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FOUNDATION, ALGEBRAIC, AND ANALYTICAL METHODS IN SOFT COMPUTING

Pseudo-fractional operators of variable order and applications

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Abstract

The fractional calculus provides, over the decades, new tools based on formulations of definitions and discussions of properties, which allows greater connections with other areas. As highlighted, two pillars well-founded and built over these years, the first we highlight the fractional calculus that addresses integrals and derivatives of a non-variable order. Second, a natural consequence of the classic fractional calculus, they investigated the possibility of fractional integrals and derivatives of a variable order, although more restricted when discussing basic and fundamental properties of the fractional calculus. From these two pillars and the g-calculus theory (pseudo-analysis), a third pillar started to be built, although recently, but there are already some interesting results. In this sense, in the present paper, we present new extensions of pseudo-fractional operators for integral and derivative in the sense of g-calculus and investigate some essential properties of fractional calculus. In order to elucidate the results discussed, we present an application involving the ψ-pseudo fractional integral inequality of Chebyshev.

Keywords Pseudo-addition · Pseudo-multiplication · Pseudo-fractional integrals · Pseudo-fractional derivatives · Variable order

1 Introduction

Differential and integral calculus has a very important and very important role in the history of mathematics. Since its first ideas discussed by Leibniz and Newton, until today its impacts in several areas of science are addressed. However, at the same time that the first ideas of differential and integral calculus were being addressed, I also had certain doubts and questions about it. One of them was what would happen if n were $\frac{1}{2}$. This questioning was through a letter from L'Hospital to Leibniz. At the moment, there was no answer to such a question, but the answer that in the future, important consequences would bring science Leibniz (1849, 1962a, b). Today, 326 years after his first ideas, the natural question that arises is: will Leibniz's prophecy about fractional calculus really be consumed? Without a doubt, today fractional calculus is an area of mathematics that has a very important role and relevance both in the theoretical sense and in the sense that it involves applications Nikan et al. (2021); Hassani et al. (2021); Almeida et al. (2021); Nemati et al. (2019); Yang et al. (2019); Diethelm and Ford (2002); Frederico and Torres (2008); Frederico and Lazo (2016); Lazo et al. (2019); Sousa et al. (2021, 2020); Boudjerida et al. (2020); Xia and Chai (2018); Yang and Wang (2019); Abbas et al. (2015) and references therein.

Currently there is a wide class of integrals and fractional derivatives, each proposed for some reason and motivation Oliveira and Capelas (2018); Kilbas et al. (2006); Sousa and Capelas (2018); Oliveira and Capelas (2019); Almeida (2017); Jarad et al. (2012). However, since there are several definitions, what to know and how to choose a fractional derivative for a given problem? It is not an easy task. In this sense, noted by the Caputo, Riemann–Liouville fractional

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derivatives and Hilfer fractional derivative itself, in 2018 Sousa and Oliveira Sousa and Capelas (2018), introduced the so-called ψ-Hilfer fractional derivative, which contains a vast number of particular cases in a single operator. In fact, this property is of great relevance and allows you to discuss a certain problem and analyze which best fractional derivative is suitable for the proposed problem, from the variations of $\beta \to 0$, $\beta \to 1$ e of the particular choice of $\psi(t) = t$.

On the other hand, it was noted that it was possible to extend the classic fractional derivatives whose fractional order is a number. Then, they realized that by imposing the variable order of the operators (integral and derivative), it would be possible to extend the operators and obtain new properties, in particular, to model phenomena in which the classic case did not obtain more precise results Almeida and Torres (2015); Tavares et al. (2018); Almeida et al. (2018); Tavares et al. (2015); Song and Zhang (2019). However, when discussing these operators, it is noted that some properties are lost, because the variable order of the operators (integral and derivative). Over the past few years, Almeida, Delfin and Tavares Tavares et al. (2016); Almeida (2017), investigated the Caputo and Hadamard fractional derivatives of varying order and proposed new properties. In this sense, the results of the fractional calculus of variable order extended the theory of the calculus of variations Almeida et al. (2019). There are several problems that are addressed via fractional derivatives of variable order whose results are relevant, some of which can be obtained at. Motivated by the ψ-Hilfer fractional derivative and the theory of fractional derivatives of variable order, Sousa et al. Sousa et al. (2020), proposed a new extension and discussed important properties, in particular involving approximations for the fractional derivative.

In addition to the fractional calculus being a generalization of the integer order calculus, we also have another branch of mathematics that also generalizes the integer order calculus, and is called g-calculus, which works with the tools of pseudo-operators, pseudo-addition, pseudo-multiplication, among other tools that can be obtained in the works Kuich (1986); Mesiar and Rybárik (1993); Pap (1993, 2002, 2005); Pap and Štrboja (2010); Pap et al. (2014). The natural question that arises and arises from the history between differential and integral calculus with fractional calculus, is the following: will it be if the first ideas of fractional pseudo-operators have already been discussed: the answer is yes. Some works in this sense are found at. However, this area is recent and few results are found in the literature Babakhani et al. (2018); Hosseini et al. (2016). The number of definitions of fractional integrals and derivatives via pseudo-operators is very restricted and when compared with the classic fractional calculus, this number becomes even more restricted. In this sense, since over the years the number of definitions will certainly grow, then in 2020 Sousa et al. Sousa et al. (2020), motivated by this question and by other open questions, especially on the occasion, in discussing variational problems, introduced a ψ-Hilfer fractional derivative, a new fractional pseudo-operator and discussed essential properties for the theory. As the area is still under construction, few results are found, in particular, involving the theory of differential equations Hosseini et al. (2016); Yadollahzadeh et al. (2019); Vanterler et al. (2020); Sousa et al. (2020); Agahi and Alipour (2017); Abundo and Pirozzi (2019); Hosseini et al. (2016); Agahi et al. (2019).

One of the problems when discussing via fractional operators of varying order is the fact that some properties are lost, however others are relevant, such as polynomial operators approaches are one of them. Adding the pseudo-operator theory to the fractional calculus of variable order, some results are not trivial to solve, however, the results are extremely important and of great relevance to solve some open problems. According to the history of fractional calculus of integer order and variable order, and of g-calculus and fractional g-calculus of integer order, a natural consequence of the construction of the theory, is to discuss new operators whose fractional order is a variable.

As can be seen in the history of fractional calculus, so far countless fractional operators have been introduced, due to some motivation be it in the theoretical sense or in the sense of applications. But the natural question that arises is: is it still interesting to propose a new fractional operator? What important consequence does this new operator provide for the theory? Another important factor that can be considered, is that depending on the area, the fractional calculus does not yet impact, in particular, the g-calculus.

Motivated by these questions and other open and relevant questions in the g-calculus theory, in this present paper we contribute significantly and positively to this new and rich area that has recently appeared, that is, fractional pseudo-operators in g-calculus theory. So, in order to elucidate what are the main results obtained in this paper, we point out below, namely:

(1) Through the ψ-Riemann–Liouville fractional integral of variable order and g-integral, we introduced a new extension for these operators, called ψ-Riemann–Liouville pseudo- fractional integral of variable order.
(2) Motivated by item 1, by the fractional derivatives ψ-Caputo, ψ-Riemann–Liouville (variable order) and g-derivative, we investigated two new extensions for joining these operators.
(3) Some properties of pseudo-operators are investigated.
(4) Through the ψ-Riemann–Liouville pseudo-fractional integral, we investigate new Chebyshev inequalities.

One of the consequences that the results provide to this new theory that is still under construction is the possibility of discussing results involving polynomials that can be approximated by means of one of these extensions of fractional derivatives of variable order (pseudo-operators). We can also highlight another natural consequence, that is, a new tool that allows us to discuss and attack open variational problems.

In this sense, the work is organized as follows. Section 2, we present the mathematical background, that is, some classical definitions of integral and fractional derivatives of variable order, as well as auxiliary results of the theory of fractional calculus of variable order. In Sect. 3, we intend to give a small approach on definitions and results of the g-calculus theory (pseudo-analysis), essential in the construction of the operators investigated here. In Sect. 4, we present our first main contribution of this paper, that is, we present a new extension of pseudo-operators (differentiation and integration) for fractional pseudo-operators. In this sense, some classic results from the fractional calculus theory are presented. In Sect. 5, we performed an application involving the ψ-pseudo-fractional integral inequality of Chebyshev.

2 Mathematical background—auxiliary results

In this section, we present some variations of definitions of fractional integrals and derivatives of variable order. Some results are discussed with their respective evidence, others are just mentioned.

First, we start with the fundamental idea of Riemann–Liouville fractional integral of a variable order with respect to another function.

Definition 2.1 Sousa et al. (2020) Let $\alpha(x)$ with domain (either a finite or an infinite interval $x \in [a, b]$), taking value on the open interval $[0, 1]$, $f \in L^1([a, b], \mathbb{R})$ and $\psi \in C^1([a, b], \mathbb{R})$ an increasing function. The fractional integral of variable order of a function $f \in L^1([a, b], \mathbb{R})$ with respect to another function ψ, of variable order $\alpha(x)$, on the left and the right, are given by

$$\mathbb{I}_{a+}^{\alpha(x);\psi} f(x) = \frac{1}{\Gamma(\alpha(x))} \int_a^x \psi'(t)(\psi(x) - \psi(t))^{\alpha(x)-1} f(t)\mathrm{d}t$$

and

$$\mathbb{I}_{b-}^{\alpha(x);\psi} f(x) = \frac{1}{\Gamma(\alpha(x))} \int_x^b \psi'(t)(\psi(t) - \psi(x))^{\alpha(x)-1} f(t)\mathrm{d}t.$$

Definition 2.2 Sousa et al. (2020) Let a, b be two reals with $0 < a < b$ and $f, \psi \in C^n([a, b], \mathbb{R})$ two functions such that ψ is increasing monotonically and $\psi'(t) \neq 0$, $\forall t \in [a, b]$.

(1) The type I (left-sided and right-sided) ψ-Riemann–Liouville derivatives of variable order $\alpha(x)$, $0 < \alpha(x) < 1$, are defined by

$$\mathbb{D}_{a+}^{\alpha(x);\psi} f(x) = \frac{1}{\Gamma(1-\alpha(x))} \left(\frac{1}{\psi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right) \times \int_a^x \psi'(t)(\psi(x) - \psi(t))^{-\alpha(x)} f(t)\mathrm{d}t$$

and

$$\mathbb{D}_{b-}^{\alpha(x);\psi} f(x) = \frac{1}{\Gamma(1-\alpha(x))} \left(-\frac{1}{\psi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right) \times \int_x^b \psi'(t)(\psi(t) - \psi(x))^{-\alpha(x)} f(t)\mathrm{d}t;$$

(2) The type II (left-sided and right-sided) ψ-Riemann–Liouville derivatives of variable order $\alpha(x)$, $0 < \alpha(x) < 1$, are defined by

$$\mathcal{D}_{a+}^{\alpha(x);\psi} f(x) = \left(\frac{1}{\psi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right) \frac{1}{\Gamma(1-\alpha(x))} \times \int_a^x \psi'(t)(\psi(x) - \psi(t))^{-\alpha(x)} f(t)\mathrm{d}t$$

and

$$\mathcal{D}_{b-}^{\alpha(x);\psi} f(x) = \left(-\frac{1}{\psi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right) \frac{1}{\Gamma(1-\alpha(x))} \times \int_x^b \psi'(t)(\psi(t) - \psi(x))^{-\alpha(x)} f(t)\mathrm{d}t.$$

Definition 2.3 Sousa et al. (2020) Let a, b be two reals with $0 < a < b$ and $f, \psi \in C^n([a, b], \mathbb{R})$ two functions such that ψ is increasing monotonically and $\psi'(t) \neq 0$, $\forall t \in [a, b]$.

