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OPTIMIZATION

A modified VIKOR method for group decision-making based on aggregation operators for hesitant intuitionistic fuzzy linguistic term sets

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Abstract
The methodology of managing hesitancy situations is improving step by step with new basics and tools which have their specific characteristics. As a rule, aggregation operators can undoubtedly deal with the data in a definite way. Employing hesitant intuitionistic fuzzy linguistic term sets (HIFLTSs), the adaptability in creating assessment data under uncertainty can be accomplished to a bigger degree than either intuitionistic linguistic sets or hesitant fuzzy linguistic term sets do. In this view, this paper provides a multi-criteria group decision-making (MCGDM) method based on new aggregation operators for HIFLTSs and the proposed modified VIKOR method. Besides, some operational laws for HIFLTSs are studied with their important properties, which is followed by the definitions of some aggregation operators, including the hesitant intuitionistic fuzzy linguistic weighted averaging (HIFLWA) operator and the hesitant intuitionistic fuzzy linguistic weighted geometric (HIFLWG) operator. The hesitant intuitionistic fuzzy linguistic individual regret (HIFLIR) measure and hesitant intuitionistic fuzzy linguistic compromise (HIFLC) measure are established with the help of hesitant intuitionistic fuzzy group utility measure. After that, by using the proposed aggregation operators, we develop a modified VIKOR method in the context of HIFLTSs. The outcome of this research is ranking and selecting of best alternative with the help of modified VIKOR method based on aggregation operators for HIFLTSs. A numerical problem is provided to verify the proposed approach, and its accuracy and effectiveness have been demonstrated through the comparative analysis of the modified VIKOR method with the TOPSIS method for the selection of the best alternative. This research study showed that the proposed approach can describe the fuzziness and uncertainty of experts more relevantly and the ranking results calculated with the modified VIKOR method are effective and reliable.

Keywords Hesitant fuzzy sets · Hesitant fuzzy linguistic term sets · Hesitant intuitionistic fuzzy linguistic sets · MCGDM · VIKOR method

1 Introduction

Decision-making particularly provides a gateway in handling complex issues related to daily life situations with the help of multicriteria decision-making (MCDM) techniques. The most satisfactory alternative is chosen from a set of alternatives inside seeing conflicting criteria with a high level of satisfaction. Several alternatives are interpreted with the help of decision-makers (DMs) opinions. There is no limit of criterion in MCDM, but the main thing is that how the various alternatives are integrated or evaluated. Although many researchers have achieved much in this regard while dealing with MCDM. MCGDM is also a valuable technique Geng et al. (2017); Wang et al. (2019); Yeh et al. (2014) for dealing with human activities and handling problems faced by them in their lives. In some circumstances, we have to consult a wide range of information to overcome the real-life issues as these problems are out of the range of individual experts as in many complex problems Xu (2000). For

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this, the selection of experts from different fields is necessary, depending upon their work experiences and educational backgrounds. In this way, the required alternative is achieved from a set of available alternatives Zhou and Chen (2012); Zhou et al. (2012). During the selection of alternatives, the DMs judgments and preferences are important while dealing with such ambiguous situations and conflicting criteria Sengupta and Pal (2009). Within sight of conflicting criteria, the DMs may have different thinking and observations for different alternatives, and their opinions may be different from one another. Therefore, it is not easy to achieve consistent results obtained from several possible values Liao and Xu (2013) . The human hesitancy in decision-making process can be reduced to a larger extent by utilizing Torra (2010) idea of the hesitant fuzzy set (HFS). In a hesitant fuzzy set, several different membership values are assigned to an alternative in the presence of conflicting criteria.

For a DM, hesitancy is quite common during the thinking process and a lot of terms while providing his/her opinions for evaluation of an alternative. It is difficult for the DM to give a point of view in numeric form in the presence of hesitancy. By motivating the concept of Torra's HFSs Torra (2010), Rodríguez et al. (2011a) developed hesitant fuzzy linguistic term sets (HFLTSs) which deals with such situation properly. In HFLTSs, the subjective decisions of people can be spoken to in an alternate having adaptable structure and have demonstrated their validity in real decision-making processes. In the wake of characterizing the idea of HFLTS and activities over HFLTS, Rodríguez et al. (2011b) utilized it in linguistic expressions. The DMs feel more comfortable by expressing their views in linguistic expressions in MCDM models as it is close to the human intellectual area. Rodríguez et al. (2013) utilized context-free grammars for HFLTS and developed a decision-making model with the help of linguistic information, which is acceptable for human intellectual area. Liao et al. (2014) presented the essential properties of HFLTS and built up a group of closeness and distance measure for HFLTS and used it into MCDM issues all the more successfully. Later on, Wei et al. (2014) proposed comparison techniques and aggregation theory, which is extensively used for HFLTSs. The TOPSIS method for HFLTSs was utilized by Beg and Rashid (2013) in which the expert's opinion was taken as HFLTSs. The HFLTSs are changed into fuzzy envelopes by utilizing Liu and Rodríguez (2014) TOPSIS-based technique. Wu et al. (2019) developed TOPSIS and VIKOR-based models for HFLTSs.

For dealing with MCDM issues with linguistic data, the important is that how to aggregate linguistic data with a high level of satisfaction for selection of best alternative. Therefore, numerous operators have been developed for dealing with linguistic data, like numerical aggregation operators Yager (1988), linguistic aggregation operators Delgado et al. (1992) and many more. The linguistic data, as well as their numerical weights, were proposed by Yager (1998) by utilizing linguistic weighted median operator. There are several aggregation operators for dealing HFLTSs properly in decision-making problems Gou and Xu (2016); Xia and Xu (2011). For an aggregation of HFLTSs, the min-upper and max-lower operators are utilized by Rodríguez et al. (2011b). Lately, plenty of researchers have contemplated HFLTSs from various points and built up aggregation theory for HFLTSs Wei et al. (2014). In the continuation, we develop some useful operational laws and aggregation weighted operators for HIFLTSs that can be used to aggregate the DMs preferences to solve real-life MCGDM problems.

During the selection of linguistic terms, the DMs may hesitate as in the case of HFLTSs because it contains only membership values. To conquer this circumstance, the idea of HIFLTS was presented by Beg and Beg and Rashid (2014) which deals with fuzziness in a proper way. They also developed TOPSIS method for HIFLTS with the assessment of limited DMs for the determination of the best alternative. In recent time, the study of MCDM problems overgrowing for HIFLTS rather than HFLTS, as it deals appropriately with fuzziness. Therefore, the use of HIFLTS can be more productive to deal with the problems of MCDM all the more proficiently. The MCDM problems are handled by utilizing new definitions with corresponding operations developed by Zhou et al. (2016) for HIFLTS more efficiently. Many MCDM methods have been developed for dealing with HIFLTSs for selection of best alternative like outranking method proposed by Faizi et al. (2017) and ELECTRE-based outranking method proposed by Rashid et al. (2018) for MCGDM. Furthermore, Faizi et al. (2018) proposed many distance measures for HIFLTSs by utilizing risk factor parameter.

When we resolve the hassle of the way to explicit assessment information, we want to have a look at the way to cope with the received information. In this case, the aggregation of decision information is a key step to provide a reasonable and satisfactory tools to solve the problems of decision-making. The aggregation operators have advantages to solve the MCGDM problems, in order that they are becoming an increasing number of attentions and feature additionally have become a warm studies topic. For example, Zhu et al. (2016) extended the power operators to the linguistic hesitant fuzzy environment and established a series of linguistic hesitant fuzzy power aggregation operators. Zhou et al. (2015) introduced the operational laws and comparative methods of hesitant intuitionistic fuzzy numbers and used them to investigate the aggregation operators and the approximate consistency tests. Mahboob et al. (2021) proposed an optimization preference-based approach with hesitant intuitionistic linguistic distribution in group decision-making. Similarly, the complex situation can be handled with the right decision with the help of efficient MCDM methods. In fuzzy set theory, many MCDM methods have been introduced and then extended according to the extensions of this set. With the headway of the new systems and methodologies in MCDM to achieve optimal solutions, methods like TOPSIS, AHP, ANP, VIKOR, etc., were developed and modified in the context of various extensions of fuzzy sets. The VIKOR technique is well known for legitimately managing the ambiguities. The VIKOR method provides a compromise solution while dealing with MCDM problems Opricovic (1998); Opricovic and Tzeng (2002). There are many applications of VIKOR method in real-life situations like mountain destination selection Opricovic and Tzeng (2004), alternate bus fuel modes selection (Tzeng et al., 2005), financial evaluation performance Yalcin et al. (2012), waste management Gündoğdu et al. (2019), material selection for automotive piston components Dev et al. (2020) and many more. No doubt that VIKOR method is a powerful tool in handling MCDM problem carrying conflicting criteria. By taking into account the above motivations, firstly, we develop some operational laws for HIFLTSs and used them to develop some aggregation operators for HIFLTSs to aggregate hesitant intuitionistic fuzzy linguistic information. Secondly, the classical VIKOR method is modified in the context of HIFLTSs for the selection of best alternative in MCGDM problems. The novelty of this paper is as follows.

1. To introduce some operational laws for HIFLTSs (Beg and Rashid (2014)) along with some important properties with necessary proofs.
2. To aggregate hesitant intuitionistic fuzzy linguistic information, the HIFLWA operator and HIFLWG operator are developed with some important proofs.
3. To develop a modified VIKOR method in the context of HIFLTSs. For this, the hesitant intuitionistic fuzzy group utility measure is developed by using HIFLTSs and then, the HIFLIR measure and HIFLC measure are established so that the alternatives can be ranked with the help of all these measures.
4. The validity of the presented work is certified by applying it to handle an MCGDM situation by considering a numerical problem from a daily life situation. The same numerical problem is demonstrated with the TOPSIS method for the superiority of the modified VIKOR method for the selection of the best alternative.

This paper is divided into the following sections for simplicity: Sect. 2 comprises of fundamental concepts like linguistic term set, HFS, HFLTS and HIFLTS. Section 3 builds operational laws for HIFLTSs and provides some useful properties with necessary proofs. In Sect. 4, the VIKOR method for MCGDM problem for HIFLTSs is developed. A numerical-based problem from daily life is presented in Sect. 5 by utilizing the proposed technique. The paper finishes in Sect. 6 with end comments.

2 Preliminaries

The linguistic term set, HFLTS and HIFLTS with their fundamental concepts and operations are quickly explored in this section.

Definition 2.1 Herrera et al. (1996) Let $S = \{s_t | t = 0, ..., g\}$ be a finite linguistic term set having odd cardinality, where each s_t $(0 \leq t \leq g)$ indicates a possible value for the linguistic variable. This linguistic term set has the following characteristics:

For any $s_\alpha, s_\beta \in S$,

1. Negation operator: $neg(s_\alpha) = s_\beta$, where $\alpha + \beta = g$;
2. Ordered set: If $s_\alpha \leq s_\beta \Leftrightarrow \alpha \leq \beta$, then:
   (a) Maximum operator: $max(s_\alpha, s_\beta) = s_\alpha$, if $s_\beta \leq s_\alpha$;
   (b) Minimum operator: $min(s_\alpha, s_\beta) = s_\alpha$, if $s_\alpha \leq s_\beta$.

