



Quantification of emotions in decision making

V. I. Yukalov^{1,2}

Accepted: 15 October 2021 / Published online: 5 November 2021

© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2021

Abstract

The problem of quantification of emotions in the choice between alternatives is considered. The alternatives are evaluated in a dual manner. From one side, they are characterized by rational features defining the utility of each alternative. From the other side, the choice is affected by emotions labeling the alternatives as attractive or repulsive, pleasant or unpleasant. A decision maker needs to make a choice taking into account both these features, the utility of alternatives and their attractiveness. The notion of utility is based on rational grounds, while the notion of attractiveness is vague and rather is based on irrational feelings. A general method, allowing for the quantification of the choice combining rational and emotional features, is described. Despite that emotions seem to avoid precise quantification, their quantitative evaluation is possible at the aggregate level. The analysis of a series of empirical data demonstrates the efficiency of the approach, including the realistic behavioral problems that cannot be treated by the standard expected utility theory.

Keywords Emotions in decision making · Quantification of emotions · Behavioral probability · Dual choice · Affective computing · Artificial intelligence

1 Introduction

The problem of making a choice between alternatives is a core of decision theory and its numerous applications in economics, finances, and the operation of intelligence, whether artificial or human. The most well-developed procedure of decision making is based on the expected utility theory formalized by von Neumann and Morgenstern (1953). However, as is well known, it is rather a rare occasion when decisions are made on the basis of purely rational grounds estimating the alternative utility. Almost always the choice is essentially affected by emotions, and humans do not strictly follow the prescriptions of the expected utility theory, which results in numerous paradoxes and often does not allow even for qualitative predictions. To take into account behavioral effects related to the influence of emotions and other subjective biases, various so-called non-expected utility theories were suggested by replacing the expected utility with specially

constructed functionals invented for the purpose of a posteriori interpretation of one or just a few phenomena. A list of such non-expected utility theories can be found in the review by Machina (2008).

However, non-expected utility theories are descriptive requiring fitting of several parameters from particular experimental data. In addition, spoiling the structure of the expected utility leads to the appearance of inconsistencies and new paradoxes producing more problems than it resolves (Safra and Segal 2008; Birnbaum 2008; Al-Najjar and Weinstein 2009a, b).

The major problem in describing real-life decision making is caused by the difficulty of quantifying such behavioral phenomena as emotions. This is because subjective emotions are not precisely defined in explicit mathematical terms, contrary to such a crisp notion as utility that can be evaluated on rational grounds. Therefore, emotions increase the uncertainty that always exists in any choice, when decision makers evaluate the features of the given alternatives (Scherer and Moors 2019). When analyzing alternatives, decision makers experience different feelings, emotions, and subconscious intuitive movements (Kahneman 1982; Picard 1997; Minsky 2006; Plessner et al. 2008). This is why, even choosing between seemingly well-formulated lotteries, humans often do not obey the normative prescriptions of utility theory, but

✉ V. I. Yukalov
yukalov@theor.jinr.ru

¹ Bogolubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Russia 141980

² Instituto de Física de São Carlos, Universidade de São Paulo, CP 369, São Carlos, São Paulo 13560-970, Brazil

make decisions qualitatively contradicting the latter (Kahneman and Tversky 1979).

It is important to differentiate two sides in the problem of emotion quantification. One side is the assessment of emotions experienced by a subject as reactions on external events, e.g., hearing voice or looking at pictures. The arising emotions can include happiness, anger, pleasure, disgust, fear, sadness, astonishment, pain, and so on. The severity or intensity of such emotions can be estimated by studying the expressive forms manifesting themselves in motor reactions, such as facial expressions, pantomime, and general motor activity, and by measuring physiological reactions, such as the activity of the sympathetic and parasympathetic parts of the autonomic nervous system, as well as the activity of the endocrine glands. Vegetative manifestations of emotions can be noticed by studying changes in the electrical resistance of the skin, the frequency and strength of heart contractions, blood pressure, skin temperature, hormonal and chemical composition of the blood, and like that. Several methods of appraising particular emotions in separate setups have been considered (Amjadzadeh and Ansari-Asl 2017; Vartanov and Vartanova 2018; Scherer and Moors 2019; Vartanov et al. 2020; Wang et al. 2021).

The other, principally different, side of emotion evaluation concerns the study of the influence of emotions on taking decisions by subjects. It is generally accepted that human decisions are not purely rational, but emotions do play a great role in decision making. However, the quantitative influence of emotions on the process of decision making remains yet an unsolved problem.

The present paper studies the second problem: How it would be possible to assess the influence of emotions on decisions taken by humans? We do not consider the somatic or physiological effects produced by emotions, but we aim at analyzing how subjective emotions, arising in the process of decision making, influence the resulting decisions.

Subjectivity in decision making arises because of uncertainty in the suggested choice. This uncertainty can be of dual nature. From one side, there is the usual probabilistic uncertainty based on deliberations related to the alternative utility. From the other side, there is an uncertainty in the choice due to the subjective feelings that are not regulated by rational rules. Emotions can be separated into three classes. One class contains, loosely speaking, positive features, such as “good”, “pleasant”, “attractive” and like that. The second class is composed of negative characterizations, such as “bad”, “unpleasant”, and “repulsive”. And the third class is intermediate, comprising neutral definitions expressing indifference with respect to the alternatives under consideration.

Despite the subjectiveness of emotions, their influence in the choice between alternatives sometimes can be quantified. Of course, this looks to be impossible for a particular deci-

sion maker and for each separate choice procedure. Yet, it turns out that quantification is admissible at the aggregate level for a typical decision maker representing the average characteristics of a large group of decision makers.

The formulation of explicit mathematical rules allowing for the selection of an optimal alternative under vague uncertainty due to the influence of emotions, is not merely useful for characterizing human decision making, but it is compulsory for the realization of affective computing (Picard 1997) and for overcoming the challenge of creating artificial intelligence (Russel and Norvig 2016; Poole and Mackworth 2017; Neapolitan and Jiang 2018). The achievement of human-level machine intelligence is a principal goal of artificial intelligence since its inception.

The process of decision making, actually, consists of two sides that can conditionally be named rational and irrational. The rational side describes the comparative usefulness of the considered alternatives, while the irrational side is due to emotions making the decision process less predictable. It is the rational-irrational duality that makes the quantification of the decision-making process so difficult.

The distinction between rational and irrational has been extensively discussed in the literature on dual processes (Sun 2002; Paivio 2007; Evans 2007; Stanovich 2011; Kahneman D 2011) according to which the procedure of taking decisions in human brains can be treated as a result of two different processes that can be called rational (logical, controlled, regulated, deterministic, slow, defined by clear rules) and irrational (intuitive, uncontrolled, emotional, random, fast, defined in a fuzzy manner). These processes may proceed in parallel or in turn, but in any case they act differently (Milner and Goodale 2008; Kahneman 2011).

It is important to stress that the differentiation of mental processes onto rational and irrational has the meaning for the moment of taking a decision (Ariely 2008). It is a psychological distinction but not a philosophical one. It is clear that giving a specially invented philosophical definition it is straightforward to include afterward all intuitive and emotional effects into the rank of rational just giving a definition that rational is all what leads to the desired goal. Then, illogical uncontrolled feelings that occasionally lead to the goal should be termed rational, and vice versa logical conclusions that occasionally miss the goal should be named irrational (Searle 2001; Julmi 2019). The philosophical definition of rational has the meaning only afterward, when the goal has been reached. Only then it becomes clear what was leading to the goal and what was not.

Moreover, the philosophical definition of rational as what leads to the goal is ambiguous. For instance, assume that your goal is to become rich. The easiest way to become rich is to steal. Hence, to steal is rational. But then you are caught by police and put into jail. To be jailed was not your goal. Hence, to steal is not rational. So it is not clear, is it rational or not,

while from the psychological point of view there is no ambiguity. An action that is logically and explicitly formulated is psychologically rational. The psychological definition is based on real physiological processes in the brain, while the philosophical definition is not uniquely defined and depends on interpretations.

In what follows, we distinguish rational from irrational as it is accepted in decision making, where rational is what can be explicitly formulated, based on clear rules, deterministic, logical, prescriptive, normative, while irrational is the opposite to rational (Ariely 2008; Zafirovski 2012), being intuitive, uncontrolled, emotional, random, defined in a fuzzy way.

The dual nature of decision making, comprising the rational-irrational duality, suggests that this duality could be mathematically represented by a theory that naturally includes some kind of duality in its basis. The proper candidate for this could be quantum theory, with its particle-wave duality. A consistent approach realizing this analogy, by treating decision making as the procedure of quantum measurements, is the recently developed quantum decision theory (Yukalov and Sornette 2008, 2009a, b, 2011, 2014, 2016, 2018; Yukalov 2020, 2021a).

However, mathematical techniques of quantum theory are not customary for the majority of people. Therefore, it would be desirable to develop a theory that could incorporate the achievements of quantum decision theory at the same time avoiding mathematical complications of quantum techniques and the language of quantum theory so unfamiliar for the majority of researchers. The development of such an approach and its farther elaboration is the goal of the present paper. Specifically, the new results of the present article are as follows.

- (i) The axiomatic formulation of dual decision theory, taking account of cognition-emotion duality, that is rational-irrational duality in decision making, and comprising the main points of quantum decision theory, but without involving any quantum formulae.
- (ii) The derivation, without appealing to quantum theory, of the non-informative prior estimate for attraction factor measuring the typical influence of emotions on the process of decision making.
- (iii) Illustration of a simple rule for distinguishing emotionally attractive and repulsive characteristics of the considered alternatives in the case of highly uncertain lotteries of the Kahneman–Tversky type.
- (iv) Analysis of empirical data confirming that the typical influence of emotions in decision making composes 25%.