(1) The type I (left-sided and right-sided) ψ-Caputo derivatives of variable order $\alpha(x)$, $0 < \alpha(x) < 1$, are defined by

$$^{C}\mathbb{D}_{a+}^{\alpha(x);\psi} f(x) = \frac{1}{\Gamma(1-\alpha(x))} \left(\frac{1}{\psi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right) \times \int_a^x \psi'(t)(\psi(x) - \psi(t))^{-\alpha(x)}[f(t) - f(a)]\mathrm{d}t$$

and

$$
\begin{aligned}
&{}^{C}\mathbb{D}_{b-}^{\alpha(x);\psi} f(x)\\
&= \frac{1}{\Gamma(1-\alpha(x))}\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)\\
&\quad\times\int_x^b \psi'(t)(\psi(t)-\psi(x))^{-\alpha(x)}[f(t)-f(b)]dt;
\end{aligned}
$$

(2) The type II (left-sided and right-sided) ψ-Caputo derivatives of variable order $\alpha(x)$, $0<\alpha(x)<1$, are defined by

$$
\begin{aligned}
&{}^{C}\mathcal{D}_{a+}^{\alpha(x);\psi} f(x)\\
&= \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)\frac{1}{\Gamma(1-\alpha(x))}\\
&\quad\times\int_a^x \psi'(t)(\psi(x)-\psi(t))^{-\alpha(x)}[f(t)-f(a)]dt
\end{aligned}
$$

and

$$
\begin{aligned}
&{}^{C}\mathcal{D}_{b-}^{\alpha(x);\psi} f(x)\\
&= \left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)\frac{1}{\Gamma(1-\alpha(x))}\\
&\quad\times\int_x^b \psi'(t)(\psi(t)-\psi(x))^{-\alpha(x)}[f(t)-f(b)]dt.
\end{aligned}
$$

(3) The type III (left-sided and right-sided) ψ-Caputo derivatives of variable order $\alpha(x)$, $0<\alpha(x)<1$, are defined by

$$
\begin{aligned}
&{}^{C}\mathscr{D}_{a+}^{\alpha(x);\psi} f(x)\\
&= \frac{1}{\Gamma(1-\alpha(x))}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{-\alpha(x)}\\
&\quad\times\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)f(t)dt
\end{aligned}
$$

and

$$
\begin{aligned}
&{}^{C}\mathscr{D}_{b-}^{\alpha(x);\psi} f(x)\\
&= \frac{1}{\Gamma(1-\alpha(x))}\int_x^b \psi'(t)(\psi(t)-\psi(x))^{-\alpha(x)}\\
&\quad\times\left(-\frac{1}{\psi'(t)}\frac{d}{dt}\right)f(t)dt.
\end{aligned}
$$

Remark 2.4 Note that from the interpolation between the ψ-Caputo and ψ-Riemann–Liouville fractional derivatives, the ψ-Hilfer fractional derivative is obtained, all in variable order. Consequently, a wide class of particular cases, it is possible to obtain, from the particular choice of $\psi(t)$ and the limits of $\beta\to 1$ e $\beta\to 0$.

The next two results are about calculating the fractional derivatives of a variable order of power functions.

Lemma 2.5 *Let $\beta>0$ and $f(x)=[\psi(x)-\psi(a)]^{\beta}$, then*

(a)
$$
\begin{aligned}
&\mathbb{D}_{a+}^{\alpha(x);\psi}[(\psi(x)-\psi(a))^{\beta}]\\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha(x)+1)}[\psi(x)-\psi(a)]^{\beta-\alpha(x)}\\
&\quad-\frac{\alpha'(x)}{\psi'(x)}\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha(x)+2)}\\
&\quad\times[\psi(x)-\psi(a)]^{\beta-\alpha(x)+1}\Big\{\ln[\psi(x)-\psi(a)]\\
&\quad+\Psi[1-\alpha(x)]-\Psi[\beta-\alpha(x)+2]\Big\}
\end{aligned}
$$

(b)
$$
\begin{aligned}
&\mathcal{D}_{a+}^{\alpha(x);\psi}[\psi(x)-\psi(a)]^{\beta}\\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha(x)+1)}[\psi(x)-\psi(a)]^{\beta-\alpha(x)}\\
&\quad-\frac{\alpha'(x)}{\psi'(x)}\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha(x)+2)}[\psi(x)-\psi(a)]^{\beta-\alpha(x)+1}\\
&\quad\times\Big\{\ln[\psi(x)-\psi(a)]-\Psi[\beta-\alpha(x)+2]\Big\}
\end{aligned}
$$

Proof (a) By definition, we have

$$
\begin{aligned}
&\mathbb{D}_{a+}^{\alpha(x);\psi}[(\psi(x)-\psi(a))^{\beta}]\\
&= \frac{1}{\Gamma(1-\alpha(x))}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)\int_a^x \psi'(x)\\
&\quad\times(\psi(x)-\psi(t))^{-\alpha(x)}(\psi(t)-\psi(a))^{\beta}dt.
\end{aligned}
$$

With the change of variables $u=[\psi(t)-\psi(a)]/[\psi(x)-\psi(a)]$ and with the help of the beta function, we prove that

$$
\begin{aligned}
&\mathbb{D}_{a+}^{\alpha(x);\psi}[(\psi(x)-\psi(a))^{\beta}]\\
&= \frac{1}{\Gamma(1-\alpha(x))}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)\\
&\quad\times\left\{[\psi(x)-\psi(a)]^{1-\alpha(x)+\beta}\int_0^1(1-u)^{-\alpha(x)}u^{\beta}du\right\}\\
&= \frac{1}{\Gamma(1-\alpha(x))}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)\\
&\quad\times\left\{\frac{\Gamma(1-\alpha(x))\Gamma(\beta+1)}{\Gamma(\beta-\alpha(x)+2)}[\psi(x)-\psi(a)]^{1-\alpha(x)+\beta}\right\}\\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha(x)+1)}[\psi(x)-\psi(a)]^{\beta-\alpha(x)}\\
&\quad-\frac{\alpha'(x)}{\psi'(x)}\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha(x)+2)}[\psi(x)-\psi(a)]^{\beta-\alpha(x)+1}\\
&\quad\times\Big\{\ln[\psi(x)-\psi(a)]+\Psi[1-\alpha(x)]\\
&\quad-\Psi[\beta-\alpha(x)+2]\Big\}.
\end{aligned}
$$

The proof of item (b) is analogous to that of item (a). □

Lemma 2.6 *Let $\beta > 0$ and $f(x) = [\psi(x) - \psi(a)]^{\beta}$, then*

(a) $$ {}^{\mathrm{C}}\mathbb{D}_{a+}^{\alpha((x);\psi}[(\psi(x)-\psi(a))^{\beta}] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha(x)+1)}[\psi(x)-\psi(a)]^{\beta-\alpha(x)} - \frac{\alpha'(x)}{\psi'(x)}\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha(x)+2)}[\psi(x)-\psi(a)]^{\beta-\alpha(x)+1} \times \left\{ \ln[\psi(x)-\psi(a)] + \Psi[1-\alpha(x)] - \Psi[\beta-\alpha(x)+2] \right\}. $$

(b) $$ {}^{\mathrm{C}}\mathcal{D}_{a+}^{\alpha(x);\psi}[\psi(x)-\psi(a)]^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha(x)+1)}[\psi(x)-\psi(a)]^{\beta-\alpha(x)} - \frac{\alpha'(x)}{\psi'(x)}\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha(x)+2)}[\psi(x)-\psi(a)]^{\beta-\alpha(x)+1} \times \left\{ \ln[\psi(x)-\psi(a)] - \Psi[\beta-\alpha(x)+2] \right\}. $$

(c) $$ {}^{\mathrm{C}}\mathscr{D}_{a+}^{\alpha(x);\psi}[\psi(x)-\psi(a)]^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha(x)+1)}[\psi(x)-\psi(a)]^{\beta-\alpha(x)}. $$

Proof (c) We proved only item (c). Using the Definition of ${}^{\mathrm{C}}\mathscr{D}_{a+}^{\alpha(x);\psi}$ and the integral what we calculated in Lemma 2.5, we obtain

$$ {}^{\mathrm{C}}\mathscr{D}_{a+}^{\alpha(x);\psi}[\psi(x)-\psi(a)]^{\beta} = \frac{1}{\Gamma(1-\alpha(x))}\int_a^x \psi'(t)(\psi(x) -\psi(t))^{-\alpha(x)}\left(\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)[(\psi(t)-\psi(a))^{\beta}]\mathrm{d}t = \frac{\Gamma(\beta+1)}{\Gamma(\beta)\Gamma(1-\alpha(x))}\int_a^x \psi'(t)(\psi(x) -\psi(t))^{-\alpha(x)}(\psi(t)-\psi(a))^{\beta-1}\mathrm{d}t = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha(x)+1)}(\psi(x)-\psi(a))^{\beta-\alpha(x)}. $$ □

The next results are relations between operators of fractional integration and derivation, and integration by parts. For these results we use the reference.

Theorem 2.7 *Sousa et al. (2020) The following relations the left fractional operators hold:*

$$ {}^{\mathrm{C}}\mathbb{D}_{a+}^{\alpha(x);\psi}f(x) = {}^{\mathrm{C}}\mathscr{D}_{a+}^{\alpha(x);\psi}f(x) + \frac{\alpha'(x)}{\psi'(x)\Gamma(2-\alpha(x))} \times \int_a^x (\psi(x)-\psi(t))^{1-\alpha(x)} \times \left[\frac{1}{1-\alpha(x)} - \ln(\psi(x)-\psi(t))\right] f'(t)\mathrm{d}t $$

and

$$ {}^{\mathrm{C}}\mathbb{D}_{a+}^{\alpha(x);\psi}f(x) = {}^{\mathrm{C}}\mathcal{D}_{a+}^{\alpha(x);\psi}f(x) - \frac{\alpha'(x)}{\psi'(x)}\frac{\Psi(1-\alpha(x))}{\Gamma(1-\alpha(x))} \times \int_a^x \psi'(t)(\psi(x)-\psi(t))^{-\alpha(x)}[f(t)-f(a)]\mathrm{d}t. $$

Theorem 2.8 *Sousa et al. (2020) Let $f \in C^1([a,b],\mathbb{R})$ and $x \in [a,b]$. Then, we have*

(a) $\mathbb{D}_{a+}^{\alpha(x);\psi}f(x) = \mathcal{D}_{a+}^{\alpha(x);\psi}f(x) = 0$ *at* $x = a$;

(b) $\mathbb{D}_{b-}^{\alpha(x);\psi}f(x) = \mathcal{D}_{b-}^{\alpha(x);\psi}f(x) = 0$ *at* $x = b$.

Theorem 2.9 *Sousa et al. (2020) Let $f \in C^1([a,b],\mathbb{R})$ and $x \in [a,b]$. Then, we have*

(a) ${}^{\mathrm{C}}\mathbb{D}_{a+}^{\alpha(x);\psi}f(x) = {}^{\mathrm{C}}\mathcal{D}_{a+}^{\alpha(x);\psi}f(x) = {}^{\mathrm{C}}\mathscr{D}_{a+}^{\alpha(x);\psi}f(x) = 0$ *at* $x = a$;

(b) ${}^{\mathrm{C}}\mathbb{D}_{b-}^{\alpha(x);\psi}f(x) = {}^{\mathrm{C}}\mathcal{D}_{b-}^{\alpha(x);\psi}f(x) = {}^{\mathrm{C}}\mathscr{D}_{b-}^{\alpha(x);\psi}f(x) = 0$ *at* $x = b$.

With some computations, a relationship between the ψ-Riemann–Liouville and the ψ-Caputo fractional derivatives is easily deduced:

$$ \mathbb{D}_{a+}^{\alpha(x);\psi}f(x) = {}^{\mathrm{C}}\mathbb{D}_{a+}^{\alpha(x);\psi}f(x) + \frac{f(a)}{\Gamma(1-\alpha(x))}\left(\frac{1}{\psi'(x)}\frac{d}{\mathrm{d}x}\right) \times \int_a^x \psi'(t)(\psi(x)-\psi(t))^{-\alpha(x)}\mathrm{d}t = {}^{\mathrm{C}}\mathbb{D}_{a+}^{\alpha(x);\psi}f(x) + \frac{f(a)}{\Gamma(1-\alpha(x))}(\psi(x)-\psi(a))^{-\alpha(x)} + \frac{f(a)\,\alpha'(x)}{\psi'(x)\Gamma(2-\alpha(x))}(\psi(x)-\psi(a))^{1-\alpha(x)} \times \left[\frac{1}{1-\alpha(x)} - \ln(\psi(x)-\psi(a))\right] $$

and

$$ \mathcal{D}_{a+}^{\alpha(x);\psi}f(x) = {}^{\mathrm{C}}\mathcal{D}_{a+}^{\alpha(x);\psi}f(x) + f(a)\left(\frac{1}{\psi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right)\frac{1}{\Gamma(1-\alpha(x))} \times \int_a^x \psi'(t)(\psi(x)-\psi(t))^{-\alpha(x)}\mathrm{d}t = {}^{\mathrm{C}}\mathcal{D}_{a+}^{\alpha(x);\psi}f(x) + \frac{f(a)}{\Gamma(1-\alpha(x))}(\psi(x)-\psi(a))^{-\alpha(x)} $$

$$+\frac{f(a)\,\alpha'(x)}{\psi'(x)\Gamma(2-\alpha(x))}(\psi(x)-\psi(a))^{1-\alpha(x)}$$
$$\times\Big[\Psi(2-\alpha(x))-\ln(\psi(x)-\psi(a))\Big].$$