For any two linguistic terms $s_\alpha, s_\beta \in S$ and $\lambda, \lambda_1, \lambda_2 \in [0, 1]$, the operational laws in Xu (2004) were described as follows:

1. $s_\alpha \oplus s_\beta = s_\beta \oplus s_\alpha = s_{\alpha+\beta}$;
2. $s_\alpha \otimes s_\beta = s_\beta \otimes s_\alpha = s_{\alpha\beta}$;
3. $\lambda s_\alpha = s_{\lambda\alpha}$;
4. $s_\alpha^\lambda = s_{\alpha^\lambda}$;
5. $(\lambda_1 + \lambda_2)\ s_\alpha = \lambda_1 s_\alpha + \lambda_2 s_\alpha$;
6. $\lambda(s_\alpha + s_\beta) = \lambda s_\alpha + \lambda s_\beta$.

The discrete linguistic term set S was extended to a continuous linguistic term set $\bar{S}$, by Xu Xu (2004, 2008), such that $\bar{S} = \{s_t | 0 \leq t \leq g\}$. The linguistic variable s_t is called an original linguistic variable if $s_t \in S$, and a virtual linguistic variable if $s_t \notin S$. The original linguistic variable is directly provided by the DM during the assessment process, and the virtual linguistic variable is used only during the computation process.

2.1 Hesitant fuzzy linguistic term set

Definition 2.2 Rodríguez et al. (2013) Let S be linguistic term set. A HFLTS, H_s, is an ordered finite subset of the set of consecutive linguistic terms of S .

Liao et al. (2015) further refined the definition of HFLTS as follows:

Definition 2.3 Liao et al. (2015) Let $x_i \epsilon X$ $(i = 1, 2, ..., N)$ be fixed and S be a linguistic term set. Mathematically, a

HFLTS H_s on X can be defined as:

$$H_s = \{\langle x_i, h_s(x_i)\rangle \mid x_i \epsilon X\},$$

where $h_s(x_i)$ is a set of some values in the linguistic term set S and can be expressed as:

$$h_s(x_i) = \left\{s_{\phi_l}(x_i) \mid s_{\phi_l} \epsilon S,\ l = 1, 2, ...L\right\},$$

where $h_s(x_i)$ denote the possible degrees of linguistic variable x_i to the linguistic term set S and L is the number of linguistic terms in $h_s(x_i)$.

2.2 Hesitant intuitionistic fuzzy linguistic term set

Definition 2.4 Beg and Rashid (2014): A HIFLTS on X consists of a couple of functions $m(x)$ and $n(x)$ that when applied to X returns ordered finite subset of the set of consecutive linguistic terms of $S = \{s_\alpha \mid \alpha = 0, 1, ..., g\}$, which can be represented as the following mathematical symbols:

$$I_S = \{\langle x, m_S(x), n_S(x)\rangle \mid x \epsilon X\},$$

where $m_S(x)$ and $n_S(x)$ are subsets of the consecutive linguistic term sets, denoting the possible membership degrees and non-membership degrees of the element $x \epsilon X$ to the set I_S, with the condition that

$$s_0 \leq \max(m_S(x)) \oplus \min(n_S(x)) \leq s_g \text{ and}$$
$$s_0 \leq \min(m_S(x)) \oplus \max(n_S(x)) \leq s_g.$$

For comfort, we use the notation for hesitant intuitionistic fuzzy linguistic element (HIFLE) as $I_S = \{m_S, n_S\}$ and $\#m_S$ denotes the number of linguistic variables in m_S.

Definition 2.5 Let $I_S^1 = \left\{m_S^1, n_S^1\right\}$ and $I_S^2 = \left\{m_S^2, n_S^2\right\}$ be two HIFLEs where $m_S^1 = \{s_{a_1} \mid s_{a_1} \epsilon S, a_1 = 1, 2, ...\#m_S^1\}$, $n_S^1 = \{s_{b_1} \mid s_{b_1} \epsilon S, b_1 = 1, 2, ...\#n_S^1\}$, $m_S^2 = \{s_{a_2} \mid s_{a_2} \epsilon S, a_2 = 1, 2, ...\#m_S^2\}$ and $n_S^2 = \{s_{b_2} \mid s_{b_2} \epsilon S, b_2 = 1, 2, ...\#n_S^2\}$ are HFLTSs in the form of membership and non-membership functions. Suppose $\#m_S^1 = \#m_S^2$ and $\#n_S^1 = \#n_S^2$, then the hesitant intuitionistic Euclidean distance measure between I_S^1 and I_S^2 can be defined as follows:

$$d\left(I_S^1, I_S^2\right) = \frac{1}{L}\sqrt{\sum_{a_1=a_2=1}^{\#m_S^1=\#m_S^2}(a_1 - a_2)^2 + \sum_{b_1=b_2=1}^{\#n_S^1=\#n_S^2}(b_1 - b_2)^2}$$

where L is the product of number of elements in m_S^1, n_S^1 and S. We can easily observe that d satisfies the following conditions:

1. $0 \leq d\left(I_S^1, I_S^2\right) \leq 1$;
2. $d\left(I_S^1, I_S^2\right) = 0$ iff $I_S^1 = I_S^2$;
3. $d\left(I_S^1, I_S^2\right) = d\left(I_S^2, I_S^1\right)$.

3 Some operational laws for HIFLTEs

We present some operational laws and their related properties with necessary proofs for HIFLEs in this section.

Definition 3.1 Let S be a linguistic term set and $I_S^1 = \left\{m_S^1, n_S^1\right\}$ and $I_S^2 = \left\{m_S^2, n_S^2\right\}$ be two HIFLEs as described above. Let $I_S = \{m_S, n_S\}$ be another HIFLE where $m_S = \{s_a \mid s_a \epsilon S, a = 1, 2, ... \neq m_S\}$ and $n_S = \{s_b \mid s_b \epsilon S, b = 1, 2, ... \neq n_S\}$. Then, for any positive real number λ, we define

1. $I_S^1 \oplus I_S^2 = \left\{\bigcup_{a_1, a_2 \epsilon [0,g]}\left(s_{a_1 + a_2 - \frac{a_1 a_2}{g}}\right), \bigcup_{b_1, b_2 \epsilon [0,g]}\left(s_{\frac{b_1 b_2}{g}}\right)\right\}$;
2. $I_S^1 \otimes I_S^2 = \left\{\bigcup_{a_1, a_2 \epsilon [0,g]}\left(s_{\frac{a_1 a_2}{g}}\right), \bigcup_{b_1, b_2 \epsilon [0,g]}\left(s_{b_1 + b_2 - \frac{b_1 b_2}{g}}\right)\right\}$;
3. $I_S^1 \ominus I_S^2 = \left\{\bigcup_{a_1, a_2 \epsilon [0,g]}\left(s_{g\left(\frac{a_1 - a_2}{g - a_2}\right)}\right), \bigcup_{b_1, b_2 \epsilon [0,g]}\left(s_{g\left(\frac{b_1}{b_2}\right)}\right)\right\}$;
4. $I_S^1 \oslash I_S^2 = \left\{\bigcup_{a_1, a_2 \epsilon [0,g]}\left(s_{g\left(\frac{a_1}{a_2}\right)}\right), \bigcup_{b_1, b_2 \epsilon [0,g]}\left(s_{g\left(\frac{b_1 - b_2}{g - b_2}\right)}\right)\right\}$;
5. $\lambda I_S = \left\{\bigcup_{s_a \epsilon m_S}\left(s_{g(1-(1-\frac{a}{g})^\lambda)}\right), \bigcup_{s_b \epsilon n_S}\left(s_{g(\frac{b}{g})^\lambda}\right)\right\}$;
6. $(I_S)^\lambda = \left\{\bigcup_{s_a \epsilon m_S}\left(s_{g(\frac{a}{g})^\lambda}\right), \bigcup_{s_b \epsilon n_S}\left(s_{g(1-(1-\frac{b}{g})^\lambda)}\right)\right\}$;
7. $(I_S)^c = \{n_S, m_S\}$.

Theorem 3.2 *Let S be a linguistic term set, $I_S = \{m_S, n_S\}$, $I_S^1 = \left\{m_S^1, n_S^1\right\}$ and $I_S^2 = \left\{m_S^2, n_S^2\right\}$ be three HIFLEs and $\lambda, \lambda_1, \lambda_2 \succ 0$ be three positive real numbers. Then, I_S, I_S^1 and I_S^2 satisfy the following properties.*

1. $I_S^1 \oplus I_S^2 = I_S^2 \oplus I_S^1$;
2. $I_S^1 \otimes I_S^2 = I_S^2 \otimes I_S^1$;
3. $(I_S^1 \ominus I_S^2) \oplus I_S^2 = I_S^1$ *for* $I_S^1 \geq I_S^2$ *where* $s_{a_2} \neq s_g$, $s_{b_2} \neq s_0$;
4. $(I_S^1 \oslash I_S^2) \otimes I_S^2 = I_S^1$ *for* $I_S^1 \leq I_S^2$ *where* $s_{a_2} \neq s_0$, $s_{b_2} \neq s_g$;
5. $\lambda(I_S^1 \oplus I_S^2) = \lambda I_S^1 \oplus \lambda I_S^2$;
6. $\lambda(I_S^1 \ominus I_S^2) = \lambda I_S^1 \ominus \lambda I_S^2$ *for* $I_S^1 \geq I_S^2$ *where* $s_{a_2} \neq s_g$, $s_{b_2} \neq s_0$;
7. $(I_S^1 \otimes I_S^2)^\lambda = (I_S^1)^\lambda \otimes (I_S^2)^\lambda$;
8. $(I_S^1 \oslash I_S^2)^\lambda = (I_S^1)^\lambda \oslash (I_S^2)^\lambda$ *for* $I_S^1 \leq I_S^2$ *where* $s_{a_2} \neq s_0$, $s_{b_2} \neq s_g$;
9. $\lambda I_S^2 \oplus \lambda I_S^2 = (\lambda_1 + \lambda_2) I_S^2$;
10. $\lambda_1 I_S \ominus \lambda_2 I_S = (\lambda_1 - \lambda_2) I_S$;
11. $(I_S)^{\lambda_1} \otimes (I_S)^{\lambda_2} = (I_S)^{\lambda_1 + \lambda_2}$;
12. $(I_S)^{\lambda_1} \oslash (I_S)^{\lambda_2} = (I_S)^{\lambda_1 - \lambda_2}$.

Proof The first two proofs are evident and therefore excluded.