The plan of the paper is as follows. The approach to be formulated possesses two major features. First, it is probabilistic, which requires to define the corresponding

probability measure. Second, it is dual, aiming at taking into consideration rational as well as irrational characteristics of alternatives. In Sect. 2, the rules for defining the rational probabilistic choice, describing the utility of the alternatives, are formulated. In Sect. 3, it is shown how it is possible to characterize the emotional attractiveness of alternatives. In Sect. 4, the behavioral probability is defined, combining the rational utility measure and the irrational emotional characteristic, called attraction factor. The properties of the attraction factor are considered in Sect. 5, where the typical value of the attraction factor is found to be 1/4, which is called the quarter law. This value allows for the estimation of non-informative priors for the attraction factors describing the influence of emotions at the aggregate level. Section 6 shows how the attraction factors for multiple alternatives can be estimated. Section 7, by the example of binary decision tasks, illustrates that the definition of attraction factors, generally speaking, is contextual. This is because the attraction factors can contain parameters whose values need to be adjusted for describing a particular set of decision problems, which limits their use for other sets of decision tasks. In Sect. 8, the method of defining the attraction factor structure in the case of two alternatives with equal or very close utilities is described. Section 9 considers difficult choice tasks in the case of the Kahneman–Tversky lotteries, whose utilities are either exactly equal or very close to each other, so that the standard utility theory is not applicable. A method is suggested estimating the quality of the lotteries and their attractiveness and giving good quantitative predictions at the aggregate level. In Sect. 10, the analysis of a large set of lotteries is given demonstrating the validity of the quarter law at the aggregate level. Section 11 concludes.

2 Probabilistic uncertainty

The main task of decision theory is to describe the process of choice between a given set of alternatives

$$\mathcal{A} = \{A_n : n = 1, 2, \dots, N_A\}. \quad (1)$$

Each alternative can be characterized from two sides, from the rational point of view of its usefulness and, from the other side, following irrational feelings and emotions.

In this and the following sections, we describe a new approach to decision making, taking into account the rational reasoning by estimating the utility of the considered alternatives as well as the presence of irrational emotions accompanying the choice.

Even when there exist rational logical arguments explaining the utility of the given alternatives, not all subjects incline to prefer a single alternative, but always an alternative A_n , with a clearly defined utility, is selected only by a fraction

$f(A_n)$ of decision makers (Slovic and Tversky 1974), which can be termed *rational fraction*.

In the present section, the definition of the rational fraction is formulated and its properties are described.

Definition 1 A rational fraction $f(A_n)$ is the fraction of decision makers that would choose the alternative A_n provided their decisions would be based solely on rational grounds. The rational fraction is semi-positive and normalized

$$\sum_{n=1}^{N_A} f(A_n) = 1, \quad 0 \leq f(A_n) \leq 1. \tag{2}$$

The rational fraction represents the classical probability, with its standard properties, including the additivity with respect to mutually exclusive alternatives,

$$f\left(\bigcup_n A_n\right) = \sum_n f(A_n). \tag{3}$$

Definition 2 An alternative A_1 is called more useful than A_2 if and only if

$$f(A_1) > f(A_2). \tag{4}$$

Two alternatives, A_1 and A_2 , are equally useful if and only if

$$f(A_1) = f(A_2). \tag{5}$$

The rational fraction $f(A_n)$ shows how useful the alternative is, because of which it can be called the *utility fraction*.

The ideas of decision theory are typically illustrated by the choice between lotteries. Let the alternatives be represented by the lotteries

$$A_n = \{x_i, p_n(x_i) : i = 1, 2, \dots, N_n\}, \tag{6}$$

which are the probability distributions over payoffs x_i , with $p_n(x_i)$ being the payoff probabilities that can be either objective (von Neumann and Morgenstern 1953) or subjective (Savage 1954). The lottery utility is quantified by the expected utility

$$U(A_n) = \sum_i u(x_i) p_n(x_i), \tag{7}$$

where $u(x)$ is a utility function. More generally, it is possible to introduce a utility functional

$$U(A_n) = \sum_i u(x_i) w(p_n(x_i)),$$

with $w(p_n(x_i))$ being a postulated weighting function (Kahneman and Tversky 1979).

The rational fraction, associated with the expected utility, should satisfy the limiting conditions

$$\begin{aligned} f(A_n) &\rightarrow 1, & U(A_n) &\rightarrow \infty, \\ f(A_n) &\rightarrow 0, & U(A_n) &\rightarrow -\infty, \end{aligned} \tag{8}$$

whose meaning is clear. An explicit form of the rational fraction can be done by the Luce rule (Luce 1959; Luce and Raiffa 1989) according to which if an alternative A_n is characterized by an attribute a_n , then the weight of this alternative can be defined as:

$$f(A_n) = \frac{a_n}{\sum_{n=1}^{N_A} a_n} \quad (a_n \geq 0). \tag{9}$$

When the expected utilities of all lotteries are semi-positive, the attribute values can be defined by these utilities

$$a_n = U(A_n), \quad U(A_n) \geq 0, \tag{10}$$

while when the expected utilities are negative, the attribute values are defined by the inverse quantities

$$a_n = \frac{1}{|U(A_n)|}, \quad U(A_n) < 0. \tag{11}$$

In the case of mixed utility signs, it is straightforward to shift the utilities by a minimal available wealth making these utilities semi-positive.

Definition 3 When alternatives are represented by lotteries, the rational fractions can be defined as:

$$f(A_n) = \frac{U(A_n)}{\sum_{n=1}^{N_A} U(A_n)} \quad (U(A_n) \geq 0) \tag{12}$$

for semi-positive utilities and as

$$f(A_n) = \frac{|U(A_n)|^{-1}}{\sum_{n=1}^{N_A} |U(A_n)|^{-1}} \quad (U(A_n) < 0) \tag{13}$$

for negative utilities.

Generally speaking, as the utility $U(A_n)$, one can imply either the standard expected utility (von Neumann and Morgenstern 1953), or a utility functional, for instance as is used in the prospect theory (Kahneman and Tversky 1979). It is also possible to define the rational fraction as the minimizer of an information functional, as has been done for resolving the St. Petersburg paradox (Yukalov 2021b).

The so-defined rational fraction quantifies the fraction of decision makers choosing an alternative being based only on rational arguments of the alternative utility. In other words,

it is the probability that an alternative would be chosen by decision makers, provided they are purely rational.

3 Emotional uncertainty

In addition to the probabilistic uncertainty that can be quantified by the rational fraction, there exists an emotional uncertainty ascribing to the alternatives such vague emotional characteristics that do not seem to allow for a quantification. In the simplest case, these characteristics can be separated into three classes of different quality. One quality class includes such specifications as “positive”, “good”, “pleasant”, and “attractive”, while the other is composed of such depictions as “negative”, “bad”, “unpleasant”, and “repulsive”. The third, intermediate class qualifies the related alternatives as “neutral” or “indifferent” with respect to their attractiveness. The principal question is how it would be possible to describe in mathematical terms and quantify these classes of emotional uncertainty?

We shall denote the set of alternatives pertaining to the positive quality class as \mathcal{A}_+ , while the set of alternatives pertaining to the negative quality class, as \mathcal{A}_- . The set of alternatives from the neutral quality class is denoted by \mathcal{A}_0 . Let the emotional attractiveness of an alternative A_n be represented by an attraction factor $q(A_n)$. For a positive, attractive alternative, the attraction factor is positive, for a negative, repulsive alternative, it is negative, and for a neutral alternative, it is zero. The absolute value of the attraction factor is limited by one.

Definition 4 The attraction factor pertaining to a positive, negative or neutral quality class, respectively, varies in the intervals

$$\begin{aligned} 0 < q(A_n) \leq 1 \quad (A_n \in \mathcal{A}_+) , \\ -1 \leq q(A_n) < 0 \quad (A_n \in \mathcal{A}_-) , \\ q(A_n) = 0 \quad (A_n \in \mathcal{A}_0) . \end{aligned} \tag{14}$$

Definition 5 An alternative A_1 is more attractive than A_2 if and only if

$$q(A_1) > q(A_2) . \tag{15}$$

Conversely, an alternative A_2 is more repulsive than A_1 . Two alternatives are said to be equally attractive, or equally repulsive, if and only if

$$q(A_1) = q(A_2) . \tag{16}$$

Recall that quality, or attractiveness, is a vague subjective notion which can be interpreted as that the attraction factor is a random quantity varying in the frame of its quality class.

The qualities “attractive” or “repulsive” are subjective, being associated with concrete decision makers. Moreover, they can change for the same decision maker taking decisions at different moments of time or under different circumstances; hence, they are contextual (Helland 2018).

4 Behavioral probability

In real life, humans make decisions taking into account rational arguments, at the same time being influenced by irrational feelings and emotions. This implies that both quantities, the rational fraction and attraction factor define the probability $p(A_n)$ of choosing alternatives by decision makers. Thus, the probability $p(A_n)$ of choosing an alternative A_n embodies both a rational evaluation of the alternative utility as well as reflects the emotional attitude of decision makers toward the considered alternatives. This rational-irrational duality is typical for the behavior of real-life decision makers, because of which the probability $p(A_n)$ can be called *behavioral probability*.

When looking for the form of this probability, it is necessary to keep in mind that the rational decision making has to be a particular case of the more general process encompassing both rational and irrational sides of decision making. That is, when irrational effects become not important, the choice becomes purely rational. This requirement can be written as a limiting condition.

Correspondence principle Rational decision making is a particular case of behavioral decision making, when irrational effects play no role:

$$p(A_n) \rightarrow f(A_n) , \quad q(A_n) \rightarrow 0 . \tag{17}$$

The behavior of decision makers reflects the superposition of rational and irrational sides of consciousness. In other words, the real-life behavior is a superposition of cognition and emotions. This suggests the following axiom.

Axiom 1. Behavioral probability is the sum of a rational fraction and of an attraction factor:

$$p(A_n) = f(A_n) + q(A_n) , \tag{18}$$

with $p(A_n)$ being semi-positive and normalized,

$$\sum_{n=1}^{N_A} p(A_n) = 1 , \quad 0 \leq p(A_n) \leq 1 . \tag{19}$$

The condition of additivity is not required, so that, in general, the probability measure $\{p(A_n)\}$ is not necessarily additive.

The rational part of the behavioral probability is explicitly defined by the rational fraction, while the irrational part is characterized by a quantity represented by the attraction

factor. The behavioral probability, being a superposition of two terms, reflects the existence in life of rational-irrational duality, or cognition-emotion duality, or utility-attractiveness duality. When dealing with empirical data, the probability $p(A_n)$ describes the total fraction of decision makers preferring the given alternative.