Theorem 2.10 *Sousa et al. (2020) Let $0<\alpha(x)<1-\frac{1}{n}$ for all $x\in[a,b]$ with a number $n\in\mathbb{N}$ greater than or equal to two, and $\psi'(t)\neq 0$. If $f\in C'([a,b],\mathbb{R})$, $h\in C([a,b],\mathbb{R})$ and $\mathbb{I}_{b-}^{1-\alpha(x);\psi}h\in AC[a,b]$, then*

$$\int_a^b h(x)\,{}^{\mathrm{C}}\mathcal{D}_{a+}^{\alpha(x);\psi}f(x)\mathrm{d}x$$
$$=f(x)\,\mathbb{I}_{b-}^{1-\alpha(x);\psi}\left(\frac{h(x)}{\psi'(x)}\right)$$
$$\times\Big|_a^b+\int_a^b\psi'(x)\mathcal{D}_{b-}^{\alpha(x);\psi}\left(\frac{h(x)}{\psi'(x)}\right)f(x)\mathrm{d}x$$

and

$$\int_a^b h(x)\,{}^{\mathrm{C}}\mathcal{D}_{b-}^{\alpha(x);\psi}f(x)\mathrm{d}x$$
$$=-f(x)\,\mathbb{I}_{a+}^{1-\alpha(x);\psi}\left(\frac{h(x)}{\psi'(x)}\right)$$
$$\times\Big|_a^b+\int_a^b\psi'(x)\mathcal{D}_{a+}^{\alpha(x);\psi}\left(\frac{h(x)}{\psi'(x)}\right)f(x)\mathrm{d}x.$$

Theorem 2.11 *Sousa et al. (2020) Let $0<\alpha(x)<1-\frac{1}{n}$ for all $x\in[a,b]$ with a number $n\in\mathbb{N}$ greater than or equal to two, and $\psi'(t)\neq 0$. If $f\in C'([a,b],\mathbb{R})$, $h\in C([a,b],\mathbb{R})$ and $\mathbb{I}_{b-}^{1-\alpha(x);\psi}h\in AC[a,b]$, then*

$$\int_a^b h(x)\,\mathcal{D}_{a+}^{\alpha(x);\psi}f(x)\mathrm{d}x$$
$$=\left(\frac{h(x)}{\psi'(x)}\right)\mathbb{I}_{b-}^{1-\alpha(x);\psi}f(x)$$
$$\times\Big|_a^b+\int_a^b\mathbb{I}_{b-}^{1-\alpha(x);\psi}f(x)\cdot\psi'(x)f_{\psi^-}\left(\frac{h(x)}{\psi'(x)}\right)\mathrm{d}x$$

and

$$\int_a^b h(x)\,\mathcal{D}_{b-}^{\alpha(x);\psi}f(x)\mathrm{d}x$$
$$=\left(\frac{h(x)}{\psi'(x)}\right)\mathbb{I}_{a+}^{1-\alpha(x);\psi}f(x)$$
$$\times\Big|_a^b+\int_a^b\mathbb{I}_{a+}^{1-\alpha(x);\psi}f(x)\cdot\psi'(x)f_{\psi^+}\left(\frac{h(x)}{\psi'(x)}\right)\mathrm{d}x.$$

3 Pseudo-analysis

In this section, we summarize some properties of the pseudo-analysis Babakhani et al. (2018); Mesiar and Rybárik (1993); Pap (1993); Pap and Štrboja (2010); Pap et al. (2014).

Let $[a,b]\subset[-\infty,+\infty]$. The full order on $[a,b]$ will be denoted by $\preceq$.

Definition 3.1 Pap and Štrboja (2010) A binary operation $\oplus$ on $[a,b]$ is pseudo-addition if it is commutative, non-decreasing (with respect to $\preceq$), continuous, associative, and with a zero (neutral) element denoted by $\mathbf{0}$. Let $[a,b]_+=\{x\,|\,x\in[a,b],\mathbf{0}\preceq x\}$. A binary operation $\odot$ on $[a,b]$ is pseudo-multiplication if it is commutative, positively non-decreasing, i.e., $x\preceq y$ implies $x\odot z\preceq y\odot z$ for all $z\in[a,b]_+$, associative and with a unit element $\mathbf{1}\in[a,b]$, i.e., for each $x\in[a,b]$, $\mathbf{1}\odot x=x$. Also, $\mathbf{0}\odot x=\mathbf{0}$ and that $\odot$ is distributive over $\oplus$, i.e.,

$$x\odot(y\oplus z)=(x\odot y)\oplus(x\odot z).$$

The structure $([a,b],\oplus,\odot)$ is a semiring (see Kuich (1986)).

Definition 3.2 Pap et al. (2014) An important class of pseudo-operations $\oplus$ and $\odot$ is when these are defined by a monotone and continuous function $g:[a,b]\to[0,\infty]$, i.e., pseudo-operations $\oplus$ and $\odot$ are given with

$$x\oplus y=g^{-1}(g(x)+g(y))\quad\text{and}$$
$$x\odot y=g^{-1}(g(x)\,g(y)).$$

Definition 3.3 Pap et al. (2014) Let X be a non-empty set and $\mathcal{A}$ be a σ-algebra of subsets of a set X. A set $\mu:\mathcal{A}\to[a,b]$ is called a σ-$\oplus$-measure if it satisfies the following conditions:

(1) $\mu(\emptyset)=\mathbf{0}$;

(2) $\mu:\left(\bigcup_{i=1}^{\infty}\mathcal{A}_i\right)=\bigoplus_{i=1}^{\infty}\mu(\mathcal{A}_i)$ holds for any sequence $\{\mathcal{A}_i\}_{i\in\mathbb{N}}$ of pairwise disjoint sets from $\mathcal{A}$.

Definition 3.4 Pap et al. (2014) Let pseudo-operations $\oplus$ and $\odot$ are defined by a monotone and continuous function $g:[a,b]\to[0,\infty]$. The g-integral for a measurable function $f:[c,d]\to[a,b]$ is given by

$$\int_{[c,d]}^{\oplus}f\odot\mathrm{d}x=g^{-1}\left(\int_c^d g(f(x))\mathrm{d}x\right).$$

Definition 3.5 Pap (1993) Let g be the additive generator of the strict pseudo-addition $\oplus$ on $[a, b]$ such that g is continuously differentiable on (a, b). The corresponding pseudo-multiplication $\odot$ will always be defined as $u \odot v = g^{-1}(g(u)g(v))$. If the function f is differentiable on (c, d) and has the same monotonicity as the function g, then the g-derivative of f at the point $x \in (c, d)$ is defined by

$$\frac{\mathrm{d}^{\oplus} f(x)}{\mathrm{d}x} = g^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}x} g(f(x))\right).$$

Also, if there exists the n-g-derivative of f, then

$$\frac{\mathrm{d}^{(n)\oplus} f(x)}{\mathrm{d}x} = g^{-1}\left(\frac{\mathrm{d}^n}{\mathrm{d}x^n} g(f(x))\right).$$

Definition 3.6 Mesiar and Rybárik (1993) Let g be a generator of a pseudo-addition $\oplus$ on interval $[-\infty, +\infty]$. Binary operation $\ominus$ and $\oslash$ on $[-\infty, +\infty]$ defined by the formulas:

$$x \ominus y = g^{-1}(g(x) - g(y)) \quad x \oslash y = g^{-1}\left(\frac{g(x)}{g(y)}\right),$$

if expressions $g(x) - g(y)$ and $\dfrac{g(x)}{g(y)}$ have sense are said to be the pseudo-subtraction and pseudo-division consistent with the pseudo-addition $\oplus$.

Definition 3.7 Mesiar and Rybárik (1993) Let $g : [-\infty, +\infty] \to [-\infty, +\infty]$ be a continuous, strictly increasing and odd function such that $g(0) = 0$, $g(1) = 1$, $g(+\infty) = +\infty$. The system of pseudo-arithmetical operations $\{\oplus, \ominus, \odot, \oslash\}$ generated by this function is said to be the consistent system.

Theorem 3.8 *Agahi et al. (2015) (Chebyshev's inequality for the first class of pseudo-integrals) Let $u, v : [0, 1] \to [a, b]$ be two measurable functions and let a generator $g : [a, b] \to [0, \infty)$ f the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function. If u and v are comonotone, the inequality*

$$\int_{[0,1]}^{\oplus} (u \odot v) \odot d\mu \geq \left(\int_{[0,1]}^{\oplus} u \odot d\mu\right) \odot \left(\int_{[0,1]}^{\oplus} v \odot d\mu\right) \tag{3.1}$$

holds and the reverse inequality holds whenever u and v are counter monotone functions.

The second class of pseudo-integrals is when $x \oplus y = \sup(x, y)$ and $x \odot y = g^{-1}(g(x)\, g(y))$, the pseudo-integral for a equation $f : \mathbb{R} \to [a, b]$ is given by

$$\int_{\mathbb{R}}^{\oplus} f(x) \odot d\mu = \sup_{x \in \mathbb{R}} \sup \left(f(x) \odot \widetilde{\psi}(x)\right) \tag{3.2}$$

where function $\widetilde{\psi}$ defines sup-measure μ. Any sup-measure generated as essential supremum of a continuous density can be obtained as a limit of pseudo-additive measures with respect to generated pseudo-addition.

We denote by m the usual Lebesgue measure on $\mathbb{R}$. We have

$$\begin{aligned} \mu(A) &= ess \sup_m (x \mid x \in A) \\ &= \sup \{a / m(\{x \mid x \in A,\ x > a\}) > 0\}. \end{aligned} \tag{3.3}$$

Theorem 3.9 *Agahi et al. (2015) Let μ be a sup-measure on $([0, \infty], B([0, \infty]))$, where $B([0, \infty])$ is the Borel σ-algebra on $[0, \infty]$, $\mu(A) = ess \sup_m (\widetilde{\psi}(x) \mid x \in A)$, and $\widetilde{\psi} : [0, \infty] \to [0, \infty]$ is a continuous density. Then for any pseudo-addition $\oplus$ with a generator g there exists a family $\{\mu_\lambda\}$ of $\oplus_\lambda-$measure on $([0, \infty], B([0, \infty]))$ where $\oplus_\lambda$ is generated by g^λ (the function g of the power λ), $\lambda \in (0, \infty)$ such that $\lim_{\lambda \to \infty} \mu_\lambda = \mu$.*

Theorem 3.10 *Agahi et al. (2015) Let $([0, \infty], \sup \odot)$ be a semiring with $\odot$ with a generator g, i.e., we have $x \odot y = g^{-1}(g(x)\, g(y))$ for every $x, y \in [a, b]$. Let μ be the same as in* Theorem 3.9. *Then there exists a family $\{\mu_\lambda\}$ of $\oplus_\lambda-$measures where $\oplus_\lambda$ is generated by g^λ, $\lambda \in (0, \infty)$ such that for every continuous function $f : [0, \infty] \to [0, \infty]$*

$$\begin{aligned} \int^{\sup} f \odot d\mu &= \lim_{\lambda \to \infty} \int^{\oplus_\lambda} f \odot d\mu_\lambda \\ &= \lim_{\lambda \to \infty} \left(g^\lambda\right)^{-1} \left(\int g^\lambda(f(x))\, dx\right). \end{aligned} \tag{3.4}$$

Theorem 3.11 *Agahi et al. (2015) (Chebyschev's inequality for the second class of pseudo-integrals). Let $u, v : [0, 1] \to [a, b]$ be two continuous functions and $\odot$ is represented by an increasing multiplicative generator g and μ be the same as in* Theorem 3.9. *If u and v comonotone, then the inequality*

$$\int_{[0,1]}^{\sup} (u \odot v) \odot d\mu \geq \left(\int_{[0,1]}^{\sup} u \odot d\mu\right) \left(\int_{[0,1]}^{\sup} v \odot d\mu\right) \tag{3.5}$$

holds and the reverse inequality holds whenever u and v are counter monotone functions.

4 Pseudo-fractional operators of variable order

In this section, we present the main contribution of this paper, that is, we extend the results of [13] and the definitions presented in Sect. 3. Then, we begin to introduce the Riemann–Liouville pseudo-fractional integral.

Definition 4.1 Let a, b be two reals with $0 < a < b$ and a generator $g : [a, b] \to [0, \infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function. Also let ψ be an increasing and positive function on $(a, b]$, having a continuous derivative ψ' on (a, b) with $\psi'(x) \neq 0$. The ψ-Riemann–Liouville pseudo-fractional integrals (left-sided and right-sided) of variable order $\alpha(x)$ of a measurable function $f : [a, b] \to [a, b]$ are defined as

$$\begin{aligned}&\mathbb{I}^{\alpha(x);\psi}_{\oplus,\odot;a+} f(x)\\ &:= g^{-1}\left(\mathbb{I}^{\alpha(x);\psi}_{a+} g(f(x))\right)\\ &= \int^{\oplus}_{[a,x]} \left[g^{-1}\left(\frac{\psi'(t)(\psi(x)-\psi(t))^{\alpha(x)-1}}{\Gamma(\alpha(x))}\right)\right.\\ &\left.\quad \times \odot f(t)\right] \odot \mathrm{d}t\end{aligned}$$

and

$$\begin{aligned}&\mathbb{I}^{\alpha(x);\psi}_{\oplus,\odot;b-} f(x)\\ &:= g^{-1}\left(\mathbb{I}^{\alpha(x);\psi}_{b-} g(f(x))\right)\\ &= \int^{\oplus}_{[x,b]} \left[g^{-1}\left(\frac{\psi'(t)(\psi(t)-\psi(x))^{\alpha(x)-1}}{\Gamma(\alpha(x))}\right)\right.\\ &\left.\quad \times \odot f(t)\right] \odot \mathrm{d}t,\end{aligned}$$

respectively, where $0 < \alpha(x) < 1$, $\forall x \in [a, b]$.