3. $(I_S^1 \ominus I_S^2) \oplus I_S^2 = I_S^1$

$$I_S^1 \ominus I_S^2 = \left\{ \bigcup_{a_1, a_2 \epsilon [0,g]} \left(s_{g(\frac{a_1 - a_2}{g - a_2})} \right), \bigcup_{b_1, b_2 \epsilon [0,g]} \left(s_{g(\frac{b_1}{b_2})} \right) \right\}$$

$$(I_S^1 \ominus I_S^2) \oplus I_S^2$$

$$= \left\{ \bigcup_{a_1, a_2 \epsilon [0,g]} \left(s_{g(\frac{a_1 - a_2}{g - a_2})} \right), \bigcup_{b_1, b_2 \epsilon [0,g]} \left(s_{g(\frac{b_1}{b_2})} \right) \right\} \oplus \{m_S^2, n_S^2\}$$

$$= \left\{ \bigcup_{a_1, a_2 \epsilon [0,g]} \left(s_{g(\frac{a_1 - a_2}{g - a_2})} \right) \oplus m_S^2, \bigcup_{b_1, b_2 \epsilon [0,g]} \left(s_{g(\frac{b_1}{b_2})} \right) \otimes n_S^2 \right\}$$

$$= \left\{ \underset{a_1, a_2 \epsilon [0,g]}{U} \left(s_{g(\frac{a_1 - a_2}{g - a_2}) + a_2 - \frac{g\left(\frac{a_1 - a_2}{g - a_2}\right) a_2}{g}} \right), \underset{b_1, b_2 \epsilon [0,g]}{U} \left(s_{g \frac{\left(\frac{b_1}{b_2}\right) b_2}{g}} \right) \right\}$$

$$= \left\{ \bigcup_{a_1, a_2 \epsilon [0,g]} \left(s_{\frac{g a_1 - g a_2 + g a_2 - a_2^2 - a_1 a_2 + a_2^2}{g - a_2}} \right), \bigcup_{s_{b_1} \epsilon n_S^1} (s_{b_1}) \right\}$$

$$= \left\{ \bigcup_{s_{a_1} \epsilon m_S^1} (s_{a_1}), \bigcup_{s_{b_1} \epsilon n_S^1} (s_{b_1}) \right\}$$

$$= \{m_S^1, n_S^1\}$$

$$= I_S^1.$$

4. $I_S^1 \oslash I_S^2 = \left\{ \bigcup_{a_1, a_2 \epsilon [0,g]} \left(s_{g(\frac{a_1}{a_2})} \right), \bigcup_{b_1, b_2 \epsilon [0,g]} \left(s_{g(\frac{b_1 - b_2}{g - b_2})} \right) \right\}$

$$(I_S^1 \oslash I_S^2) \otimes I_S^2$$

$$= \left\{ \bigcup_{a_1, a_2 \epsilon [0,g]} \left(s_{g(\frac{a_1}{a_2})} \right), \bigcup_{b_1, b_2 \epsilon [0,g]} \left(s_{g(\frac{b_1 - b_2}{g - b_2})} \right) \right\} \otimes \left\{ m_S^2, n_S^2 \right\}$$

$$= \left\{ \bigcup_{a_1, a_2 \epsilon [0,g]} \left(s_{g(\frac{a_1}{a_2})} \right) \right.$$

$$\left. \otimes m_S^2, \bigcup_{b_1, b_2 \epsilon [0,g]} \left(s_{g(\frac{b_1 - b_2}{g - b_2})} \right) \oplus n_S^2 \right\}$$

$$= \left\{ \bigcup_{a_1, a_2 \epsilon [0,g]} \left(s_{g \frac{\left(\frac{a_1}{a_2}\right) a_2}{g}} \right), \bigcup_{b_1, b_2 \epsilon [0,g]} \left(s_{g(\frac{b_1 - b_2}{g - b_2}) + b_2 - \frac{g\left(\frac{b_1 - b_2}{g - b_2}\right) b_2}{g}} \right) \right\}$$

$$= \left\{ \bigcup_{s_{a_1} \epsilon m_S^1} \left(\{s_{a_1}\} \right), , \bigcup_{b_1, b_2 \epsilon [0,g]} \left(s_{\frac{g b_1 - g b_2 + g b_2 - b_2^2 - b_1 b_2 + b_2^2}{g - b_2}} \right) \right\}$$

$$= \left\{ \bigcup_{s_{a_1} \epsilon m_S^1} (s_{a_1}), \bigcup_{s_{b_1} \epsilon n_S^1} (s_{b_1}) \right\}$$

$$= \{m_S^1, n_S^1\}$$

$$= I_S^1.$$

5. $I_S^1 \oplus I_S^2 = \left\{ \bigcup_{a_1, a_2 \epsilon [0,g]} \left(s_{a_1 + a_2 - \frac{a_1 a_2}{g}} \right), \bigcup_{b_1, b_2 \epsilon [0,g]} \left(s_{(\frac{b_1 b_2}{g})} \right) \right\}$

$$\lambda(I_S^1 \oplus I_S^2)$$

$$= \left\{ \bigcup_{a_1, a_2 \epsilon [0,g]} \left(s_{g\left(1 - \left(1 - \frac{a_1 + a_2 - \frac{a_1 a_2}{g}}{g}\right)\right)^{\lambda}} \right), \bigcup_{b_1, b_2 \epsilon [0,g]} \left(s_{g\left(\frac{\frac{b_1 b_2}{g}}{g}\right)^{\lambda}} \right) \right\}$$

$$= \left\{ \bigcup_{a_1, a_2 \epsilon [0,g]} \left(s_{g\left(1 - \left(\frac{g^2 - a_1 g - a_2 g + a_1 a_2}{g^2}\right)\right)^{\lambda}} \right), \bigcup_{b_1, b_2 \epsilon [0,g]} \left(s_{g\left(\frac{b_1 b_2}{g^2}\right)^{\lambda}} \right) \right\}$$

$$= \left\{ \bigcup_{a_1, a_2 \epsilon [0,g]} \left(s_{g\left(1 - \left(\frac{(g - a_1) - a_2 (g - a_1)}{g^2}\right)\right)^{\lambda}} \right), \bigcup_{b_1, b_2 \epsilon [0,g]} \left(s_{g\left((\frac{b_1}{g})^{\lambda} (\frac{b_2}{g})^{\lambda}\right)} \right) \right\}$$

$$= \left\{ \bigcup_{a_1, a_2 \epsilon [0,g]} \left(s_{g\left(1 - \left((\frac{g - a_1}{g})(\frac{g - a_2}{g})\right)\right)^{\lambda}} \right), \bigcup_{b_1, b_2 \epsilon [0,g]} \left(s_{g\left((\frac{b_1}{g})^{\lambda} (\frac{b_2}{g})^{\lambda}\right)} \right) \right\}$$

$$= \left\{ \bigcup_{a_1, a_2 \epsilon [0,g]} \left(s_{g\left(1 - \left((\frac{g - a_1}{g})^{\lambda} (\frac{g - a_2}{g})\right)^{\lambda}\right)} \right), \bigcup_{b_1, b_2 \epsilon [0,g]} \left(s_{g\left((\frac{b_1}{g})^{\lambda} (\frac{b_2}{g})^{\lambda}\right)} \right) \right\}$$

$$= \lambda I_S^1 \oplus \lambda I_S^2.$$

6. $(I_S^1 \ominus I_S^2) = \left\{ \bigcup_{a_1, a_2 \epsilon [0,g]} \left(s_{g(\frac{a_1 - a_2}{g - a_2})} \right), \bigcup_{b_1, b_2 \epsilon [0,g]} \left(s_{g(\frac{b_1}{b_2})} \right) \right\}$

$$\lambda(I_S^1 \ominus I_S^2)$$

$$= \lambda \left\{ \bigcup_{a_1, a_2 \epsilon [0,g]} \left(s_{g(\frac{a_1 - a_2}{g - a_2})} \right), \bigcup_{b_1, b_2 \epsilon [0,g]} \left(s_{g(\frac{b_1}{b_2})} \right) \right\}$$

$$= \left\{ \bigcup_{a_1, a_2 \epsilon [0,g]} \left(s_{g(1 - (1 - \frac{g(\frac{a_1 - a_2}{g - a_2})}{g}))^{\lambda}} \right), \bigcup_{b_1, b_2 \epsilon [0,g]} \left(s_{g(\frac{g \frac{b_1}{b_2}}{g})^{\lambda}} \right) \right\}$$

$$= \left\{ \bigcup_{a_1, a_2 \epsilon [0,g]} \left(s_{g(1 - (\frac{a_1 - a_2}{g - a_2}))^{\lambda}} \right), \bigcup_{b_1, b_2 \epsilon [0,g]} \left(s_{g(\frac{b_1}{b_2})^{\lambda}} \right) \right\}$$

$$= \left\{ \bigcup_{a_1, a_2 \epsilon [0,g]} \left(s_{g(1 - (\frac{a_1 - a_2}{g - a_2}))^{\lambda}} \right), \bigcup_{b_1, b_2 \epsilon [0,g]} \left(s_{g(\frac{b_1}{b_2})^{\lambda}} \right) \right\}$$

$$= \lambda I_S^1 \ominus \lambda I_S^2.$$

7. $I_S^1 \otimes I_S^2 = \left\{ \bigcup_{a_1, a_2 \epsilon [0,g]} \left(s_{\frac{a_1 a_2}{g}} \right), \bigcup_{b_1, b_2 \epsilon [0,g]} \left(s_{b_1 + b_2 - \frac{b_1 b_2}{g}} \right) \right\}$

$$(I_S^1 \otimes I_S^2)^{\lambda}$$

$$= \left\{ \bigcup_{a_1, a_2 \epsilon [0,g]} \left(s_{\left(\frac{\frac{a_1 a_2}{g}}{g}\right)^{\lambda}} \right), \bigcup_{b_1, b_2 \epsilon [0,g]} \left(s_{g\{1 - (1 - \frac{b_1 + b_2 - \frac{b_1 b_2}{g}}{g})^{\lambda}\}} \right) \right\}$$

$$= \left\{ \bigcup_{a_1, a_2 \epsilon [0,g]} \left(s_{\left(\frac{a_1 a_2}{g^2}\right)^{\lambda}} \right), \bigcup_{b_1, b_2 \epsilon [0,g]} \left(s_{g\{1 - (1 - \frac{b_1 + b_2 - \frac{b_1 b_2}{g}}{g})^{\lambda}\}} \right) \right\}$$

$$= (I_S^1)^{\lambda} \otimes (I_S^2)^{\lambda}.$$

8. $I_S^1 \oslash I_S^2$

$$= \left\{ \bigcup_{a_1, a_2 \epsilon [0,g]} \left(s_{g(\frac{a_1}{a_2})} \right), \bigcup_{b_1, b_2 \epsilon [0,g]} \left(s_{g(\frac{b_1 - b_2}{g - b_2})} \right) \right\}$$

$$(I_S^1 \oslash I_S^2)^{\lambda}$$

$$= \left\{ \bigcup_{a_1, a_2 \epsilon [0,g]} \left(s_{g(\frac{a_1}{a_2})^{\lambda}} \right), \bigcup_{b_1, b_2 \epsilon [0,g]} \left(s_{g\{1 - (1 - (\frac{b_1 - b_2}{g - b_2})^{\lambda}\}} \right) \right\}$$

$$= (I_S^1)^{\lambda} \oslash (I_S^2)^{\lambda}.$$

9. $\lambda_1 I \oplus \lambda_2 I = \left\{ \underset{s_a \epsilon m}{\cup} \left(s_{g(\frac{a}{g})^{\lambda_1}} \right), \underset{s_b \epsilon n}{\cup} \left(s_{g(1-(1-\frac{b}{g})^{\lambda_1})} \right) \right\}$

$\oplus \left\{ \underset{s_a \epsilon m}{\cup} \left(s_{g(\frac{a}{g})^{\lambda_2}} \right), \underset{s_b \epsilon n}{\cup} \left(s_{g(1-(1-\frac{b}{g})^{\lambda_2})} \right) \right\}$

$= \left\{ \underset{s_a \epsilon m}{\cup} \left(s_{g(\frac{a}{g})^{\lambda_1} + g(\frac{a}{g})^{\lambda_2} - \frac{g(\frac{a}{g})^{\lambda_1} g(\frac{a}{g})^{\lambda_2}}{g}} \right), \underset{s_b \epsilon n}{\cup} \left(s_{\frac{g(1-(1-\frac{b}{g})^{\lambda_1} g(1-(1-\frac{b}{g})^{\lambda_2}}{g}} \right) \right\}$

