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FOUNDATIONS

Single axiomatic characterization of a hesitant fuzzy generalization of rough approximation operators

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Abstract
Hesitant fuzzy set is a natural generalization of the classical fuzzy set. A hesitant fuzzy set on a universe of discourse is in terms of a function that when applied to the universe returns a finite subset of [0, 1]. Since the axiomatic method of approximation operator is of great significance in the research of the mathematical structure of rough set theory, it is a fundamental problem in axiomatic method to find the minimum set of abstract axioms. This paper first introduces the basic concepts, properties and related operations of hesitant fuzzy set, hesitant fuzzy rough set and hesitant fuzzy rough approximation operator. Secondly, by defining inner product, outer product and by exploring their related properties, the single axiomatization problem of the classical hesitant fuzzy rough approximation operator is solved. Furthermore, we study the single axiomatization of hesitant fuzzy rough approximation operators derived from serial, reflexive, symmetric and transitive hesitant fuzzy relations, respectively. Finally, we compare and analyze the advantages and disadvantages of hesitant fuzzy set, fuzzy rough set and hesitant fuzzy rough set through some cases.

Keywords Hesitant fuzzy approximation operators · Hesitant fuzzy relation · Hesitant fuzzy rough sets

1 Introduction

Pawlak, a Polish mathematician, first proposed rough set theory in 1982, which is an efficient scientific way for modeling and processing incomplete and uncertain information (Pawlak 1982, 1991). As the basic structure of rough set theory, approximation space is made up of a universe of discourse and a binary relation imposed on it. The most important concepts in rough set theory are the upper and lower approximation operators derived from the approximation space. In rough set data analysis, two methods are commonly used to develop approximation operators, i.e. constructive method and axiomatic method. In the constructive method, binary relation, partition, covering, neighborhood system and Boolean subalgebra are used as the original concepts, then these concepts are used to construct approximation operators. Compared with construction method, axiomatic method, also known as algebraic method, does not take approximation space as the basic concept. On the contrary, it abstracts the upper and lower approximation operators into original concepts. In this method, most approximation operators generated in constructive methods are described by a set of axioms. As another mathematical tool to deal with fuzzy and uncertain knowledge, fuzzy set theory was first established by Zadeh, an American cybernetic expert, in 1965 (Zadeh 1965). After that, the fusion of rough set and fuzzy set has become one of the hot research directions in the processing of intelligence information in recent years (Liang et al. 2019).

In the study of fuzzy rough sets, Morsi and Yakout (1998) first used axiomatic method to study fuzzy approximation operators. Wu et al. (2003) studied the axiomatization of fuzzy rough approximation operators derived from general fuzzy relations, and obtained the axioms set describing various fuzzy rough approximation operators. Mi et al. (2008) and Wu (2011) characterized the dual fuzzy rough approximation operators defined by general trigonometric

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modules and anti trigonometric modules. She and Wang (2009) obtained the axiomatic characterization of fuzzy rough approximation operators defined by residuated lattices. Zhou et al. (2009) studied the intuitionistic fuzzy rough approximation operators by using axiomatic approach. The axiomatic characterization of intuitionistic fuzzy rough set based on two universe was given by Zhang et al. (2012). Furthermore, Zhang et al. (2019) studied axiomatic characterization of approximation operators based on covering, and recently, Pang et al. (2019) proposed the axiomatizations of L-fuzzy rough approximation operators based on three kinds of new L-fuzzy relations. Shao et al. (2019) gave the axiomatic characterizations of adjoint generalized (dual) concept systems. Zhao and Li (2018) studied the axiomatization on generalized neighborhood system-based rough sets. And Gao et al. (2018) studied axiomatic approaches to rough approximation operators via ideal on a complete completely distributive lattice.

In 2013, Liu (2013) first introduced inner product operation in rough set theory, with this notion, he studied single axiomatic characterization of rough approximation operators. Later, Wu (2017), Wu and Xu (2016), Wu et al. (2019) also gave a single axiomatic characterization of fuzzy rough approximation operators determined by dual triangular norms. By using single axioms, Pang and Mi (2020) characterized L-rough approximate operators with respect to various types of L-relations. Recently, Wang and Gong (2020) proposed single axioms for (S, T)-fuzzy rough approximation operators with fuzzy product operations.

Meanwhile, fuzzy rough set theory is further extended to hesitant fuzzy rough set theory Torra (2010). For the first time, Yang and Song (2014) considered the axiomatic characterization of the hesitation fuzzy rough approximation operators derived respectively from a general hesitation fuzzy relation and the serial, reflexive, symmetric and transitive hesitation fuzzy relations. However, the inclusion relation between two hesitant fuzzy sets defined in Yang and Song (2014) does not necessarily satisfy antisymmetric property, in other words, for any two hesitant fuzzy sets A and B, if $B \subseteq A$ and $A \subseteq B$, there is not necessarily an equation $A = B$ hold. In order to make up for this problem, Zhang et al. (2019) improved Yang's model and proposed a new hesitant fuzzy rough set model.