Remark 4.2 Sousa et al. (2020)

(1) If $\alpha(x) = \alpha$, with $0 < \alpha < 1$ be a constant, we have the ψ-Riemann–Liouville pseudo-fractional integrals of order α

$$\begin{aligned}&\mathbb{I}^{\alpha;\psi}_{\oplus,\odot;a+} f(x)\\ &= \int^{\oplus}_{[a,x]} \left[g^{-1}\left(\frac{\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}}{\Gamma(\alpha)}\right)\right.\\ &\left.\quad \times \odot f(t)\right] \odot \mathrm{d}t\end{aligned}$$

and

$$\begin{aligned}&\mathbb{I}^{\alpha;\psi}_{\oplus,\odot;b-} f(x)\\ &= \int^{\oplus}_{[x,b]} \left[g^{-1}\left(\frac{\psi'(t)(\psi(t)-\psi(x))^{\alpha-1}}{\Gamma(\alpha)}\right)\right.\\ &\left.\quad \times \odot f(t)\right] \odot \mathrm{d}t.\end{aligned}$$

(2) If $g(x) = x$, we have the ψ-Riemann–Liouville fractional integrals of order $0 < \alpha(x) < 1$, given by

$$\mathbb{I}^{\alpha(x);\psi}_{a+} f(x) = \int_a^t \frac{\psi'(t)(\psi(x)-\psi(t))^{\alpha(x)-1}}{\Gamma(\alpha(x))} f(t)\mathrm{d}t$$

and

$$\mathbb{I}^{\alpha(x);\psi}_{b-} f(x) = \int_t^b \frac{\psi'(t)(\psi(t)-\psi(x))^{\alpha(x)-1}}{\Gamma(\alpha(x))} f(t)\mathrm{d}t$$

(3) If $\alpha(x) = 1$ and $\psi(x) = x$, we have the pseudo-integrals given by (note that $g^{-1}(1) = 1$)

$$\mathbb{I}_{\oplus,\odot;a+} f(x) = \int^{\oplus}_{[a,x]} f(t) \odot \mathrm{d}t$$

and

$$\mathbb{I}_{\oplus,\odot;b-} f(x) = \int^{\oplus}_{[x,b]} f(t) \odot \mathrm{d}t.$$

(4) Note that the possibilities that operators $\mathbb{I}^{\alpha(x),\psi}_{\oplus,\odot;a+}$ and $\mathbb{I}^{\alpha(x),\psi}_{\oplus,\odot;b-}$ allow for the discussion of particular cases are numerous, that is, in this single operator we have the following branches: pseudo-operator, fractional pseudo-operator, fractional calculus with respect to another function and ψ-fractional calculus of varying order.

The law of exponents is not always valid for ψ-Riemann–Liouville pseudo-fractional integrals of variable order $\alpha(x)$, i.e.,

$$\mathbb{I}^{\alpha(x);\psi}_{\oplus,\odot;a+}\mathbb{I}^{\beta(x);\psi}_{\oplus,\odot;a+} f(x) \neq \mathbb{I}^{\alpha(x)+\beta(x);\psi}_{\oplus,\odot;a+} f(x) \quad \text{and}$$
$$\mathbb{I}^{\alpha(x);\psi}_{\oplus,\odot;b-}\mathbb{I}^{\beta(x);\psi}_{\oplus,\odot;b-} f(x) \neq \mathbb{I}^{\alpha(x)+\beta(x);\psi}_{\oplus,\odot;b-} f(x).$$

Example 4.3 Let $g(x) = x$ and $f(x) = (\psi(x) - \psi(a))^{\gamma-1}$ with $\gamma > 0$. The corresponding pseudo-operations are $x \oplus y = x + y$ and $x \odot y = xy$. Then, the left-sided ψ-Riemann–Liouville pseudo-fractional integral of variable order $\alpha(x)$, $0 < \alpha(x) < 1$, of f is

$$\begin{aligned}&\mathbb{I}^{\alpha(x);\psi}_{\oplus,\odot;a+}[(\psi(x)-\psi(a))^{\gamma-1}]\\ &= g^{-1}\left(\mathbb{I}^{\alpha(x);\psi}_{a+} g\left((\psi(x)-\psi(a))^{\gamma-1}\right)\right)\\ &= g^{-1}\left(\mathbb{I}^{\alpha(x);\psi}_{a+} [\psi(x)-\psi(a)]^{\gamma-1}\right)\\ &= g^{-1}\left(\frac{\Gamma(\gamma)}{\Gamma(\alpha(x)+\gamma)} (\psi(x)-\psi(a))^{\alpha(x)+\gamma-1}\right)\\ &= \frac{\Gamma(\gamma)}{\Gamma(\alpha(x)+\gamma)} (\psi(x)-\psi(a))^{\alpha(x)+\gamma-1}.\end{aligned}$$

Remark 4.4 One of the problems when discussing examples involving pseudo-operators, in this particular case, pseudo-integral, is the difficulty imposed by g, with the particular choice of $f(x)$.

Definition 4.5 Let a generator $g : [a, b] \to [0, \infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function. Also let ψ be an increasing and positive function on $(a, b]$, having a continuous derivative ψ' on (a, b) and $\psi'(x) \neq 0$. The ψ-Riemann–Liouville pseudo-fractional derivatives (left-sided and right-sided) of variable order $\alpha(x)$, where $0 < \alpha(x) < 1$, of a measurable function $f : [a, b] \to [a, b]$:

(1) Type I are defined by

$$\begin{aligned}
&\mathbb{D}^{\alpha(x);\psi}_{\oplus,\odot;a+} f(x)\\
&:= g^{-1}\left(\mathbb{D}^{\alpha(x);\psi}_{a+} f(x)\right)\\
&= g^{-1}\left(\frac{1}{\Gamma(1-\alpha(x))}\right) \odot g^{-1}\left(f_{\psi^+}\right) \odot \int^{\oplus}_{[a,x]}\\
&\times \left[g^{-1}\left(\psi'(t)(\psi(x)-\psi(t))^{-\alpha(x)}\right) \odot f(t)\right] \odot \mathrm{d}t
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{D}^{\alpha(x);\psi}_{\oplus,\odot;b-} f(x)\\
&:= g^{-1}\left(\mathbb{D}^{\alpha(x);\psi}_{b-} f(x)\right)\\
&= g^{-1}\left(\frac{1}{\Gamma(1-\alpha(x))}\right) \odot g^{-1}\left(f_{\psi^-}\right) \odot \int^{\oplus}_{[x,b]}\\
&\times \left[g^{-1}\left(\psi'(t)(\psi(t)-\psi(x))^{-\alpha(x)}\right) \odot f(t)\right] \odot \mathrm{d}t.
\end{aligned}$$

(2) Type II are defined by

$$\begin{aligned}
&\mathcal{D}^{\alpha(x);\psi}_{\oplus,\odot;a+} f(x)\\
&:= g^{-1}\left(\mathcal{D}^{\alpha(x);\psi}_{a+} f(x)\right)\\
&= g^{-1}\left(f_{\psi^+}\right) \odot g^{-1}\left(\frac{1}{\Gamma(1-\alpha(x))}\right) \odot \int^{\oplus}_{[a,x]}\\
&\times \left[g^{-1}\left(\psi'(t)(\psi(x)-\psi(t))^{-\alpha(x)}\right) \odot f(t)\right] \odot \mathrm{d}t
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{D}^{\alpha(x);\psi}_{\oplus,\odot;b-} f(x)\\
&:= g^{-1}\left(\mathcal{D}^{\alpha(x);\psi}_{b-} f(x)\right)\\
&= g^{-1}\left(f_{\psi^-}\right) \odot g^{-1}\left(\frac{1}{\Gamma(1-\alpha(x))}\right) \odot \int^{\oplus}_{[x,b]}\\
&\times \left[g^{-1}\left(\psi'(t)(\psi(t)-\psi(x))^{-\alpha(x)}\right) \odot f(t)\right] \odot \mathrm{d}t.
\end{aligned}$$

Definition 4.6 Let a generator $g : [a, b] \to [0, \infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function. Also let ψ be an increasing and positive function on $(a, b]$, having a continuous derivative ψ' on (a, b) and $\psi'(x) \neq 0$. The ψ-Caputo pseudo-fractional derivatives (left-sided and right-sided) of variable order $\alpha(x)$, where $0 < \alpha(x) < 1$, of a measurable function $f : [a, b] \to [a, b]$:

(1) Type I are defined by

$$\begin{aligned}
&{}^{\mathsf{C}}\mathbb{D}^{\alpha(x);\psi}_{\oplus,\odot;a+} f(x)\\
&:= g^{-1}\left({}^{\mathsf{C}}\mathbb{D}^{\alpha(x);\psi}_{a+} f(x)\right)\\
&= g^{-1}\left(\frac{1}{\Gamma(1-\alpha(x))}\right) \odot g^{-1}\left(f_{\psi^+}\right) \odot \int^{\oplus}_{[a,x]}\\
&\times \left[g^{-1}\left(\psi'(t)(\psi(x)-\psi(t))^{-\alpha(x)}\right) \odot \right.\\
&\left. \times [f(t)-f(a)]\right] \odot \mathrm{d}t
\end{aligned}$$

and

$$\begin{aligned}
&{}^{\mathsf{C}}\mathbb{D}^{\alpha(x);\psi}_{\oplus,\odot;b-} f(x)\\
&:= g^{-1}\left({}^{\mathsf{C}}\mathbb{D}^{\alpha(x);\psi}_{b-} f(x)\right)\\
&= g^{-1}\left(\frac{1}{\Gamma(1-\alpha(x))}\right) \odot g^{-1}\left(f_{\psi^-}\right) \odot \int^{\oplus}_{[x,b]}\\
&\times \left[g^{-1}\left(\psi'(t)(\psi(t)-\psi(x))^{-\alpha(x)}\right) \odot \right.\\
&\left. \times [f(t)-f(b)]\right] \odot \mathrm{d}t.
\end{aligned}$$

(2) Type II are defined by

$$\begin{aligned}
&{}^{\mathsf{C}}\mathcal{D}^{\alpha(x);\psi}_{\oplus,\odot;a+} f(x)\\
&:= g^{-1}\left({}^{\mathsf{C}}\mathcal{D}^{\alpha(x);\psi}_{a+} f(x)\right)\\
&= g^{-1}\left(f_{\psi^+}\right) \odot g^{-1}\left(f_{\psi^+}\right) \odot \int^{\oplus}_{[a,x]}\\
&\times \left[g^{-1}\left(\psi'(t)(\psi(x)-\psi(t))^{-\alpha(x)}\right) \odot \right.\\
&\left. \times [f(t)-f(a)]\right] \odot \mathrm{d}t
\end{aligned}$$

and

$$
\begin{aligned}
&{}^{\mathsf{C}}\mathcal{D}^{\alpha(x);\psi}_{\oplus,\odot;b-}f(x)\\
&:= g^{-1}\left({}^{\mathsf{C}}\mathcal{D}^{\alpha(x);\psi}_{b-}f(x)\right)\\
&= g^{-1}\left(f_{\psi^-}\right)\odot g^{-1}\left(f_{\psi^-}\right)\odot\int^{\oplus}_{[x,b]}\\
&\quad\times\left[g^{-1}\left(\psi'(t)(\psi(t)-\psi(x))^{-\alpha(x)}\right)\odot\right.\\
&\quad\left.\times[f(t)-f(b)]\right]\odot \mathrm{d}t.
\end{aligned}
$$

(3) Type III are defined by

$$
\begin{aligned}
&{}^{\mathsf{C}}\mathscr{D}^{\alpha(x);\psi}_{\oplus,\odot;a+}f(x)\\
&:= g^{-1}\left({}^{\mathsf{C}}\mathscr{D}^{\alpha(x);\psi}_{a+}f(x)\right)\\
&= g^{-1}\left(\frac{1}{\Gamma(1-\alpha(x))}\right)\odot\int^{\oplus}_{[a,x]}\\
&\quad\times\left[g^{-1}\left(\psi'(t)(\psi(x)-\psi(t))^{-\alpha(x)}\right)\odot\right.\\
&\quad\left.\times f^{\oplus}_{\psi^+}f(t)\right]\odot \mathrm{d}t
\end{aligned}
$$

and

$$
\begin{aligned}
&{}^{\mathsf{C}}\mathscr{D}^{\alpha(x);\psi}_{\oplus,\odot;b-}f(x)\\
&:= g^{-1}\left({}^{\mathsf{C}}\mathscr{D}^{\alpha(x);\psi}_{b-}f(x)\right)\\
&= g^{-1}\left(\frac{1}{\Gamma(1-\alpha(x))}\right)\odot\int^{\oplus}_{[x,b]}\\
&\quad\times\left[g^{-1}\left(\psi'(t)(\psi(t)-\psi(x))^{-\alpha(x)}\right)\odot\right.\\
&\quad\left.\times f^{\oplus}_{\psi^-}f(t)\right]\odot \mathrm{d}t,
\end{aligned}
$$

where $f^{\oplus}_{\psi^+}f(x) = \left(\frac{1}{\psi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\oplus}f(x) = g^{-1}\left[\left(\frac{1}{\psi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right)g(f(x))\right]$ and $f^{\oplus}_{\psi^-}f(x) = \left(-\frac{1}{\psi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\oplus}f(x) = g^{-1}\left[\left(-\frac{1}{\psi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right)g(f(x))\right]$.