$= \left\{ \underset{s_a \epsilon m}{\cup} \left(s_{g(\frac{a}{g})^{\lambda_1} + g(\frac{a}{g})^{\lambda_2} - \frac{g(\frac{a}{g})^{\lambda_1} g(\frac{a}{g})^{\lambda_2}}{g}} \right), \underset{s_b \epsilon n}{\cup} \left(s_{g(1-(1-\frac{b}{g})^{\lambda_1}(1-(1-\frac{b}{g})^{\lambda_2}} \right) \right\}$

$= (\lambda_1 + \lambda_2) I.$

10. $\lambda_1 I \ominus \lambda_2 I$

$= \left\{ \underset{s_a \epsilon m_S}{\cup} \left(s_{g(\frac{a}{g})^{\lambda_1}} \right), \underset{s_b \epsilon n_S}{\cup} \left(s_{g(1-(1-\frac{b}{g})^{\lambda_1}} \right) \right\}$

$\ominus \left\{ \underset{s_a \epsilon m_S}{\cup} \left(s_{g(\frac{a}{g})^{\lambda_2}} \right), \underset{s_b \epsilon n_S}{\cup} \left(s_{g(1-(1-\frac{b}{g})^{\lambda_2}} \right) \right\}$

$= \left\{ \underset{s_a \epsilon m_S}{\cup} \left(s_{g \frac{g(\frac{a}{g})^{\lambda_1} - g(\frac{a}{g})^{\lambda_2}}{g - g(\frac{a}{g})^{\lambda_2}}} \right), \underset{s_b \epsilon n_S}{\cup} \left(s_{g(\frac{g(1-(1-\frac{b}{g})^{\lambda_1}}{g(1-(1-\frac{b}{g})^{\lambda_2}})} \right) \right\}$

$= \left\{ \underset{s_a \epsilon m_S}{\cup} \left(s_{g \frac{(\frac{a}{g})^{\lambda_1} - (\frac{a}{g})^{\lambda_2}}{g - g(\frac{a}{g})^{\lambda_2}}} \right), \underset{s_b \epsilon n_S}{\cup} \left(s_{g \frac{1-(1-\frac{b}{g})^{\lambda_1}}{1-(1-\frac{b}{g})^{\lambda_2}}} \right) \right\}$

$= (\lambda_1 - \lambda_2) I.$

12. $(I_S)^{\lambda_1} \oslash (I_S)^{\lambda_2}$

$= \left\{ \underset{a \epsilon [0,g]}{\cup} \left(s_{g(\frac{a}{g})^{\lambda_1}} \right), \underset{b \epsilon [0,g]}{\cup} \left(s_{g(1-(1-\frac{b}{g})^{\lambda_1})} \right) \right\}$

$\oslash \left\{ \underset{a \epsilon [0,g]}{\cup} \left(s_{g(\frac{a}{g})^{\lambda_2}} \right), \underset{b \epsilon [0,g]}{\cup} \left(s_{g(1-(1-\frac{b}{g})^{\lambda_2})} \right) \right\}$

$= \left\{ \underset{a \epsilon [0,g]}{\cup} \left(s_{g(\frac{g(\frac{a}{g})^{\lambda_1}}{g(\frac{a}{g})^{\lambda_2}})} \right), \underset{b \epsilon [0,g]}{\cup} \left(s_{g \frac{g(1-(1-\frac{b}{g})^{\lambda_1}) - g(1-(1-\frac{b}{g})^{\lambda_2})}{g - g(1-(1-\frac{b}{g})^{\lambda_2})}} \right) \right\}$

$= \left\{ \underset{a \epsilon [0,g]}{\cup} \left(s_{g(\frac{(\frac{a}{g})^{\lambda_1}}{(\frac{a}{g})^{\lambda_2}})} \right), \underset{b \epsilon [0,g]}{\cup} \left(s_{g \frac{(1-(1-\frac{b}{g})^{\lambda_1}) - (1-(1-\frac{b}{g})^{\lambda_2}}{1-(1-(1-\frac{b}{g})^{\lambda_2})}} \right) \right\}$

$= \left\{ \underset{a \epsilon [0,g]}{\cup} \left(s_{g(\frac{a}{g})^{\lambda_1 - \lambda_2}} \right), \underset{b \epsilon [0,g]}{\cup} \left(s_{g(1-(1-\frac{b}{g})^{\lambda_1 - \lambda_2})} \right) \right\}$

$= (I_S)^{\lambda_1 - \lambda_2}.$

11. $(I_S)^{\lambda_1} \otimes (I_S)^{\lambda_2} = \left\{ \underset{a \epsilon [0,g]}{\cup} \left(s_{g(\frac{a}{g})^{\lambda_1}} \right), \underset{b \epsilon [0,g]}{\cup} \left(s_{g(1-(1-\frac{b}{g})^{\lambda_1})} \right) \right\} \otimes \left\{ \underset{a \epsilon [0,g]}{\cup} \left(s_{g(\frac{a}{g})^{\lambda_2}} \right), \underset{b \epsilon [0,g]}{\cup} \left(s_{g(1-(1-\frac{b}{g})^{\lambda_2})} \right) \right\}$

$= \left\{ \underset{a \epsilon [0,g]}{\cup} \left(s_{\frac{g(\frac{a}{g})^{\lambda_1} g(\frac{a}{g})^{\lambda_2}}{g}} \right), \underset{b \epsilon [0,g]}{\cup} \left(s_{g(1-(1-\frac{b}{g})^{\lambda_1}) + g(1-(1-\frac{b}{g})^{\lambda_2}) - \frac{g(1-(1-\frac{b}{g})^{\lambda_1}) g(1-(1-\frac{b}{g})^{\lambda_2})}{g}} \right) \right\}$

$= \left\{ \underset{a \epsilon [0,g]}{\cup} \left(s_{g(\frac{a}{g})^{\lambda_1 + \lambda_2}} \right), \underset{b \epsilon [0,g]}{\cup} \left(s_{g(1-(1-\frac{b}{g})^{\lambda_1}) + g(1-(1-\frac{b}{g})^{\lambda_2}) - (1-(1-\frac{b}{g})^{\lambda_1}) g(1-(1-\frac{b}{g})^{\lambda_2})} \right) \right\}$

$= \left\{ \underset{a \epsilon [0,g]}{\cup} \left(s_{g(\frac{a}{g})^{\lambda_1 + \lambda_2}} \right), \underset{b \epsilon [0,g]}{\cup} \left(s_{g(1-(1-\frac{b}{g})^{\lambda_1} + 1-(1-\frac{b}{g})^{\lambda_2} - (1-(1-\frac{b}{g})^{\lambda_2} - (1-\frac{b}{g})^{\lambda_1} + (1-\frac{b}{g})^{\lambda_2}(1-\frac{b}{g})^{\lambda_1})} \right) \right\}$

$= \left\{ \underset{a \epsilon [0,g]}{\cup} \left(s_{g(\frac{a}{g})^{\lambda_1 + \lambda_2}} \right), \underset{b \epsilon [0,g]}{\cup} \left(s_{g(1-(1-\frac{b}{g})^{\lambda_1} + 1-(1-\frac{b}{g})^{\lambda_2} - 1 + (1-\frac{b}{g})^{\lambda_2} + (1-\frac{b}{g})^{\lambda_1} - (1-\frac{b}{g})^{\lambda_2}(1-\frac{b}{g})^{\lambda_1}))} \right) \right\}$

$= \left\{ \underset{a \epsilon [0,g]}{\cup} s \left(g(\frac{a}{g})^{\lambda_1 + \lambda_2} \right), \underset{b \epsilon [0,g]}{\cup} \left(s_{g(1-(1-\frac{b}{g})^{\lambda_1 + \lambda_2})} \right) \right\}$

$= (I_S)^{\lambda_1 + \lambda_2}.$

Definition 3.3 Let $(I_S^1, I_S^2, ...I_S^n)$ be a set of HIFLEs and $w = (w_1, w_2, ...w_n)^t$ be a weight vector of $\{I_S^j\}_{j=1}^n$ with $0 \le w_j \le 1$ such that $\sum_{j=1}^n w_j = 1$. Then, the HIFLWA operator is defined as follows:

$$(I_S^1, I_S^2, ...I_S^n) = \oplus_{j=1}^n (w_j I_S^j).$$

Theorem 3.4 *Let* $I_j = \{m_{S_j}, n_{S_j}\}$ *be a collection of HIFLEs where* $m_{S_j} = \{s_{a_j} |_{a_j} \epsilon[0, g]\}$ *and* $n_{S_j} = \{s_{b_j} |_{b_j} \epsilon[0, g]\}$, $(j = 1, 2, 3, ...n)$*, then*

$$HIFLWA\left(I_S^1, I_S^2, ...I_n\right) = \oplus_{j=1}^n w_j I_S^j$$
$$= \left\{ \cup_{a_j \epsilon[0,g]} \left(s_{g\left(1-\prod_{j=1}^n (1-\frac{a_j}{g})^{w_j}\right)} \right), \cup_{b_j \epsilon[0,g]} \left(s_{g\prod_{j=1}^n \left(\frac{b_j}{g}\right)^{w_j}} \right) \right\}.$$

Proof By the operational laws for HIFLEs $I_S^1 = \{m_S^1, n_S^1\}$ and $I_S^2 = \{m_S^2, n_S^2\}$ as discussed in Definition 3.1, we have

$$w_1 I_S^1 = \left\{ \cup_{s_{a_1} \epsilon[0,g]} \left(s_{g\left(1-\left(1-\frac{a_1}{g}\right)^{w_1}\right)} \right), \cup_{s_{b_1} \epsilon[0,g]} \left(s_{g(\frac{b_1}{g})^{w_1}} \right) \right\}$$
$$w_2 I_S^2 = \left\{ \cup_{s_{a_2} \epsilon[0,g]} \left(s_{g\left(1-\left(1-\frac{a_2}{g}\right)^{w_2}\right)} \right), \cup_{s_{b_2} \epsilon[0,g]} \left(s_{g(\frac{b_2}{g})^{w_2}} \right) \right\}$$

By using mathematical induction, we have for $n = 2$,

$$w_1 I_S^1 \oplus w_2 I_S^2$$
$$= \left\{ \cup_{s_{a_1} \epsilon[0,g]} \left(s_{g\left(1-\left(1-\frac{a_1}{g}\right)^{w_1}\right)} \right), \cup_{s_{b_1} \epsilon[0,g]} \left(s_{g(\frac{b_1}{g})^{w_1}} \right) \right\}$$
$$\oplus \left\{ \cup_{s_{a_2} \epsilon[0,g]} \left(s_{g\left(1-\left(1-\frac{a_2}{g}\right)^{w_2}\right)} \right), \cup_{s_{b_2} \epsilon[0,g]} \left(s_{g(\frac{b_2}{g})^{w_2}} \right) \right\}$$
$$= \left\{ \begin{array}{l} \cup_{s_{a_1}, s_{a_2} \epsilon[0,g]} \left(s_{1-(1-\frac{a_1}{g})^{w_1} + 1 - (1-\frac{a_2}{g})^{w_2} - \frac{\left(1-(1-\frac{a_1}{g})^{w_1}\right)\left(1-(1-\frac{a_2}{g})^{w_2}\right)}{g}} \right), \\ \cup_{s_{b_1}, s_{b_2} \epsilon[0,g]} \left(s_{\frac{(\frac{b_1}{g})^{w_1} (\frac{b_2}{g})^{w_2}}{g}} \right) \end{array} \right\}$$
$$= \left\{ \cup_{a_j \epsilon[0,g]} \left(s_{g\left(1-\prod_{j=1}^2 (1-\frac{a_j}{g})^{w_j}\right)} \right), \cup_{b_j \epsilon[0,g]} \left(s_{g\prod_{j=1}^2 \left(\frac{b_j}{g}\right)^{w_j}} \right) \right\}$$

We can easily verify that the conditions $\max(I_S^1) + \min(I_S^2) \le g$ and $\min(I_S^1) + \max(I_S^2) \le g$ always hold.

