalized decision matrix $NW = [w_i \times \tilde{\tilde{n}}_{ij}]_{r\times s} = [\tilde{\tilde{v}}_{ij}]$, $\left(\tilde{\tilde{v}}_{ij} = \left\langle \{v_{ij}^{(1)}, v_{ij}^{(2)}, \ldots, v_{ij}^{(m)}\}; \{\gamma_{ij}^{(1)}, \gamma_{ij}^{(2)}, \ldots, \gamma_{ij}^{(n)}\} \right\rangle\right)$ will be constructed.

Step 6 Subjective options Two subjective alternatives PIS and NIS must be determined using the proposed ranking function in this paper:

$$\begin{aligned} A^+ &= \{\tilde{\tilde{p}}_1^+, \tilde{\tilde{p}}_2^+, \ldots, \tilde{\tilde{p}}_s^+\} \\ &= \{(max_i \tilde{\tilde{v}}_{ij} | j \in B)\&(min_i \tilde{\tilde{v}}_{ij} | j \in C)\}, \end{aligned}$$

$$\begin{aligned} A^- &= \{\tilde{\tilde{p}}_1^-, \tilde{\tilde{p}}_2^-, \ldots, \tilde{\tilde{p}}_s^-\} \\ &= \{(min_i \tilde{\tilde{v}}_{ij} | j \in B)\&(max_i \tilde{\tilde{v}}_{ij} | j \in C)\}, \end{aligned}$$

in which,

$$max_i \tilde{\tilde{v}}_{ij} = \left\langle \{max_i v_{ij}^{(1)}, max_i v_{ij}^{(2)}, \ldots, max_i v_{ij}^{(m)}\}; \{max_i \gamma_{ij}^{(1)}, max_i \gamma_{ij}^{(2)}, \ldots, max_i \gamma_{ij}^{(n)}\} \right\rangle,$$

and

$$min_i \tilde{\tilde{v}}_{ij} = \left\langle \{min_i v_{ij}^{(1)}, min_i v_{ij}^{(2)}, \ldots, min_i v_{ij}^{(m)}\}; \{min_i \gamma_{ij}^{(1)}, min_i \gamma_{ij}^{(2)}, \ldots, min_i \gamma_{ij}^{(n)}\} \right\rangle. \quad (1)$$

Step 7 Distancing phase Utilizing one of the proposed distance functions in this paper, the distance between each alternative and two subjective alternatives must be computed:

$$S_i^+ = \sum_{j=1}^{s} d(\tilde{\tilde{v}}_{ij} - \tilde{\tilde{p}}_j^+), \qquad S_i^- = \sum_{j=1}^{s} d(\tilde{\tilde{v}}_{ij} - \tilde{\tilde{p}}_j^-), \quad i = 1, 2, \ldots, r.$$

Step 8 Relative distance Computing the relative distances $R_i = \frac{S_i^-}{S_i^+ + S_i^-}$, $i = 1, 2, \ldots, r$, and rank them.

Step 9 Ranking phase Reorder the candidates C_i according to order of R_i, $(i = 1, 2, \ldots, r)$.

5 Numerical example

Example 2 (Energy project selection) (Xu and Zhang 2012) Let there exist five energy projects (candidates) $C_i (i = 1, 2, 3, 4, 5)$ to be ranked based on attributes: a_1 (technological), a_2 (environmental), a_3 (socio-political), and a_4 (economic). It is clear that in the real world, different factors influence a decision. In this example, considering the different effects of such projects on the environment, people, research units and industries, in addition to the opinions of experts, the opinions of various social groups, universities and NGOs are also used. Given that historical data, different reports from different social/scientific/economical/NGOs (non-governmental organization) sources, and even existing documents state different values for each option against each criterion, the director not only wants to use all of them without eliminating some of them in decision making, but also asks the evaluators (experts) to express their opinions and even increase the number of these values, as $h(A) = \{a_1, a_2, \ldots, a_m\}$. Then, decision makers express their hesitation (satisfaction) degrees of these values with a finite set of values between zero and one, as in $mh(A) = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$, $\gamma_i \in [0, 1]$. It means that each evaluation value is a GHFNs $\tilde{\tilde{A}} = \langle \{a_1, a_2, \ldots, a_m\}; \{\gamma_1, \gamma_2, \ldots, \gamma_n\} \rangle$.