Example 4.7 Let $g(x)=x^{\beta}$ and $f(x)=[\psi(x)-\psi(a)]$ with $\beta>0$. The corresponding pseudo-operations are $x\oplus y=\sqrt[\beta]{x^{\beta}+y^{\beta}}$ and $x\odot y = xy$. Then the left-sided ψ-Riemann–Liouville pseudo-fractional derivative of variable order $\alpha(x)$, where $0<\alpha(x)<1$, of f is

$$
\begin{aligned}
&\mathbb{D}^{\alpha(x);\psi}_{\oplus,\odot;a+}[\psi(x)-\psi(a)]\\
&= g^{-1}\left(\mathbb{D}^{\alpha(x);\psi}_{a+}g(f(x))\right)\\
&= g^{-1}\left(\mathbb{D}^{\alpha(x);\psi}_{a+}[\psi(x)-\psi(a)]^{\beta}\right)\\
&= g^{-1}\left\{\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha(x)+1)}(\psi(x)-\psi(a))^{\beta-\alpha(x)}\right.\\
&\quad-\frac{\alpha'(x)}{\psi'(x)}\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha(x)+2)}(\psi(x)-\psi(a))^{\beta-\alpha(x)+1}\\
&\quad\times\left[\Psi(1-\alpha(x))-\Psi(\beta-\alpha(x)+2)\right.\\
&\quad\left.\left.+\ln(\psi(x)-\psi(a))\right]\right\}\\
&= \left(\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha(x)+1)}(\psi(x)-\psi(a))^{\beta-\alpha(x)}\right.\\
&\quad-\frac{\alpha'(x)}{\psi'(x)}\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha(x)+2)}\\
&\quad\times(\psi(x)-\psi(a))^{\beta-\alpha(x)+1}\\
&\quad\times\left[\Psi(1-\alpha(x))-\Psi(\beta-\alpha(x)+2)\right.\\
&\quad\left.\left.+\ln(\psi(x)-\psi(a))\right]\right)^{1/\beta}.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&\mathcal{D}^{\alpha(x);\psi}_{\oplus,\odot;a+}[\psi(x)-\psi(a)]\\
&= g^{-1}\left(\mathcal{D}^{\alpha(x);\psi}_{a+}g(f(x))\right)\\
&= g^{-1}\left(\mathcal{D}^{\alpha(x);\psi}_{a+}[\psi(x)-\psi(a)]^{\beta}\right)\\
&= g^{-1}\left\{\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha(x)+2)}(\psi(x)-\psi(a))^{\beta-\alpha(x)}\right.\\
&\quad-\frac{\alpha'(x)}{\psi'(x)}\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha(x)+2)}(\psi(x)-\psi(a))^{\beta-\alpha(x)+1}\\
&\quad\left.\times\left[\ln(\psi(x)-\psi(a))-\Psi(\beta-\alpha(x)+2)\right]\right\}\\
&= \left(\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha(x)+1)}(\psi(x)-\psi(a))^{\beta-\alpha(x)}\right.\\
&\quad-\frac{\alpha'(x)}{\psi'(x)}\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha(x)+2)}\\
&\quad\times(\psi(x)-\psi(a))^{\beta-\alpha(x)+1}\\
&\quad\left.\times\left[\ln(\psi(x)-\psi(a))-\Psi(\beta-\alpha(x)+2)\right]\right)^{1/\beta}.
\end{aligned}
$$

Example 4.8 Let $g(x)=x^\beta$ and $f(x)=[\psi(x)-\psi(a)]$ with $\beta>0$. The corresponding pseudo-operations are $x\oplus y=\sqrt[\beta]{x^\beta+y^\beta}$ and $x\odot y=xy$. Then the left-sided ψ-Caputo pseudo-fractional derivative of variable order $\alpha(x)$, where $0<\alpha(x)<1$, of f is

$$
\begin{aligned}
&{}^{\mathsf{C}}\mathbb{D}^{\alpha(x);\psi}_{\oplus,\odot;a+}[\psi(x)-\psi(a)]\\
&=\sqrt[\beta]{\begin{aligned}&\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha(x)+1)}(\psi(x)-\psi(a))^{\beta-\alpha(x)}\\&-\frac{\alpha'(x)}{\psi'(x)}\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha(x)+2)}\\&\times(\psi(x)-\psi(a))^{\beta-\alpha(x)+1}\\&\times\Big[\Psi(1-\alpha(x))-\Psi(\beta-\alpha(x)+2)\\&+\ln(\psi(x)-\psi(a))\Big]\end{aligned}};
\end{aligned}
$$

$$
\begin{aligned}
&{}^{\mathsf{C}}\mathcal{D}^{\alpha(x);\psi}_{\oplus,\odot;a+}[\psi(x)-\psi(a)]\\
&=\sqrt[\beta]{\begin{aligned}&\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha(x)+1)}(\psi(x)-\psi(a))^{\beta-\alpha(x)}\\&-\frac{\alpha'(x)}{\psi'(x)}\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha(x)+2)}\\&\times(\psi(x)-\psi(a))^{\beta-\alpha(x)+1}\\&\times\Big[\ln(\psi(x)-\psi(a))-\Psi(\beta-\alpha(x)+2)\Big]\Big]\Big\}\end{aligned}};
\end{aligned}
$$

$$
\begin{aligned}
&{}^{\mathsf{C}}\mathscr{D}^{\alpha(x);\psi}_{\oplus,\odot;a+}[\psi(x)-\psi(a)]\\
&=\sqrt[\beta]{\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha(x)+1)}(\psi(x)-\psi(a))^{\beta-\alpha(x)}}.
\end{aligned}
$$

Theorem 4.9 *Let g be a generator of a pseudo-addition $\oplus$ on the interval $[-\infty,\infty]$. Then for $0<\alpha(x)<1$, the following relations between the left pseudo-fractional operators hold:*

$$
\begin{aligned}
&{}^{\mathsf{C}}\mathbb{D}^{\alpha(x);\psi}_{\oplus,\odot;a+}f(x)\\
&={}^{\mathsf{C}}\mathscr{D}^{\alpha(x);\psi}_{a+}f(x)\oplus\Big\{g^{-1}[\alpha'(x)]\oslash\\
&\times\Big[g^{-1}(\psi'(x))\odot g^{-1}\big[\Gamma(2-\alpha(x)\big]\Big]\odot\int_{[a,x]}^{\oplus}\\
&\times\Big[g^{-1}\Big((\psi(x)-\psi(t))^{1-\alpha(x)}\Big[\frac{1}{1-\alpha(x)}\\
&-\ln(\psi(x)-\psi(t))\Big]\Big)\odot\frac{\mathrm{d}^{\oplus}}{\mathrm{d}t}f(t)\Big]\odot\mathrm{d}t\Big\}
\end{aligned}
$$

and

$$
\begin{aligned}
&{}^{\mathsf{C}}\mathbb{D}^{\alpha(x);\psi}_{\oplus,\odot;a+}f(x)\\
&={}^{\mathsf{C}}\mathcal{D}^{\alpha(x);\psi}_{\oplus,\odot;a+}f(x)\ominus\Big\{\Big[\Big(g^{-1}[\alpha'(x)]\odot g^{-1}\\
&\times[\Psi(1-\alpha(x))]\Big)\oslash\Big(g^{-1}[\psi'(x)]\odot g^{-1}\\
&\times[\Gamma(1-\alpha(x))]\Big)\Big]\odot\int_{[a,x]}^{\oplus}\Big[g^{-1}\Big(\psi'(t)(\psi(x)\\
&-\psi(t))^{-\alpha(x)}\Big)\odot[f(t)-f(a)]\Big]\odot\mathrm{d}t\Big\}.
\end{aligned}
$$

Proof Using the definition for ${}^{\mathsf{C}}\mathbb{D}^{\alpha(x);\psi}_{\oplus,\odot;a+}$ and Theorem 2.7, we can write

$$
\begin{aligned}
&{}^{\mathsf{C}}\mathbb{D}^{\alpha(x);\psi}_{\oplus,\odot;a+}f(x)\\
&=g^{-1}\Big({}^{\mathsf{C}}\mathbb{D}^{\alpha(x);\psi}_{a+}g(f(x))\Big)\\
&=g^{-1}\Big({}^{\mathsf{C}}\mathscr{D}^{\alpha(x);\psi}_{a+}g(f(x))\\
&+\frac{\alpha'(x)}{\psi'(x)\Gamma(2-\alpha(x))}\int_a^x(\psi(x)-\psi(t))^{1-\alpha(x)}\\
&\times\Big[\frac{1}{1-\alpha(x)}-\ln(\psi(x)-\psi(t))\Big]g'(f(t))\mathrm{d}t\Big)\\
&=g^{-1}\Big\{g\Big(g^{-1}\Big({}^{\mathsf{C}}\mathscr{D}^{\alpha(x);\psi}_{a+}g(f(x))\Big)\Big)\\
&+g\Big[g^{-1}\Big(\frac{\alpha'(x)}{\psi'(x)\Gamma(2-\alpha(x))}\int_a^x(\psi(x)-\psi(t))^{1-\alpha(x)}\\
&\times\Big[\frac{1}{1-\alpha(x)}-\ln(\psi(x)-\psi(t))\Big]g'(f(t))\mathrm{d}t\Big)\Big]\Big\}\\
&=g^{-1}\Big({}^{\mathsf{C}}\mathscr{D}^{\alpha(x);\psi}_{a+}g(f(x))\Big)\oplus g^{-1}\\
&\times\Big(\frac{\alpha'(x)}{\psi'(x)\Gamma(2-\alpha(x))}\int_a^x(\psi(x)-\psi(t))^{1-\alpha(x)}\\
&\times\Big[\frac{1}{1-\alpha(x)}-\ln(\psi(x)-\psi(t))\Big]g'(f(t))\mathrm{d}t\Big)\\
&=g^{-1}\Big({}^{\mathsf{C}}\mathscr{D}^{\alpha(x);\psi}_{a+}g(f(x))\Big)\oplus g^{-1}\\
&\times\Big\{g\Big[g^{-1}\Big(\frac{\alpha'(x)}{\psi'(x)\Gamma(2-\alpha(x))}\Big)\Big]\Big[g\Big(g^{-1}\Big(\int_a^x(\psi(x)\\
&-\psi(t))^{1-\alpha(x)}\Big[\frac{1}{1-\alpha(x)}-\ln(\psi(x)-\psi(t))\Big]\\
&\times g'(f(t))\mathrm{d}t\Big)\Big)\Big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= g^{-1}\left({}^{\mathsf{C}}\mathscr{D}_{a+}^{\alpha(x);\psi} g(f(x))\right) \\
&\oplus\left\{g^{-1}\left(\frac{\alpha'(x)}{\psi'(x)\Gamma(2-\alpha(x))}\right) \odot g^{-1}\right. \\
&\times\left(\int_a^x (\psi(x)-\psi(t))^{1-\alpha(x)}\right. \\
&\left.\left.\times\left[\frac{1}{1-\alpha(x)} - \ln(\psi(x)-\psi(t))\right] g'(f(t))\mathrm{d}t\right)\right\} \\
&= g^{-1}\left({}^{\mathsf{C}}\mathscr{D}_{a+}^{\alpha(x);\psi} g(f(x))\right) \\
&\oplus\left\{g^{-1}\left[\frac{g\big(g^{-1}\big(\alpha'(x)\big)\big)}{g\big(g^{-1}\big(\psi'(x)\Gamma(2-\alpha(x))\big)\big)}\right] \odot g^{-1}\right. \\
&\times\left(\int_a^x g\left[g^{-1}\left((\psi(x)-\psi(t))^{1-\alpha(x)}\right.\right.\right. \\
&\left.\left.\left.\left.\times\left[\frac{1}{1-\alpha(x)} - \ln(\psi(x)-\psi(t))\right] g'(f(t))\right)\mathrm{d}t\right]\right)\right\} \\
&= {}^{\mathsf{C}}\mathscr{D}_{a+}^{\alpha(x);\psi} f(x) \oplus \left\{g^{-1}[\alpha'(x)] \oslash\right. \\
&\left.\times\left[g^{-1}\left(\psi'(x)\Gamma(2-\alpha(x))\right)\right]\right\} \odot \int_{[a,x]}^{\oplus} g^{-1} \\
&\times\left((\psi(x)-\psi(t))^{1-\alpha(x)}\right. \\
&\left.\times\left[\frac{1}{1-\alpha(x)} - \ln(\psi(x)-\psi(t))\right] g'(f(t))\right)\mathrm{d}t \\
&= {}^{\mathsf{C}}\mathscr{D}_{a+}^{\alpha(x);\psi} f(x) \oplus \left\{g^{-1}[\alpha'(x)] \oslash\right. \\
&\left.\times\left[g^{-1}\left(g(g^{-1}(\psi'(x)))g\left(g^{-1}[\Gamma(2-\alpha(x))]\right)\right)\right]\right\} \odot \\
&\times\int_{[a,x]}^{\oplus} g^{-1}\left[g\left(g^{-1}\left((\psi(x)-\psi(t))^{1-\alpha(x)}\right.\right.\right. \\
&\left.\left.\times\left[\frac{1}{1-\alpha(x)} - \ln(\psi(x)-\psi(t))\right]\right)\right) \\
&\left.\times g\left(g^{-1}\left(g'(f(t))\right)\right)\right]\mathrm{d}t\Big\} = {}^{\mathsf{C}}\mathscr{D}_{a+}^{\alpha(x);\psi} f(x) \\
&\oplus\left\{g^{-1}[\alpha'(x)] \oslash \left[g^{-1}(\psi'(x)) \odot\right.\right. \\
&\left.\left.\times g^{-1}\left[\Gamma(2-\alpha(x)]\right]\right\} \odot \int_{[a,x]}^{\oplus} \\
&\times\left[g^{-1}\left((\psi(x)-\psi(t))^{1-\alpha(x)}\left[\frac{1}{1-\alpha(x)}\right.\right.\right. \\
&\left.\left.\left. - \ln(\psi(x)-\psi(t))\right]\right) \odot \frac{\mathrm{d}^{\oplus}}{\mathrm{d}t} f(t)\right] \odot \mathrm{d}t.
\end{aligned}
$$

