



Improvement of the approximations and accuracy measure of a rough set using somewhere dense sets

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Abstract

Rough set theory is a non-statistical approach to handle uncertainty and uncertain knowledge. It is characterized by two methods called classification (lower and upper approximations) and accuracy measure. The closeness of notions and results in topology and rough set theory motivates researchers to explore the topological aspects and their applications in rough set theory. To contribute to this area, this paper applies a topological concept called “somewhere dense sets” to improve the approximations and accuracy measure in rough set theory. We firstly discuss further topological properties of somewhere dense and cs -dense sets and give explicitly formulations to calculate S -interior and S -closure operators. Then, we utilize these two sets to define new concepts in rough set context such as SD -lower and SD -upper approximations, SD -boundary region, and SD -accuracy measure of a subset. We establish the fundamental properties of these concepts as well as show their relationships with the previous ones. In the end, we compare the current method of approximations with the previous ones and provide two examples to elucidate that the current method is more accurate.

Keywords Somewhere dense set · Lower and upper approximations · Accuracy measure · Interior and closure operators

1 Introduction

In a simple expression, the phenomenon or problem is vague if it contains some elements or components which we cannot decide where are they? inside or outside. Rough set theory initiated by Pawlak (1982, 1991) is one of the followed techniques to deal with vagueness (uncertainty) of information systems data and imperfect knowledge. This theory starts from an equivalence relation that is the basis to define the concepts of lower and upper approximations. However, an equivalence relation seems a very strict condition that limits the applications of the rough set theory. This leads to study the approximation operators using specific kinds of binary relations (Abo-Tabl 2013) or arbitrary binary relations by many authors (El-Bably and Al-shami 2021).

The boundary region is calculated as the difference between the upper and lower approximations, and according to the boundary region is empty or not, the sets are classified to a crisp (exact) set or a rough (inexact) set. As we know, the

increase in boundary region hampers appropriate decision-making; therefore, we mainly aim to reduce the boundary region by decreasing the upper approximation and increasing the lower approximation. Another important concept of the rough set theory is the accuracy measure of a set which expresses the degree of completeness of our knowledge. The accuracy measure of a set shows how large the boundary region is; however, it does not say anything of the structure of the boundary region. On the other hand, the approximations provide some insights into the structure of the boundary region without information of its size.

The concepts in rough set theory such as membership of elements and inclusion and equality relations cannot be determined in the absolute sense, but on the information which we know about them. For example, the sets with different elements can be equal in the rough sense if they have the same lower and/or upper approximations. This thought refers to the topological structure when we compare between sets in terms of their interior and closure points, not to the elements they consist of. Skowron (1988) and Wiweger (1989) first investigated the role of topological aspects in rough sets. Lashin et al. (2005) introduced a topology using binary relations and applied to generalize the basic rough set concepts. One of the good achievements of solving the missing attribute

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values problem by using the topological notions was given in Salama (2010). In fact, a combination of rough set theory and topological theory became the main goal of many studies, for example the interaction between rough sets and digital topology Abo-Tabl (2014), separation axioms of approximation spaces (Al-shami et al. 2021), approximations generated from rough sets and ideals (Hosny et al. 2021), topological properties of rough approximations (Kondo and Dudek 2006; Kozae et al. 2007), continuous functions between approximation spaces (Salama et al. 2021), rough set and topological algebra (Wu et al. 2008), topological methods induced from covering rough sets (Zhu 2007). This interaction also included some generalizations of topology such as minimal structure (Azzam et al. 2020) and bitopology (Salama 2020b). Some interesting applications to supply chain management were given in Chen et al. (2021), Wang et al. (2020), Xiao et al. (2021).

The notion of near open sets (it also called generalized open set) is a major area of research in the classical topology. They are utilized to initiate a wider classes of topological structures and extended some topological concepts such as compact and connected spaces and separation axioms. The most celebrated types of near open sets are those formulated by using the closure and interior operators such as α -open, pre-open, semi-open, b -open and β -open sets. Quite recently, Al-shami (2017) investigated a wider class of near open sets, namely somewhere dense sets, and then he with Al-shami and Noiri (2019) applied to define new types of continuous maps.

In 2014, El-Monsef et al. (2014) exploited some near open sets to define new types of approximations in the case of fore-set and after-set. Amer et al. (2017) and Salama (2018) generalized these approximations and presented several types of j -near approximations. These approximations consider a vital methods to reduces the boundary region using topological concepts. In 2018, Hosny (2018) defined new four approximations by use of the concepts of $\delta\beta$ -open sets and \bigwedge_β -sets. Recently, Salama (2020a) defined the concept of higher order sets as a new class of new types of near open and closed sets. He initiated this class by much iteration of topological closure and interior operations for a given set. This manuscript contributes to this direction by making use of the concept of somewhere dense sets. We initiate new rough set models and present some comparisons which illustrate that our approach produces approximations and accuracy measures better than those given in El-Monsef et al. (2014), Amer et al. (2017). Moreover, we prove that our approach keeps most properties of Pawlak’s approximations as illustrated in Proposition 7 and 8.

The remainder of this manuscript are organized as follows. In Sect. 2, we recall some definitions and properties of rough sets and topological spaces that help the reader to well understand this manuscript. In Sect. 3, we explore more

properties of somewhere dense and cs -dense sets. In Sect. 4, we apply somewhere dense sets to establish new kinds of approximation operators. We scrutinize the main properties of these approximation operators, and demonstrate that they produce accuracy measures greater than that of Amer et al. (2017). Finally, we give some conclusions and make a plan for future works in Sect. 5.

2 Preliminaries

Now, we proceed to give the main notions in rough set theory. We begin with the fundamental concepts in this field that owes to the idea of classifying a set by another two sets.

Definition 1 (Pawlak 1982, 1991) Let \mathcal{R} be an equivalence relation (i.e., reflexive, symmetric and transitive) on a finite set $E \neq \emptyset$. We associate every $X \subseteq E$ with two subsets.

$$\underline{\mathcal{R}}(X) = \bigcup \{A \in E/\mathcal{R} : A \subseteq X\}, \text{ and}$$

$$\overline{\mathcal{R}}(X) = \bigcup \{A \in E/\mathcal{R} : A \cap X \neq \emptyset\}$$

The two sets $\underline{\mathcal{R}}(X)$ and $\overline{\mathcal{R}}(X)$ are, respectively, called lower and upper approximations of X .

From now on, we consider E a nonempty finite set, unless stated otherwise.

Proposition 1 (Pawlak 1982, 1991) Let \mathcal{R} be an equivalence relation on E and $U, V \subseteq E$. Then, the following properties hold.

- | | |
|---|--|
| (L1) $\underline{\mathcal{R}}(U) \subseteq U$ | (U1) $U \subseteq \overline{\mathcal{R}}(U)$ |
| (L2) $\underline{\mathcal{R}}(\emptyset) = \emptyset$ | (U2) $\overline{\mathcal{R}}(\emptyset) = \emptyset$ |
| (L3) $\underline{\mathcal{R}}(E) = E$ | (U3) $\overline{\mathcal{R}}(E) = E$ |
| (L4) If $U \subseteq V$, | (U4) If $U \subseteq V$, |
| then $\underline{\mathcal{R}}(U) \subseteq \underline{\mathcal{R}}(V)$ | then $\overline{\mathcal{R}}(U) \subseteq \overline{\mathcal{R}}(V)$ |
| (L5) $\underline{\mathcal{R}}(U \cap V) = \underline{\mathcal{R}}(U) \cap \underline{\mathcal{R}}(V)$ | (U5) $\overline{\mathcal{R}}(U \cap V) \subseteq \overline{\mathcal{R}}(U) \cap \overline{\mathcal{R}}(V)$ |
| (L6) $\underline{\mathcal{R}}(U) \cup \underline{\mathcal{R}}(V) \subseteq \underline{\mathcal{R}}(U \cup V)$ | (U6) $\overline{\mathcal{R}}(U \cup V) = \overline{\mathcal{R}}(U) \cup \overline{\mathcal{R}}(V)$ |
| (L7) $\underline{\mathcal{R}}(U^c) = (\overline{\mathcal{R}}(U))^c$ | (U7) $\overline{\mathcal{R}}(U^c) = (\underline{\mathcal{R}}(U))^c$ |
| (L8) $\underline{\mathcal{R}}(\underline{\mathcal{R}}(U)) = \underline{\mathcal{R}}(U)$ | (U8) $\overline{\mathcal{R}}(\overline{\mathcal{R}}(U)) = \overline{\mathcal{R}}(U)$ |
| (L9) $\underline{\mathcal{R}}((\underline{\mathcal{R}}(U))^c) = (\overline{\mathcal{R}}(U))^c$ | (U9) $\overline{\mathcal{R}}((\overline{\mathcal{R}}(U))^c) = (\underline{\mathcal{R}}(U))^c$ |
| (L10) $\forall K \in E/\mathcal{R} \Rightarrow \underline{\mathcal{R}}(K) = K$ | (U10) $\forall K \in E/\mathcal{R} \Rightarrow \overline{\mathcal{R}}(K) = K$ |

As we know that the core of Pawlak approach is the equivalence relation which makes some problems and limitations for theoretical and practical aspects. Therefore, many proposals have been made to overcome these obstacles and generalize this approach. One of them is idea of N_j -neighborhoods which were defined for any binary relation as follows.