Thus, each element of the decision matrix, as shown in Table 2, has its own interpretation, and consists of two parts: the first part contains the values of direct evaluation, and the second part includes the degrees of skepticism of experts from the first part. For instance, the real part of $\tilde{\tilde{A}}_{11} = \langle \{8, 7, 9, 10\}; \{0.3, 0.4, 0.5\} \rangle$ shows the technological advantages of the first energy project based on the available documents and reports provided by various units. Which, of course, after studying and evaluating by the experts, the allocation of low levels of satisfaction indicates that they do not welcome the reports. The other given information in Table 2 has also similar interpretations.

Mathematical processing of these given GHFNs in Table 2 requires to adjust their both parts. This is done in pessimistic way in this article. Then, hesitant fuzzy subjective options PIS and NIS can be determined as

$$\begin{aligned} \text{PIS} = \Big(&\langle \{8, 8, 8, 8, 9, 10\}; \{0.6, 0.6, 0.6, 0.7, 0.9\} \rangle, \\ &\langle \{9, 9, 9, 9, 9, 10\}; \{0.6, 0.6, 0.7, 0.8, 0.9\} \rangle \\ &\langle \{4, 4, 5, 6, 8, 9\}; \{0.7, 0.7, 0.7, 0.8, 0.9\} \rangle, \\ &\langle \{5, 5, 5, 6, 7, 10\}; \{0.6, 0.6, 0.6, 0.8, 0.9\} \rangle \Big), \end{aligned}$$

$$\begin{aligned} \text{NIS} = \Big(&\langle \{1, 1, 2, 2, 3, 7\}; \{0.1, 0.3, 0.3, 0.3, 0.5\} \rangle, \\ &\langle \{2, 2, 4, 5, 7, 7\}; \{0.1, 0.1, 0.2, 0.4, 0.7\} \rangle \\ &\langle \{1, 1, 1, 1, 3, 8\}; \{0.1, 0.1, 0.1, 0.1, 0.5\} \rangle, \\ &\langle \{1, 1, 2, 3, 4, 5\}; \{0.3, 0.3, 0.3, 0.4, 0.6\} \rangle \Big). \end{aligned}$$

Now, utilizing the proposed distance measure in this paper, the distance of each candidate $C_i, i = 1, 2, \ldots, 5$, from both PIS and NIS options, which is displayed by d_i^+ and d_i^-, respectively, must be calculated:

$$\begin{aligned} &S_1^+ = 3.1135, \; S_2^+ = 2.3991, \; S_3^+ = 3.5197, \\ &S_4^+ = 2.7723, \; S_5^+ = 1.8767, \\ &S_1^- = 2.5738, \; S_2^- = 3.5154, \\ &S_3^- = 2.0993, \; S_4^- = 3.2047, \\ &S_5^- = 3.5605. \end{aligned}$$