Let the theorem hold for $n = k$, i.e.,

$$HIFLWA\left(I_S^1, I_S^2, ..., I_S^k\right)$$
$$= \oplus_{j=1}^k w_j I_S^j$$
$$= \left\{ \cup_{a_j \epsilon[0,g]} \left(s_{g\left(1-\prod_{j=1}^k (1-\frac{a_j}{g})^{w_j}\right)} \right), \cup_{b_j \epsilon[0,g]} \left(s_{g\prod_{j=1}^k \left(\frac{b_j}{g}\right)^{w_j}} \right) \right\}$$

To prove it for $n = k + 1$, consider

$$\left\{ \cup_{a_j \epsilon[0,g]} \left(s_{g\left(1-\prod_{j=1}^k (1-\frac{a_j}{g})^{w_j}\right)} \right), \cup_{b_j \epsilon[0,g]} \left(s_{g\prod_{j=1}^k \left(\frac{b_j}{g}\right)^{w_j}} \right) \right\} \oplus w_1 I_S^1$$
$$= \left\{ \cup_{a_j \epsilon[0,g]} \left(s_{g\left(1-\prod_{j=1}^k (1-\frac{a_j}{g})^{w_j}\right)} \right), \cup_{b_j \epsilon[0,g]} \left(s_{g\prod_{j=1}^k \left(\frac{b_j}{g}\right)^{w_j}} \right) \right\}$$
$$\oplus \left\{ \cup_{s_{a_1} \epsilon[0,g]} \left(s_{g\left(1-(1-\frac{a_1}{g})^{w_1}\right)} \right), \cup_{s_{b_1} \epsilon[0,g]} \left(s_{g(\frac{b_1}{g})^{w_1}} \right) \right\}$$
$$= \left\{ \begin{array}{l} \cup_{a_j \epsilon[0,g]} \left(s_{g\left(1-\prod_{j=1}^k (1-\frac{a_j}{g})^{w_j}\right) + g\left(1-(1-\frac{a_1}{g})^{w_1}\right) - \frac{1}{g}\left(g\left(1-\prod_{j=1}^k (1-\frac{a_j}{g})^{w_j}\right)\right)\left(g\left(1-(1-\frac{a_1}{g})^{w_1}\right)\right)} \right), \\ \cup_{b_j \epsilon[0,g]} \left(s_{\frac{1}{g}\left(g\prod_{j=1}^k \left(\frac{b_j}{g}\right)^{w_j}\right)\left(g(\frac{b_1}{g})^{w_1}\right)} \right) \end{array} \right\}$$

$$= \left\{ \bigcup_{a_j \in [0,g]} \left(s_{g\left(1-\prod_{j=1}^{k}(1-\frac{a_j}{g})^{w_j}\right)\left[1-1-\left(1-\frac{a_1}{g}\right)^{w_1}\right]+g\left(1-(1-\frac{a_1}{g})^{w_1}\right)} \right), \bigcup_{b_j \in [0,g]} \left(s_{g\left(\prod_{j=1}^{k}\left(\frac{b_j}{g}\right)^{w_j}\right)\left(\frac{b_1}{g}\right)^{w_1}} \right) \right\}$$

$$= \left\{ \bigcup_{a_j \in [0,g]} \left(s_{g\left(1-\prod_{j=1}^{k}(1-\frac{a_j}{g})^{w_j}\right)\left(1-\frac{a_1}{g}\right)^{w_1}+g\left(1-(1-\frac{a_1}{g}\right)^{w_1}} \right), \bigcup_{b_j \in [0,g]} \left(s_{g\left(\prod_{j=1}^{k+1}\left(\frac{b_j}{g}\right)^{w_j}\right)} \right) \right\}$$

$$= \left\{ \bigcup_{a_j \in [0,g]} \left(s_{g\left(\left(1-\frac{a_1}{g}\right)^{w_1}-\prod_{j=1}^{k+1}(1-\frac{a_j}{g})^{w_j}+1-\left(1-\frac{a_1}{g}\right)^{w_1}\right)} \right), \bigcup_{b_j \in [0,g]} \left(s_{g\left(\prod_{j=1}^{k+1}\left(\frac{b_j}{g}\right)^{w_j}\right)} \right) \right\}$$

$$= \left\{ \bigcup_{a_j \in [0,g]} \left(s_{g\left(1-\prod_{j=1}^{k+1}(1-\frac{a_j}{g})^{w_j}\right)} \right), \bigcup_{b_j \in [0,g]} \left(s_{g\left(\prod_{j=1}^{k+1}\left(\frac{b_j}{g}\right)^{w_j}\right)} \right) \right\}$$

The theorem holds for $n = k + 1$, hence by principle of mathematical induction, it also holds for all $n \in N^+$.

Note: In particular, if $w = \left(\frac{1}{n}, ..., \frac{1}{n}\right)^t$ such that $\sum_{j=1}^{n} w_j = 1$, then the HIFLWA operator is reduced to HIFLA operator as follows:

$$HIFLA\left(I_S^1, I_S^2, ...I_n\right) = \frac{1}{n}\left(\bigoplus_{j=1}^{n} I_S^j\right) = \frac{1}{n}\left\{ \bigcup_{a_j \in [0,g]} \left(s_{g\left(1-\prod_{j=1}^{n}(1-\frac{a_j}{g})\right)} \right), \bigcup_{b_j \in [0,g]} \left(s_{g\prod_{j=1}^{n}\left(\frac{b_j}{g}\right)} \right) \right\}.$$

Definition 3.5 Let $(I_S^1, I_S^2, ...I_S^n)$ be a set of HIFLEs and $w = (w_1, w_2, ...w_n)^t$ be a weight vector of $\{I_S^j\}_{j=1}^n$ with $0 \le w_j \le 1$ such that $\sum_{j=1}^{n} w_j = 1$.Then, the HIFLWG operator is defined as follows:

$$(I_S^1, I_S^2, ...I_S^n) = \bigoplus_{j=1}^{n} (I_S^j)^{w_j}.$$

Theorem 3.6 *Let $I_j = \left\{m_{S_j}, n_{S_j}\right\}$ be a collection of HIFLEs where $m_{S_j} = \left\{s_{a_j} \mid_{a_j} \epsilon[0, g]\right\}$ and $n_{S_j} = \left\{s_{b_j} \mid_{b_j} \epsilon[0, g]\right\}$, $(j = 1, 2, 3, ...n)$, then*

$$HIFLWG\left(I_S^1, I_S^2, ...I_n\right) = \bigotimes_{j=1}^{n} \left(I_S^j\right)^{w_j} = \left\{ \bigcup_{a_j \in [0,g]} \left(s_{g\prod_{j=1}^{n}\left(\frac{a_j}{g}\right)^{w_j}} \right), \bigcup_{b_j \in [0,g]} \left(s_{g\left(1-\prod_{j=1}^{n}(1-\frac{b_j}{g})^{w_j}\right)} \right) \right\}.$$

Proof By the operational laws for HIFLEs $I_S^1 = \{m_S^1, n_S^1\}$ and $I_S^2 = \{m_S^2, n_S^2\}$ as discussed in Definition 3.1, we have

$$(I_S^1)^{w_1} = \left\{ \bigcup_{s_{a_1} \in [0,g]} \left(s_{g(\frac{a_1}{g})^{w_1}} \right), \bigcup_{s_{b_1} \in [0,g]} \left(s_{g\left(1-\left(1-\frac{b_1}{g}\right)^{w_1}\right)} \right) \right\}$$

$$(I_S^2)^{w_2} = \left\{ \bigcup_{s_{a_2} \in [0,g]} \left(s_{g(\frac{a_2}{g})^{w_2}} \right), \bigcup_{s_{b_2} \in [0,g]} \left(s_{g\left(1-\left(1-\frac{b_2}{g}\right)^{w_2}\right)} \right) \right\}$$

By using mathematical induction, we have for $n = 2$,

$$(I_S^1)^{w_1} \otimes (I_S^2)^{w_2} = \left\{ \bigcup_{s_{a_1} \in [0,g]} \left(s_{g(\frac{a_1}{g})^{w_1}} \right), \bigcup_{s_{b_1} \in [0,g]} \left(s_{g\left(1-\left(1-\frac{b_1}{g}\right)^{w_1}\right)} \right) \right\} \otimes \left\{ \bigcup_{s_{a_2} \in [0,g]} \left(s_{g(\frac{a_2}{g})^{w_2}} \right), \bigcup_{s_{b_2} \in [0,g]} \left(s_{g\left(1-\left(1-\frac{b_2}{g}\right)^{w_2}\right)} \right) \right\}$$

$$= \left\{ \bigcup_{s_{a_1}, s_{a_2} \in [0,g]} \left(s_{\frac{(\frac{a_1}{g})^{w_1}(\frac{a_2}{g})^{w_2}}{g}} \right), \bigcup_{s_{b_1}, s_{b_2} \in [0,g]} \left(s_{1-(1-\frac{b_1}{g})^{w_1}+1-(1-\frac{b_2}{g})^{w_2}-\frac{\left(1-(1-\frac{b_1}{g})^{w_1}\right)\left(1-(1-\frac{b_2}{g})^{w_2}\right)}{g}} \right) \right\}$$

$$= \left\{ \bigcup_{a_j \in [0,g]} \left(s_{g\prod_{j=1}^{2}\left(\frac{a_j}{g}\right)^{w_j}} \right), \bigcup_{b_j \in [0,g]} \left(s_{g\left(1-\prod_{j=1}^{2}(1-\frac{b_j}{g})^{w_j}\right)} \right) \right\}$$

We can easily verify that the conditions $\max(I_S^1) + \min(I_S^2) \le g$ and $\min(I_S^1) + \max(I_S^2) \le g$ always hold.