Definition 2 (El-Monsef et al. 2014; Allam et al. 2005, 2006; Yao 1996, 1998) Let \mathcal{R} be a binary relation on E . The j -neighborhoods of $v \in E$ (denoted by $N_j(v)$) are defined for each $j \in \{r, l, < r \rangle, < l \rangle, i, u, < i \rangle, < u \rangle\}$ as follows:

- (i) $N_r(v) = \{w \in E : v\mathcal{R}w\}$.
- (ii) $N_l(v) = \{w \in E : w\mathcal{R}v\}$.
- (iii)
$$N_{\langle r \rangle}(v) = \begin{cases} \bigcap_{v \in N_r(w)} N_r(w) & : \text{there exists } N_r(w) \\ & \text{containing } v \\ \emptyset & : \text{Otherwise} \end{cases}$$
- (iv)
$$N_{\langle l \rangle}(v) = \begin{cases} \bigcap_{v \in N_l(w)} N_l(w) & : \text{there exists } N_l(w) \\ & \text{containing } v \\ \emptyset & : \text{Otherwise} \end{cases}$$
- (v) $N_i(v) = N_r(v) \cap N_l(v)$.
- (vi) $N_u(v) = N_r(v) \cup N_l(v)$.
- (vii) $N_{\langle i \rangle}(v) = N_{\langle r \rangle}(v) \cap N_{\langle l \rangle}(v)$.
- (viii) $N_{\langle u \rangle}(v) = N_{\langle r \rangle}(v) \cup N_{\langle l \rangle}(v)$.

From now on, we consider $j \in \{r, l, \langle r \rangle, \langle l \rangle, i, u, \langle i \rangle, \langle u \rangle\}$, unless stated otherwise.

Definition 3 Let \mathcal{R} be a binary relation on E and $\phi_j : E \rightarrow 2^E$ be a map which associates each $v \in E$ with its j -neighborhood in 2^E . The triple (E, \mathcal{R}, ϕ_j) is called a j -neighborhood space (in short, j -NS).

Remark 1 Neighborhoods system originally comes from topology; study rough sets in view of different kinds of neighborhoods system attracted the attention of several researchers. In this regard, many types of neighborhood systems were defined and discussed such as containment neighborhoods (Al-shami 2021), E_j -neighborhoods (Al-shami et al. 2021), core neighborhoods (Mareay 2016), and remote neighborhood (Sun et al. 2019).

Recall that a topology on a nonempty set E is a family of subsets of E which is closed under arbitrary union and finite intersection. Some authors do not approve writing a condition of belonging of the universal and empty sets to topology's conditions because the universal set comes from the empty intersection, and the empty set comes from the empty union.

If every open subset of a topological space is also closed, then a topology is called a clopen topology. A topology is called disconnected if it can be written as a union of two disjoint open sets except for the empty and universal sets, and

a topology is called extremally disconnected if the closure of any open set is open.

The result below explains one of the most important and interesting methods of generating topological structures by use of the notion of neighborhood systems; also, it allows a wide interaction between the concepts of rough set theory and topological space.

Theorem 1 (El-Monsef et al. 2014) If (E, \mathcal{R}, ϕ_j) is a j -NS, then a class $\theta_j = \{U \subseteq E : N_j(v) \subseteq U \text{ for each } v \in U\}$ is a topology on E for each j .

Definition 4 A subset A of a j -NS (E, \mathcal{R}, ϕ_j) is called a j -open set if $A \in \theta_j$. Its complement is called a j -closed set.

We denote the family of all j -closed sets by Γ_j . That is, $\Gamma_j = \{V \subseteq E : V^c \in \theta_j\}$.

The following two approximations are formulated with a topological flavor.

Definition 5 (El-Monsef et al. 2014) The j -lower and j -upper approximations of a subset A of a j -NS (E, \mathcal{R}, ϕ_j) are formulated, respectively, by

$$\underline{\mathcal{R}}_j(A) = \bigcup \{U \in \theta_j : U \subseteq A\}, \text{ and}$$

$$\overline{\mathcal{R}}_j(A) = \bigcap \{V \in \Gamma_j : A \subseteq V\}$$

One easily note that $\underline{\mathcal{R}}_j(A)$ and $\overline{\mathcal{R}}_j(A)$ are, respectively, the interior and closure points of A in a topological space (E, θ_j) ; hence, we sometimes write $\underline{\mathcal{R}}_j(A) = int_j(A)$ and $\overline{\mathcal{R}}_j(A) = cl_j(A)$, where int_j and cl_j denote the interior and closure operators in a topological space (E, θ_j) , and j denotes a type of neighborhood given in 2 that generating a topology θ_j on E .

Definition 6 (El-Monsef et al. 2014) The j -boundary, j -positive and j -negative regions and j -accuracy measure of a subset A of a j -NS (E, \mathcal{R}, ϕ_j) are formulated, respectively, by

$$B_j(A) = \overline{\mathcal{R}}_j(A) \setminus \underline{\mathcal{R}}_j(A),$$

$$POS_j(A) = \underline{\mathcal{R}}_j(A),$$

$$NEG_j(A) = E \setminus \overline{\mathcal{R}}_j(A), \text{ and}$$

$$M_j(A) = \frac{|\underline{\mathcal{R}}_j(A)|}{|\overline{\mathcal{R}}_j(A)|} \text{ provided that } \overline{\mathcal{R}}_j(A) \neq \emptyset.$$

It is clear that $M_j(A)$ is greater or equal to zero and less or equal to one for each $A \subseteq E$.

In 1963, Levine (1963) defined the concept of semiopen set which represents a wider class of open sets. After that, topologists initiated many wider classes of open sets, the most celebrated types of them are mentioned in the definition below and Definition 11; each one of these classes is called a

class of near open sets. Many topological concepts, notions and properties were presented and discussed again using near open sets which make a central area of topological researches. Note that near open sets were formulated by use of the closure and interior operators.

Definition 7 A subset A of a topological space (E, ζ) is said to be:

- (i) semiopen (Levine 1963) if $A \subseteq cl(int(A))$; Equivalently, there is an open set U such that $U \subseteq A \subseteq cl(U)$.
- (ii) preopen (Mashhour et al. 1982) if $A \subseteq int(cl(A))$; Equivalently, there is an open set U such that $A \subseteq U \subseteq cl(A)$.
- (iii) α -open (Njastad 1965) if $A \subseteq int(cl(int(A)))$; Equivalently, if it is both semiopen and preopen set.
- (iv) b -open (Andrijevic 1996) if $A \subseteq int(cl(A)) \cup cl(int(A))$ (it is also called γ -open set).
- (v) β -open (El-Monsef et al. 1983) if $A \subseteq cl(int(cl(A)))$ (sometime, it is called semi preopen set).

The complement of the above sets is, respectively, called semiclosed, preclosed, α -closed, b -closed and β -closed sets.

In a similar way, the above near open sets were formulated and studied in a j -NS.

Definition 8 (El-Monsef et al. 2014) A subset A of a j -NS (E, \mathcal{R}, ϕ_j) is said to be $j\alpha$ -open (resp. j -semiopen, j -preopen, jb -open, $j\beta$ -open) if $A \subseteq int_j(cl_j(int_j(A)))$ (resp. $A \subseteq cl_j(int_j(A))$, $A \subseteq int_j(cl(A))$, $A \subseteq int_j(cl_j(A)) \cup cl_j(int_j(A))$, $A \subseteq cl_j(int_j(cl_j(A)))$).