By these values, the relative distance of all candidates is computed as

$$R_1 = 0.4525, R_2 = 0.5944, R_3 = 0.3736, R_4 = 0.5362, R_5 = 0.6548.$$

According to R_i' values, candidates will be ranked as

$$C_3 \prec C_1 \prec C_4 \prec C_2 \prec C_5.$$

Table 2 Decision matrix of energy project selection problem

Candidates	a_1	a_2
C_1	$\langle\{8, 7, 9, 10\}; \{0.3, 0.4, 0.5\}\rangle$	$\langle\{7, 2, 9, 4, 5\}; \{0.9, 0.8, 0.1, 0.7\}\rangle$
C_2	$\langle\{10, 9, 8\}; \{0.3, 0.5\}\rangle$	$\langle\{7, 5\}; \{0.9, 0.7, 0.5, 0.6, 0.2\}\rangle$
C_3	$\langle\{9, 2, 3, 7, 5\}; \{0.6, 0.7\}\rangle$	$\langle\{3, 8, 6, 5, 7, 2\}; \{0.9, 0.6\}\rangle$
C_4	$\langle\{2, 7, 9, 5, 1\}; \{0.3, 0.4, 0.8, 0.7\}\rangle$	$\langle\{9, 10\}; \{0.2, 0.4, 0.7\}\rangle$
C_5	$\langle\{7, 2, 3\}; \{0.3, 0.9, 0.6, 0.7, 0.1\}\rangle$	$\langle\{4, 5\}; \{0.8, 0.4, 0.6, 0.7\}\rangle$
Weight	0.2341	0.2474
Candidates	a_3	a_4
C_1	$\langle\{1, 8, 3\}; \{0.2, 0.4, 0.5\}\rangle$	$\langle\{7, 9, 5\}; \{0.3, 0.9, 0.5, 0.7\}\rangle$
C_2	$\langle\{8, 4, 5, 1, 9\}; \{0.8, 0.6, 0.5, 0.1\}\rangle$	$\langle\{10, 3, 7, 6\}; \{0.3, 0.4, 0.7\}\rangle$
C_3	$\langle\{6, 8, 3, 5\}; \{0.3, 0.5, 0.7\}\rangle$	$\langle\{9, 8, 3, 5\}; \{0.6, 0.4\}\rangle$
C_4	$\langle\{3, 9, 5\}; \{0.8, 0.1\}\rangle$	$\langle\{1, 2, 3, 4, 5\}; \{0.9, 0.8, 0.6\}\rangle$
C_5	$\langle\{8, 9, 6, 4, 5\}; \{0.9, 0.8, 0.7\}\rangle$	$\langle\{7, 2, 9, 4, 6\}; \{0.9, 0.6, 0.3, 0.7\}\rangle$
Weight	0.3181	0.2004

5.1 Validity discussion

Wang and Triantaphyllou (2008) proposed three criteria test to check and demonstrating the feasibility of the introduced MCDM methods: 1-Replacing the non-optimal option with a worse option has no effect on the superior option. 2- Transitivity property is satisfied in an effective MCDM method. 3- Integrating the resulted rankings by solving the obtained subproblems from decomposition of the initial problem is the same as the initial ranking of the main problem.

It is easy to discuss these criteria test: A worse option than a non-optimal one is more near to NIS solution than it, and simultaneously is more far from PIS solution. So, the best option will be unchanged. The final ranking orders of options are determined by ranking of their scores, which are positive real values and obtained at the end of TOPSIS method. Therefore, the criteria 2 and 3 are true due to the properties of the real numbers.

5.2 Comparison analysis

In this subsection, by considering different scenarios for given data in decision matrix and applying some other methods, the results will be compared with the method proposed by the article.

I. *Consider only the real part of the given GHFNs in Table 2.*
It means, each element of the resulting decision matrix is a finite set of real values. First, we calculate the average of each matrix element and replace it. In this way, we will have a new decision matrix based on definite numbers. Then, by applying the TOPSIS method on this matrix, we rank the options: $C_1 \prec C_5 \prec C_3 \prec C_4 \prec C_2$.

II. *Consider only the membership part of the given GHFNs in Table 2.*
Hesitant fuzzy TOPSIS has been proposed by Xu and Zhang (2012) to solve MADM problems in which the uncertainty is modeled via the hesitant fuzzy sets. Xu and Zhang (2012), consider HFSs, i.e., only the second part of each element given in Table 2 and solved Example 2. They ranked candidates like: $C_4 \prec C_1 \prec C_2 \prec C_3 \prec C_5$.

A direct comparison of these two types of rankings with that obtained based on applying GHFNs may not be the right thing to do, because the data they use are inherently different despite the similarities.

III. *Change the given GHFNs in Table 2 to GHFNs with equal values in their real part.*
Let us pave the way for a better comparison. Suppose the real part of all the given GHFNs in Table 2 changed to be equal and solve the problem again using TOPSIS method. The new obtained ranking order in this case is quite consistent with what was previously obtained in Xu and Zhang (2012). It is showed that the proposed method in this article is an extension of the existing methods in a new environment.

Example 3 (Adapted from Keikha (2021a)) In this Example, we investigate a MADM problem, in which the evaluation value of options against criteria has been modeled through hesitant fuzzy numbers as in Table 3.