The alternatives A_n from the set \mathcal{A} acquire the following properties understood as the corresponding relations between their probabilities.

- (i) *Ordering*: for any two alternatives A_1 and A_2 , one of the relations necessarily holds: either $A_1 < A_2$, in the sense that $p(A_1) < p(A_2)$, or $A_1 \leq A_2$, when $p(A_1) \leq p(A_2)$, or $A_1 > A_2$, if $p(A_1) > p(A_2)$, or $A_1 \geq A_2$, when $p(A_1) \geq p(A_2)$, or $A_1 \sim A_2$, if $p(A_1) = p(A_2)$.
- (ii) *Linearity*: the relation $A_1 \leq A_2$, implying $p(A_1) \leq p(A_2)$, means that $A_2 \geq A_1$, in the sense that $p(A_2) \geq p(A_1)$.
- (iii) *Transitivity*: for any three alternatives, such that $A_1 \leq A_2$, with $p(A_1) \leq p(A_2)$, and $A_2 \leq A_3$, when $p(A_2) \leq p(A_3)$, it follows that $A_1 \leq A_3$, in the sense that $p(A_1) \leq p(A_3)$.
- (iv) *Completeness*: the set of alternatives \mathcal{A} contains a minimal A_{\min} and a maximal A_{\max} elements, for which $p(A_{\min}) = \min_n p(A_n)$ and, respectively, $p(A_{\max}) = \max_n p(A_n)$. The ordered set of these alternatives is called a complete lattice.

Relations between behavioral probabilities determine preference relations between the alternatives.

Definition 6 An alternative A_1 is called preferable to A_2 if and only if

$$p(A_1) > p(A_2) \quad (A_1 > A_2) . \tag{20}$$

Two alternatives A_1 and A_2 are indifferent if and only if

$$p(A_1) = p(A_2) \quad (A_1 \sim A_2) . \tag{21}$$

Definition 7 The alternative A_{opt} is called optimal if and only if it corresponds to the maximal behavioral probability,

$$p(A_{\text{opt}}) = \max_n p(A_n) . \tag{22}$$

It is clear that an alternative can be more useful but not preferable, since its behavioral probability consists of a rational fraction and an irrational attraction factor. An alternative A_1 is preferable to A_2 , implying that $p(A_1) > p(A_2)$, then and only then when

$$f(A_1) - f(A_2) > q(A_2) - q(A_1) . \tag{23}$$

Both quantities, the rational fraction and attraction factor, are important in the process of taking decisions.

5 Attraction factor

Although the attraction factor is a random quantity, it possesses, on average, some general properties that, because of their importance, are formulated as theorems.

Theorem 1 *The attraction factor $q(A_n)$ varies in the interval*

$$- f(A_n) \leq q(A_n) \leq 1 - f(A_n) \tag{24}$$

and satisfies the alternation law

$$\sum_{n=1}^{N_A} q(A_n) = 0 . \tag{25}$$

Proof These properties follow directly from the definition of the behavioral probability (18), its semi-definiteness and normalization (19), and from the semi-definiteness and normalization of the rational fraction (2). □

Irrational feelings and emotions, playing a very important role in decision making, are characterized by the attraction factor. Strictly speaking, the attraction factor $q(A_n)$ is a random quantity that varies for different people and different conditions. Despite that it is random, it enjoys some specific features that can be used for estimating the non-informative priors quantifying this factor.

Recall that being random does not prevent the quantity from possessing well-defined properties on average. In decision making, this means that although the attraction factor is difficult to define for a single decision maker and a single choice, but it may enjoy quite explicit properties at the aggregate level as an average for a large group of decision makers and over several choices. Such averages, playing the role of non-informative priors, could allow us to evaluate typical attraction factors, even having no detailed information on each of the separate decision makers.

Definition 8 If a quantity y is given on an interval $[a, b]$, the average of y is defined as the arithmetic average

$$\bar{y} \equiv \frac{a + b}{2} . \tag{26}$$

In the case when the boundaries a and b themselves are the quantities given on the intervals $[a_1, a_2]$ and, respectively, $[b_1, b_2]$, the non-informative prior for y is the arithmetic average

$$\bar{y} \equiv \frac{\bar{a} + \bar{b}}{2} \left(\bar{a} \equiv \frac{a_1 + a_2}{2} , \bar{b} \equiv \frac{b_1 + b_2}{2} \right) . \tag{27}$$

Theorem 2 Quarter law *The average value for the attraction factor $q(A_n)$ in the positive quality class is*

$$\bar{q}(A_n) = \frac{1}{4} \quad (A_n \in \mathcal{A}_+) \tag{28}$$

and in the negative quality class, it is

$$\bar{q}(A_n) = -\frac{1}{4} \quad (A_n \in \mathcal{A}_-). \tag{29}$$

Proof According to Theorem 1, the attraction factor is defined on the interval $[-f(A_n), 1 - f(A_n)]$. Hence, for the positive quality class it is given on the interval $[0, 1 - f(A_n)]$ and for the negative quality class, on the interval $[-f(A_n), 0]$. By the definition of the averages, we have for the positive quality class

$$\bar{q}(A_n) = \frac{1 - \bar{f}(A_n)}{2} \quad (A_n \in \mathcal{A}_+), \tag{30}$$

while for the negative quality class

$$\bar{q}(A_n) = -\frac{\bar{f}(A_n)}{2} \quad (A_n \in \mathcal{A}_-). \tag{31}$$

Since $f(A_n)$ is defined on the interval $[0, 1]$, its non-informative prior is the average $\bar{f}(A_n) = 1/2$. This gives the averages (28) and (29). \square

In this way, the average behavioral probability of choosing an alternative A_n by a group of decision makers can be estimated by the expression

$$p(A_n) = f(A_n) \pm 0.25, \tag{32}$$

provided the probability properties (18) are preserved.

The attraction factor, being a random quantity, varies for different agents as well as for the same decision maker at different moments of time. Therefore, the same decision task, with the same utility factors, even for the same pool of subjects may be accompanied by different behavioral probabilities (Murphy and Fu 2018), hence different attraction factors. Such variations is a kind of random noise. Numerous empirical data show that these variations lead to statistical errors of about 0.1 (Murphy and ten Brincke 2018). This implies that if the difference $p(A_n) - f(A_n)$ is smaller than 0.1, then the attractiveness of the alternative pertains to the neutral class and the attraction factor can be set to zero.

6 Multiple alternatives

The estimate for the non-informative prior of the attraction factor, derived above, is especially useful for the case of

choosing between two alternatives. In the case, where there are many alternatives in the set \mathcal{A} , it is possible to more precisely estimate typical attraction factors, playing the role of non-informative priors.

Suppose N_A alternatives can be classified according to the level of their attractiveness, so that

$$q(A_n) > q(A_{n+1}) \quad (n = 1, 2, \dots, N_A - 1). \tag{33}$$

Let the nearest to each other attraction factors $q(A_n)$ and $q(A_{n+1})$ be separated by a typical gap

$$\Delta \equiv q(A_n) - q(A_{n+1}). \tag{34}$$

And let us accept that the average over the set \mathcal{A} absolute value of the attraction factor can be estimated by the non-informative prior $\bar{q} = 1/4$, so that

$$\bar{q} \equiv \frac{1}{N_A} \sum_{n=1}^{N_A} |q(A_n)| = \frac{1}{4}. \tag{35}$$

Theorem 3 For a set \mathcal{A} of N_A alternatives, under conditions (33)–(35), the non-informative priors for the attraction factors are

$$\begin{aligned} q(A_n) &= \frac{N_A - 2n + 1}{2N_A} \quad (N_A \text{ even}), \\ q(A_n) &= \frac{N_A(N_A - 2n + 1)}{2(N_A^2 - 1)} \quad (N_A \text{ odd}), \end{aligned} \tag{36}$$

depending on whether N_A is even or odd.

Proof In accordance with conditions (33)–(35), we can write

$$q(A_n) = q(A_1) - (n - 1)\Delta. \tag{37}$$

From the alternation law (25), it follows

$$q(A_1) = \frac{N_A - 1}{2} \Delta. \tag{38}$$

Using the definition of the average \bar{q} in equation (35) gives the gap

$$\Delta = \begin{cases} 4\bar{q}/N_A, & N_A \text{ even} \\ 4\bar{q}N_A/(N_A^2 - 1), & N_A \text{ odd} \end{cases}, \tag{39}$$

depending on whether the number of alternatives N_A is even or odd. Then, expression (38) becomes

$$q(A_1) = \begin{cases} 2\bar{q}(N_A - 1)/N_A, & N_A \text{ even} \\ 2\bar{q}N_A/(N_A + 1), & N_A \text{ odd} \end{cases}. \tag{40}$$

Using (37), we get

$$q(A_n) = \begin{cases} 2\bar{q}(N_A + 1 - 2n)/N_A, & N_A \text{ even} \\ 2\bar{q}N_A(N_A + 1 - 2n)/(N_A^2 - 1), & N_A \text{ odd} \end{cases} \quad (41)$$

In view of equality (35), we have $\bar{q} = 1/4$. Then, expression (40) leads to

$$q(A_1) = \begin{cases} (N_A - 1)/2N_A, & N_A \text{ even} \\ N_A/2(N_A + 1), & N_A \text{ odd} \end{cases} \quad (42)$$

And finally, Eq. (41) results in the answer (36). \square

As applications of the above theorem, let us give some examples. Thus for a set of two alternatives, we have the already known values of the non-informative priors for the attraction factors

$$\{q(A_n) : n = 1, 2\} = \left\{ \frac{1}{4}, -\frac{1}{4} \right\} \quad (43)$$

For three alternatives, we find

$$\{q(A_n) : n = 1, 2, 3\} = \left\{ \frac{3}{8}, 0, -\frac{3}{8} \right\} \quad (44)$$

Respectively, for the set of four alternatives, we obtain the quality factors

$$\{q(A_n) : n = 1, 2, 3, 4\} = \left\{ \frac{3}{8}, \frac{1}{8}, -\frac{1}{8}, -\frac{3}{8} \right\} \quad (45)$$

7 Binary alternatives

The case of binary alternatives is, probably, the most often considered in applications, being a typical choice problem. In previous sections, a method of evaluating the average value of the attraction factors is described. As is shown, the typical attraction factor can be estimated, despite that, in general, attractiveness seems to be a vague notion. The natural question arises whether it would be feasible to give a more detailed assessment of attractiveness. This problem has been discussed for the case of two alternatives, when it has been necessary to choose between two lotteries (Favre et al. 2016; Vincent et al. 2016; Ferro et al. 2021; Zhang and Kjellström 2021).