Let the theorem hold for $n = k$, i.e.,

$$
\begin{aligned}
&HIFLWG\left(I_S^1, I_S^2, ..., I_S^k\right)\\
&= \underset{j=1}{\overset{k}{\otimes}}\left(I_S^j\right)^{w_j}\\
&= \left\{\bigcup_{a_j\epsilon[0,g]}\left(s_{g\prod_{j=1}^{k}\left(\frac{a_j}{g}\right)^{w_j}}\right), \bigcup_{b_j\epsilon[0,g]}\left(s_{g\left(1-\prod_{j=1}^{k}(1-\frac{b_j}{g})^{w_j}\right)}\right)\right\}
\end{aligned}
$$

To prove it for $n = k + 1$, consider

$$
\begin{aligned}
&\left\{\bigcup_{a_j\epsilon[0,g]}\left(s_{g\prod_{j=1}^{k}\left(\frac{a_j}{g}\right)^{w_j}}\right), \bigcup_{b_j\epsilon[0,g]}\left(s_{g\left(1-\prod_{j=1}^{k}(1-\frac{b_j}{g})^{w_j}\right)}\right)\right\}\otimes\left(I_S^1\right)^{w_1}\\
&= \left\{\bigcup_{a_j\epsilon[0,g]}\left(s_{g\prod_{j=1}^{k}\left(\frac{a_j}{g}\right)^{w_j}}\right), \bigcup_{b_j\epsilon[0,g]}\left(s_{g\left(1-\prod_{j=1}^{k}(1-\frac{b_j}{g})^{w_j}\right)}\right)\right\}\\
&\otimes\left\{\bigcup_{s_{a_1}\epsilon[0,g]}\left(s_{g(\frac{a_1}{g})^{w_1}}\right), \bigcup_{s_{b_1}\epsilon[0,g]}\left(s_{g\left(1-(1-\frac{b_1}{g})^{w_1}\right)}\right)\right\}\\
&= \left\{\begin{array}{l}\bigcup_{a_j\epsilon[0,g]}\left(s_{\frac{1}{g}\left(g\prod_{j=1}^{k}\left(\frac{a_j}{g}\right)^{w_j}\right)\left(g(\frac{a_1}{g})^{w_1}\right)}\right),\\ \bigcup_{b_j\epsilon[0,g]}\left(s_{g\left(1-\prod_{j=1}^{k}(1-\frac{b_j}{g})^{w_j}\right)+g\left(1-(1-\frac{b_1}{g})^{w_1}\right)-\frac{1}{g}\left(g\left(1-\prod_{j=1}^{k}(1-\frac{b_j}{g})^{w_j}\right)\right)\left(g\left(1-(1-\frac{b_1}{g})^{w_1}\right)\right)}\right)\end{array}\right\}\\
&= \left\{\begin{array}{l}\bigcup_{a_j\epsilon[0,g]}\left(s_{g\left(\prod_{j=1}^{k}\left(\frac{a_j}{g}\right)^{w_j}\right)\left(\frac{a_1}{g}\right)^{w_1}}\right),\\ \bigcup_{b_j\epsilon[0,g]}\left(s_{g\left(1-\prod_{j=1}^{k}(1-\frac{b_j}{g})^{w_j}\right)\left[1-1-\left(1-\frac{b_1}{g}\right)^{w_1}\right]+g\left(1-(1-\frac{b_1}{g})^{w_1}\right)}\right)\end{array}\right\}\\
&= \left\{\begin{array}{l}\bigcup_{a_j\epsilon[0,g]}\left(s_{g\left(\prod_{j=1}^{k+1}\left(\frac{a_j}{g}\right)^{w_j}\right)}\right),\\ \bigcup_{b_j\epsilon[0,g]}\left(s_{g\left(1-\prod_{j=1}^{k}(1-\frac{b_j}{g})^{w_j}\right)\left(1-\frac{b_1}{g}\right)^{w_1}+g(1-(1-\frac{b_1}{g})^{w_1}}\right)\end{array}\right\}\\
&= \left\{\bigcup_{a_j\epsilon[0,g]}\left(s_{g\left(\prod_{j=1}^{k+1}\left(\frac{a_j}{g}\right)^{w_j}\right)}\right), \bigcup_{b_j\epsilon[0,g]}\left(s_{g\left(\left(1-\frac{b_1}{g}\right)^{w_1}-\prod_{j=1}^{k+1}(1-\frac{b_j}{g})^{w_j}+1-\left(1-\frac{b_1}{g}\right)^{w_1}\right)}\right)\right\}\\
&= \left\{\bigcup_{a_j\epsilon[0,g]}\left(s_{g\left(\prod_{j=1}^{k+1}\left(\frac{a_j}{g}\right)^{w_j}\right)}\right), \bigcup_{b_j\epsilon[0,g]}\left(s_{g\left(1-\prod_{j=1}^{k+1}(1-\frac{b_j}{g})^{w_j}\right)}\right)\right\}
\end{aligned}
$$

The theorem holds for $n = k + 1$, hence by principle of mathematical induction, it also holds for all $n \in N^+$.

Note: In particular, if $w = \left(\frac{1}{n}, ..., \frac{1}{n}\right)^t$ such that $\sum_{j=1}^{n} w_j = 1$, then the HIFLWG is simply reduced to hesitant intuitionistic fuzzy linguistic geometric operator HIFLGO as follows:

$$HIFLG\left(I_S^1, I_S^2, ...I_n\right) = \left(\overset{n}{\underset{j=1}{\otimes}} I_S^j\right)^{\frac{1}{n}}$$

$$= \left\{\bigcup_{a_j \epsilon [0,g]}\left(s_{g\prod_{j=1}^{n}\left(\frac{a_j}{g}\right)}\right), \bigcup_{b_j \epsilon [0,g]}\left(s_{g\left(1-\prod_{j=1}^{n}(1-\frac{b_j}{g})\right)}\right)\right\}^{\frac{1}{n}}.$$

□

4 The VIKOR method of MCGDM with HIFLTSs

In this section, we employ the proposed aggregation operators for HIFLTSs to solve the MCGDM problems by using the VIKOR method. Specifically, a modified VIKOR method in the context of hesitant intuitionistic fuzzy linguistic information is introduced. For this, the HIFLIR and HIFLC measures are designed with the help of hesitant intuitionistic fuzzy group utility measure so that the decision-making process can go toward a compromise solution. The proposed methodology is described with the details below.

Suppose $A = \{A_1, A_2, ..., A_r\}$ is the set of alternatives and $D = \{D_1, D_2, ..., D_k\}$ the set of DMs. Suppose the alternatives are evaluated by the DMs with respect to several criteria $c_j (j = 1, 2, ..., n)$. Let $w = (w_1, w_2, ..., w_n)^T$ be the weighting vector of the criteria, satisfying $\sum_{j=1}^{n} w_j = 1$, where $0 \le w_j \le 1,\ j = 1, 2, ..., n$. The VIKOR method is performed with the help of above assumptions by using the following 5 steps:

Step 1. The hesitant intuitionistic fuzzy linguistic judgment matrices are constructed by utilizing the experts opinions.

It is commonly very simple for a DM to provide his/her evaluations by using the linguistic data. Such linguistic data can be changed into HIFLTSs with the help of context free grammar as used by Rodríguez et al. (2013) for HFLTSs. Let $I_s^t (t = 1, 2, ..., k)$ be a judgment matrix comprising the data in the form of HIFLTSs as given below:

$$I_s^t = \begin{bmatrix} I_{s_{11}}^t & I_{s_{12}}^t & \cdots & I_{s_{1n}}^t \\ I_{s_{21}}^t & I_{s_{22}}^t & \cdots & I_{s_{2n}}^t \\ \vdots & \vdots & \ddots & \vdots \\ I_{s_{r1}}^t & I_{s_{r2}}^t & \cdots & I_{s_{rn}}^t \end{bmatrix}$$

Each $I_{s_{ij}}^t = \left\{m_{s_{ij}}^t, n_{s_{ij}}^t\right\}$ is HIFLTS indicating the degree that the alternative A_i satisfying the criterion c_j, where $m_{s_{ij}}^t = \cup_{s_{a_l^{ij}} \in m_{s_{ij}}^t}\left(s_{a_l^{ij}} \epsilon S \mid l = 1, 2, ..., \neq m_{s_{ij}}^t\right)$ and $n_{s_{ij}}^t = \cup_{s_{b_p^{ij}} \in m_{s_{ij}}^t}\left(s_{b_p^{ij}} \epsilon S \mid p = 1, 2, ..., \neq n_{s_{ij}}^t\right)$.

Step 2. In this step, the construction of aggregated hesitant intuitionistic fuzzy linguistic judgment matrix I_s is carried out by using the aggregation rules as proposed in Definition 3.3. The resultant I_s matrix is obtained as follows:

$$I_s = \begin{bmatrix} I_{s_{11}} & I_{s_{12}} & \cdots & I_{s_{1n}} \\ I_{s_{21}} & I_{s_{22}} & \cdots & I_{s_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ I_{s_{r1}} & I_{s_{r2}} & \cdots & I_{s_{rn}} \end{bmatrix},$$

where each $I_{s_{ij}} = \left\{m_{s_{ij}}, n_{s_{ij}}\right\}$ is HIFLTS, $m_{s_{ij}} = \cup_{s_{a_l^{ij}} \in m_{s_{ij}}^t}\left(s_{a_l^{ij}} \epsilon S \mid l = 1, 2, ..., \neq m_{s_{ij}}\right)$ and $n_{s_{ij}} = \cup_{s_{b_p^{ij}} \in m_{s_{ij}}^t}\left(s_{b_p^{ij}} \epsilon S \mid p = 1, 2, ..., \neq n_{s_{ij}}\right)$.

Step 3. The hesitant intuitionistic linguistic positive ideal solution (PIS) H_{j^+} and negative ideal solution (NIS) H_{j^-} are constructed as follows: $H_{j^+} = \left\{I_{s_{1+}}, I_{s_{2+}}, ..., I_{s_{n+}}\right\}$ and $H_{j^-} = \left\{I_{s_{1-}}, I_{s_{2-}}, ..., I_{s_{n-}}\right\}$

where

$$I_{s_{j+}} = \begin{cases} \left\{\left(\max_{\substack{t=1 \\ i=1,...,r}}^{k} m_{s_{ij}}^{t+} = \max_{\substack{t=1 \\ i=1,...,r}}^{k}\left(s_{m_l^{ij}}\right)\right), \left(\min_{\substack{t=1 \\ i=1,...,r}}^{k} n_{s_{ij}}^{t} = \min_{\substack{t=1 \\ i=1,...,r}}^{k}\left(s_{n_l^{ij}}\right)\right)\right\} \\ \text{for benefit criterion } c_j \\ \left\{\left(\min_{\substack{t=1 \\ i=1,...,r}}^{k} m_{s_{ij}}^{t-} = \min_{\substack{t=1 \\ i=1,...,r}}^{k}\left(s_{m_l^{ij}}\right)\right), \left(\max_{\substack{t=1 \\ i=1,...,r}}^{k} n_{s_{ij}}^{-} = \max_{\substack{t=1 \\ i=1,...,r}}^{k}\left(s_{n_l^{ij}}\right)\right)\right\} \\ \text{for cost criterion } c_j \end{cases} \tag{4.1}$$

$$I_{s_{j-}} = \begin{cases} \left\{\left(\min_{\substack{t=1 \\ i=1,...,r}}^{k} m_{s_{ij}}^{t+} = \min_{\substack{t=1 \\ i=1,...,r}}^{k}\left(s_{m_l^{ij}}\right)\right), \left(\max_{\substack{t=1 \\ i=1,...,r}}^{k} n_{s_{ij}}^{t} = \max_{\substack{t=1 \\ i=1,...,r}}^{k}\left(s_{n_l^{ij}}\right)\right)\right\} \\ \text{for benefit criterion } c_j \\ \left\{\left(\max_{\substack{t=1 \\ i=1,...,r}}^{k} m_{s_{ij}}^{t-} = \max_{\substack{t=1 \\ i=1,...,r}}^{k}\left(s_{m_l^{ij}}\right)\right), \left(\min_{\substack{t=1 \\ i=1,...,r}}^{k} n_{s_{ij}}^{-} = \min_{\substack{t=1 \\ i=1,...,r}}^{k}\left(s_{n_l^{ij}}\right)\right)\right\} \\ \text{for cost criterion } c_j \end{cases} \tag{4.2}$$

for $j = 1, 2...., n$.