The complement of A is called $j\alpha$ -closed (resp. j -semiclosed, j -preclosed, jb -closed, $j\beta$ -closed) sets.

- Remark 2** (i) The families of $j\alpha$ -open, j -semiopen, j -preopen, jb -open and $j\beta$ -open subsets of (E, θ_j) are, respectively, denoted by $\alpha O(\theta_j)$, $semi O(\theta_j)$, $pre O(\theta_j)$, $bO(\theta_j)$ and $\beta O(\theta_j)$
- (ii) The families of $j\alpha$ -closed, j -semiclosed, j -preclosed, jb -closed and $j\beta$ -closed subsets of (E, θ_j) are, respectively, denoted by $\alpha C(\Gamma_j)$, $semi C(\Gamma_j)$, $pre C(\Gamma_j)$, $bC(\Gamma_j)$ and $\beta C(\Gamma_j)$

Similarly to Definition 5, the authors of Amer et al. (2017), Salama (2018) introduced the concepts of j -near lower and j -near upper approximations of a subset using $j\alpha$ -open and $j\alpha$ -closed, j -semiopen and j -semiclosed, j -preopen and j -preclosed, jb -open and jb -closed, and $j\beta$ -open and $j\beta$ -closed sets; also, they formulated the concepts of j -near boundary, j -near positive and j -near negative regions and j -near accuracy measure.

Definition 9 (Amer et al. 2017; Salama 2018) For each $k \in \{\alpha, semi, pre, b, \beta\}$, the jk -near lower and jk -near upper

approximations of a subset A of a j -NS (E, \mathcal{R}, ϕ_j) are formulated, respectively, by

$$\underline{\mathcal{R}}_j^k(A) = \bigcup \{U \in kO(\theta_j) : U \subseteq A\} = k - int_j(A), \text{ and}$$

$$\overline{\mathcal{R}}_j^k(A) = \bigcap \{V \in kC(\Gamma_j) : A \subseteq V\} = k - cl_j(A)$$

From now on, we consider $k \in \{\alpha, semi, pre, \beta, b\}$, unless stated otherwise.

Definition 10 (Amer et al. 2017; Salama 2018) The jk -near boundary, jk -near positive and jk -near negative regions and jk -near accuracy measure of a subset A of a j -NS (E, \mathcal{R}, ϕ_j) are formulated, respectively, by

$$B_j^k(A) = \overline{\mathcal{R}}_j^k(A) \setminus \underline{\mathcal{R}}_j^k(A),$$

$$POS_j^k(A) = \underline{\mathcal{R}}_j^k(A),$$

$$NEG_j^k(A) = E \setminus \overline{\mathcal{R}}_j^k(A), \text{ and}$$

$$M_j^k(A) = \frac{|\underline{\mathcal{R}}_j^k(A)|}{|\overline{\mathcal{R}}_j^k(A)|} \text{ provided that } \overline{\mathcal{R}}_j^k(A) \neq \emptyset.$$

It is clear that $M_j^k(A)$ is greater or equal to zero and less or equal to one for each $A \subseteq E$.

Definition 11 (Al-shami 2017) A subset A of (X, ζ) is said to be somewhere dense if $int(cl(A)) \neq \emptyset$. Its complement is said to be cs -dense.

Theorem 2 (Al-shami 2017) A subset H of (X, ζ) is cs -dense iff there is a closed set $F \neq X$ such that $int(H) \subseteq F$.

Definition 12 (Al-shami 2017) For a subset A of (X, ζ) , the S -interior of A (briefly, $Sint(A)$) is the union of all somewhere dense sets that are contained in A , the S -closure of A (briefly, $Scl(A)$) is the intersection of all cs -dense sets containing A .

In Al-shami (2017); Al-shami and Noiri (2019), the authors revealed the main properties of somewhere dense and cs -dense sets. They proved that every nonempty α -open (preopen, semi open, b -open and β -open) set is somewhere dense, but the converse need not be true. Also, they demonstrated some unique characteristics of somewhere dense sets which it does not own by the previous ones such that every set in a topological space is somewhere dense or cs -dense, every superset of a somewhere dense set is somewhere dense, and every subset of a cs -dense set is cs -dense. It is worth noting that a family of somewhere dense sets forms a generalized topology, whereas the other families of near open sets forms a supra topology.

Definition 13 (Al-shami 2017) A topological space (X, ζ) is called ST_1 if for every $v \neq w \in X$, ζ contains two somewhere dense sets such that one of them containing v but not w , and the other containing w but not v .

3 Further properties of somewhere dense sets

In this section, we explore more properties of somewhere dense sets which help us to establish some properties of the approximations and accuracy measure of a set in the next section.

Proposition 2 *The S-interior of any subset H of (X, ζ) is given by the following formulation.*

$$Sint(H) = \begin{cases} \emptyset : H \text{ is only cs - dense} \\ H : \text{otherwise} \end{cases}$$

Proof Let H be only a cs-dense set. Suppose that $Sint(H) \neq \emptyset$. Then, $int(H) \neq \emptyset$; therefore, $cl(int(H)) \neq \emptyset$. Thus, H is also a somewhere dense set. This contradicts assumption. Hence, $Sint(H) = \emptyset$, as required. In the remaining two cases, H is somewhere dense; hence, $Sint(H) = H$. \square

Corollary 1 *For any subset H of (X, ζ), we have $Sint[Sint(H)] = Sint(H)$.*

Proof It is clear that $Sint(H) = \emptyset$ implies that $Sint[Sint(H)] = \emptyset$. If $Sint(H) \neq \emptyset$, then it follows from the above proposition that $Sint(H) = H$. Hence, $Sint[Sint(H)] = Sint(H)$, as required. \square

Proposition 3 *The S-closure of any subset H of (X, ζ) is given by the following formulation.*

$$Scl(H) = \begin{cases} X : H \text{ is only somewhere dense} \\ H : \text{otherwise} \end{cases}$$

Proof Let H be only a somewhere dense set. Then, $H \neq \emptyset$. Suppose that $Scl(H) = F \neq X$. Therefore, F is a cs-dense set containing H. This implies that H is also cs-dense, but this contradicts assumption. Hence, $Scl(H) = X$, as required. In the remaining two cases, H is cs-dense; hence, $Scl(H) = H$. \square

Corollary 2 *For any subset H of (X, ζ), we have $Scl[Scl(H)] = Scl(H)$.*

Proof If $Scl(H) = X$, then $Scl[Scl(H)] = X$. If $Scl(H) \neq X$, then $Scl(H)$ is a proper cs-dense subset of (X, ζ). It follows from the above proposition that $Scl[Scl(H)] = Scl(H)$, as required. \square

Proposition 4 *Every singleton subset of a disconnected topological space is a cs-dense set.*

Proof Let (X, ζ) be disconnected. Then, there exists a proper nonempty subset H of X which is both open and closed. This means that there exist proper closed subsets M and N of X such that $int(H) \subseteq M$ and $int(H^c) \subseteq N$. Therefore, for each $v \in X$ either $v \in H$ or $v \in H^c$. Thus, $int(\{v\}) \subseteq M$ or $int(\{v\}) \subseteq N$. Hence, $\{v\}$ is a cs-dense set. \square

Corollary 3 *Every disconnected topological space is an ST_1 -space.*

Proof Let $v \neq w$ in a disconnected topological space (X, ζ). Then, $\{v\}$ and $\{w\}$ are cs-dense sets. Obviously, $\{v\}^c$ and $\{w\}^c$ are somewhere dense sets containing w and v, respectively, such that $v \notin \{v\}^c$ and $w \notin \{w\}^c$. Hence, the proof is complete. \square

Theorem 3 *If (U, ζ_U) is an open subspace of (X, ζ) and $F \subseteq X$, then $U \cap cl(F) = cl_U(U \cap F)$, where $cl_U(U \cap F)$ is the closure of $U \cap F$ in the subspace (U, ζ_U).*

Proof For any $v \notin U \cap cl(F)$, one of the following cases holds.

- (i) Either $v \notin U$. This directly leads to that $v \notin cl_U(U \cap F)$.
- (ii) Or $v \in U$. In this case, note that $v \notin cl(F)$. Consequently, $G \cap F = \emptyset$ for some open subsets G of (X, ζ) containing v. It is clear that $G \cap U$ is an open set containing v such that $(G \cap U) \cap (F \cap U) = \emptyset$; thus, $v \notin cl_U(U \cap F)$.