Utilizing weighted hesitant arithmetic averaging (WHAA) operator, each row of Table 3 has been aggregated to a single HFN. Then, the derived HFNs have been ranked in ascending order due to their scores (see Table 4), and candidates ranked as $C_3 \prec C_1 \prec C_5 \prec C_2 \prec C_4$. Since HFNs are tools

Table 3 The hesitant fuzzy numbers decision matrix of Example 3

Candidates	a_1	a_2	a_3	a_4
C_1	$\langle 4; \{0.3, 0.4, 0.5, 0.7\}\rangle$	$\langle 7; \{0.1, 0.7, 0.8, 0.9\}\rangle$	$\langle 3; \{0.2, 0.4, 0.5, 0.7\}\rangle$	$\langle 8; \{0.3, 0.5, 0.6, 0.9\}\rangle$
C_2	$\langle 6; \{0.3, 0.8, 0.8, 0.9\}\rangle$	$\langle 3; \{0.5, 0.6, 0.9, 0.9\}\rangle$	$\langle 7; \{0.4, 0.5, 0.6, 0.8\}\rangle$	$\langle 2; \{0.3, 0.6, 0.7, 0.9\}\rangle$
C_3	$\langle 4; \{0.1, 0.2, 0.3, 0.2\}\rangle$	$\langle 6; \{0.3, 0.4, 0.2, 0.3\}\rangle$	$\langle 7; \{0.3, 0.5, 0.1, 0.1\}\rangle$	$\langle 6; \{0.2, 0.4, 0.1, 0.1\}\rangle$
C_4	$\langle 2; \{0.3, 0.4, 0.7, 0.8\}\rangle$	$\langle 3; \{0.2, 0.4, 0.7, 0.4\}\rangle$	$\langle 8; \{0.1, 0.8, 0.5, 0.5\}\rangle$	$\langle 7; \{0.6, 0.8, 0.9, 0.8\}\rangle$
C_5	$\langle 9; \{0.1, 0.3, 0.6, 0.7\}\rangle$	$\langle 1; \{0.4, 0.6, 0.7, 0.8\}\rangle$	$\langle 3; \{0.7, 0.8, 0.9, 0.8\}\rangle$	$\langle 2; \{0.3, 0.6, 0.7, 0.9\}\rangle$
Weight	0.3	0.1	0.4	0.2

Table 4 The aggregated hesitant fuzzy decision matrix of Example 3

Alternatives	WHAA values	Score values (SV)
C_1	$\langle 4.7; \{0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}\rangle$	2.82
C_2	$\langle 5.3; \{0.5, 0.6, 0.7, 0.8, 0.9\}\rangle$	3.71
C_3	$\langle 5.8; \{0.2, 0.3, 0.4, 0.5\}\rangle$	2.03
C_4	$\langle 5.5; \{0.6, 0.7, 0.8, 0.9\}\rangle$	4.125
C_5	$\langle 4.4; \{0.7, 0.8, 0.9\}\rangle$	3.52

for uncertainty modeling, they are not error free. From the point of view of error analysis, with increasing the number of mathematical operations, the accumulation and propagation of error also increases during the calculations. Therefore, methods that use fewer operations are more appropriate. The WHAA method, in addition to having a relatively large number of mathematical operations to align the row data, also depends on the score value of the resulted HFN, which is actually an approximation of it.

Let us consider each given HFN in Table 2 as a GHFN, in which its real part containing a single value. Utilizing TOPSIS method resulted the following ranking:

$$C_1 \prec C_5 \prec C_3 \prec C_4 \prec C_2.$$

As it is observed, these two ranking orders are completely different, but have similarities, like the options available in the top and bottom half of the rankings. It is natural to have differences in results because the methods used and the type of data processing are fundamentally different. However, the method proposed in this paper produces more reliable results due to the use of less arithmetic operations (only distance function without any extra approximation such as score value is used) in the TOPSIS method. What can be said about the difference between the proposed method and other similar methods is that in this method, decisions are made according to the existing reality. Because the use of generalized hesitant fuzzy numbers causes the data to enter the processing phase without any change, while the use of fuzzy numbers or similar ones leads to the use of infinite values and infinite membership degrees. Another important advantage of this approach is that it transforms a group decision problem into an individual decision problem without the need to change the nature of the problem. And even the greater participation of people in the group not only does not cause any challenge, but also leads to better decisions. But it should be noted that we are at the beginning of the road and as we will see in the conclusion section, many mathematical concepts still need to be developed.