Note that $I_{s_{j+}}$ and $I_{s_{j-}}$ are HIFLTSs which contain single linguistic term in the membership and non-membership functions, respectively.

Step 4. The hesitant intuitionistic fuzzy group utility measure is calculated as

$$HIFLGU_i = \sum_{j=1}^{n} w_j \frac{d\left(Hj^+,\ I_{s_{ij}}\right)}{d\left(Hj^+, Hj^-\right)}$$

where $w_j (j = 1, 2, ...n)$ is weight of the criteria $c_j (j = 1, 2, ...n)$ satisfying $\sum_{j=1}^{n} w_j = 1, 0 \le w_j \le 1,\ j = 1, 2, ..., n$ and d is the hesitant intuitionistic Euclidean distance measure which can be calculated by using Definition 2.5. Similarly, the hesitant intuitionistic individual regret measure $HIFLIR_i$ is calculated as follows:

$$HIFLIR_i = \max\left(w_j \frac{d\left(Hj^+,\ I_{s_{ij}}\right)}{d\left(Hj^+, Hj^-\right)}\right)$$

Finally, the hesitant intuitionistic fuzzy linguistic compromise measure $HIFLC_i$ for alternative A_i is then computed as follows:

$$HIFLC_i = \theta \frac{HIFLGU_i - HIFLGU^+}{HIFLGU^- - HIFLGU^+} + (1-\theta)\frac{HIFLIR_i - HIFLIR^+}{HIFLIR^- - HIFLIR^+}$$

where $HIFLGU^+ = \min(HIFLGU_i)$, $HIFLGU^- = \max(HIFLGU_i)$, $HIFLIR^+ = \min(HIFLIR_i)$ and $HIFLIR^- = \max(HIFLIR_i)$, $i = 1, 2, ...r$.

The parameter θ values lies between [0, 1] and is termed as decision mechanism index, and it has the following assumptions with respect to the values of θ. If $\theta > 0.5$, then maximum group benefits in terms of decision-making are achieved. If $\theta = 0.5$, this shows that decision-making are toward compromise solution. If $\theta < 0.5$, then minimum individual regret values are achieved in terms of decision-making. In general in VIKOR method, the value of θ is taken 0.5 by compromising maximum group benefit and minimum value for individual regret. The smaller the value of $HIFLC_i$, the best the ranking of the alternative A_i. The alternative $A^{(1)}$ which is ranked first by the measure $\min\{HIFLC_i \mid i = 1, 2, ..., r\}$ is proposed as compromise if the following two conditions are satisfied Yang et al. (2009):

(i) (Acceptable advantage): $HIFLC(A^{(2)}) - HIFLC(A^{(1)}) \geq \frac{1}{r-1}$, where $A^{(2)}$ representing second ranking alternative and r showing overall alternatives.

(ii) (Acceptable stability in decision-making): Alternative $A^{(1)}$ should likewise be the best ranked by $\{HIFLGU_i$ or/and $HIFLIR_i \mid i = 1, 2, ..., r\}$.

5 A numerical example

To check the adequacy and commonsense advantages for the proposed methodology, a numerical model is exhibited in this section. A detailed comparison analysis of the modified VIKOR method with the TOPSIS method is also established to justify the validity of the proposed work in managing the uncertain and vague behaviors of the DMs with HIFLTSs.

A family intended to build a building for their business purposes, and they are searching for an all around presumed development organization. The family got four recommendations in response in the form of alternatives A_1, A_2, A_3 and A_4. The development organization have set their decisions in the form of five criteria:

c_1 : project cost,
c_2 : quality of the construction material and machines,
c_3 : working hours,
c_4 : odds of specialized hazard and
c_5 : warranty of quality work.

Three DMs D_1, D_2 and D_3 are requested to evaluate the four proposals by using linguistic term set $S = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\}$ = {nothing, very low, low, medium, high, very high, perfect}. Assume the weighting vector of the five criteria is $w = (0.10, 0.50, 0.20, 0.10, 0.10)$.

Step 1. The three experts, respectively, provide their opinions in the form of HIFLTSs about the performance of $A_i\,(i = 1, 2, 3, 4)$ with respect to the criteria $c_j\,(j = 1, 2, 3, 4, 5)$ and as a result, the intuitionistic fuzzy linguistic decision matrices are then constructed as appeared in Tables 1, 2, and 3.

Step 2. The aggregated hesitant intuitionistic fuzzy linguistic judgment matrix $H = [H_{ij}]_{5\times4}$ is computed where,

$$H_{11} = \{(s_{5.1667}, s_{5.3333}, s_{5.4444}, s_{5.5556}, s_{5.5833}, s_{5.6667}, s_{5.7222}, s_{5.7778}), (s_0, s_{0.0278}, s_{0.0556}, s_{0.0833}, s_{0.1111}, s_{0.1667})\}$$

$$H_{21} = \{(s_{5.1111}, s_{5.3333}, s_{5.5}, s_{5.5556}, s_{5.6667}, s_{5.75}), (s_0, s_{0.0278}, s_{0.0556})\}$$

$$H_{31} = \{(s_{5.3333}, s_{5.5}, s_{5.5556}, s_{5.6667}, s_{5.75}, s_{5.7778}, s_{5.8333}), (s_{0.0278}, s_{0.0556})\}$$

$$H_{41} = \{(s_1, s_{1.8333}, s_2, s_{2.5278}, s_{2.6667}, s_{3.2222}), (s_{0.25}, s_{0.3333}, s_{0.4444}, s_{0.5}, s_{0.6667}, s_{0.75}, s_{0.8889}, s_1, s_{1.3333})\}$$

$$H_{51} = \{(s_{2.6667}, s_{3.2222}, s_{3.3333}, s_{3.5}, s_{3.7778}, s_{3.9167}, s_4, s_{4.3333}), s_{0.1667}, s_{0.2222}, s_{0.25}, s_{0.3333}, s_{0.4444}, s_{0.5}, s_{0.6667})\}$$

$$H_{12} = \{(s_6), (s_0)\}$$

$$H_{22} = \{(s_{3.2222}, s_{3.7778}, s_{3.9167}, s_{4.2222}, s_{4.3333}, s_{4.6111}, s_{4.6667}, s_{4.8889}, s_{5.1111}), (s_{0.1667}, s_{0.2222}, s_{0.25}, s_{0.3333}, s_{0.4444}, s_{0.5}, s_{0.6667}, s_{0.8889})\}$$

$$H_{32} = \{(s_{5.5556}, s_{5.6667}, s_{5.7778}, s_{5.8333}, s_{5.8889}, s_{5.9167}), (s_0, s_{0.0278}, s_{0.0556})\}$$

$$H_{42} = \{(s_{5.3333}, s_{5.5}, s_{5.5556}, s_{5.6667}, s_{5.75}, s_{5.7778}, s_{5.8333}), (s_0, s_{0.0278}, s_{0.0556}, s_{0.1111})\}$$

$$H_{52} = \{(s_{3.2222}, s_{3.7778}, s_{3.9167}, s_{4.2222}, s_{4.3333}, s_{4.6667}), (s_0, s_{0.0556}, s_{0.0833}, s_{0.1111}, s_{0.1667}, s_{0.2222}, s_{0.25}, s_{0.3333}, s_{0.4444}, s_{0.5}, s_{0.6667})\}$$

$$H_{13} = \{(s_{3.3333}, s_{3.7778}, s_4, s_{4.2222}, s_{4.3333}, s_{4.5}, s_{4.6667}, s_{4.75}, s_5), (s_0, s_{0.0556}, s_{0.0833}, s_{0.1111}, s_{0.1667})\}$$

$$H_{23} = \{(s_{5.6667}, s_{5.7778}, s_{5.8333}, s_{5.8889}, s_{5.9167}, s_{5.9444}), (s_0, s_{0.0278})\}$$

$$H_{33} = \{(s_{2.6667}, s_{3.2222}, s_{3.3333}, s_{3.5}, s_{3.7778}, s_{3.9167}, s_4, s_{4.3333}), (s_0, s_{0.2500}, s_{0.3333}, s_{0.4444})\}$$

Table 1 Decision matrix I_s^1 provided by DM_1

	A_1	A_2	A_3	A_4
c_1	$\{(s_3, s_4), (s_1, s_2)\}$	$\{(s_6), (s_0)\}$	$\{(s_0, s_1, s_2), (s_2, s_3)\}$	$\{(s_1, s_2), (s_0, s_1)\}$
c_2	$\{(s_2, s_3), (s_0, s_1)\}$	$\{(s_2, s_3, s_4), (s_1, s_2)\}$	$\{(s_3, s_4), (s_0, s_1)\}$	$\{(s_0, s_1), (s_2, s_3)\}$
c_3	$\{(s_4, s_5), (s_1)\}$	$\{(s_2, s_3), (s_1, s_2)\}$	$\{(s_0, s_1), (s_3, s_4)\}$	$\{(s_0, s_1), (s_2, s_3)\}$
c_4	$\{(s_1, s_2), (s_1, s_2, s_3)\}$	$\{(s_3, s_4), (s_1, s_2)\}$	$\{(s_1, s_2), (s_2, s_3)\}$	$\{(s_3), (s_1, s_2)\}$
c_5	$\{(s_0, s_1), (s_3, s_4)\}$	$\{(s_1, s_2), (s_2, s_3, s_4)\}$	$\{(s_5), (s_0, s_1)\}$	$\{(s_4, s_5), (s_0, s_1)\}$

Table 2 Decision matrix I_s^2 provided by DM_2

	A_1	A_2	A_3	A_4
c_1	$\{(s_4, s_5), (s_0, s_1,)\}$	$\{(s_1, s_2), (s_3, s_4)\}$	$\{(s_2, s_3), (s_1, s_2)\}$	$\{(s_6), (s_0)\}$
c_2	$\{(s_2, s_3), (s_1, s_2)\}$	$\{(s_1, s_2), (s_2, s_3, s_4)\}$	$\{(s_4, s_5), (s_0, s_1)\}$	$\{(s_2, s_3), (s_1, s_2, s_3)\}$
c_3	$\{(s_3, s_4), (s_1, s_2)\}$	$\{(s_4, s_5), (s_0, s_1)\}$	$\{(s_1, s_2), (s_3, s_4)\}$	$\{(s_1, s_2), (s_1, s_2, s_3)\}$
c_4	$\{(s_0, s_1), (s_3, s_4)\}$	$\{(s_4, s_5), (s_0, s_1,)\}$	$\{(s_2, s_3), (s_1, s_2)\}$	$\{(s_3, s_4), (s_0, s_1, s_2)\}$
c_5	$\{(s_1, s_2), (s_2, s_3)\}$	$\{(s_2, s_3), (s_1, s_2, s_3)\}$	$\{(s_4, s_5), (s_0, s_1)\}$	$\{(s_4, s_5), (s_1)\}$