Hence, $cl_U(U \cap F) \subseteq U \cap cl(F)$.

On the other hand, let $v \notin cl_U(U \cap F)$. Then, a subspace (U, ζ_U) contains an open set G such that $v \in G$ and $G \cap (U \cap F) = \emptyset$. By hypotheses, U is an open subset of X; therefore, G is also an open subset of (X, ζ). It is clear that $(G \cap U) \cap F = \emptyset$. Then, $v \notin U \cap cl(F)$. Thus, $U \cap cl(F) \subseteq cl_U(U \cap F)$. Hence, we obtain the desired result. \square

Corollary 4 *Let U be a dense subset of (X, ζ) containing H. Then, H is a somewhere dense subset of (X, ζ) iff H is a somewhere dense subset of (U, ζ_U).*

Proof To prove the necessary part, let H be a somewhere dense subset of (X, ζ). Then, there exists a nonempty open set G such that $G \subseteq cl(H)$. Now, there exists an open set $F \in \zeta_U$ such that $F = G \cap U \subseteq cl(H) \cap U \subseteq cl_U(H)$. Since U is dense, then $F \neq \emptyset$. Thus, H is a somewhere dense subset of (U, ζ_U).

One can prove the sufficient part using a similar technique. \square

Proposition 5 *The families of α-open and semi-open subsets of an extremally disconnected topological space are identical.*

Proof It is well known that every α-open set is semi-open. To prove the converse, let H be a semi-open set. Then, $H \subseteq cl(int(H))$. By hypothesis of extremally disconnectedness, $cl(int(H))$ is an open set; therefore, $int(cl(int(H))) = cl(int(H))$. Thus, $H \subseteq int(cl(int(H)))$, and this means that H is an α-open set. Hence, the proof is finished. \square

Proposition 6 *The following properties hold in a finite clopen topological space.*

- (i) *The family of preopen (b-open, β -open and somewhere dense) set is the power set of the universal set.*
- (ii) *The families of open and semi-open sets are identical.*

Proof (i): It is clear that $cl(H)$ is a closed set for every subset H of (E, θ_j) . By hypothesis of a clopen topology $int(cl(H)) = cl(H) \supseteq H$; therefore, $H \subseteq int(cl(H))$. This means that H is a preopen set. Thus, the collection of preopen sets is $P(E)$. It is well known that: pre-open set $\implies \beta$ -open set \implies somewhere dense set. This implies that the families of b -open, β -open and somewhere dense set are also $P(E)$. Hence, the proof is finished.

(ii): It is obvious. □

4 Approximations using somewhere dense sets

In this section, we employ the concepts of somewhere dense and cs -dense sets to define new approximations in rough set content, namely jSD -near lower and jSD -near upper approximations. Based on them, we introduce the concepts of jSD -near boundary, jSD -near positive and jSD -near negative regions and jSD -near accuracy measure. We prove their main properties with the help of some examples. We complete this section by comparing the new approximations with the previous ones and showing the importance of the current approximations in improving the accuracy measure.

Definition 14 A subset A of a j -NS (E, \mathcal{R}, ϕ_j) is said to be j -somewhere dense if $int_j(cl_j(A)) \neq \emptyset$. The complement of A is called jcs -dense.

The families of j -somewhere dense and jcs -dense sets are, respectively, denoted by $SD(\theta_j)$ and $CS(\Gamma_j)$.

Definition 15 The jSD -near lower and jSD -near upper approximations of a subset A of a j -NS (E, \mathcal{R}, ϕ_j) are formulated, respectively, by

$$\underline{\mathcal{R}}_j^{SD}(A) = \bigcup \{U \in SD(\theta_j) : U \subseteq A\} = Sint_j(A), \text{ and}$$

$$\overline{\mathcal{R}}_j^{SD}(A) = \bigcap \{V \in CS(\Gamma_j) : A \subseteq V\} = Scl_j(A)$$

The following two results present the main features of the above two approximations.

Proposition 7 *Let (E, \mathcal{R}, ϕ_j) be a j -NS and $X, Y \subseteq E$. Then, the following properties hold.*

- (i) $\underline{\mathcal{R}}_j^{SD}(X) \subseteq X$.

- (ii) $\underline{\mathcal{R}}_j^{SD}(\emptyset) = \emptyset$.
- (iii) $\underline{\mathcal{R}}_j^{SD}(E) = E$.
- (iv) If $X \subseteq Y$, then $\underline{\mathcal{R}}_j^{SD}(X) \subseteq \underline{\mathcal{R}}_j^{SD}(Y)$.
- (v) $\underline{\mathcal{R}}_j^{SD}(X \cap Y) \subseteq \underline{\mathcal{R}}_j^{SD}(X) \cap \underline{\mathcal{R}}_j^{SD}(Y)$.
- (vi) $\underline{\mathcal{R}}_j^{SD}(X) \cup \underline{\mathcal{R}}_j^{SD}(Y) \subseteq \underline{\mathcal{R}}_j^{SD}(X \cup Y)$.
- (vii) $\underline{\mathcal{R}}_j^{SD}(X^c) = (\overline{\mathcal{R}}_j^{SD}(X))^c$.
- (viii) $\underline{\mathcal{R}}_j^{SD}(\underline{\mathcal{R}}_j^{SD}(X)) = \underline{\mathcal{R}}_j^{SD}(X)$.

Proof (i) According to Definition 15, we obtain $\underline{\mathcal{R}}_j^{SD}(X) = \bigcup \{U \in SD(\theta_j) : U \subseteq X\} \subseteq X$.

(ii) We know that \emptyset is not a somewhere dense set, also, $\bigcup_{i \in I} A_i = \emptyset$, where I is an empty index set. So that $\underline{\mathcal{R}}_j^{SD}(\emptyset) = \emptyset$, as required.

(iii) Since E is a somewhere dense set, $E \subseteq \underline{\mathcal{R}}_j^{SD}(E)$. It follows from (i) above that $\underline{\mathcal{R}}_j^{SD}(E) \subseteq E$. Hence, $\underline{\mathcal{R}}_j^{SD}(E) = E$.

(iv) Since $X \subseteq Y$, we obtain $\bigcup \{U \in SD(\theta_j) : U \subseteq X\} \subseteq \bigcup \{U \in SD(\theta_j) : U \subseteq Y\}$. Therefore, $\underline{\mathcal{R}}_j^{SD}(X) \subseteq \underline{\mathcal{R}}_j^{SD}(Y)$.

(v) Since $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$, it follows from (iv) above that $\underline{\mathcal{R}}_j^{SD}(X \cap Y) \subseteq \underline{\mathcal{R}}_j^{SD}(X)$ and $\underline{\mathcal{R}}_j^{SD}(X \cap Y) \subseteq \underline{\mathcal{R}}_j^{SD}(Y)$. Hence, we obtain the desired result.

(vi) One can prove it following similar arguments given in (v).

(vii) $\underline{\mathcal{R}}_j^{SD}(X^c) = \bigcup \{U \in SD(\theta_j) : U \subseteq X^c\} = (\bigcap \{U^c \in CS(\Gamma_j) : X \subseteq U^c\})^c = (\overline{\mathcal{R}}_j^{SD}(X))^c$.

(viii) It follows from (i) and (iv) above that $\underline{\mathcal{R}}_j^{SD}(\underline{\mathcal{R}}_j^{SD}(X)) \subseteq \underline{\mathcal{R}}_j^{SD}(X)$. On the other hand, let $v \in \underline{\mathcal{R}}_j^{SD}(X)$. Then, there is a $U \in SD(\theta_j)$ such that $v \in U \subseteq X$. This implies that $U \subseteq \underline{\mathcal{R}}_j^{SD}(X)$. Therefore, $v \in \underline{\mathcal{R}}_j^{SD}(\underline{\mathcal{R}}_j^{SD}(X))$. Hence, the proof is complete. □

To elucidate that the inclusion relations are proper in the properties (i) and (iv-vi) of the above proposition, we provide the following example.