6 Conclusion

This article desires to present an exquisite MADM method for solving the decision-making problems under the hesitant and uncertain environments. The uncertainty of the data in this paper is handled with the help of the generalized hesitant fuzzy information which consists of two disjoint finite sets: real and membership parts. In addition to keeping the nature of the information intact, this type of numbers enables us to represent the opinions of a group of experts on a single subject from the perspective of a particular criterion by means of a GHFN. In other words, a MAGDM problem can easily be expressed in the form of a typical MADM problem. Given that this concept is at the beginning of a long journey, the development of some mathematical tools is a prerequisite for the practical development of the GHFNs. Maintenance the advantages of it, this paper focuses around to proposed a way to compute the distances between GHFNs by utilizing three useful distance measures, i.e., Hamming distance, Euclidean distance, and generalized distance measures, individually. Next, using these definitions, we updated the common TOPSIS method for solving MADM problems with generalized hesitant fuzzy numbers. The advantage of the presented method is that it eschews the wrong decisions based on the small changes in entrance information. Finally, by solving numerical examples, we explained and compared the proposed method.

What is certain is that much work still needs to be done theoretically in this area, such as entropy measure, similarity measure, norm-based operators of GHFNs, and so on. In the future and in the field of application, many decision-making methods such as analytic hierarchy process AHP/ANP method, Choquet integral method, VIKOR method, ELECTRE method (Tzeng and Huang 2011) will have to be updated to use GHFNs. Mathematical development of linear programming methods, data envelopment analysis, graph theory, game theory and mechanism design (Marek and Peter 2021), brain hemorrhage patients(Garg and Kaur 2020), natural language processing (Nagarhalli et al. 2021), etc., can be considered by researchers, and the necessary conditions should be provided for applying GHFNs in these fields.

Funding The authors have not disclosed any funding.

Data availability The author A. Keikha declares that this research received no specific grant from any funding agency.

Declarations

Conflict of interest Author A. Keikha declares that he has no conflict of interest.

References

Abualigah L, Diabat A, Sumari P, Gandomi AH (2021) Applications, deployments, and integration of internet of drones (IoD): a review. IEEE Sens J 21(22):25532–25546

Abualigah L, Diabat A, Mirjalili SA, Elaziz MA, Gandomi AH (2021) The arithmetic optimization algorithm. Comput Methods Appl Mech Eng 376

Abualigah L, Elaziz MA, Sumari P, Geem ZW, Gandomi AH (2022) Reptile search algorithm (RSA): a nature-inspired meta-heuristic optimizer. Exp Syst Appl 191

Abualigah L, Yousri D, Elaziz MA, Evees AA, Al-qaness MAA, Gandomi AH (2021) Aquila optimizer: a novel meta-heuristic optimization algorithm. Comput Indus Eng 157

Aggrawal N (2021) Decision making model based on fuzzy AHP and TOPSIS using weakest t-norm operations. Ann Optim Theory Pract. https://doi.org/10.22121/AOTP.2021.271299.1061

Al-shami TM, Ibrahim HZ, Alzahrani S, Azzam AA, EL-Maghrabi AI (2022) SR-Fuzzy sets and their applications to weighted aggregated operators in decision-making. J Funct Spaces

Al-shami TM (2021) Improvement of the approximations and accuracy measure of a rough set using somewhere dense sets. Soft Comput 25(23):14449–14460

Atanassov KT (1983) Intuitionistic fuzzy sets. In: Sgurev V (ed) VII ITKR'S session. Sofia, Jone

Atef M, Ali MI, Al-shami TM (2021) Fuzzy soft covering-based multi-granulation fuzzy rough sets and their applications. Comput Appl Math 40:4. https://doi.org/10.1007/s40314-021-01501-x

Bellman RE, Zadeh LA (1970) Decision-making in a fuzzy environment. Manage Sci 17(4):141–164

Deli I (2020) A TOPSIS method by using generalized trapezoidal hesitant fuzzy numbers and application to a robot selection problem. J Intell Fuzzy Syst 38(1):779–793

Deli I, Karaaslan F (2021) Generalized trapezoidal hesitant fuzzy numbers and their applications to multi criteria decision-making problems. Soft Comput 25:1017–1032