Let us consider the choice between two lotteries, A and B , whose utilities are $U(A)$ and $U(B)$, respectively. These quantities can represent either the standard expected utility (von Neumann and Morgenstern 1953) or other utility functionals employed in decision theory, e.g., the utility functional of

prospect theory (Kahneman and Tversky 1979; Tversky and Kahneman 1992). The attraction factor in the form:

$$\begin{aligned} q(A) &= \min\{\varphi(A), \varphi(B)\} \tanh\{a[U(A) - U(B)]\}, \\ q(B) &= -q(A), \end{aligned} \quad (46)$$

has been considered (Vincent et al. 2016; Ferro et al. 2021), where

$$\begin{aligned} \varphi(A) &= \frac{1}{Z} e^{\beta U(A)}, \quad \varphi(B) = \frac{1}{Z} e^{\beta U(B)}, \\ Z &= e^{\beta U(A)} + e^{\beta U(B)}. \end{aligned}$$

It has been used for characterizing the series of 91 binary decision tasks (lotteries), where the choice is made by the pool of 142 subjects (Murphy and ten Brincke 2018). The parameters of the attraction factor are fitted so that to optimally agree with the given experimental data. It is shown (Vincent et al. 2016; Ferro et al. 2021) that the decision theory with this attraction factor better describes the empirical data than the stochastic cumulative prospect theory (Tversky and Kahneman 1992) and than the stochastic rank-dependent utility theory (Quiggin 1982).

As is clear, the parameters calibrated so that to optimally describe a given set of lotteries may be not appropriate for another set of lotteries. In that sense, each combination of parameters is contextual, being suitable for a particular set of decision tasks, but not necessarily adequate for other groups of decision problems. This can be easily understood noticing that the attraction factor (46) becomes zero, when the utilities $U(A)$ and $U(B)$ coincide, or it becomes negligible when these utilities are close to each other. At the same time very close, or equal utilities often correspond to high uncertainty in the choice, which results in large attraction factors. A typical example of such a situation has been illustrated by Kahneman and Tversky (1979) for a set of lotteries with close or coinciding expected utilities.

The choice between two alternatives with equal or close utilities is a kind of the ‘‘Buridan’s donkey problem’’ (Kane 2005) that can be solved only considering emotions.

8 Buridan’s donkey problem

Emotions are characterized by attraction factors. Hence, to quantify emotions means the necessity of evaluating the values of the attraction factors for the considered alternatives. As is explained in Sect. 5, the first step in this estimation is the formulation of the quarter law stating that the non-informative prior for the magnitude of the attraction factor for either positive or negative quality classes is ± 0.25 . However, we need to define how the actual classification could be realized, so that each alternative would be associated with

the corresponding quality class, either positive or negative. This problem is typical for the research area known as soft computing aspiring to find methods that tolerate imprecision and uncertainty of fuzzy notions to achieve tractability and robustness allowing for quantitative conclusions (Clocksin 2003; de Silva 2003; Jamshidi 2003).

Below we suggest an algorithm that is applicable to that situation of close utilities of lotteries, whether with gains or with losses. This algorithm can be justified on the basis of studies in experimental neuroscience, which have discovered that, when making a choice, the main and foremost attention of decision makers is directed toward the payoff probabilities (Kim et al. 2012). This implies that subjects evaluate higher the probabilities than the related payoffs (Yukalov and Sornette 2014, 2018). In mathematical terms, this can be formulated as the existence of different types of scaling for the alternative quality with respect to payoff utility and payoff probability. Say, the payoff utility is scaled linearly, while the payoff probability, exponentially.

To be explicit, let us consider a simple case of two alternatives, one lottery

$$A_1 = \{u, p \mid 0, 1 - p\}, \tag{47}$$

with a payoff utility u and a related probability p , and the other lottery

$$A_2 = \left\{ \lambda u, \frac{p}{\lambda} \mid 0, 1 - \frac{p}{\lambda} \right\}, \tag{48}$$

whose payoff utility and probability are scaled in such a way that the expected utilities of both lotteries are equal,

$$U(A_1) = U(A_2) = up.$$

Then, the corresponding rational fractions coincide, $f(A_1) = f(A_2) = 1/2$, and one cannot choose a preferable alternative being based on rational arguments. This is a typical example of a series of lotteries considered by Kahneman and Tversky (1979). However, empirical studies show that subjects do make clear preferences between the lotteries, depending on their payoffs and probabilities. This implies that subjects are able to intuitively classify the lotteries into positive (attractive) or negative (repulsive).

Since “quality” or “attractiveness” are vague notions, it would be tempting to accomplish the quality classification by means of words. For instance, we could accept that between two lotteries that one is of better quality, or more attractive, that yields a more certain gain or less certain loss. Often this is a reasonable way of classification, although not always.

Suppose the lottery A_1 is quite certain, which implies that the payoff probability is in the interval $1/2 < p \leq 1$. Hence, the average probability is $p = 3/4$, which is appreciated by people as highly certain (Hillson 2003, 2019). Let the

scaling with $\lambda > 1$ be such that the payoff utility increases, while its probability diminishes. When λ is not large, subjects do prefer the more certain lottery A_1 . However, strongly increasing the payoff utility may attract more people, despite a small payoff probability, as has been confirmed by real-life lotteries (Rabin 2000).

In order to describe the method of classification of alternatives into positive (attractive) or negative (repulsive) quality classes, let us introduce the *quality functional* $Q(A_n)$. The fact that decision makers in their choice pay the main and foremost attention to the payoff probabilities (Kim et al. 2012) is formalized by a linear dependence of the quality functional with respect to the payoff utility and by an exponential dependence with respect to the payoff probability. For the case of the lotteries (47) and (48), this implies the quality functionals

$$Q(A_1) = ub^p, \quad Q(A_2) = \lambda ub^{p/\lambda}.$$

When the scaling λ is of order one and A_1 is more certain, subjects consider the more certain lottery as more attractive, which means that $Q(A_1)$ is larger than $Q(A_2)$. But if the payoff probability is diminished by an order, which assumes $\lambda = 10$, while the payoff utility increases by an order, then the lottery A_1 can become less attractive than A_2 , so that $Q(A_1)$ becomes smaller than $Q(A_2)$. The change of attractiveness occurs where $Q(A_1) = Q(A_2)$. The latter equality gives the expression for the base b that for $p = 3/4$ and $\lambda = 10$ yields

$$b = \lambda^{\lambda/(\lambda-1)p} = 30. \tag{49}$$

The above arguments give a clue allowing us to define the quality functional for any lottery.

Definition 9 The quality functional of an arbitrary lottery is

$$Q(A_n) = \sum_i u(x_i) 30^{p_n(x_i)}. \tag{50}$$

Comparing the quality functionals of different lotteries, we can meet the case, where these functionals are equal, but the lotteries differ from each other by the gain-loss number difference

$$N(A_n) = N_+(A_n) - N_-(A_n) \tag{51}$$

between the number of admissible gains $N_+(A_n)$ and the number of possible losses $N_-(A_n)$. A typical example is the comparison of the lottery A_1 defined in (47) and the lottery

$$A_3 = \{u_1, p \mid u_2, p \mid 0, 1 - 2p\}, \tag{52}$$

in which $u_1 + u_2 = u$. Then, the related quality functionals are equal

$$Q(A_3) = u_1 b^p + u_2 b^p = u b^p = Q(A_1) ,$$

where $b = 30$. If $u_n > 0$, then the lottery A_1 possesses only one admissible gain and no losses, while the lottery A_3 , two gains and also no losses. Hence, $N(A_1) = 1$ and $N(A_3) = 2$. Since $N(A_3)$ is larger than $N(A_1)$, the lottery A_3 is treated as more attractive. Similarly, for the lotteries with losses, where $u_n < 0$, and quality functionals are equal, the gain-loss number difference $N(A_1) = -1$ is larger than $N(A_3) = -2$, so that the lottery A_1 with a smaller number of losses is more attractive.

Definition 10 A lottery A_1 is of better quality, or more attractive, than A_2 , so that $q(A_1) > q(A_2)$, if either

$$Q(A_1) > Q(A_2) , \tag{53}$$

or if

$$Q(A_1) = Q(A_2) , \quad N(A_1) > N(A_2) . \tag{54}$$

If some lotteries A_1 and A_2 cannot be classified as more or less attractive, they are said to be of equal quality, or equally attractive, so that $q(A_1) = q(A_2)$. If there are only two of these lotteries, then the alternation law $q(A_1) + q(A_2) = 0$ implies $q(A_1) = q(A_2) = 0$. In that case, the lotteries are in the neutral quality class.

If the alternative A_1 is more attractive than A_2 , then the related behavioral probabilities can be estimated as:

$$p(A_1) = f(A_1) + 0.25 , \quad p(A_2) = f(A_2) - 0.25 , \tag{55}$$

where $f(A_n)$ are rational fractions. Here the inequality $0 \leq p(A_n) \leq 1$ is assumed, which can be formalized by the definition

$$p(A_n) = \text{Ret}_{[0,1]} \{ f(A_n) \pm 0.25 \} ,$$

where the retract function is defined as:

$$\text{Ret}_{[0,1]} z = \begin{cases} 0, & z < 0 \\ z, & 0 \leq z \leq 1 \\ 1, & z > 1 \end{cases} .$$

Recall that the above expressions estimate the aggregate fractions of decision makers averaged over many subjects and a set of choices. For a single decision maker, the attraction factor is a random quantity. However, the average attraction factor and, respectively, the average behavioral probability can be estimated according to rules (55). The rational fraction $f(A_n)$ shows the fraction (frequentist probability) of

decision makers that would choose the corresponding alternative on the basis of only rational rules. While the behavioral probability $p(A_n)$ defines the real total fraction of decision makers actually choosing A_n , taking into account both the rational utility as well as the irrational emotional attractiveness of the alternatives.