Example 1 Consider a j -NS (E, \mathcal{R}, ϕ_j) is the same as given in Example 5. In the case of $j = r$ (one can similarly check the other cases of j), we have $\theta_r = \{\emptyset, E, \{y\}, \{x, y\}, \{v, w\}, \{v, w, y\}\}$ and $\Gamma_r = \{\emptyset, E, \{x\}, \{x, y\}, \{v, w\}, \{v, w, x\}\}$. Let $V = \{v, x\}$, $W = \{v, w\}$, $X = \{x, y\}$ and $Y = \{x, y\}$. By calculation, we obtain $\underline{\mathcal{R}}_r^{SD}(V) = V$, $\underline{\mathcal{R}}_r^{SD}(W) = W$, $\underline{\mathcal{R}}_r^{SD}(X) = \emptyset$ and $\underline{\mathcal{R}}_r^{SD}(Y) = Y$. Now, we note the following.

- (i) $\underline{\mathcal{R}}_r^{SD}(X) \not\subseteq X$.
- (ii) $\underline{\mathcal{R}}_r^{SD}(X) \subseteq \underline{\mathcal{R}}_r^{SD}(W)$, but $X \not\subseteq W$.
- (iii) $\underline{\mathcal{R}}_r^{SD}(V) \cap \underline{\mathcal{R}}_r^{SD}(Y) = \{x\} \not\subseteq \underline{\mathcal{R}}_r^{SD}(V \cap Y) = \emptyset$.

$$(iv) \mathcal{R}_r^{SD}(W \cup X) = \{v, w, x\} \not\subseteq \mathcal{R}_r^{SD}(W) \cup \mathcal{R}_r^{SD}(X) = \{v, w\}.$$

Proposition 8 Let (E, \mathcal{R}, ϕ_j) be a j -NS and $X, Y \subseteq E$. Then, the following properties hold.

- (i) $X \subseteq \overline{\mathcal{R}}_j^{SD}(X)$.
- (ii) $\overline{\mathcal{R}}_j^{SD}(\emptyset) = \emptyset$.
- (iii) $\overline{\mathcal{R}}_j^{SD}(E) = E$.
- (iv) If $X \subseteq Y$, then $\overline{\mathcal{R}}_j^{SD}(X) \subseteq \overline{\mathcal{R}}_j^{SD}(Y)$.
- (v) $\overline{\mathcal{R}}_j^{SD}(X \cap Y) \subseteq \overline{\mathcal{R}}_j^{SD}(X) \cap \overline{\mathcal{R}}_j^{SD}(Y)$.
- (vi) $\overline{\mathcal{R}}_j^{SD}(X) \cup \overline{\mathcal{R}}_j^{SD}(Y) \subseteq \overline{\mathcal{R}}_j^{SD}(X \cup Y)$.
- (vii) $\overline{\mathcal{R}}_j^{SD}(X^c) = (\overline{\mathcal{R}}_j^{SD}(X))^c$.
- (viii) $\overline{\mathcal{R}}_j^{SD}(\overline{\mathcal{R}}_j^{SD}(X)) = \overline{\mathcal{R}}_j^{SD}(X)$.

Proof Following similar technique given in the proof of Proposition 7. □

To elucidate that the inclusion relations are proper in the properties (i) and (iv-vi) of the above proposition, we provide the following example.

Example 2 Let $V = \{x, y\}$, $W = \{v, w\}$, $X = \{y\}$ and $Y = \{v, w, y\}$ be subsets of a j -NS (E, \mathcal{R}, ϕ_j) which given in Example 5. By calculation, we obtain $\overline{\mathcal{R}}_r^{SD}(V) = V$, $\overline{\mathcal{R}}_r^{SD}(W) = W$, $\overline{\mathcal{R}}_r^{SD}(X) = X$ and $\overline{\mathcal{R}}_r^{SD}(Y) = E$. Now, we note the following.

- (i) $Y \not\subseteq \overline{\mathcal{R}}_r^{SD}(Y)$.
- (ii) $\overline{\mathcal{R}}_r^{SD}(V) \subseteq \overline{\mathcal{R}}_r^{SD}(Y)$, but $V \not\subseteq Y$.
- (iii) $\overline{\mathcal{R}}_r^{SD}(V) \cap \overline{\mathcal{R}}_r^{SD}(Y) = V \not\subseteq \mathcal{R}_r^{SD}(V \cap Y) = \{y\}$.
- (iv) $\overline{\mathcal{R}}_r^{SD}(W \cup X) = E \not\subseteq \mathcal{R}_r^{SD}(W) \cup \mathcal{R}_r^{SD}(X) = \{v, w, x\}$.

Remark 3 The equality relations hold in the properties (v) and (vi) which, respectively, given in Proposition 7 and Proposition 8 if (E, θ_j) is strongly hyperconnected (i.e., a set is dense \iff it is nonempty open).

The next result is the key point to improve the accuracy measure by use of the concept of near open sets; it also explains the grades of accuracy measure according to the different types of near open sets.

Theorem 4 Let (E, \mathcal{R}, ϕ_j) be a j -NS and $X \subseteq E$. Then,

$$\underline{\mathcal{R}}_j(X) \subseteq \underline{\mathcal{R}}_j^k(X) \subseteq \underline{\mathcal{R}}_j^{SD}(X) \subseteq X \subseteq \overline{\mathcal{R}}_j^{SD}(X) \subseteq \overline{\mathcal{R}}_j^k(X) \subseteq \overline{\mathcal{R}}_j(X).$$

Proof It is well known that the collection of k -open subsets of (X, θ_j) contains a topology θ_j ; therefore, $\underline{\mathcal{R}}_j(X) \subseteq \underline{\mathcal{R}}_j^k(X)$. Also, it was showed in Al-shami (2017) that the collection of somewhere dense subsets of (X, θ_j) contains the collection of k -open subsets of (X, θ_j) for each k ; therefore, $\underline{\mathcal{R}}_j^k(X) \subseteq \underline{\mathcal{R}}_j^{SD}(X)$. It follows from Proposition 7 that $\underline{\mathcal{R}}_j^{SD}(X) \subseteq X$. Hence, $\underline{\mathcal{R}}_j(X) \subseteq \underline{\mathcal{R}}_j^k(X) \subseteq \underline{\mathcal{R}}_j^{SD}(X) \subseteq X$. Following similar arguments, one can prove that $X \subseteq \overline{\mathcal{R}}_j^{SD}(X) \subseteq \overline{\mathcal{R}}_j^k(X) \subseteq \overline{\mathcal{R}}_j(X)$. □

Definition 16 A subset X of a j -NS (E, \mathcal{R}, ϕ_j) is called jSD -exact if $\underline{\mathcal{R}}_j^{SD}(X) = \overline{\mathcal{R}}_j^{SD}(X) = X$. Otherwise, it is called a jSD -rough set.

It is clear that every j -near exact set is jSD -exact, but the converse is not always true. We give the next example to support this result.

Example 3 Let $X = \{v, x, y\}$ be a subset of a j -NS (E, \mathcal{R}, ϕ_r) given in Example 5. On the one hand, we note that $\underline{\mathcal{R}}_r^{SD}(X) = \overline{\mathcal{R}}_r^{SD}(X) = X$. Then, X is an rSD -exact set. On the other hand, $\underline{\mathcal{R}}_r^{semi}(X) = \{x, y\} \neq \overline{\mathcal{R}}_r^{semi}(X) = E$ and $\underline{\mathcal{R}}_r^\alpha(X) = \{x, y\} \neq \overline{\mathcal{R}}_r^\alpha(X) = E$. Then, X is neither an $rsemi$ -exact set nor an $r\alpha$ -exact set.

Definition 17 The jSD -near boundary, jSD -near positive, and jSD -near negative regions and jSD -near accuracy measure of a subset A of a j -NS (E, \mathcal{R}, ϕ_j) are formulated, respectively, by

$$\begin{aligned} B_j^{SD}(A) &= \overline{\mathcal{R}}_j^{SD}(A) \setminus \underline{\mathcal{R}}_j^{SD}(A), \\ POS_j^{SD}(A) &= \underline{\mathcal{R}}_j^{SD}(A), \\ NEG_j^{SD}(A) &= E \setminus \overline{\mathcal{R}}_j^{SD}(A), \text{ and} \\ M_j^{SD}(A) &= \frac{|\underline{\mathcal{R}}_j^{SD}(A)|}{|\overline{\mathcal{R}}_j^{SD}(A)|} \text{ provided that } \overline{\mathcal{R}}_j^{SD}(A) \neq \emptyset. \end{aligned}$$

It is clear that $M_j^{SD}(A)$ is greater or equal to zero and less or equal to one for each $A \subseteq E$.