Denoeux T (2014) Dempster-Shafer theory: introduction, connections with rough sets and application to clustering, invited talk. In: 9th international conference on rough sets and knowledge technology (RSKT 2014), Shanghai, China, October 24–26

Faizi S, Rashid T, Sałabun W, Zafar S, Watrobski J (2018) Decision making with uncertainty using hesitant fuzzy sets. Int J Fuzzy Syst 20(1):93–103

Garg H, Kaur G (2020) Quantifying gesture information in brain hemorrhage patients using probabilistic dual hesitant fuzzy sets with unknown probability information. Comput Indus Eng 140

Garg H, Keikha A, Nehi HM (2020) Multiple attribute decision making problem using TOPSIS and Choquet integral with hesitant fuzzy number information. Math Probl Eng. https://doi.org/10.1155/2020/9874951

Glaz AL, Haralambous Y, Kim-Dufor D, Lenca R, Billot R, Ryan TC, Marsh J, DeVylder J, Walter M, Berrouiguet S, Lemey C (2021) Machine learning and natural language processing in mental health: systematic review. J Med Internet Res 23:5. https://doi.org/10.2196/15708

Guha D, Chakraborty D (2010) A new approach to fuzzy distance measure and similarity measure between two generalized fuzzy numbers. Appl Soft Comput J 10(1):90–99

Guha D, Chakraborty D (2010) A theoretical development of distance measure for intuitionistic fuzzy numbers. Int J Math Math Sci

Ibrahim H. Z, Al-shami T. M, Elbarbary O. G (2021) (3, 2)-Fuzzy sets and their applications to topology and optimal choices. Comput Intell Neurosci (Article ID 1272266, 14 pages)

Karnik NN, Mendel JM (2001) Centroid of a type-2 fuzzy set. Inf Sci 132:195–220

Keikha A (2021) Archimedean t-norm and t-conorm-based aggregation operators of HFNs, with the approach of improving Education. Int J Fuzzy Syst. https://doi.org/10.1007/s40815-021-01137-3

Keikha A (2021) Generalized hesitant fuzzy numbers: introducing, arithmetic operations, aggregation operators, and an application. Int J Intell Syst (accepted). https://doi.org/10.1002/int.22604

Keikha A (2021) Introducing a new type of HFSs and their application in solving MADM. J Intell Fuzzy Syst 1–12:1

Klir G. J (2006) Uncertainty and information': foundations of generalized information theory. John Wiley & Sons, Inc

Li D, Zeng W, Li J (2015) New distance and similarity measures on hesitant fuzzy sets and their applications in multiple criteria decision making. Eng Appl Artif Intell 40:11–16

Li D, Zeng W, Zhao Y (2015) Note on distance measure of hesitant fuzzy sets. Inf Sci 321:103–115

Liao HC, Xu ZS (2013) A VIKOR-based method for hesitant fuzzy multi criteria decision making. Fuzzy optimization decision making 12:373–392

Liao HC, Xu ZS (2014) Subtraction and division operations over hesitant fuzzy sets. J Intell Fuzzy Syst 27(1):65–72

Liao HC, Xu ZS (2014) Some new hybrid weighted aggregation operators under hesitant fuzzy multi-criteria decision making environment. J Intell Fuzzy Syst 26(4):1601–1617

Liao HC, Xu ZS (2015) Extended hesitant fuzzy hybrid weighted aggregation operators and their application in decision making. Soft Comput 19(9):2551–2564

Liao HC, Xu ZS (2017) Hesitant fuzzy decision making methodologies and applications. Springer, Singapore

Liao HC, Xu ZS, Xia MM (2014) Multiplicative consistency of hesitant fuzzy preference relation and its application in group decision making. Int J Inf Technol Decis Mak 13(1):47–76

Liao HC, Xu ZS, Zeng XJ (2015) Novel correlation coefficients between hesitant fuzzy sets and their application in decision making. Knowl-Based Syst 82:115–127

Liao H, Wu X, Keikha A, Hafezalkotob A (2018) Power average-based score function and extension rule of hesitant fuzzy set and the hesitant power average operators. J Intell Fuzzy Syst 35(3):3873–3882

Marek P, Peter T (2021) A theory of simplicity in games and mechanism design . University of Zurich, Department of Economics, Working Paper No. 393. Available at SSRN. https://doi.org/10.2139/ssrn.3881072