□

Theorem 4.10 *Let g be a generator of a pseudo-addition $\oplus$ on the interval $[-\infty,\infty]$. Then for $0<\alpha(x)<1$ and $x\in[a,b]$, we have*

(a) $\mathbb{D}_{\oplus,\odot;a+}^{\alpha(x);\psi} f(x) = \mathcal{D}_{\oplus,\odot;a+}^{\alpha(x);\psi} f(x) = \mathbf{0}$;

(b) $\mathbb{D}_{\oplus,\odot;b-}^{\alpha(x);\psi} f(x) = \mathcal{D}_{\oplus,\odot;b-}^{\alpha(x);\psi} f(x) = \mathbf{0}$.

Proof For the first equality, one has

$$
\begin{aligned}
&\left\|\mathbb{D}_{\oplus,\odot;a+}^{\alpha(x);\psi} f(x)\right\| \\
&= \left\|g^{-1}\left(\mathbb{D}_{a+}^{\alpha(x);\psi} g(f(x))\right)\right\| \\
&= \left\|g^{-1}\left[\frac{1}{\Gamma(1-\alpha(x))} f_{\psi^+}\int_a^x \psi'(t)\right.\right. \\
&\left.\left.\times(\psi(x)-\psi(t))^{-\alpha(x)} g(f(t))\mathrm{d}t\right]\right\| \\
&\le \left\|g^{-1}\left[\frac{1}{\Gamma(1-\alpha(x))} f_{\psi^+} g(f(x))\right.\right. \\
&\left.\left.\times\int_a^x \psi'(t)(\psi(x)-\psi(t))^{-\alpha(x)}\mathrm{d}t\right]\right\| \\
&= \left\|g^{-1}\left[g\left(g^{-1}\left(\frac{1}{\Gamma(1-\alpha(x))}\right)\right)g\right.\right. \\
&\times\left(g^{-1}\left(f_{\psi^+} g(f(x))\right)\right)g\left(g^{-1}\left(\int_a^x \psi'(t)\right.\right. \\
&\left.\left.\left.\left.\times(\psi(x)-\psi(t))^{-\alpha(x)}\mathrm{d}t\right)\right)\right]\right\| \\
&= \left\|g^{-1}\left(\frac{1}{\Gamma(1-\alpha(x))}\right) \odot g^{-1}(f_{\psi^+}) \odot g^{-1}(g(f(x)))\right. \\
&\left.\times \odot g^{-1}\left[\int_a^x \psi'(t)(\psi(x)-\psi(t))^{-\alpha(x)}\mathrm{d}t\right]\right\| \\
&\le \left\|g^{-1}\left(\frac{1}{\Gamma(1-\alpha(x))}\right)\right\| \odot \left\|g^{-1}(f_{\psi^+})\right\| \odot \\
&\times\|f(x)\| \odot \left\|g^{-1}\left(\int_a^x \psi'(t)\right.\right. \\
&\left.\left.\times(\psi(x)-\psi(t))^{-\alpha(x)}\mathrm{d}t\right)\right\| \\
&= \left\|g^{-1}\left(\frac{1}{\Gamma(1-\alpha(x))}\right)\right\| \odot \left\|g^{-1}(f_{\psi^+})\right\| \odot \\
&\times\|f(x)\| \odot \left\|g^{-1}\left(\frac{(\psi(x)-\psi(a))^{1-\alpha(x)}}{1-\alpha(x)}\right)\right\|,
\end{aligned}
$$

which is zero at $x=a$. Here zero is $\mathbf{0}=g^{-1}(0)$. The proof that the $\mathcal{D}_{\oplus,\odot;a+}^{\alpha(x);\psi}$ also vanish at the end point $t=a$ follows by similar arguments. □

Theorem 4.11 *Let g be a generator of a pseudo-addition $\oplus$ on the interval $[-\infty,\infty]$. Then for $0<\alpha(x)<1$ and $x\in[a,b]$, we have*

(a) ${}^{\mathsf{C}}\mathbb{D}_{\oplus,\odot;a+}^{\alpha(x);\psi} f(x) = {}^{\mathsf{C}}\mathcal{D}_{\oplus,\odot;a+}^{\alpha(x);\psi} f(x) = {}^{\mathsf{C}}\mathscr{D}_{\oplus,\odot;a+}^{\alpha(x);\psi} f(x) = \mathbf{0}$;

(b) ${}^{\mathrm{C}}\mathbb{D}^{\alpha(x);\psi}_{\oplus,\odot;b-}f(x) = {}^{\mathrm{C}}\mathcal{D}^{\alpha(x);\psi}_{\oplus,\odot;b-}f(x) = {}^{\mathrm{C}}\mathscr{D}^{\alpha(x);\psi}_{\oplus,\odot;b-}f(x) = \mathbf{0}$.

Proof For the third equality, we have

$$\begin{aligned}
&\left\|{}^{\mathrm{C}}\mathscr{D}^{\alpha(x);\psi}_{\oplus,\odot;a+}f(x)\right\| \\
&= \left\|g^{-1}\left({}^{\mathrm{C}}\mathscr{D}^{\alpha(x);\psi}_{a+}g(f(x))\right)\right\| \\
&= \left\|g^{-1}\left(\frac{1}{\Gamma(1-\alpha(x))}\right)\odot g^{-1}\right. \\
&\quad \left.\times\left(\int_a^x \psi'(t)(\psi(x)-\psi(t))^{-\alpha(x)} f_{\psi+}g(f(t))\mathrm{d}t\right)\right\| \\
&\le \left\|g^{-1}\left(\frac{1}{\Gamma(1-\alpha(x))}\right)\right\|\odot\left\|g^{-1}\left(f_{\psi+}g(f(x))\right.\right. \\
&\quad \left.\left.\times\int_a^x \psi'(t)(\psi(x)-\psi(t))^{-\alpha(x)}\mathrm{d}t\right)\right\| \\
&\le \left\|g^{-1}\left(\frac{1}{\Gamma(1-\alpha(x))}\right)\right\|\odot\left\|f^{\oplus}_{\psi+}f(x)\right\|\odot \\
&\quad \times\left\|g^{-1}\left(\int_a^x \psi'(t)(\psi(x)-\psi(t))^{-\alpha(x)}\mathrm{d}t\right)\right\| \\
&= g^{-1}\left(\frac{1}{\Gamma(1-\alpha(x))}\right)\odot f^{\oplus}_{\psi+}f(x)\odot \\
&\quad \times\left\|g^{-1}\left(\frac{(\psi(x)-\psi(a))^{1-\alpha(x)}}{1-\alpha(x)}\right)\right\|,
\end{aligned}$$

which is zero at $x = a$, where zero is $g^{-1}(0) = \mathbf{0}$. For the first equality, we use Theorem 4.9,

$$\begin{aligned}
&\left\|{}^{\mathrm{C}}\mathbb{D}^{\alpha(x);\psi}_{\oplus,\odot;a+}f(x)\right\| \\
&\le \left\|{}^{\mathrm{C}}\mathscr{D}^{\alpha(x);\psi}_{\oplus,\odot;a+}f(x)\right\| \\
&\quad + \left\|g^{-1}[\alpha'(x)]\oslash\left[g^{-1}(\psi'(x))\odot g^{-1}\left[\Gamma(2-\alpha(x))\right]\right]\right\| \\
&\quad \odot\left\|\frac{\mathrm{d}^{\oplus}}{\mathrm{d}x}f(x)\right\|\odot\left\|g^{-1}\left(\int_a^x (\psi(x)-\psi(t))^{1-\alpha(x)}\right.\right. \\
&\quad \left.\left.\times\left[\frac{1}{1-\alpha(x)}-\ln(\psi(x)-\psi(t))\mathrm{d}t\right]\right)\right\|.
\end{aligned}$$

We need to show that the following integrals,

$$\frac{1}{1-\alpha(x)}\int_a^x (\psi(x)-\psi(t))^{1-\alpha(x)}\mathrm{d}t \quad \text{and}$$
$$\int_a^x (\psi(x)-\psi(t))^{1-\alpha(x)}\ln(\psi(x)-\psi(t))\mathrm{d}t$$

are zero at the point $x = a$. The first integral can be solved using integration by parts,

$$\begin{aligned}
&\frac{1}{1-\alpha(x)}\int_a^x (\psi(x)-\psi(t))^{1-\alpha(x)}\mathrm{d}t \\
&= \frac{1}{1-\alpha(x)}\int_a^x \frac{(\psi(x)-\psi(t))^{1-\alpha(x)}}{\psi'(t)}\psi'(t)\mathrm{d}t \\
&= \frac{1}{(1-\alpha(x))(2-\alpha(x))}\left[\frac{(\psi(x)-\psi(a))^{2-\alpha(x)}}{\psi'(a)}\right. \\
&\quad \left.-\int_a^x (\psi(x)-\psi(t))^{2-\alpha(x)}\frac{\psi''(t)}{(\psi'(t))^2}\mathrm{d}t\right],
\end{aligned} \tag{4.1}$$

which is zero at $x = a$. On the other hand, since $\ln(\psi(x) - \psi(t)) \le \ln(\psi(b) - \psi(a))$ for all $a \le t \le x \le b$, we have

$$\begin{aligned}
&\ln(\psi(b)-\psi(a))\int_a^x (\psi(x)-\psi(t))^{1-\alpha(x)}\mathrm{d}t \\
&= \frac{\ln(\psi(b)-\psi(a))}{(2-\alpha(x))}\left[\frac{(\psi(x)-\psi(a))^{2-\alpha(x)}}{\psi'(a)}\right. \\
&\quad \left.-\int_a^x (\psi(x)-\psi(t))^{2-\alpha(x)}\frac{\psi''(t)}{(\psi'(t))^2}\mathrm{d}t\right].
\end{aligned} \tag{4.2}$$

Taking $x = a$ in Eq.(4.1) and Eq.(4.2), we concluded that

$$\left\|{}^{\mathrm{C}}\mathbb{D}^{\alpha(x);\psi}_{\oplus,\odot;a+}f(x)\right\| \le \left\|{}^{\mathrm{C}}\mathscr{D}^{\alpha(x);\psi}_{\oplus,\odot;a+}f(x)\right\|.$$

In this sense, we concluded the proof of item a. Similarly, it directly follows item b. □