Proposition 9 Let (E, \mathcal{R}, ϕ_j) be a j -NS and $X \subseteq E$. Then,

- (i) $B_j^{SD}(X) \subseteq B_j^k(X) \subseteq B_j(X)$.
- (ii) $M_j(X) \leq M_j^k(X) \leq M_j^{SD}(X)$.

Proof The proof of (i) immediately follows from Theorem 4.

To prove (ii): Theorem 4 shows that $\underline{\mathcal{R}}_j^k(X) \subseteq \underline{\mathcal{R}}_j^{SD}(X)$ and $\overline{\mathcal{R}}_j^{SD}(X) \subseteq \overline{\mathcal{R}}_j^k(X)$. Then, $|\underline{\mathcal{R}}_j^k(X)| \leq |\underline{\mathcal{R}}_j^{SD}(X)|$ and $|\overline{\mathcal{R}}_j^{SD}(X)| \leq |\overline{\mathcal{R}}_j^k(X)|$. This means that $|\underline{\mathcal{R}}_j^k(X)| \times |\overline{\mathcal{R}}_j^{SD}(X)| \leq |\underline{\mathcal{R}}_j^{SD}(X)| \times |\overline{\mathcal{R}}_j^k(X)|$. By dividing the both

sides by $|\overline{\mathcal{R}}_j^{SD}(X)| \times |\overline{\mathcal{R}}_j^k(X)|$, we obtain the following inequality

$$\frac{|\underline{\mathcal{R}}_j^k(X)|}{|\overline{\mathcal{R}}_j^k(X)|} \leq \frac{|\underline{\mathcal{R}}_j^{SD}(X)|}{|\overline{\mathcal{R}}_j^{SD}(X)|} \tag{1}$$

Following similar arguments, we obtain the following inequality.

$$\frac{|\underline{\mathcal{R}}_j(X)|}{|\overline{\mathcal{R}}_j(X)|} \leq \frac{|\underline{\mathcal{R}}_j^k(X)|}{|\overline{\mathcal{R}}_j^k(X)|} \tag{2}$$

It follows from the two equalities 1 and 2 that

$$\frac{|\underline{\mathcal{R}}_j(X)|}{|\overline{\mathcal{R}}_j(X)|} \leq \frac{|\underline{\mathcal{R}}_j^k(X)|}{|\overline{\mathcal{R}}_j^k(X)|} \leq \frac{|\underline{\mathcal{R}}_j^{SD}(X)|}{|\overline{\mathcal{R}}_j^{SD}(X)|}$$

Hence, we prove the desired result. □

Proposition 10 A subset A of a j -NS (E, \mathcal{R}, ϕ_j) is jSD -exact iff $B_j^{SD}(A) = \emptyset$.

Proof Necessity: Let A be a jSD -exact set. Then, $B_j^{SD}(A) = \overline{\mathcal{R}}_j^{SD}(A) \setminus \underline{\mathcal{R}}_j^{SD}(A) = \overline{\mathcal{R}}_j^{SD}(A) \setminus \overline{\mathcal{R}}_j^{SD}(A) = \emptyset$.

Sufficiency: If $B_j^{SD}(A) = \emptyset$, then $\overline{\mathcal{R}}_j^{SD}(A) \setminus \underline{\mathcal{R}}_j^{SD}(A) = \emptyset$. Since $\underline{\mathcal{R}}_j^{SD}(A) \subseteq \overline{\mathcal{R}}_j^{SD}(A)$, then $\overline{\mathcal{R}}_j^{SD}(A) = \underline{\mathcal{R}}_j^{SD}(A)$. Thus, A is jSD -exact. □

Proposition 11 The SD -near accuracy measure of any subset X of (E, θ_j) is given by the following formulation.

$$M_j^{SD}(X) = \begin{cases} 1 : X \text{ is somewhere dense and } cs - \text{dense} \\ 0 : X \text{ is only } cs - \text{dense} \\ \frac{|X|}{|E|} : X \text{ is only somewhere dense} \end{cases}$$

Proof Let X be a nonempty subset of (E, θ_j) . Then, we have only the following three cases.

Case 1: If X is somewhere dense and cs -dense, then it follows from Proposition 2 and 3 that $Sint(X) = Scl(X) = X$. Therefore,

$$M_j^{SD}(X) = \frac{\underline{\mathcal{R}}_r^{SD}(X)}{\overline{\mathcal{R}}_r^{SD}(X)} = \frac{|Sint(X)|}{|Scl(X)|} = 1.$$

Case 2: If X is only cs -dense, then it follows from Proposition 2 and 3 that $Sint(X) = \emptyset$ and $Scl(X) = X$. Therefore,

$$M_j^{SD}(X) = \frac{\underline{\mathcal{R}}_r^{SD}(X)}{\overline{\mathcal{R}}_r^{SD}(X)} = \frac{|Sint(X)|}{|Scl(X)|} = 0.$$

Case 3: If X is only somewhere dense, then it follows from Proposition 2 and 3 that $Sint(X) = X$ and $Scl(X) = E$. Therefore,

$$M_j^{SD}(X) = \frac{\underline{\mathcal{R}}_r^{SD}(X)}{\overline{\mathcal{R}}_r^{SD}(X)} = \frac{|Sint(X)|}{|Scl(X)|} = \frac{|X|}{|E|}.$$

□

The above property is one of the unique characteristics of somewhere dense sets. To validate this matter, consider the following example in the cases of $k = \alpha$, *semi* and $j = r$.

Example 4 It was showed in Example 3 that $\underline{\mathcal{R}}_r^\alpha(X) = \underline{\mathcal{R}}_r^{semi}(X) = \{x, y\}$ and $\overline{\mathcal{R}}_r^\alpha(X) = \overline{\mathcal{R}}_r^{semi}(X) = E$. Then, $M_r^\alpha(X) = M_r^{semi}(X) = \frac{1}{2} \neq \frac{|X|}{|E|} = \frac{|3|}{|4|}$.

In Amer et al. (2017), Salama (2018), the authors discussed different methods to approximate sets, and they showed that the best one of these approximations is obtained by β -open sets. They justified that by decreasing (or canceling) the boundary regions to a minimum. However, we explain in the following that we can still improve the approximations of a set more than those obtained using β -open sets using the concept of somewhere dense sets.

Example 5 If (E, \mathcal{R}, ϕ_j) is a j -NS, where $\mathcal{R} = \{(v, v), (v, w), (w, v), (x, y)\}$ is a binary relation on $E = \{v, w, x, y\}$. Then,

- (i) $\theta_r = \{\emptyset, E, \{y\}, \{x, y\}, \{v, w\}, \{v, w, y\}\}$.
- (ii) $\theta_l = \{\emptyset, E, \{x\}, \{x, y\}, \{v, w\}, \{v, w, x\}\}$.
- (iii) $\theta_{<r>} = \{\emptyset, E, \{v\}, \{x\}, \{y\}, \{v, x\}, \{v, y\}, \{v, w\}, \{x, y\}, \{v, w, x\}, \{v, w, y\}, \{v, x, y\}\}$.
- (iv) $\theta_{<l>} = \{\emptyset, E, \{v\}, \{x\}, \{y\}, \{v, x\}, \{v, y\}, \{v, w\}, \{x, y\}, \{v, w, x\}, \{v, w, y\}, \{v, x, y\}\}$.
- (v) $\theta_u = \{\emptyset, E, \{v, w\}, \{x, y\}\}$.
- (vi) $\theta_i = \{\emptyset, E, \{x\}, \{y\}, \{x, y\}, \{v, w\}, \{v, w, x\}, \{v, w, y\}\}$.
- (vii) $\theta_{<u>} = \{\emptyset, E, \{v\}, \{x\}, \{y\}, \{v, x\}, \{v, y\}, \{x, y\}, \{v, w\}, \{v, w, x\}, \{v, w, y\}, \{v, x, y\}\}$.
- (viii) $\theta_{<i>} = \{\emptyset, E, \{v\}, \{x\}, \{y\}, \{v, x\}, \{v, y\}, \{x, y\}, \{v, w\}, \{v, w, x\}, \{v, w, y\}, \{v, x, y\}\}$.

It is easy to calculate Γ_j from θ_j for each j as follows.