9 Kahneman–Tversky lotteries

By a number of examples, Kahneman and Tversky (1979) have shown that the expected utility theory in many cases does not work at all, so that decision makers do not decide according to utility theory, because of the very close lottery utilities, and even choose the alternatives that should be neglected according to the utility theory prescriptions. In their experiments, the number of participants was about 100. The typical statistical error was close to ± 0.1 . Payoffs below are given in monetary units, whose measures are of no importance when using dimensionless rational fractions.

Below, we show that the method described above correctly predicts the aggregate choice, giving good quantitative estimates for behavioral probabilities. Rational fractions are calculated by formulas of Sect. 2. For simplicity, the linear utility function $u(x)$ is accepted. The attraction factor is represented by its non-informative prior, with the sign prescribed by the lottery quality functional defined in Sect. 8. For brevity, we use the notation $Q(A_n) \equiv Q_n$.

For the convenience of the reader, we summarize the formulae that are used below in characterizing the lotteries. The rational utility fraction is calculated according to the definition in Sect. 2 as

$$f(L_n) = \frac{U(L_n)}{\sum_n U(L_n)} \quad (U(L_n) \geq 0)$$

for semi-positive expected utilities and as

$$f(L_n) = \frac{|U(L_n)|^{-1}}{\sum_n |U(L_n)|^{-1}} \quad (U(L_n) < 0)$$

for negative expected utilities, where the latter are given by the expression

$$U(L_n) = \sum_i x_i p_n(x_i) .$$

The quality functional $Q_n = Q(L_n)$ is defined in (50).

Choice 1. Consider two lotteries

$$L_1 = \{2.5, 0.33 \mid 2.4, 0.66 \mid 0, 0.01\} , \quad L_2 = \{2.4, 1\} .$$

The rational fractions are $f(L_1) = 0.501$ and $f(L_2) = 0.499$, so that the first lottery should be chosen on the

rational grounds. However, the lottery quality functionals $Q_1 = 30.3$ and $Q_2 = 72$ show that the second lottery, being more certain, is more attractive, since $Q_2 > Q_1$. Hence, $q(L_2) > q(L_1)$, and involving the non-informative prior, we have $q(L_1) = -0.25$, while $q(L_2) = 0.25$. This gives the behavioral probabilities

$$p(L_1) = 0.25, \quad p(L_2) = 0.75,$$

according to which the second lottery is optimal. This is in agreement with the empirical results

$$p_{\text{exp}}(L_1) = 0.18, \quad p_{\text{exp}}(L_2) = 0.82.$$

The more certain, but less useful lottery is chosen.

Choice 2. One chooses between the lotteries

$$L_1 = \{2.5, 0.33 \mid 0, 0.67\}, \quad L_2 = \{2.4, 0.34 \mid 0, 0.66\}.$$

The rational fractions are close to each other, $f(L_1) = 0.503$ and $f(L_2) = 0.497$. At the first glance, it is difficult to say which of the lotteries is more attractive, since the first lottery has a slightly larger payoff, while the second is a little more certain. But the lottery qualities $Q_1 = 7.68$ and $Q_2 = 7.63$ show that the first lottery is a bit more attractive. Hence, $q(L_1) = 0.25$ and $q(L_2) = -0.25$. Then, the behavioral probabilities are

$$p(L_1) = 0.75, \quad p(L_2) = 0.25$$

which is well comparable with the experimental data

$$p_{\text{exp}}(L_1) = 0.83, \quad p_{\text{exp}}(L_2) = 0.17.$$

This is an example, where the majority prefer a less certain, but more useful lottery.

Choice 3. Considering the lotteries

$$L_1 = \{4, 0.8 \mid 0, 0.2\}, \quad L_2 = \{3, 1\},$$

one sees that the first lottery, although being less certain, is more useful, having a larger rational fraction $f(L_1) = 0.516$ while $f(L_2) = 0.484$. But its quality is lower than that of the second lottery, $Q_1 = 60.8$, while $Q_2 = 90$. This means that the second lottery is more attractive, because of which $q(L_1) = -0.25$ and $q(L_2) = 0.25$. As a result, the behavioral probabilities are

$$p(L_1) = 0.27, \quad p(L_2) = 0.73,$$

being close to the experimentally observed

$$p_{\text{exp}}(L_1) = 0.20, \quad p_{\text{exp}}(L_2) = 0.80.$$

Here the more certain, although less useful lottery is chosen.

Choice 4. For the lotteries

$$L_1 = \{4, 0.20 \mid 0, 0.80\}, \quad L_2 = \{3, 0.25 \mid 0, 0.75\},$$

the rational fractions are again close to each other, as in the previous case, $f(L_1) = 0.516$ and $f(L_2) = 0.484$. But the quality of the first lottery is higher than that of the second, $Q_1 = 7.9$, but $Q_2 = 7.02$. This makes the first lottery more attractive, with $q(L_1) = 0.25$ and $q(L_2) = -0.25$. And the choice reverses, as compared to the previous case,

$$p(L_1) = 0.77, \quad p(L_2) = 0.23,$$

in agreement with the empirical results

$$p_{\text{exp}}(L_1) = 0.65, \quad p_{\text{exp}}(L_2) = 0.35.$$

Again, a less certain, although more useful, lottery is chosen.

Choice 5. Between the lotteries

$$L_1 = \{6, 0.45 \mid 0, 0.55\}, \quad L_2 = \{3, 0.9 \mid 0, 0.10\},$$

it is difficult to choose which is better. The first lottery suggests a twice larger payoff, while the second, twice higher payoff probability. The utility of both the lotteries is the same, with the same rational fractions $f(L_1) = f(L_2) = 0.5$. However, the lottery qualities are different, $Q_1 = 27.7$, while $Q_2 = 64.1$, showing that the second lottery is more attractive, which gives $q(L_1) = -0.25$ and $q(L_2) = 0.25$. Therefore, the behavioral probabilities become

$$p(L_1) = 0.25, \quad p(L_2) = 0.75.$$

And the empirical data are

$$p_{\text{exp}}(L_1) = 0.14, \quad p_{\text{exp}}(L_2) = 0.86.$$

More certain lottery is chosen.

Choice 6. The lotteries

$$L_1 = \{6, 0.001 \mid 0, 0.999\}, \quad L_2 = \{3, 0.002 \mid 0, 0.998\},$$

have the same rational fractions $f(L_1) = f(L_2) = 0.5$. But the quality of the first lottery is higher than that of the second, $Q_1 = 6.02$, while $Q_2 = 3.02$. That is, the first lottery is more attractive, so that $q(L_1) = 0.25$ and $q(L_2) = -0.25$. This yields the behavioral probabilities

$$p(L_1) = 0.75, \quad p(L_2) = 0.25,$$

practically coinciding with the experimental data

$$p_{\text{exp}}(L_1) = 0.73, \quad p_{\text{exp}}(L_2) = 0.27.$$

Between two equally useful lotteries, the less certain is chosen.

Choice 7. For the lotteries

$$L_1 = \{6, 0.25 \mid 0, 0.75\},$$

$$L_2 = \{4, 0.25 \mid 2, 0.25 \mid 0, 0.5\},$$

the rational fractions are again the same, which does not make it possible to choose on the basis of utility, $f(L_1) = f(L_2) = 0.5$. Although the lottery qualities are equal, $Q_1 = Q_2 = 14$, but the second lottery suggests a larger choice of gains, $N(L_2) = 2 > N(L_1) = 1$, which makes it more attractive, with $q(L_1) = -0.25$ and $q(L_2) = 0.25$. As a result, the behavioral probabilities read as

$$p(L_1) = 0.25, \quad p(L_2) = 0.75.$$

The empirical data are

$$p_{\text{exp}}(L_1) = 0.18, \quad p_{\text{exp}}(L_2) = 0.82.$$

Among seemingly equally useful lotteries, the choice is made under the influence of the attraction factor.

Choice 8. Considering the lotteries

$$L_1 = \{5, 0.001 \mid 0, 0.999\}, \quad L_2 = \{0.005, 1\},$$

we see that they are of equal utility, with the rational fractions $f(L_1) = f(L_2) = 0.5$. But the lottery qualities are essentially different, $Q_1 = 5.02$ and $Q_2 = 0.15$, which defines the attraction factors $q(L_1) = 0.25$ and $q(L_2) = -0.25$. Then, the behavioral probabilities are

$$p(L_1) = 0.75, \quad p(L_2) = 0.25.$$

This is very close to the empirical data

$$p_{\text{exp}}(L_1) = 0.72, \quad p_{\text{exp}}(L_2) = 0.28.$$

Again, this is an example, when a less certain lottery is chosen among two equally useful lotteries.

Choice 9. The lotteries

$$L_1 = \{10, 0.5 \mid 0, 0.5\}, \quad L_2 = \{5, 1\}$$

have equal utilities, with the rational fractions $f(L_1) = f(L_2) = 0.5$. The lottery qualities $Q_1 = 54.8$ and $Q_2 = 150$ show that the second lottery is more attractive; hence, $q(L_1) = -0.25$ and $q(L_2) = 0.25$. Therefore, the behavioral probabilities become

$$p(L_1) = 0.25, \quad p(L_2) = 0.75.$$

The empirical probabilities are

$$p_{\text{exp}}(L_1) = 0.16, \quad p_{\text{exp}}(L_2) = 0.84.$$

The second lottery is more certain, although has a smaller payoff.