Theorem 4.12 *Let g be a generator of a pseudo-addition $\oplus$ on the interval $[-\infty, \infty]$. Then for $0 < \alpha(x) < 1$, a relationship between the ψ-Riemann–Liouville and the ψ-Caputo pseudo-fractional derivatives of variable order is given by*

$$\begin{aligned}
&\mathbb{D}^{\alpha(x);\psi}_{\oplus,\odot;a+}f(x) \\
&= \mathbb{D}^{\alpha(x);\psi}_{\oplus,\odot;a+}f(x)\oplus\left\{\left[f(a)\odot g^{-1}\right.\right. \\
&\quad \left.\left.\times\left((\psi(x)-\psi(a))^{-\alpha(x)}\right)\right]\oslash\left[g^{-1}\left[\Gamma(1-\alpha(x))\right]\right]\right\} \\
&\quad \oplus\left\{\left[f(a)\odot g^{-1}(\alpha'(x))\odot g^{-1}\right.\right. \\
&\quad \times\left((\psi(x)-\psi(a))^{1-\alpha(x)}\right)\left]\oslash\left[g^{-1}\left(\psi'(x)\right)g^{-1}\right.\right. \\
&\quad \left.\left.\times\left[\Gamma(2-\alpha(x))\right]\right]\right\}\odot g^{-1}\left[\frac{1}{1-\alpha(x)}\right. \\
&\quad \left.-\ln(\psi(x)-\psi(a))\right]
\end{aligned}$$

and

$$\begin{aligned}&\mathcal{D}^{\alpha(x);\psi}_{\oplus,\odot;a+}f(x)\\&={}^{\mathsf{C}}\mathcal{D}^{\alpha(x);\psi}_{\oplus,\odot;a+}f(x)\oplus\Big\{f(a)\odot\Big[g^{-1}\\&\times((\psi(x)-\psi(a))^{-\alpha(x)})\oslash g^{-1}(\Gamma(1-\alpha(x)))\Big]\Big\}\\&\oplus\Big\{\Big[f(a)\odot g^{-1}(\alpha'(x))\odot\\&\times g^{-1}\Big((\psi(x)-\psi(a))^{1-\alpha(x)}\Big)\Big]\oslash\\&\times\Big[g^{-1}\Big(\psi'(x)\Big)g^{-1}\big[\Gamma(2-\alpha(x))\big]\Big]\Big\}\odot g^{-1}\\&\times\Big[\Psi(2-\alpha(x))-\ln(\psi(x)-\psi(a))\Big].\end{aligned}$$

Proof

$$\begin{aligned}&\mathbb{D}^{\alpha(x);\psi}_{\oplus,\odot;a+}f(x)\\&=g^{-1}\left(\mathbb{D}^{\alpha(x);\psi}_{a+}g(f(x))\right)\\&=g^{-1}\Big\{\mathbb{D}^{\alpha(x);\psi}_{a+}g(f(x))\\&+\frac{g(f(a))}{\Gamma(1-\alpha(x))}(\psi(x)-\psi(a))^{-\alpha(x)}\\&+\frac{g(f(a))\alpha'(x)}{\psi'(x)\Gamma(2-\alpha(x))}(\psi(x)\\&-\psi(a))^{1-\alpha(x)}\Big[\frac{1}{1-\alpha(x)}-\ln(\psi(x)-\psi(a))\Big]\Big\}\\&=g^{-1}\Big\{g\Big(g^{-1}\Big(\mathbb{D}^{\alpha(x);\psi}_{a+}g(f(x))\Big)\Big)\\&+g\Big(g^{-1}\Big(\frac{g(f(a))}{\Gamma(1-\alpha(x))}(\psi(x)-\psi(a))^{-\alpha(x)}\Big)\Big)\\&+g\Big(g^{-1}\Big(\frac{g(f(a))\alpha'(x)}{\psi'(x)\Gamma(2-\alpha(x))}(\psi(x)\\&-\psi(a))^{1-\alpha(x)}\Big[\frac{1}{1-\alpha(x)}-\ln(\psi(x)-\psi(a))\Big]\Big)\Big)\Big\}\\&=g^{-1}\left(\mathbb{D}^{\alpha(x);\psi}_{a+}g(f(x))\right)\oplus g^{-1}\\&\times\Big(\frac{g(f(a))}{\Gamma(1-\alpha(x))}(\psi(x)-\psi(a))^{-\alpha(x)}\Big)\\&\oplus g^{-1}\Big(\frac{g(f(a))\alpha'(x)}{\psi'(x)\Gamma(2-\alpha(x))}(\psi(x)-\psi(a))^{1-\alpha(x)}\\&\times\Big[\frac{1}{1-\alpha(x)}-\ln(\psi(x)-\psi(a))\Big]\Big)\\&=\mathbb{D}^{\alpha(x);\psi}_{\oplus,\odot;a+}f(x)\oplus g^{-1}\\&\times\Big[\frac{g\big(g^{-1}\big(g(f(a))(\psi(x)-\psi(a))^{-\alpha(x)}\big)\big)}{g\big(g^{-1}\big(\Gamma(1-\alpha(x))\big)\big)}\Big]\oplus g^{-1}\\&\times\Big[\frac{g\big(g^{-1}\big(g(f(a))g\big(g^{-1}\big(\alpha'(x)\big)\big)(\psi(x)-\psi(a))^{1-\alpha(x)}\big)\big)}{g\big(g^{-1}\big(\psi'(x)\big)\big)g\big(g^{-1}\big(\Gamma(2-\alpha(x))\big)\big)}\\&\times g\Big(g^{-1}\Big[\frac{1}{1-\alpha(x)}-\ln(\psi(x)-\psi(a))\Big]\Big)\Big]\\&=\mathbb{D}^{\alpha(x);\psi}_{\oplus,\odot;a+}f(x)\oplus\Big\{\Big[f(a)\odot g^{-1}\\&\times\Big((\psi(x)-\psi(a))^{-\alpha(x)}\Big)\Big]\oslash\Big[g^{-1}\Big[\Gamma(1-\alpha(x))\Big]\Big]\Big\}\\&\oplus\Big\{\Big[f(a)\odot g^{-1}(\alpha'(x))\odot g^{-1}\\&\times\Big((\psi(x)-\psi(a))^{1-\alpha(x)}\Big)\Big]\oslash\Big[g^{-1}\Big(\psi'(x)\Big)g^{-1}\\&\times\big[\Gamma(2-\alpha(x))\big]\Big]\Big\}\odot g^{-1}\Big[\frac{1}{1-\alpha(x)}\\&-\ln(\psi(x)-\psi(a))\Big].\end{aligned}$$

□

To conclude this section, we discuss the g-integration by parts for the ψ-Caputo pseudo-fractional derivative and ψ-Riemann–Liouville pseudo-fractional.

Theorem 4.13 *Let f be a measurable function on $[a,b]$ and g a generator of pseudo-addition $\oplus$ on interval $[-\infty,\infty]$. Also let $0<\alpha(x)<1-\frac{1}{n}$ for all $x\in[a,b]$ with a number $n\in\mathbb{N}$ greater than or equal to two, and $\psi'(t)\neq 0$, then we have*

$$\begin{aligned}&\int_{[a,b]}^{\oplus}\Big[h(x)\odot\Big({}^{\mathsf{C}}\mathcal{D}^{\alpha(x);\psi}_{\oplus,\odot;a+}f(x)\Big)\Big]\odot\mathrm{d}x\\&=f(x)\odot\mathbb{I}^{1-\alpha(x);\psi}_{\oplus,\odot;b-}\Big[h(x)\oslash g^{-1}(\psi'(x))\Big]\Big|_a^b\\&\oplus\int_{[a,b]}^{\oplus}\Big[g^{-1}(\psi'(x))\odot\Big(\mathcal{D}^{\alpha(x);\psi}_{\oplus,\odot;b-}\\&\times\Big[h(x)\oslash g^{-1}(\psi'(x))\Big]\Big)\odot f(x)\Big]\odot\mathrm{d}x.\end{aligned}$$

and

$$\begin{aligned}&\int_{[a,b]}^{\oplus}\Big[h(x)\odot{}^{\mathsf{C}}\mathcal{D}^{\alpha(x);\psi}_{\oplus,\odot;b-}f(x)\Big]\odot\mathrm{d}x\\&=f(x)\odot\mathbb{I}^{1-\alpha(x);\psi}_{\oplus,\odot;a+}\Big[h(x)\oslash g^{-1}(\psi'(x))\Big]\Big|_a^b\\&\oplus\int_{[a,b]}^{\oplus}\Big[g^{-1}(\psi'(x))\odot\\&\times\Big(\mathcal{D}^{\alpha(x);\psi}_{\oplus,\odot;a+}\Big[h(x)\oslash g^{-1}(\psi'(x))\Big]\Big)\odot f(x)\Big]\odot\mathrm{d}x.\end{aligned}$$

Proof By the definition of $\mathbb{D}^{\alpha(x);\psi}_{\oplus,\odot;a+}$ and using Theorem 2.10, we have

$$\int_{[a,b]}^{\oplus}\left[h(x)\odot\left({}^{\mathsf{C}}\mathcal{D}^{\alpha(x);\psi}_{\oplus,\odot;a+}f(x)\right)\right]\odot\mathrm{d}x$$
$$=g^{-1}\left\{\int_a^b g\left[h(x)\left({}^{\mathsf{C}}\mathcal{D}^{\alpha(x);\psi}_{\oplus,\odot;a+}f(x)\right)\right]\mathrm{d}x\right\}$$
$$=g^{-1}\left\{\int_a^b g\left[g^{-1}(g(h(x)))g^{-1}\times\left(g\left({}^{\mathsf{C}}\mathcal{D}^{\alpha(x);\psi}_{\oplus,\odot;a+}f(x)\right)\right)\right]\mathrm{d}x\right\}$$
$$=g^{-1}\left\{\int_a^b\left[g(h(x))g\left({}^{\mathsf{C}}\mathcal{D}^{\alpha(x);\psi}_{\oplus,\odot;a+}f(x)\right)\right]\mathrm{d}x\right\}$$
$$=g^{-1}\left\{\int_a^b\left[g(h(x))g\left(g^{-1}\times\left({}^{\mathsf{C}}\mathcal{D}^{\alpha(x);\psi}_{a+}g(f(x))\right)\right)\right]\mathrm{d}x\right\}$$
$$=g^{-1}\left\{\int_a^b\left[g(h(x))\left({}^{\mathsf{C}}\mathcal{D}^{\alpha(x);\psi}_{a+}g(f(x))\right)\right]\mathrm{d}x\right\}$$
$$=g^{-1}\left\{g(f(x))\,\mathbb{I}^{1-\alpha(x);\psi}_{b-}\left(\frac{g(h(x))}{\psi'(x)}\right)\times\Big|_a^b+\int_a^b\psi'(x)\mathcal{D}^{\alpha(x);\psi}_{b-}\left(\frac{g(h(x))}{\psi'(x)}\right)g(f(x))\mathrm{d}x\right\}$$
$$=g^{-1}\left\{g(f(x))\,g\left(g^{-1}\left(\mathbb{I}^{1-\alpha(x);\psi}_{b-}\left(\frac{g(h(x))}{\psi'(x)}\right)\Big|_a^b\right)\right)+g\left(g^{-1}\left(\int_a^b\psi'(x)\mathcal{D}^{\alpha(x);\psi}_{b-}\left(\frac{g(h(x))}{\psi'(x)}\right)\times g(f(x))\mathrm{d}x\right)\right)\right\}$$
$$=f(x)\odot g^{-1}\left\{\mathbb{I}^{1-\alpha(x);\psi}_{b-}g\left[g^{-1}\times\left(\frac{g(h(x))}{g(g^{-1}(\psi'(x)))}\right)\right]\Big|_a^b\right\}$$
$$\oplus g^{-1}\left[\int_a^b g\left(g^{-1}\left(\psi'(x)\right)\right)g\times\left(g^{-1}\left(\mathcal{D}^{\alpha(x);\psi}_{b-}\left(\frac{g(h(x))}{\psi'(x)}\right)\right)\right)g(f(x))\mathrm{d}x\right]$$
$$=f(x)\odot\mathbb{I}^{1-\alpha(x);\psi}_{\oplus,\odot;b-}\left[h(x)\oslash g^{-1}(\psi'(x))\right]\Big|_a^b$$
$$\oplus\int_{[a,b]}^{\oplus}\left[g^{-1}(\psi'(x))\odot g^{-1}\left(\mathcal{D}^{\alpha(x);\psi}_{b-}g\times\left(g^{-1}\left(\left(\frac{g(h(x))}{\psi'(x)}\right)\right)\right)\right)\odot f(x)\right]\odot\mathrm{d}x$$
$$=f(x)\odot\mathbb{I}^{1-\alpha(x);\psi}_{\oplus,\odot;b-}\left[h(x)\oslash g^{-1}(\psi'(x))\right]\Big|_a^b$$
$$\oplus\int_{[a,b]}^{\oplus}\left[g^{-1}(\psi'(x))\odot\left(\mathcal{D}^{\alpha(x);\psi}_{\oplus,\odot;b-}\left[h(x)\oslash g^{-1}\times(\psi'(x))\right]\right)\odot f(x)\right]\odot\mathrm{d}x.$$

□

Theorem 4.14 *Let f be a measurable function on $[a,b]$ and g a generator of pseudo-addition $\oplus$ on interval $[-\infty,\infty]$. Also let $0<\alpha(x)<1-\frac{1}{n}$ for all $x\in[a,b]$ with a number $n\in\mathbb{N}$ greater than or equal to two, and $\psi'(t)\neq 0$, then we have*

$$\int_{[a,b]}^{\oplus}\left[h(x)\odot\left(\mathcal{D}^{\alpha(x);\psi}_{\oplus,\odot;a+}f(x)\right)\right]\odot\mathrm{d}x$$
$$=\left[h(x)\oslash g^{-1}(\psi'(x))\right]\odot\mathbb{I}^{1-\alpha(x);\psi}_{\oplus,\odot;b-}f(x)\Big|_a^b$$
$$\oplus\int_{[a,b]}^{\oplus}\left[\left(\mathbb{I}^{1-\alpha(x);\psi}_{\oplus,\odot;b-}f(x)\odot g^{-1}\times(\psi'(x))\odot\left(f^{\oplus}_{\psi-}\left[h(x)\oslash g^{-1}(\psi'(x))\right]\right)\right]\odot\mathrm{d}x\right.$$

and

$$\int_{[a,b]}^{\oplus}\left[h(x)\odot\left(\mathcal{D}^{\alpha(x);\psi}_{\oplus,\odot;b-}f(x)\right)\right]\odot\mathrm{d}x$$
$$=\left[h(x)\oslash g^{-1}(\psi'(x))\right]\odot\mathbb{I}^{1-\alpha(x);\psi}_{\oplus,\odot;a+}f(x)\Big|_a^b$$
$$\oplus\int_{[a,b]}^{\oplus}\left[\left(\mathbb{I}^{1-\alpha(x);\psi}_{\oplus,\odot;a+}f(x)\odot g^{-1}\times(\psi'(x))\odot\left(f^{\oplus}_{\psi+}\left[h(x)\oslash g^{-1}(\psi'(x))\right]\right)\right]\odot\mathrm{d}x.\right.$$

Proof The theorem is proved in a similar way as Theorem 2.10. □

5 Application

In this section, we will investigate the ψ-pseudo fractional integral inequality of Chebyshev.