- (i) $\Gamma_r = \{\emptyset, E, \{x\}, \{x, y\}, \{v, w\}, \{v, w, x\}\}$.
- (ii) $\Gamma_l = \{\emptyset, E, \{y\}, \{x, y\}, \{v, w\}, \{v, w, y\}\}$.
- (iii) $\Gamma_{<r>} = \{\emptyset, E, \{w\}, \{x\}, \{y\}, \{w, y\}, \{w, x\}, \{v, w\}, \{x, y\}, \{v, w, x\}, \{v, w, y\}, \{w, x, y\}\}$.
- (iv) $\Gamma_{<l>} = \{\emptyset, E, \{w\}, \{x\}, \{y\}, \{w, y\}, \{w, x\}, \{v, w\}, \{x, y\}, \{v, w, x\}, \{v, w, y\}, \{w, x, y\}\}$.
- (v) $\Gamma_u = \{\emptyset, E, \{v, w\}, \{x, y\}\}$.
- (vi) $\Gamma_i = \{\emptyset, E, \{x\}, \{y\}, \{x, y\}, \{v, w\}, \{v, w, x\}, \{v, w, y\}, \}$.

Table 1 Comparison between r -approximations, $r\beta$ -near approximations and rSD -near approximations

$P(E)$	θ_j		$\beta O(\theta_r)$		$SD(\theta_r)$	
	θ_r		\mathcal{R}_r^β		\mathcal{R}_r^{SD}	
	$\underline{\mathcal{R}}_r$	$\overline{\mathcal{R}}_r$	$\underline{\mathcal{R}}_r^\beta$	$\overline{\mathcal{R}}_r^\beta$	$\underline{\mathcal{R}}_r^{SD}$	$\overline{\mathcal{R}}_r^{SD}$
{v}	\emptyset	{v, w}	{v}	{v}	{v}	{v}
{w}	\emptyset	{v, w}	{w}	{w}	{w}	{w}
{x}	\emptyset	{x}	\emptyset	{x}	\emptyset	{x}
{y}	{y}	{x, y}	{y}	{x, y}	{y}	{y}
{v, w}	{v, w}	{v, w}	{v, w}	{v, w}	{v, w}	{v, w}
{v, x}	\emptyset	{v, w, x}	{v}	{v, x}	{v, x}	{v, x}
{v, y}	{y}	E	{v, y}	{v, x, y}	{v, y}	{v, y}
{w, x}	\emptyset	{v, w, x}	{w}	{w, x}	{w, x}	{w, x}
{w, y}	{y}	E	{w, y}	{w, x, y}	{w, y}	{w, y}
{x, y}	{x, y}	{x, y}	{x, y}	{x, y}	{x, y}	{x, y}
{v, w, x}	{v, w}	{v, w, x}	{v, w}	{v, w, x}	{v, w, x}	{v, w, x}
{v, w, y}	{v, w, y}	E	{v, w, y}	E	{v, w, y}	E
{v, x, y}	{x, y}	E	{v, x, y}	{v, x, y}	{v, x, y}	{v, x, y}
{w, x, y}	{x, y}	E	{w, x, y}	{w, x, y}	{w, x, y}	{w, x, y}
E	E	E	E	E	E	E
\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset

Bold values indicate the differences between the values

- (vii) $\Gamma_{<u>} = \{\emptyset, E, \{w\}, \{x\}, \{y\}, \{w, x\}, \{w, y\}, \{x, y\}, \{v, w\}, \{v, w, x\}, \{v, w, y\}, \{w, x, y\}\}$.
- (viii) $\Gamma_{<i>} = \{\emptyset, E, \{w\}, \{x\}, \{y\}, \{w, x\}, \{w, y\}, \{x, y\}, \{v, w\}, \{v, w, x\}, \{v, w, y\}, \{w, x, y\}\}$.

Now, it was showed in Al-shami (2017) that every α -open (preopen, semi open, β -open and b -open) set is a β -open set. To demonstrate the efficiency of the new approximations that are based on somewhere dense sets, we suffice by calculating all β -open and β -closed sets for $j = r$.

$$\beta O(\theta_r) = \{\emptyset, E, \{v\}, \{w\}, \{y\}, \{x, y\}, \{v, w\}, \{v, y\}, \{w, y\}, \{v, w, y\}, \{v, x, y\}, \{w, x, y\}\}, \text{ and}$$

$$\beta C(\theta_r) = \{\emptyset, E, \{v\}, \{w\}, \{x\}, \{x, y\}, \{v, w\}, \{w, x\}, \{v, x\}, \{v, w, x\}, \{v, x, y\}, \{w, x, y\}\}.$$

To show how the accuracy measure with respect to somewhere dense sets is greater than accuracy measure with respect to β -open sets, we first calculate the collection of somewhere dense and cs -dense subsets of θ_r .

$$SD(\theta_r) = \{\emptyset, E, \{v\}, \{w\}, \{y\}, \{x, y\}, \{v, w\}, \{v, y\}, \{w, y\}, \{v, x\}, \{w, x\}, \{v, w, y\}, \{v, w, x\}, \{v, x, y\}, \{w, x, y\}\} = P(E) \setminus \{x\}, \text{ and}$$

$$CS(\theta_r) = \{\emptyset, E, \{v\}, \{w\}, \{x\}, \{y\}, \{x, y\}, \{v, w\}, \{v, y\}, \{w, y\}, \{v, x\}, \{w, x\}, \{v, w, x\}, \{v, x, y\}, \{w, x, y\}\} = P(E) \setminus \{v, w, y\}.$$

Table 2 Comparison between r -accuracy, $r\beta$ -near accuracy and rSD -near accuracy

$P(E)$	Accuracy		
	M_r	M_r^β	M_r^{SD}
{v}	0	1	1
{w}	0	1	1
{x}	0	0	0
{y}	$\frac{1}{2}$	$\frac{1}{2}$	1
{v, w}	1	1	1
{v, x}	0	$\frac{1}{2}$	1
{v, y}	$\frac{1}{4}$	$\frac{2}{3}$	1
{w, x}	0	$\frac{1}{2}$	1
{w, y}	$\frac{1}{4}$	$\frac{2}{3}$	1
{x, y}	1	1	1
{v, w, x}	$\frac{2}{3}$	$\frac{2}{3}$	1
{v, w, y}	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$
{v, x, y}	$\frac{1}{2}$	1	1
{w, x, y}	$\frac{1}{2}$	1	1
E	1	1	1

Bold values indicate the differences between the values

In Table 1, we calculate r -lower and r -upper approximations, $r\beta$ -near lower and $r\beta$ -near upper approximations, and rSD -near lower and rSD -near upper approximations for every subset of E .

Depending on Table 1, we construct Table 2 which represents the r -accuracy, $r\beta$ -accuracy, and rSD -accuracy for every subset of E .

Table 3 Information system

Student	Subjects		
	Maths	Physics	Chemistry
s_1	Good	Excellent	Good
s_2	Very good	Very good	Very good
s_3	Very good	Good	Excellent
s_4	Good	Excellent	Good

From the above two tables which show the differences between the approximations and accuracy measure under three different cases, we note that the new approximations reduce the size of boundary regions; also, the accuracy measure by use of somewhere dense sets is higher than the other measures; the green rows in the above two tables illustrate this fact. To analyze the obtained results, note that the family of somewhere dense sets is wider than the family of nonempty α -open (semiopen, preopen, b -open, β -open) sets. This maximizes the SD -near lower approximations which are the counterparts of S -interior topological operator. In addition, the family of cs -dense sets is wider than the family of nonempty α -closed (semiclosed, preclosed, b -closed, β -closed) sets. This minimizes the SD -near upper approximations which are the counterparts of S -closure topological operator.

In what follows, we introduce a practical example to elucidate the significance of applying somewhere dense sets in the information system.

Example 6 Consider the information system as given in Table 3, where $S = \{s_1, s_2, s_3, s_4\}$, $A = \{Maths, physics, chemistry\}$, and $E = \{Good, Very good, Excellent\}$ represent, respectively, the universal set of students, the set of attributes and the set of values.

The equivalences classes of S are $\{\{s_1, s_4\}, \{s_2\}, \{s_3\}\}$. Then, θ_j in the case of $j = r$ is $\{\emptyset, \{s_2\}, \{s_3\}, \{s_2, s_3\}, \{s_1, s_4\}, \{s_1, s_2, s_4\}, \{s_1, s_3, s_4\}, S\}$. Since \mathcal{R} is an equivalence relation, then θ_j is a clopen topology; therefore $\theta_j = \Gamma_j$. It follows from Proposition 6 that the family of semi-open subsets of (S, θ_j) is θ_j , and the family of somewhere dense subsets of (S, θ_j) is the power set of S .