Choice 10. For the lotteries

$$L_1 = \{2, 0.5 \mid 1, 0.5\}, \quad L_2 = \{1.5, 1\},$$

rational fractions are equal, $f(L_1) = f(L_2) = 0.5$. The lottery qualities are $Q_1 = 16.4$ and $Q_2 = 45$. Hence, the second lottery is more attractive, which means that $q(L_1) = -0.25$ and $q(L_2) = 0.25$. Then, the behavioral probabilities are

$$p(L_1) = 0.25, \quad p(L_2) = 0.75.$$

This is very close to the experimentally found probabilities

$$p_{\text{exp}}(L_1) = 0.20, \quad p_{\text{exp}}(L_2) = 0.80,$$

actually coinciding with them within the accuracy of experiments.

Choice 11. The previous lotteries dealt with gains. Now we shall treat the lotteries with losses, which implies that the subject has to pay, that is to lose, the amount of monetary units marked as negative. Consider the lotteries

$$L_1 = \{-4, 0.8 \mid 0, 0.2\}, \quad L_2 = \{-3, 1\}.$$

The rational fraction of the second lottery is larger, $f(L_1) = 0.484$, while $f(L_2) = 0.516$. However, the first lottery is more attractive, since its quality is higher, $Q_1 = -60.8$, while $Q_2 = -90$. This tells us that $q(L_1) = 0.25$ and $q(L_2) = -0.25$, which leads to the behavioral probabilities

$$p(L_1) = 0.73, \quad p(L_2) = 0.27.$$

In experiments, the majority also choose the first lottery,

$$p_{\text{exp}}(L_1) = 0.92, \quad p_{\text{exp}}(L_2) = 0.08.$$

The situation is opposite to the case of gains. Now a lottery with a less certain loss is preferable.

Choice 12. For the lotteries

$$L_1 = \{-4, 0.2 \mid 0, 0.8\}, \quad L_2 = \{-3, 0.25 \mid 0, 0.75\},$$

the rational fractions are $f(L_1) = 0.484$ and $f(L_2) = 0.516$. The related lottery qualities read as $Q_1 = -7.9$ and

$Q_2 = -7.02$, showing that the second lottery is more attractive, with $q(L_1) = -0.25$ and $q(L_2) = 0.25$. Then, we find the behavioral probabilities

$$p(L_1) = 0.23, \quad p(L_2) = 0.77.$$

Now the majority of decision makers choose the second lottery,

$$p_{\text{exp}}(L_1) = 0.42, \quad p_{\text{exp}}(L_2) = 0.58,$$

although it suggests a more certain loss.

Choice 13. The lotteries

$$L_1 = \{-3, 0.9 \mid 0, 0.1\}, \quad L_2 = \{-6, 0.45 \mid 0, 0.55\}$$

possess equal utility, hence equal rational fractions $f(L_1) = f(L_2) = 0.5$. But the second lottery is more attractive, since its quality is higher, $Q_1 = -64.1$, while $Q_2 = -27.7$. Therefore, $q(L_1) = -0.25$ and $q(L_2) = 0.25$. The behavioral probabilities

$$p(L_1) = 0.25, \quad p(L_2) = 0.75$$

show that the second lottery is optimal, in agreement with the empirical observations,

$$p_{\text{exp}}(L_1) = 0.08, \quad p_{\text{exp}}(L_2) = 0.92.$$

The second lottery is preferred, although its loss is higher, but the loss is less certain.

Choice 14. For the lotteries

$$L_1 = \{-3, 0.002 \mid 0, 0.998\},$$

$$L_2 = \{-6, 0.001 \mid 0, 0.999\}$$

rational fractions are equal, $f(L_1) = f(L_2) = 0.5$. However, the lottery qualities $Q_1 = -3.02$ and $Q_2 = -6.02$ demonstrate that the first lottery is more attractive, so that $q(L_1) = 0.25$ and $q(L_2) = -0.25$. This results in the behavioral probabilities

$$p(L_1) = 0.75, \quad p(L_2) = 0.25$$

that are very close to the experimentally found,

$$p_{\text{exp}}(L_1) = 0.70, \quad p_{\text{exp}}(L_2) = 0.30.$$

Now, between two equally useful lotteries, the lottery suggesting a more certain loss is chosen.

Choice 15. The lotteries

$$L_1 = \{-1, 0.5 \mid 0, 0.5\}, \quad L_2 = \{-0.5, 1\}$$

also have equal rational fractions, $f(L_1) = f(L_2) = 0.5$. But the quality of the first lottery is higher, $Q_1 = -5.48$, while $Q_2 = -15$. Hence, the first lottery is more attractive, with $q(L_1) = 0.25$, but $q(L_2) = -0.25$. The resulting behavioral probabilities

$$p(L_1) = 0.75, \quad p(L_2) = 0.25$$

are in good agreement with the experimental data

$$p_{\text{exp}}(L_1) = 0.69, \quad p_{\text{exp}}(L_2) = 0.31.$$

The first lottery is preferred, although its loss is larger.

Choice 16. Among the lotteries

$$L_1 = \{-6, 0.25 \mid 0, 0.75\},$$

$$L_2 = \{-4, 0.25 \mid -2, 0.25 \mid 0, 0.5\},$$

that look similar, having the same rational fractions $f(L_1) = f(L_2) = 0.5$, and equal qualities $Q_1 = Q_2 = -14$, the second is less attractive, exhibiting a larger number of losses, $N(L_1) = -1 > N(L_2) = -2$. Therefore, $q(L_1) = 0.25$ and $q(L_2) = -0.25$. This yields the behavioral probabilities

$$p(L_1) = 0.75, \quad p(L_2) = 0.25,$$

practically coinciding with the empirical data

$$p_{\text{exp}}(L_1) = 0.70, \quad p_{\text{exp}}(L_2) = 0.30,$$

within the accuracy of experiments.

Choice 17. The lotteries

$$L_1 = \{-5, 0.001 \mid 0, 0.999\}, \quad L_2 = \{-0.005, 1\}$$

have the same utility, with equal rational fractions $f(L_1) = f(L_2) = 0.5$. But their qualities $Q_1 = -5.02$ and $Q_2 = -0.15$ show that the second lottery is more attractive, having a much larger quality. Hence, $q(L_1) = -0.25$ and $q(L_2) = 0.25$. The behavioral probabilities are

$$p(L_1) = 0.25, \quad p(L_2) = 0.75,$$

as compared with the experimental data

$$p_{\text{exp}}(L_1) = 0.17, \quad p_{\text{exp}}(L_2) = 0.83.$$

Surprisingly, the lottery with certain loss is chosen, which is explained by its higher quality.

Choice 18. For the lotteries

$$L_1 = \{-10, 0.5 \mid 0, 0.5\}, \quad L_2 = \{-5, 1\},$$

Table 1 Optimal lotteries L_{opt} from the Kahneman–Tversky set, the rational fractions for the optimal lotteries, $f(L_{\text{opt}})$, predicted behavioral probabilities $p(L_{\text{opt}})$, experimentally observed probabilities $p_{\text{exp}}(L_{\text{opt}})$, defined as the fractions of the participants choosing the optimal lottery L_{opt} , and the experimental attraction factors $q_{\text{exp}}(L_{\text{opt}})$ corresponding to the optimal lotteries

	L_{opt}	$f(L_{\text{opt}})$	$p(L_{\text{opt}})$	$p_{\text{exp}}(L_{\text{opt}})$	$q_{\text{exp}}(L_{\text{opt}})$
1	L_2	0.50	0.75	0.82	0.32
2	L_1	0.50	0.75	0.83	0.33
3	L_2	0.48	0.73	0.80	0.32
4	L_1	0.52	0.77	0.65	0.13
5	L_2	0.50	0.75	0.86	0.36
6	L_1	0.50	0.75	0.73	0.23
7	L_2	0.50	0.75	0.82	0.32
8	L_1	0.50	0.75	0.72	0.22
9	L_2	0.50	0.75	0.84	0.34
10	L_2	0.50	0.75	0.80	0.30
11	L_1	0.48	0.73	0.92	0.44
12	L_2	0.52	0.77	0.58	0.06
13	L_2	0.50	0.75	0.92	0.42
14	L_1	0.50	0.75	0.70	0.20
15	L_1	0.50	0.75	0.69	0.19
16	L_1	0.50	0.75	0.70	0.20
17	L_2	0.50	0.75	0.83	0.33
18	L_1	0.50	0.75	0.69	0.19
		0.50	0.75	0.77	0.27

At the bottom, the average values are shown

the rational fractions are equal, $f(L_1) = f(L_2) = 0.5$. But for the lottery qualities, we have $Q_1 = -54.8$ and $Q_2 = -150$. Thus, the first lottery is more attractive, hence $q(L_1) = 0.25$ and $q(L_2) = -0.25$. This gives the behavioral probabilities

$$p(L_1) = 0.75, \quad p(L_2) = 0.25,$$

in agreement with empirical data

$$p_{\text{exp}}(L_1) = 0.69, \quad p_{\text{exp}}(L_2) = 0.31.$$

Now the lottery with a larger, but less certain loss is chosen.

The results for all 18 choices between the Kahneman–Tversky lotteries are summarized in Table 1 showing which of the lotteries is optimal, that is, having the largest predicted behavioral probability

$$p(L_{\text{opt}}) \equiv \max_n p(L_n) \tag{56}$$

over the given lattice of alternatives. Also shown are the rational fractions for the optimal lottery, $f(L_{\text{opt}})$, experimental probabilities of the optimal lottery, defined as the fractions

of decision makers choosing the optimal lottery $p_{\text{exp}}(L_{\text{opt}})$, and the related empirical attraction factors

$$q_{\text{exp}}(L_{\text{opt}}) = p_{\text{exp}}(L_{\text{opt}}) - f(L_{\text{opt}}). \tag{57}$$

The results, corresponding to the non-optimal lotteries, can be easily found from the normalization conditions

$$\begin{aligned} p(L_1) + p(L_2) &= 1, & f(L_1) + f(L_2) &= 1, \\ q(L_1) + q(L_2) &= 0. \end{aligned} \tag{58}$$

At the bottom of Table 1, the average values over all 18 cases are given for the rational fraction $\bar{f}(L_{\text{opt}}) = 0.5$, predicted behavioral probability $\bar{p}(L_{\text{opt}}) = 0.75$, experimentally observed probability $\bar{p}_{\text{exp}}(L_{\text{opt}}) = 0.77$, and the experimentally observed average absolute value of the attraction factor

$$\bar{q}_{\text{exp}} = \bar{p}_{\text{exp}}(L_{\text{opt}}) - \bar{f}(L_{\text{opt}}) = 0.27. \tag{59}$$

Within the accuracy of the experiment, the predicted average behavioral probability of choosing an optimal lottery, 0.75, equals the empirical average fraction of decision makers 0.77, and the average attraction factor 0.27 practically coincides with the theoretical estimate of 0.25.