Muhammad S, Muhammad A, Noor A, Muhammad J (2020) Method of MAGDM based on Pythagorean trapezoidal uncertain linguistic hesitant fuzzy aggregation operator with Einstein operations. J Intell Fuzzy Syst 38(2):2211–2230

Nagarhalli TP, Vaze V, Rana NK (2021) Impact of machine learning in natural language processing: a review. In: 2021 third international conference on intelligent communication technologies and virtual mobile networks (ICICV), pp 1529-1534. https://doi.org/10.1109/ICICV50876.2021.9388380

Nan J, Zhang M (2014) Extensions of the TOPSIS for multiattribute decision making under intuitionistic fuzzy environment. J Inf Comput Sci 11(5):1635–1645

Palczewski K, Sałabun W (2019) The fuzzy TOPSIS applications in the last decade. Proc Comput Sci 159:2294–2303

Pollack HN (2003) Uncertain science—uncertain world. Cambridge University Press

Ranjbar M, Miri SM, Effati S (2020) Hesitant fuzzy numbers with (α, k)-cuts in compact intervals and applications. Exp Syst Appl 151

Sang X, Liu X, Cai M (2014) A distance based ranking methods for type-1 fuzzy numbers and interval type-2 fuzzy numbers. In: IEEE international conference on fuzzy systems (FUZZ-IEEE), July 6–11, Beijing, China

Sun QP, Ouyang JW (2015) Hesitant fuzzy multi attribute decision making based on TOPSIS with entropy-weighted method. Manage Sci Eng 9(3):1–6

Tong X, Yu L (2016) MADM based on distance and correlation coefficient measures with decision maker preferences under a hesitant fuzzy environment. Soft Comput 20:4449–4461

Torra V (2010) Hesitant fuzzy sets. Int J Intell Syst 25(1):529–539

Tzeng GH, Huang JJ (2011) Multiple attribute decision making methods and application. CRC

Verma R (2015) Operations over hesitant fuzzy sets: some new results. J Intell Fuzzy Syst 29(1):43–52

Verma R, Sharma BD (2013) New operations over hesitant fuzzy sets. Fuzzy Inf Eng 5(2):129–146

Wang X, Triantaphyllou E (2008) Ranking irregularities when evaluating alternatives by using some electre methods. Omega Int J Manage Sci 36:45–63

Wei G (2012) Hesitant fuzzy prioritized operators and their application to multiple attribute decision making. Knowl Based Syst 31:176–182

Wei G, Zhao X, Lin R, Wang H (2014) Approaches to hesitant fuzzy multiple attribute decision making with incomplete weight information. J Intell Fuzzy Syst 26(1):259–266

Xia MM, Xu ZS (2011) Hesitant fuzzy information aggregation in decision making. Int J Approx Reas 52(3):395–407

Xu ZS, Xia MM (2011) Distance and similarity measures on hesitant fuzzy sets. Inf Sci 181(11):2128–2138

Xu ZS, Xia MM (2011) On distance and correlation measures of hesitant fuzzy information. Int J Intell Syst 26(5):410–425

Xu ZS, Xia MM (2012) Hesitant fuzzy entropy and cross-entropy measures and their use in multi attribute decision making. Int J Intell Syst 27(9):799–22

Xu ZS, Zhang X (2012) Hesitant fuzzy multiple attribute decision making based on TOPSIS with incomplete weight information. Knowl Based Syst 52:53–64

Xu ZS, Xia MM, Chen N (2013) Some hesitant fuzzy aggregation operators with their application in decision making. Group Decis Negot 22(2):259–279

Zadeh LA (1965) Fuzzy sets. Inf Control 8:338–353

Zhang Z (2016) Several new hesitant fuzzy aggregation operators and their application to multi-criteria decision making. Proc Natl Acad Sci India Sect A 86(3)

Zhang Z (2013) Hesitant fuzzy power aggregation operators and their application to multiple attribute group decision making. Inf Sci 234:150–181

Zhang Z, Yu W, Martinez L, Gao Y (2020) Managing multigranular unbalanced hesitant fuzzy linguistic information in multiattribute large-scale group decision making: a linguistic distribution-based approach. IEEE Trans Fuzzy Syst 28(11):2875–2889