Now, we compare among Pawlak’s approach, approach given in Amer et al. (2017), Salama (2018) with respect to $k = semi$, and our approach.

Depending on Table 4, we construct Table 5 which represents the r -accuracy, $r\beta$ -near accuracy, and rSD -near accuracy for every subset of E .

From the above two tables which show the differences between the approximations and accuracy measure under three different cases, we note that the new approximations reduce the size of boundary regions; also, the accuracy measure induced from somewhere dense sets is higher than the other measures; the green rows in the above two tables illustrate this fact.

Table 4 Comparison between Pawlak’s approximations, $rsemi$ -near approximations and rSD -near approximations

$P(S)$	θ_j		$semi O(\theta_r)$		$SD(\theta_r)$	
	θ_r	$\bar{\mathcal{R}}_r$	$\underline{\mathcal{R}}_r^{semi}$	$\bar{\mathcal{R}}_r^{semi}$	$\underline{\mathcal{R}}_r^{SD}$	$\bar{\mathcal{R}}_r^{SD}$
	$\{s_1\}$	\emptyset	$\{s_1, s_4\}$	\emptyset	$\{s_1, s_4\}$	$\{s_1\}$
$\{s_2\}$	$\{s_2\}$	$\{s_2\}$	$\{s_2\}$	$\{s_2\}$	$\{s_2\}$	$\{s_2\}$
$\{s_3\}$	$\{s_3\}$	$\{s_3\}$	$\{s_3\}$	$\{s_3\}$	$\{s_3\}$	$\{s_3\}$
$\{s_4\}$	\emptyset	$\{s_1, s_4\}$	\emptyset	$\{s_1, s_4\}$	$\{s_4\}$	$\{s_4\}$
$\{s_1, s_2\}$	\emptyset	$\{s_1, s_2, s_4\}$	\emptyset	$\{s_1, s_2, s_4\}$	$\{s_1, s_2\}$	$\{s_1, s_2\}$
$\{s_1, s_3\}$	$\{s_3\}$	$\{s_1, s_3, s_4\}$	$\{s_3\}$	$\{s_1, s_3, s_4\}$	$\{s_1, s_3\}$	$\{s_1, s_3\}$
$\{s_1, s_4\}$	$\{s_1, s_4\}$	$\{s_1, s_4\}$	$\{s_1, s_4\}$	$\{s_1, s_4\}$	$\{s_1, s_4\}$	$\{s_1, s_4\}$
$\{s_2, s_3\}$	$\{s_2, s_3\}$	$\{s_2, s_3\}$	$\{s_2, s_3\}$	$\{s_2, s_3\}$	$\{s_2, s_3\}$	$\{s_2, s_3\}$
$\{s_2, s_4\}$	$\{s_2\}$	$\{s_1, s_2, s_4\}$	$\{s_2\}$	$\{s_1, s_2, s_4\}$	$\{s_2, s_4\}$	$\{s_2, s_4\}$
$\{s_3, s_4\}$	$\{s_3\}$	$\{s_1, s_3, s_4\}$	$\{s_3\}$	$\{s_1, s_3, s_4\}$	$\{s_3, s_4\}$	$\{s_3, s_4\}$
$\{s_1, s_2, s_3\}$	$\{s_2, s_3\}$	S	$\{s_2, s_3\}$	S	$\{s_1, s_2, s_3\}$	$\{s_1, s_2, s_3\}$
$\{s_1, s_2, s_4\}$	$\{s_1, s_2, s_4\}$	$\{s_1, s_2, s_4\}$	$\{s_1, s_2, s_4\}$	$\{s_1, s_2, s_4\}$	$\{s_1, s_2, s_4\}$	$\{s_1, s_2, s_4\}$
$\{s_1, s_3, s_4\}$	$\{s_1, s_3, s_4\}$	$\{s_1, s_3, s_4\}$	$\{s_1, s_3, s_4\}$	$\{s_1, s_3, s_4\}$	$\{s_1, s_3, s_4\}$	$\{s_1, s_3, s_4\}$
$\{s_2, s_3, s_4\}$	$\{s_2, s_3\}$	S	$\{s_2, s_3\}$	S	$\{s_2, s_3, s_4\}$	$\{s_2, s_3, s_4\}$
S	S	S	S	S	S	S
\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset

Bold values indicate the differences between the values

5 Conclusion

To the best of our knowledge, the tool of rough set theory is a new efficacious technique to dispose of uncertainties. In recent decades, many authors interested in the rough set theory endeavor to reduce the boundary region for the sake of increasing the accuracy measure of decision-making. One of the approaches followed is creating new types of neighborhoods and exploiting to establish new types of approximations with a view to increasing accuracy measure.

Topology forms a new type of geometry that relies on nearness or neighborhood of points instead of measuring distance between them. Classifications of sets in rough set content are based on the approximation operators which have topological properties similar to all/some properties of the interior and closure operators. Therefore, investigation rough set by use of the topological concepts is fruitful to model real-life problems such as image processing, machine learning, data mining, pattern recognition and medical events.

Although many scholars have done a great job in developing many notions of rough set content, yet there is still a lot of space in this area. In this paper, we have scrutinized further properties of somewhere dense sets which are wider than β -open sets. Then, we have employed to introduce new types of approximations, boundary region, and accuracy measure of a subset. With the help of examples, we have elucidated the relationships between them and the previous ones and investigated main properties. We conclude this manuscript by comparing the current approach of approximations with the previous ones given in El-Monsef et al. (2014), Amer et al. (2017), Salama (2018) and showing the family of somewhere dense sets utilized in our approaches is the largest granulation which ultimately made the accuracy measures are higher than the other types; see, Examples 5 and 6.

Another merit of the technique followed herein is to keep most properties of Pawlak’s approximations, whereas many properties are losing in the case of generating these approximations in a direct way from N_j -neighborhoods. In our approach, we can impose some conditions to preserve all properties of Pawlak’s approximations such as strongly hyperconnectedness. But in the general case, Pawlak’s properties $L6$ and $U6$ given in Proposition 1 are not kept by our approach which is a limitation of the technique followed in this manuscript.

In upcoming works, we shall endeavor to achieve the following goals.

- (i) Define new types of neighborhoods in rough set theory and use to define a topological structure.
- (ii) Study the approximations and accuracy measures of a set using the followed technique herein (somewhere dense sets) via topological structures induced from different

Table 5 Comparison between Pawalk’s accuracy, $rsemi$ -near accuracy and rSD -near accuracy

$P(S)$	M		
	M_r	M_r^β	M_r^{SD}
{s1}	0	0	1
{s2}	1	1	1
{s3}	1	1	1
{s4}	0	0	1
{s1, s2}	0	0	1
{s1, s3}	$\frac{1}{3}$	$\frac{1}{3}$	1
{s1, s4}	1	1	1
{s2, s3}	1	1	1
{s2, s4}	$\frac{1}{3}$	$\frac{1}{3}$	1
{s3, s4}	$\frac{1}{3}$	$\frac{1}{3}$	1
{s1, s2, s3}	$\frac{1}{2}$	$\frac{1}{2}$	1
{s1, s2, s4}	1	1	1
{s1, s3, s4}	1	1	1
{s2, s3, s4}	$\frac{1}{2}$	$\frac{1}{2}$	1
S	1	1	1

Bold values indicate the differences between the values

neighborhood systems such as E_j -neighborhoods and C_j -neighborhoods.

- (iii) Apply the current technique on soft rough set content using soft somewhere dense sets (Al-shami 2018; Al-shami et al. 2020).
- (iv) As we know the class of α -open sets forms a topological space, so that, the approximations defined by this class keep Pawlak’s properties $L6$ and $U6$ given in Proposition 1 that do not hold in the approximations given herein. On the other hand, our approach produces higher accuracy measures than approach generated by α -open sets. To obtain these two properties (best approximations and highest accuracy), we will suggest a new class of generalizations of open sets which is wider than a class of somewhere dense sets and forms a topological structure.

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Declarations

Conflict of interest The author declares that there is no conflict of interests regarding the publication of this article.

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