As the analysis of this set of choices demonstrates, it is not possible to predict the behavioral decision making of humans by considering separately either lottery utilities, payoffs, or payoff probabilities. But reliable predictions can be made by defining behavioral probabilities, including the estimates of both, rational fractions as well as attraction factors. On the aggregate level, such predictions are not merely qualitative, but provide good quantitative agreement with empirical data, involving no fitting parameters.

At the same time, the expected utility theory is not applicable to the Kahneman–Tversky lotteries, since the lottery with a higher utility is preferred only twice among 18 lotteries, that is only in the 1/9 part of the lotteries. Also, it is important to notice that the formula (46) here is not valid, as far as for the coinciding utilities it gives zero attraction factor, while the aggregate experimental data give for the attraction factor 0.27.

10 Quarter law

In the previous sections, it has been shown that the average influence of emotions in decision making can be quantified by the typical value of attraction factor, which turns out to be close to 0.25, which is termed *quarter law* and which follows from the non-informative prior estimate of Sect. 5. Thus, in the set of Kahneman–Tversky lotteries of Sect. 9 the experimentally measured average attraction factor is 0.27,

which, within the typical statistical error of 0.1, coincides with the predicted attraction factor 0.25.

In the present section, we verify the quarter law on the basis of a large set of binary lotteries studied recently (Murphy and ten Brincke 2018). In the analyzed experiment, 142 subjects were suggested a set of binary decision tasks (lotteries). The same experiment was repeated after two weeks, with randomly changing the order of the pairs of lotteries. The experiments at these two different times are referred as session 1 and session 2. There are three types of lotteries: lotteries containing only gains (all payoffs are positive), lotteries with only losses (all payoffs are negative), and mixed lotteries containing gains as well as losses. As usual, a loss implies the necessity to pay the designed amount of money. Keeping in mind the estimation of attraction factors in positive and negative quality classes, we consider the related lotteries, where the difference between the rational utility factors and the empirical choice probabilities, at least in one of the sessions, are larger than the value of the typical statistical error of 0.1 corresponding to random noise. On the basis of these lotteries, we calculate the quantities of interest and summarize the results in several tables.

Table 2 presents the results for the optimal lotteries with only gains and Table 3 shows the results for the optimal lotteries with only losses. Recall that a lottery L_1 is called optimal, as compared to a lottery L_2 if and only if the corresponding probability $p(L_1)$ is larger than $p(L_2)$. In both the cases, of either the lotteries with only gains or the lotteries with only losses, an optimal lottery is always a lottery from the positive quality class, in which $q(L_{opt}) > 0$. The situation can be different for the mixed lotteries, containing gains as well as losses. In these cases, an optimal lottery can occasionally pertain to a negative quality class. Table 4 summarizes the results for the mixed lotteries containing both gains and losses. Among these lotteries, the first sixteen examples in Table 4 are the lotteries from the positive quality class, which at the same time are the optimal lotteries. The last five cases are the lotteries that are not optimal, however being from the positive quality class.

As is seen, the value of the attraction factor in the positive or negative quality classes is ± 0.22 , which is in very good agreement with the predicted non-informative priors ± 0.25 . Thus, the quarter law provides a rather accurate estimate of the attraction factor at the aggregate level.

11 Conclusion

An approach is developed allowing for the quantification of emotions in decision making. The approach takes into account the duality of decision making, including both rational and irrational sides of decision process. The rational evaluation of alternatives is based on logical clearly pre-

Table 2 Optimal lotteries with gains

	$f(L_{opt})$	$p_1(L_{opt})$	$p_2(L_{opt})$	$q_1(L_{opt})$	$q_2(L_{opt})$
1	0.55	0.86	0.89	0.31	0.34
2	0.48	0.66	0.69	0.18	0.21
3	0.51	0.68	0.62	0.17	0.11
4	0.59	0.80	0.75	0.22	0.17
5	0.63	0.89	0.90	0.26	0.27
6	0.66	0.96	0.95	0.30	0.29
7	0.51	0.79	0.81	0.28	0.30
8	0.48	0.60	0.63	0.12	0.15
9	0.63	0.88	0.92	0.26	0.30
10	0.56	0.89	0.82	0.33	0.26
11	0.63	0.77	0.73	0.14	0.10
12	0.51	0.72	0.73	0.21	0.21
13	0.61	0.87	0.85	0.26	0.24
14	0.63	0.93	0.93	0.30	0.30
15	0.64	0.85	0.87	0.21	0.23
16	0.64	0.80	0.80	0.16	0.16
17	0.64	0.89	0.89	0.25	0.25
18	0.48	0.65	0.70	0.17	0.22
19	0.65	0.87	0.93	0.22	0.28
20	0.66	0.86	0.82	0.20	0.16
21	0.58	0.84	0.80	0.26	0.22
22	0.52	0.75	0.74	0.23	0.22
23	0.48	0.64	0.65	0.16	0.17
24	0.44	0.60	0.53	0.16	0.10
25	0.62	0.73	0.79	0.11	0.17
26	0.64	0.81	0.90	0.17	0.26
27	0.66	0.93	0.96	0.27	0.30
	0.58	0.80	0.80	0.22	0.22

The rational fraction $f(L_{opt})$ of the optimal lottery, fractions of subjects (frequentist probabilities) $p_i(L_{opt})$ choosing the optimal lottery in the session $i = 1, 2$, and the attraction factors $q_i(L_{opt})$ of the optimal lottery in the session i . At the bottom, the average values for the related quantities

scribed rules defining a rational fraction representing the probability of choosing alternatives on the basis of rational principles.

The irrational side of decision processes is due to subconscious feelings, emotions, and intuition that cannot be exactly measured for a given subject at a given moment of time, thus inducing emotional uncertainty in the process of decision making. Irrational processes are superimposed on the rational evaluation of the considered alternatives and define for each alternative a correction term called attraction factor. Since irrational processes cannot be exactly quantified, the attraction factor is a random quantity. The attraction factor can be described by linguistic characteristics that can be classified into three quality classes, briefly speaking, positive, negative, and neutral. The positive quality class includes

Table 3 Optimal lotteries with losses

	$f(L_{opt})$	$p_1(L_{opt})$	$p_2(L_{opt})$	$q_1(L_{opt})$	$q_2(L_{opt})$
1	0.52	0.77	0.75	0.25	0.23
2	0.60	0.85	0.83	0.25	0.23
3	0.53	0.72	0.71	0.19	0.18
4	0.64	0.96	0.92	0.32	0.28
5	0.55	0.70	0.68	0.15	0.13
6	0.54	0.73	0.72	0.20	0.19
7	0.63	0.79	0.84	0.16	0.21
8	0.54	0.66	0.63	0.12	0.09
9	0.56	0.80	0.89	0.24	0.33
10	0.58	0.89	0.92	0.31	0.34
11	0.49	0.66	0.71	0.17	0.22
12	0.62	0.87	0.93	0.25	0.31
13	0.55	0.79	0.74	0.24	0.19
14	0.54	0.82	0.77	0.29	0.24
15	0.53	0.65	0.70	0.12	0.17
16	0.51	0.59	0.62	0.08	0.11
17	0.56	0.79	0.86	0.23	0.30
18	0.58	0.89	0.90	0.31	0.32
19	0.61	0.76	0.74	0.15	0.13
	0.56	0.77	0.78	0.21	0.22

The rational fraction $f(L_{opt})$ of the optimal lottery, fractions of subjects (frequentist probabilities) $p_i(L_{opt})$ choosing the optimal lottery in the session $i = 1, 2$, and the attraction factors $q_i(L_{opt})$ of the optimal lottery in the session i . At the bottom, the average values for the related quantities

such specifications as attractive, pleasant, good and like that. The negative quality class comprises the features like repulsive, unpleasant, bad, and so on. The neutral quality class is intermediate, being neither positive nor negative.

The attraction factor is a variable randomly varying for different decision makers and even for the same decision maker at different moments of time. Nevertheless, being random, does not preclude this quantity to have a well-defined average value inside each of the quality classes. A theorem is proved defining the average values of the quality factor inside the positive class as $1/4$ and inside the negative class as $-1/4$. For alternatives represented by lotteries with equal or close utilities, a method is suggested ascribing each lottery to the appropriate quality class.

Being able to determine the belonging of alternatives to the related quality classes and knowing the average values of attraction factors allows us to find the average behavioral probabilities associated with the typical fractions of decision makers choosing this or that alternative.

The method is illustrated by a series of lotteries with a difficult choice, when the standard expected utility theory is not applicable, or its prescriptions contradict the choice of real humans. The empirical data confirm that the non-

Table 4 Mixed lotteries, containing gains and losses, from the positive quality class

	$f(L_+)$	$p_1(L_+)$	$p_2(L_+)$	$q_1(L_+)$	$q_2(L_+)$
1	0.40	0.69	0.66	0.29	0.26
2	0.62	0.85	0.85	0.23	0.23
3	0.67	0.87	0.82	0.20	0.15
4	0.44	0.62	0.61	0.18	0.17
5	0.50	0.64	0.54	0.15	0.05
6	0.59	0.71	0.65	0.12	0.06
7	0.54	0.69	0.63	0.16	0.10
8	0.49	0.66	0.60	0.18	0.16
9	0.57	0.87	0.85	0.30	0.28
10	0.65	0.75	0.77	0.10	0.12
11	0.52	0.77	0.70	0.26	0.19
12	0.49	0.58	0.63	0.09	0.14
13	0.55	0.87	0.92	0.32	0.37
14	0.52	0.61	0.67	0.09	0.15
15	0.53	0.80	0.83	0.27	0.30
16	0.56	0.67	0.63	0.11	0.07
17	0.00	0.27	0.27	0.27	0.27
18	0.00	0.29	0.36	0.29	0.36
19	0.00	0.30	0.45	0.30	0.45
20	0.00	0.39	0.38	0.39	0.38
21	0.00	0.37	0.35	0.37	0.35
	0.41	0.63	0.63	0.22	0.22

The rational fraction $f(L_+)$ of the lottery, fractions of subjects $p_i(L_+)$ choosing the corresponding lottery in the session $i = 1, 2$, and the attraction factors $q_i(L_+)$ of the lottery in that session i . At the bottom, the average values of the related quantities

informative prior for attraction factors provides an accurate quantification of emotions at the aggregate level.