Theorem 5.1 *Agahi et al. (2015) Let $b>a\geq 0$. Let $u,v,h_1,h_2:[0,\infty]\to[ab]$ be integrable functions and let a generator $g:[a,b]\to[0,\infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function. If u and v are comonotone, then the inequality*

$$\begin{aligned}&[(h_1 * u)(t) \odot (h_2 * v)(t)]\\&\quad\oplus [(h_1 * v)(t) \odot (h_2 * u)(t)]\\&\quad\leq [(h_2 * 1)(t) \odot (h_1 * (u \odot v))(t)]\\&\quad\oplus [(h_1 * 1)(t) \odot (h_2 * (u \odot v))(t)] \qquad (5.1)\end{aligned}$$

hold where the symbol $h_i * u$, $i = 1, 2, ..$ denote the $g-$convolution of h_i and u that are defined by [18]

$$(h_i * u)(t) = \int_{[0,t]}^{\oplus} [h_i(t-x) \odot u(x)] \odot d\mu,\ i = 1, 2$$

for all $t \in [0, \infty)$.

Taking $h_1 = h_2 = h$ in Theorem 5.1, we have the following result.

Corollary 5.2 *Agahi et al. (2015) Let $b > a \geq 0$. Let $u, v, h : [0, \infty] \to [a, b]$ be integrable functions and let a generator $g : [a, b] \to [0, \infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function. If u and v are comonotone, then the inequality*

$$\begin{aligned}&\left(\int_{[0,t]}^{\oplus} [h(t-x) \odot u(x)] \odot d\mu\right)\\&\quad\odot \left(\int_{[0,t]}^{\oplus} [h(t-x) \odot v(x)] \odot d\mu\right)\\&\quad\leq \left(\int_{[0,t]}^{\oplus} [h(t-x) \odot (u(x) \odot v(x))] \odot d\mu\right)\\&\qquad\times \left(\int_{[0,t]}^{\oplus} h(t-x) \odot d\mu\right)\end{aligned}$$

for all $t \in [0, \infty)$.

Now, we consider the second case, when $\oplus = \sup$ and $\odot = g^{-1}(g(x)\, g(y))$.

Theorem 5.3 *Agahi et al. (2015) Let $u, v, h_1, h_2 : [0, \infty] \to [a, b]$ be continuous functions and $\odot$ is represented by an increasing multiplicative generator g and μ the same as in* Theorem 3.9. *If u and v are comonotone, then the inequality*

$$\begin{aligned}&\sup([(h_1 * u)(t) \odot (h_2 * v)(t)],\\&[(h_1 * v)(t) \odot (h_2 * u)(t)])\\&\quad\leq \sup([(h_2 * 1)(t) \odot (h_1 * (u \odot v))(t)],\\&\quad[(h_1 * 1)(t) \odot (h_2 * (u \odot v))(t)]) \qquad (5.2)\end{aligned}$$

holds where

$$(h_i * u)(t) = \int_{[0,t]}^{\sup} [h_i(t-x) \odot u(x)] \odot d\mu,\ i = 1, 2$$

for all $t \in [0, \infty)$.

Taking $h_1 = h_2 = h$ in Theorem 5.3, we have the following result.

Corollary 5.4 *Agahi et al. (2015) Let $u, v, h : [0, \infty] \to [a, b]$ be continuous functions and $\odot$ is represented by an increasing multiplicative generator g and μ be the same as in* Theorem 3.9. *If u and v are comonotone, then the inequality*

$$\begin{aligned}&\left(\int_{[0,t]}^{\sup} [h(t-x) \odot u(x)] \odot d\mu\right)\\&\quad\odot \left(\int_{[0,t]}^{\sup} [h(t-x) \odot v(x)] \odot d\mu\right)\\&\quad\leq \left(\int_{[0,t]}^{\sup} [h(t-x) \odot (u(x) \odot v(x))] \odot d\mu\right)\\&\qquad\times \left(\int_{[0,t]}^{\sup} h(t-x) \odot d\mu\right)\end{aligned}$$

for all $t \in [0, \infty)$.

Corollary 5.5 *Let $u, v : [0, \infty) \to [a, b]$ be two integrable functions and let a generator $g : [a, b] \to [0, \infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function. If u and v are comonotone, then the inequality*

$$\begin{aligned}&I_{\oplus,\odot}^{\alpha(x);\psi} u(t) \odot I_{\oplus,\odot}^{\alpha(x);\psi} v(t)\\&\quad\leq \widetilde{K}(t) \odot I_{\oplus,\odot}^{\alpha(x);\psi} (u \odot v)(t) \qquad (5.3)\end{aligned}$$

holds where $\widetilde{K}(t) = g^{-1}\left(\dfrac{(\psi(t) - \psi(0))^{\alpha(x)}}{\Gamma(\alpha(x) + 1)}\right)$. Here $\Gamma(\alpha(x))$ is the Gamma function.

Proof Indeed, taking $h(t-x) = g^{-1}\left(\Gamma(\alpha(t))^{-1} \psi'(x) (\psi(t) - \psi(x))^{\alpha(t)-1}\right)$, $\alpha(x) > 0$ in Corollary 5.2, yields

$$\begin{aligned}&\left(\int_{[0,t]}^{\oplus} \left[g^{-1}\left(\Gamma(\alpha(t))^{-1} \psi'(x)\right.\right.\right.\\&\quad\left.\left.\left.\times (\psi(t) - \psi(x))^{\alpha(t)-1}\right) \odot u(x)\right] \odot dx\right)\\&\odot \left(\int_{[0,t]}^{\oplus} \left[g^{-1}\left(\Gamma(\alpha(t))^{-1} \psi'(x)\right.\right.\right.\\&\quad\left.\left.\left.\times (\psi(t) - \psi(x))^{\alpha(t)-1}\right) \odot v(x)\right] \odot dx\right)\\&\leq \left(\int_{[0,t]}^{\oplus} \left[g^{-1}\left(\Gamma(\alpha(t))^{-1} \psi'(x)\right.\right.\right.\\&\quad\left.\left.\left.\times (\psi(t) - \psi(x))^{\alpha(t)-1}\right) \odot (u \odot v)(x)\right] \odot dx\right)\end{aligned}$$

$$\odot\left(\int_{[0,t]}^{\oplus} g^{-1}\left(\Gamma\left(\alpha\left(t\right)\right)^{-1}\psi'\left(x\right)\right.\right.$$
$$\left.\left.\times\left(\psi\left(t\right)-\psi\left(x\right)\right)^{\alpha(t)-1}\right)\odot dx\right). \qquad (5.4)$$

Note that

$$\int_{[0,t]}^{\oplus} g^{-1}\left(\Gamma\left(\alpha\left(t\right)\right)^{-1}\psi'\left(x\right)\right.$$
$$\left.\times\left(\psi\left(t\right)-\psi\left(x\right)\right)^{\alpha(t)-1}\right)\odot dx$$
$$=g^{-1}\left(\frac{1}{\Gamma\left(\alpha\left(t\right)\right)}\int_0^t \psi'\left(x\right)\left(\psi\left(t\right)-\psi\left(x\right)\right)^{\alpha(t)-1}dx\right)$$
$$=g^{-1}\left(\frac{\left(\psi\left(t\right)-\psi\left(0\right)\right)^{\alpha(t)}}{\Gamma\left(\alpha\left(t\right)\right)}\right)=:\widetilde{K}\left(t\right). \qquad (5.5)$$

Substituting for inequality (5.4), we conclude that

$$\left(\int_{[0,t]}^{\oplus}\left[g^{-1}\left(\Gamma\left(\alpha\left(t\right)\right)^{-1}\psi'\left(x\right)\right.\right.\right.$$
$$\left.\left.\left.\times\left(\psi\left(t\right)-\psi\left(x\right)\right)^{\alpha(t)-1}\right)\odot u\left(x\right)\right]\odot dx\right)$$
$$\odot\left(\int_{[0,t]}^{\oplus}\left[g^{-1}\left(\Gamma\left(\alpha\left(t\right)\right)^{-1}\psi'\left(x\right)\right.\right.\right.$$
$$\left.\left.\left.\times\left(\psi\left(t\right)-\psi\left(x\right)\right)^{\alpha(t)-1}\right)\odot v\left(x\right)\right]\odot dx\right)$$
$$\leq\widetilde{K}\left(t\right)\odot\left(\int_{[0,t]}^{\oplus}\left[g^{-1}\left(\Gamma\left(\alpha\left(t\right)\right)^{-1}\psi'\left(x\right)\right.\right.\right.$$
$$\left.\left.\left.\times\left(\psi\left(t\right)-\psi\left(x\right)\right)^{\alpha(t)-1}\right)\odot\left(u\odot v\right)\left(x\right)\right]\odot dx\right).$$

□

Two interesting particular cases. First, take $\alpha(x)=\alpha$ in particular. Second, choosing $\psi(x)=x$. So, we have the following results

Corollary 5.6 *Let $u, v:[0,\infty)\to[a,b]$ be two integrable functions and let a generator $g:[a,b]\to[0,\infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function. If u and v are comonotone, then the inequality*

$$I_{\oplus,\odot}^{\alpha;\psi}u\left(t\right)\odot I_{\oplus,\odot}^{\alpha;\psi}v\left(t\right)\leq\widetilde{K}\left(t\right)\odot I_{\oplus,\odot}^{\alpha;\psi}\left(u\odot v\right)\left(t\right) \qquad (5.6)$$

holds where $\widetilde{K}\left(t\right)=g^{-1}\left(\frac{\left(\psi\left(t\right)-\psi\left(0\right)\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}\right)$. Here $\Gamma\left(\alpha\right)$ is the Gamma function.

Corollary 5.7 *Let $u, v:[0,\infty)\to[a,b]$ be two integrable functions and let a generator $g:[a,b]\to[0,\infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function. If u and v are comonotone, then the inequality*

$$I_{\oplus,\odot}^{\alpha(x)}u\left(t\right)\odot I_{\oplus,\odot}^{\alpha(x)}v\left(t\right)\leq\widetilde{K}\left(t\right)\odot I_{\oplus,\odot}^{\alpha(x)}\left(u\odot v\right)\left(t\right) \qquad (5.7)$$

holds where $\widetilde{K}\left(t\right)=g^{-1}\left(\frac{t^{\alpha(x)}}{\Gamma\left(\alpha\left(x\right)+1\right)}\right)$. Here $\Gamma\left(\alpha\left(x\right)\right)$ is the Gamma function.

Corollary 5.8 *Let $u, v:[0,\infty)\to[a,b]$ be two continuous functions and let a generator and $\odot$ is represented by an increasing multiplicative generator g and m be the same as in* Theorem 5.3. *If u and v are comonotone, then for al $\alpha\left(x\right)>0$. the inequality*

$$I_{\sup,\odot}^{\alpha(x);\psi}u\left(t\right)\odot I_{\sup,\odot}^{\alpha(x);\psi}v\left(t\right)\leq\widetilde{K}\left(t\right)\odot I_{\sup,\odot}^{\alpha(x);\psi}\left(u\odot v\right)\left(t\right) \qquad (5.8)$$

holds where $\widetilde{K}\left(t\right)=g^{-1}\left(\frac{\left(\psi\left(t\right)-\psi\left(0\right)\right)^{\alpha(x)}}{\Gamma\left(\alpha\left(x\right)+1\right)}\right)$ and for all $t\in[0,\infty)$. Here $\Gamma\left(\alpha\left(x\right)\right)$ is the Gamma function.

Proof Follow the same steps as previous Corollary 5.5 proof, through the particular choice of $h\left(t-x\right)=g^{-1}\left(\Gamma\left(\alpha\left(t\right)\right)^{-1}\psi'\left(x\right)\left(\psi\left(t\right)-\psi\left(x\right)\right)^{\alpha(t)-1}\right)$, $\alpha\left(x\right)>0$ in Corollary 5.4. □

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