Table 3 Decision matrix I_s^3 provided by DM_3

	A_1	A_2	A_3	A_4
c_1	$\{(s_1, s_2), (s_1, s_2, s_3)\}$	$\{(s_3, s_4), (s_0, s_1, s_2)\}$	$\{(s_2, s_3), (s_0, s_1)\}$	$\{(s_3, s_4), (s_1, s_2)\}$
c_2	$\{(s_4, s_5), (s_1)\}$	$\{(s_1, s_2), (s_3, s_4)\}$	$\{(s_4, s_5), (s_1)\}$	$\{(s_3, s_4), (s_0, s_1)\}$
c_3	$\{(s_2, s_3), (s_1)\}$	$\{(s_4, s_5), (s_0, s_1)\}$	$\{(s_2, s_3), (s_0, s_1)\}$	$\{(s_0, s_1, s_2), (s_2, s_3)\}$
c_4	$\{(s_0, s_1), (s_3, s_4)\}$	$\{(s_2, s_3), (s_0, s_1, s_2)\}$	$\{(s_0, s_1), (s_3, s_4)\}$	$\{(s_2, s_3, s_4), (s_0, s_1)\}$
c_5	$\{(s_2, s_3), (s_1, s_2)\}$	$\{(s_1, s_2), (s_0, s_1, s_2)\}$	$\{(s_2, s_3), (s_1, s_2)\}$	$\{(s_5), (s_1)\}$

$$H_{43} = \{(s_{2.6667}, s_{3.2222}, s_{3.3333}, s_{3.5}, s_{3.7778}, s_{3.9167}, s_4, s_{4.3333}), (s_{0.1667}, s_{0.2222}, s_{0.25}, s_{0.3333}, s_{0.4444}, s_{0.5}, s_{0.6667})\}$$

$$H_{53} = \{(s_{5.7778}, s_{5.8333}, s_{5.8889}, s_{5.9167}), (s_0, s_{0.0278}, s_{0.0556})\}$$

$$H_{14} = \{(s_6), (s_0)\}$$

$$H_{24} = \{(s_4, s_{4.3333}, s_{4.5}, s_{4.6667}, s_{4.75}, s_{4.8889}, s_5, s_{5.1667}), (s_0, s_{0.0556}, s_{0.0833}, s_{0.1111}, s_{0.1667}, s_{0.25})\}$$

$$H_{34} = \{(s_1, s_{1.8333}, s_2, s_{2.5278}, s_{2.6667}, s_{3.2222}, s_{3.3333}, s_{3.7778}), (s_{0.1111}, s_{0.1667}, s_{0.2222}, s_{0.25}, s_{0.3333}, s_{0.5}, s_{0.75})\}$$

$$H_{44} = \{(s_5, s_{5.25}, s_{5.3333}, s_{5.5}, s_{5.6667}), (s_0, s_{0.0278}, s_{0.0556}, s_{0.1111})\}$$

$$H_{54} = \{(s_{5.8889}, s_{5.9444}, s_{5.9722}), (s_0, s_{0.0278})\}$$

Step 3. For the computation of the hesitant intuitionistic linguistic PIS Hj^+ and hesitant intuitionistic linguistic NIS H_j^-, we first require to figure out the benefit and cost criteria. We can easily see that c_2, c_3 and c_5 are benefit, while c_1 and c_4 are cost criteria. The Hj^+ and H_j^- can be found by using (4.1) and (4.2) as $H_j^+ = [\{(s_{3.3333}), (s_{0.1667})\}, \{(s_{5.9444}), (s_0)\}, \{(s_{5.9167}), (s_0)\}, \{(s_1), (s_{1.3333})\}, \{(s_{5.9722}), (s_0)\}]$ $H_{j^-} = [\{(s_6), (s_0)\}, \{(s_{3.2222}), (s_{0.8889})\}, \{(s_1), (s_{0.75})\}, \{(s_{5.8333}), (s_0)\}, \{(s_{2.6667}), (s_{0.6667})\}]$

Step 4. The hesitant intuitionistic fuzzy linguistic measures $HIFLGU$, $HIFLIR_i$ and $HIFLC_i$ for alternative $A_i (i = 1, 2, ..., r)$ are calculated first and based on $HIFLC_i$, the final ranking of the alternatives is determined which can been seen in Table 4. It is clear from Table 4 that $HIFLC_3 \prec HIFLC_1 \prec HIFLC_2 \prec HIFLC_4$, $HIFLIR_3 \prec HIFLIR_1 \prec HIFLIR_2 \prec HIFLIR_4$ and $HIFLGU_3 \prec HIFLGU_1 \prec HIFLGU_2 \prec HIFLGU_4$ which concludes that the final ranking order of the alterna-

Table 4 Hesitant intuitionistic fuzzy linguistic measures and final ranking with HIFLWA operator

Alternative	$HIFLGU_i$	$HIFLIR_i$	$HIFLC_i$	Ranking
A_1	0.2950	0.1442	0.0442	2
A_2	1.4862	1.2000	0.9564	3
A_3	0.2266	0.1017	0	1
A_4	1.6067	1.2001	1	4

Table 5 Hesitant intuitionistic fuzzy linguistic measures and final ranking with HIFLWG operator

Alternative	$HIFLGU_i$	$HIFLIR_i$	$HIFLC_i$	Ranking
A_1	0.6696	0.3689	0.7150	3
A_2	0.5171	0.3103	0.2791	2
A_3	0.5497	0.2363	0.0461	1
A_4	0.8717	0.3223	0.8243	4

tives is $A3 \succ A1 \succ A2 \succ A4$ and $HIFLC_3$ attains the minimum value.

We can also observe that $HIFLC(A^{(1)}) - HIFLC(A^{(3)}) = 0.0442 < \frac{1}{r-1} = 0.3333$ (i.e., the first condition is not true) and the best alternative is also provided by $HIFLGU_i$ or $HIFLIR_i$ which is true here.

The similar steps can be performed with HIFLWG operator for this example, for saving space, we have summarized it with final results mentioned in Table 5.

As $HIFLC(A^{(1)}) - HIFLC(A^{(2)}) = 0.233 < \frac{1}{r-1} = 0.3333$ (i.e., the first condition is not true), and the best alternative is also provided by $HIFLGU_i$ or $HIFLIR_i$ which is true here.

5.1 Comparison analysis

So as to check the validity of our proposed method, the same numerical problem is additionally solved with TOPSIS method. By utilizing the proposed generalized distance measure 2.5 between two HIFLTSs of the aggregated hesitant intuitionistic fuzzy linguistic judgment matrix $H = [H_{ij}]_{5\times 4}$, the distance between each alternative A_i and the hesitant intuitionistic fuzzy linguistic positive ideal solution Hj^+ , and the distance between each alternative A_i and the hesitant intuitionistic fuzzy linguistic negative ideal solution Hj^- are calculated, respectively, as follows:

$$P_i^+ = \sum_{j=1}^{J} w_j d\left(Hj^+, I_{s_{ij}}\right),\ P_i^- = \sum_{j=1}^{J} w_j d\left(Hj^-, I_{s_{ij}}\right), \text{for } i = 1, 2, ..., r.$$

The relative closeness (RC) coefficients for each alternative A_i to the ideal solutions is calculated with the help of following formula.

$$RC(A_i) = \frac{P_i^-}{P_i^+ + P_i^-}$$

The ranking results of the alternatives are shown in Table 6. We can see that the ranking order of all the alternatives A_i $(i = 1, ..., 4)$ with respect to closeness coefficient is $A_3 \succ A_1 \succ A_2 \succ A_4$ which exactly match with the ranking order as obtained with the help of VIKOR method using HIFLWA operator (see Table 4). This shows that our proposed method is also validated.

The TOPSIS is similarly performed by using HIFLWG operator, which is summarized with final results mentioned in Table 7.

The final comparison of VIKOR and TOPSIS using HIFLWA and HIFLWG operators can be seen in Table 8 as follows:

All in all, the strengths of the proposed method are separated as follows:

(i) Two types of linguistic expressions which are simultaneously in the favor and against the alternatives in the form of HIFLTSs are provided for DMs to make evaluations under certain criteria, respectively. This can describe the fuzziness and uncertainty of experts more relevantly.

(ii) The proposed operational laws and aggregation weighted operators for HIFLTSs are very useful and effective that can be used to aggregate the DMs preferences in MCGDM problems which can demonstrate the superiority of proposed approach.

Table 6 PIS, NIS, RC coefficients and final ranking of alternatives

Alternatives	P_i^+	P_i^-	$RC(A_i)$	Ranking using TOPSIS with HIFLWA operator
A_1	0.0133	0.0603	0.8192	2
A_2	0.0596	0.0269	0.3110	3
A_3	0.0130	0.0626	0.8284	1
A_4	0.0652	0.0279	0.2999	4

Table 7 PIS, NIS, RC coefficients and the final ranking of alternatives

Alternatives	P_i^+	P_i^-	$RC(A_i)$	Ranking using TOPSIS with HIFLWG operator
A_1	0.0363	0.0427	0.5409	2
A_2	0.0275	0.0174	0.3869	4
A_3	0.0292	0.0486	0.6246	1
A_4	0.0457	0.0361	0.4415	3

Table 8 Method/Operator with final ranking of alternatives

Method/Operator	Ranking of Alternatives
VIKOR/HIFLWA	$A_3 \succ A_1 \succ A_2 \succ A_4$
VIKOR/HIFLWG	$A_3 \succ A_2 \succ A_1 \succ A_4$
TOPSIS/HIFLWA	$A_3 \succ A_1 \succ A_2 \succ A_4$
TOPSIS/HIFLWG	$A_3 \succ A_1 \succ A_4 \succ A_2$

(iii) The traditional VIKOR is extended with HIFLTSs so that DMs can judge the relative importance degree of alternatives with respect to given criteria by using linguistic phrases directly and conveniently. So, the ranking results calculated with the extended VIKOR method are effective and reliable.

6 Conclusions

To overcome circumstances in which decision-making problems utilize qualitative variables rather than numerical ones with hesitancy situations for DMs, the VIKOR method for MCGDM problems utilizing HIFLTSs has been presented in this paper. We have opted HIFLTSs for MCGDM method, which has demonstrated many recognized points of interest in speaking to human qualitative assessments during decision-making. The VIKOR method has been stretched out in this paper for HIFLTSs, to overcome the hesitancy of DMs during the decision-making processes. The HIFLWA and HIFLWG operators are introduced for HIFLTSs and applied them to solve MCGDM problem. We have also constructed the HIFLGU, HIFLIR and HIFLC measures, and then, alternatives are ranked with the help of all these measures. At last, a numerical problem was proposed to check the adequacy of the VIKOR method, and comparison analysis with TOPSIS method is carried out for validation of the final ranking of alternatives. In this study, we have affirmed that the proposed approach can handle the fuzziness and uncertainty of DMs effectively and the ranking results obtained are efficient and reliable. As far as the limitations of the research study is concerned, this paper only addresses the MCGDM problems with the proposed aggregation operators for HIFLTSs using the VIKOR method, while a comparative study of MCGDM problems with other available techniques like TODIM and ELECTRE methods is not considered. In the future, we will further extend the TODIM and ELECTRE methods to solve MCGDM problems with the help of proposed aggregation operators for HIFLTSs, which are in fact very interesting studies that deserve to be investigated on a larger scale.

Declaration

Conflict of interest The authors declare that they have no conflict of interest.

Human and animal participants This article does not contain any studies with human participants or animals performed by the author.

Consent participants It is submitted with the consent of all the authors.

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