Summarizing, the main points of the suggested approach can be formulated as follows.

- (i) Decision making is treated as a probabilistic process that can be characterized by behavioral probabilities defining the portions of decision makers choosing this or that alternative from the given set of alternatives.
- (ii) The behavioral probability, taking into account the rational-irrational or cognition-emotion duality of decision processes, describes decision making affected by emotions. The superposition of utility and attractiveness is represented as a sum of two terms, a rational fraction and an attraction factor.
- (iii) The rational fraction, having the properties of the standard additive probability, describes the fraction of decision makers that would make their choice being based solely on rational grounds, following prescribed rational rules. The rational fraction quantifies the utility of the choice.

- (iv) The attraction factor takes into account irrational effects influencing the choice, such as feelings, emotions, and biases. The attraction factor characterizes subconscious attractiveness of the considered alternatives, because of which it is called attraction factor. The attraction factor is a random quantity, varying for different subjects, different choices, and different times.
- (v) Despite being random, the attraction factor possesses well defined average features. The average values of the attraction factor for positive or negative quality classes can be defined by non-informative priors.
- (vi) The approach makes it possible to give quantitative predictions in the choice between the lotteries with emotional uncertainty, where the expected utility theory does not work. The aggregate predictions, averaged over decision makers and choices, are in good quantitative agreement with empirical data.
- (vii) Empirical data confirm the quarter law providing, at the aggregate level, an accurate evaluation of typical influence of emotions in decision making.
- (viii) The appealing feature of the approach is its straightforward axiomatic formulation employing rather simple mathematics. Although the structure of the approach is implicitly influenced by quantum theory, but it completely avoids borrowed from physics complicated quantum techniques.

Acknowledgements The author is grateful for helpful advise and useful discussions to D. Sornette and E.P. Yukalova. This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sector.

Declarations

Conflicts of interest The authors declare that they have no conflict of interest.

References

- Al-Najjar NI, Weinstein J (2009) The ambiguity aversion literature: a critical assessment. *Econ Philos* 25:249–284
- Al-Najjar NI, Weinstein J (2009) The ambiguity aversion literature: a critical assessment. *Econ Philos* 25:357–369
- Amjadzadeh M, Ansari-Asl K (2017) An innovative emotion assessment using physiological signals based on the combination mechanism. *Sci Iran D* 24:3157–3170
- Ariely D (2008) *Predictably irrational*. Harper, New York
- Birnbaum MH (2008) New paradoxes of risky decision making. *Psychol Rev* 115:463–501
- Clocksink WF (2003) Artificial intelligence and the future. *Philos Trans R Soc Lond A* 361:1721–1748
- de Silva CW (2003) The role of soft computing in intelligent machines. *Philos Trans R Soc Lond A* 361:1749–1780
- Evans J (2007) Dual-processing accounts of reasoning, judgment, and social cognition. *Ann Rev Psychol* 59:268–269
- Favre M, Wittwer A, Heinemann HR, Yukalov VI, Sornette D (2016) Quantum decision theory in simple risky choices. *PLoS One* 11:0168045
- Ferro GM, Kovalenko T, Sornette D (2021) Quantum decision theory augments rank-dependent expected utility and cumulative prospect theory. *J Econ Psychol* 86:102417
- Jamshidi M (2003) Tools for intelligent control: fuzzy controllers, neural networks and genetic algorithms. *Philos Trans R Soc Lond A* 361:1781–1808
- Julmi C (2019) When rational decision-making becomes irrational: a critical assessment and re-conceptualization of intuition effectiveness. *Bus Res* 12:291–314
- Helland IS (2018) *Epistemic processes*. Springer, Cham
- Hillson D (2003) *Effective opportunity management for projects*. Marcel Dekker, New York
- Hillson D (2019) *Capturing upside risk*. CRC Press, Boca Raton
- Kahneman D (1982) *Judgment under uncertainty, heuristics and biases*. Cambridge University, Cambridge
- Kahneman D, Tversky A (1979) Prospect theory: an analysis of decision under risk. *Econometrica* 47:263–292
- Kahneman D (2011) *Thinking fast and slow*. Farrar, Straus and Giroux, New York
- Kane R (2005) *A contemporary introduction to free will*. Oxford University, New York
- Kim BE, Seligman D, Kable JM (2012) Preference reversals in decision making under risk are accompanied by changes in attention to different attributes. *Front Neurosci* 6:109
- Luce RD (1959) *Individual choice behavior: a theoretical analysis*. Wiley, New York
- Luce RD, Raiffa R (1989) *Games and decisions: introduction and critical survey*. Dover, New York
- Machina MJ (2008) Non-expected utility theory. In: Durlauf SN, Blume LE (eds) *New Palgrave dictionary of economics*. Macmillan, New York
- Milner AD, Goodale MA (2008) Two visual systems re-viewed. *Neuropsychologia* 46:774–785
- Minsky M (2006) *The Emotion Machine*. Simon and Schuster, New York
- Murphy A, Fu L (2018) The effect of confidence in valuation estimates on arbitrage behavior and market prices. *J Behav Finance* 19:349–363
- Murphy RO, ten Brincke RHW (2018) Hierarchical maximum likelihood parameter estimation for cumulative prospect theory: improving the reliability of individual risk parameter estimates. *Manag Sci* 64:308–326
- Neapolitan RE, Jiang X (2018) *Artificial intelligence*. CRC Press, Boca Raton
- Paivio A (2007) *Mind and its evolution: a dual coding theoretical approach*. Lawrence Erlbaum Associates, Mahwah
- Picard R (1997) *Affective computing*. Massachusetts Institute of Technology, Cambridge
- Plessner H, Betsch C, Betsch T (2008) *Intuition in judgment and decision making*. Lawrence Erlbaum Associates, New York
- Poole DL, Mackworth AK (2017) *Artificial intelligence*. Cambridge University, Cambridge
- Quiggin J (1982) A theory of anticipated utility. *J Econ Behav Org* 3:323–343
- Rabin M (2000) Risk aversion and expected-utility theory: a calibration theorem. *Econometrica* 68:1281–1292
- Russel SJ, Norvig P (2016) *Artificial intelligence: a modern approach*. Pearson Education, Harlow
- Safra Z, Segal U (2008) Calibration results for non-expected utility theories. *Econometrica* 76:1143–1166
- Savage LJ (1954) *The foundations of statistics*. Wiley, New York
- Scherer KR, Moors A (2019) The emotion process: event appraisal and component differentiation. *Ann Rev Psychol* 70:719–745

- Searle JR (2001) *Rationality in action*. Massachusetts Institute of Technology, Cambridge
- Slovic P, Tversky A (1974) Who accepts Savage's axioms? *Behav Sci* 19:368–373
- Stanovich KE (2011) *Rationality and the reflective mind*. Oxford University, New York
- Sun R (2002) *Duality of the mind*. Lawrence Erlbaum Associates, Mahwah
- Tversky A, Kahneman D (1992) Advances in prospect theory: cumulative representation of uncertainty. *J Risk Uncert* 5:297–323
- Vartanov AV, Vartanova II (2018) Four-dimensional spherical model of emotion. *Proced Comput Sci* 145:604–610
- Vartanov A, Ivanov V, Vartanova I (2020) Facial expressions and subjective assessments of emotions. *Cogn Syst Res* 59:319–328
- Vincent S, Kovalenko T, Yukalov VI, Sornette D (2016) Calibration of quantum decision theory: aversion to large losses and predictability of probabilistic choices. <http://ssrn.com/abstract=2775279>
- von Neumann J, Morgenstern O (1953) *Theory of games and economic behavior*. Princeton University, Princeton
- Wang L, Liu HY, Liang WL, Zhou TH (2021) Emotional expression analysis based on fine-grade emotion quantification model via social media. In: Pan JS et al (eds) *Advances in intelligent information hiding and multimedia signal processing*. Springer, Singapore, pp 211–218
- Yukalov VI, Sornette D (2008) Quantum decision theory as quantum theory of measurement. *Phys Lett A* 372:6867–6871
- Yukalov VI, Sornette D (2009) Scheme of thinking quantum systems. *Laser Phys Lett* 6:833–839
- Yukalov VI, Sornette D (2009) Physics of risk and uncertainty in quantum decision making. *Eur Phys J B* 71:533–548
- Yukalov VI, Sornette D (2011) Decision theory with prospect interference and entanglement. *Theory Decis* 70:283–328
- Yukalov VI, Sornette D (2014) Manipulating decision making of typical agents. *IEEE Trans Syst Man Cybern Syst* 44:1155–1168
- Yukalov VI, Sornette D (2016) Quantum probability and quantum decision making. *Philos Trans R Soc A* 374:20150100
- Yukalov VI, Sornette D (2018) Quantitative predictions in quantum decision theory. *IEEE Trans Syst Man Cybern Syst* 48:366–381
- Yukalov VI (2020) Evolutionary processes in quantum decision theory. *Entropy* 22:681
- Yukalov VI (2021) Tossing quantum coins and dice. *Laser Phys* 31:055201
- Yukalov VI (2021) A resolution of St. Petersburg paradox. *J Math Econ*. <https://doi.org/10.1016/j.jmateco.2021.102537>
- Zafirovski M (2012) Beneath rational choice: elements of irrational choice theory. *Curr Sociol* 61:3–21
- Zhang C, Kjellström H (2021) A subjective model of human decision making based on quantum decision theory. [arXiv: 2101.05851](https://arxiv.org/abs/2101.05851)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.