In the process of theoretical application research, hesitant fuzzy set, fuzzy rough set and hesitant fuzzy rough set are widely used in multi-criteria decision, group decision making, multi-attribute decision making (MADM), cluster analysis and other fields. Among them, multi-attribute decision-making is widely concerned as an important part of modern decision analysis. In the process of multi-attribute decision-making, when determining the attribute value, we often encounter the situation of multiple values and hesitant among them, which leads to the attribute value expressed in the form of hesitant fuzzy element. In view of the above situation, by using TOPSIS (technology for order preference by similarity to ideal solution) and the maximizing deviation method, Xu and Zhang (2013) obtained a new method for solving MADM problem in hesitant fuzzy environment.

The determination of membership function of hesitant fuzzy set has a certain subjective apriority, while the upper and lower approximations and roughness of rough set are obtained through the calculation of objective data, so rough set theory has a certain objectivity in dealing with uncertain information. Since the two theories are highly complementary, Tian et al. (2013) defined the objective weight of the index by fuzzy rough set, then got the comprehensive weight by combining the subjective and objective weight. Finally, they applied the comprehensive weight to TOPSIS to study the MADM problem. Meanwhile, in Zhang et al. (2017), Zhang et al studied how to use hesitant fuzzy rough set model to solve MADM problem. Then, by integrating the objectivity of rough set into hesitant fuzzy environment, they made up for the deficiency of subjective apriority of hesitant fuzzy set.

Since the axiomatic method of approximation operator is of great significance to the study of the mathematical structure of rough set theory, it is a fundamental problem in axiomatic method to find the minimum set of abstract axioms. In the study of this paper, we first define novel operations of inner product and outer product between two hesitant fuzzy sets by using the disjunctive and conjunctive normal forms between two hesitant fuzzy elements. Then, by employing the new model proposed by Zhang et al. (2019), we study the single axiom characteristics of the hesitant fuzzy rough approximation operators. In the next section, we will review the basic concepts, properties and related operations about hesitant fuzzy sets, hesitant fuzzy rough sets and hesitant fuzzy approximation operators. In Sect. 3, the operations of inner product, outer product between two hesitant fuzzy sets are defined, and their related properties are examined. Then, the single axiomatization problem of the classical hesitant fuzzy rough approximation operator is solved. In Sect. 4, we further study the single axiomatization of hesitant fuzzy rough approximation operators derived from serial, reflexive, symmetric and transitive hesitant fuzzy relation, respectively. In Sect. 5, we compare and analyze hesitant fuzzy set, fuzzy rough set and hesitant fuzzy rough set through some examples. Section 6 concludes the paper with some remarks.

2 Preliminaries

This section reviews some basic concepts and properties related to hesitant fuzzy sets, hesitant fuzzy rough sets and their approximate operators.

Definition 1 (Torra 2010). Let U be a nonempty and finite universe of discourse. $A = \{\langle x, h_A(x)\rangle | x \in U\}$ is referred to as a hesitant fuzzy set on U, where h_A is the membership function of hesitant fuzzy set A that when act on a element $x \in U$ returns a finite subset of [0,1], i.e. $h_A(x)$ is a finite set of different values in [0,1], indicating the possible membership degree of x in the hesitant fuzzy set A.

Noted that, if for any $x \in U$ there is only one element in $h_A(x)$, then the hesitant fuzzy set degenerates into a fuzzy set. Therefore, hesitant fuzzy set is a generalization of a fuzzy set. For convenience, we call $h_A(x)$ the hesitant fuzzy element. The set of all hesitant fuzzy sets on U is called hesitant fuzzy power set of U, and denoted by $HF(U)$.

Definition 2 (Zhang et al. 2019) Some special hesitant fuzzy sets are defined as follows:

(1) A is referred to as an empty hesitant fuzzy set if and only if $h_A(x) = \{0\}, \forall x \in U$. In the sequel, the empty hesitant fuzzy set is denoted by $\emptyset$.
(2) A is referred to as the hesitant fuzzy universe set if and only if $h_A(x) = \{1\}, \forall x \in U$. In the sequel, the hesitant fuzzy universe set is denoted by U.
(3) A is referred to as a constant hesitant fuzzy set if and only if there exist $a_i \in [0, 1], i = 1, 2, \cdots, m$, such that $h_A(x) = \{a_1, a_2, \ldots, a_m\}$ for all $x \in U$. In the sequel, the constant hesitant fuzzy set is denoted by $\widehat{a_1 \ldots a_m}$.
(4) Given $y \in U$ and $M \subseteq U$, two special hesitant fuzzy sets 1_y and 1_M are defined respectively as follows: for $x \in U$,

$$h_{1_y}(x) = \begin{cases} \{1\} & x = y \\ \{0\} & else \end{cases}$$

$$h_{1_M}(x) = \begin{cases} \{1\} & x \in M \\ \{0\} & else \end{cases}$$

It is worth noting that different hesitant fuzzy elements may contain different numbers of values. Denote the number of values in $h_A(x)$ by $l(h_A(x))$, for correct operation, the following assumptions are given:

(1) Arrange all elements in hesitant element $h_A(x)$ in ascending order, then the kth maximum value in $h_A(x)$ is denoted by $h_A^{\sigma(k)}(x)$.
(2) If, for two hesitant fuzzy elements $h_A(x)$ and $h_B(x)$, $l(h_A(x)) \neq l(h_B(x))$, then denote by $l_{AB}(x) = \max\{l(h_A(x)), l(h_B(x))\}$. Only two hesitant fuzzy elements $h_A(x)$ and $h_B(x)$ have the same length can they compare correctly. If the number of elements in $h_A(x)$ is less than that in $h_B(x)$, extend $h_A(x)$ by repeating its maximum element until $h_A(x)$ has the same length as $h_B(x)$.

In this paper, the following definitions of operations between hesitant fuzzy elements are based on the two hesitant fuzzy elements having the same length.

Definition 3 (Zhang et al. 2019). Let U be a nonempty and finite universe of discourse, A and B are two hesitant fuzzy sets.

(1) The complement of A, denoted by A^c, is defined by: $h_{A^c}(x) = \sim h_A(x) = \{1 - h_A^{\sigma(k)}(x) | k = 1, 2 \ldots, l(h_A(x))\}$, $\forall x \in U$.
(2) The union of A and B, denoted by $A \cup B$, is defined by: $h_{A\cup B}(x) = h_A(x) \veebar h_B(x) = \{h_A^{\sigma(k)}(x) \vee h_B^{\sigma(k)}(x) | k = 1, 2 \ldots, l_{AB}(x)\}, \forall x \in U$.
(3) The intersection of A and B, denoted by $A \cap B$, is defined by: $h_{A\cap B}(x) = h_A(x) \barwedge h_B(x) = \{h_A^{\sigma(k)}(x) \wedge h_B^{\sigma(k)}(x) | k = 1, 2 \ldots, l_{AB}(x)\}, \forall x \in U$.

Theorem 1 (Zhang et al. 2019). *Suppose A and B are two hesitant fuzzy sets, then the following equations hold:*

(1) $(A \cup B)^c = A^c \cap B^c$.
(2) $(A \cap B)^c = A^c \cup B^c$.

Definition 4 (Zhang et al. 2019). Let U be the nonempty and finite universe of discourse, A and B are two hesitant fuzzy sets. We define $A \subseteq B$ if and only if $h_A(x) \preceq h_B(x)$ holds for all $x \in U$, i.e.

$$h_A(x) \preceq h_B(x) \Longleftrightarrow h_A^{\sigma(k)}(x) \leq h_B^{\sigma(k)}(x),\ k = 1, 2, \ldots, l_{AB}(x).$$

If $h_A(x) \prec h_B(x)$ holds for all $x \in U$, i.e.

$$h_A(x) \prec h_B(x) \Longleftrightarrow h_A^{\sigma(k)}(x) < h_B^{\sigma(k)}(x),\ k = 1, 2, \ldots, l_{AB}(x),$$

then we define $A \subset B$.

Obviously, the following conclusions are established for the above inclusion relation:

(1) $A \subseteq A$. (reflexivity)
(2) $A \subseteq B, B \subseteq C \Rightarrow A \subseteq C$. (transitivity)
(3) $A \subseteq B, B \subseteq A \Rightarrow A = B$. (antisymmetry)

Definition 5 (Zhang et al. 2019). Let U be a nonempty and finite universe of discourse, a hesitant fuzzy relation R on U is a hesitant fuzzy subset of $U \times U$, that is, R is defined by

$$R = \{\langle (x, y), h_R(x, y)\rangle | (x, y) \in U \times U\},$$

where the set $h_R(x, y)$ is composed of finite different values in [0, 1], and represents the possible membership degrees of the relation between x and y.

Definition 6 (Zhang et al. 2019). Let U be a nonempty and finite universe of discourse. Given a hesitant fuzzy relation R on U.

(1) If there exists a $y \in U$ such that $h_R(x, y) = \{1\}$ holds for any $x \in U$, then R is said to be serial;
(2) If the equation $h_R(x, x) = \{1\}$ holds for all $x \in U$, then R is said to be reflexive;
(3) If the equation $h_R(x, y) = h_R(y, x)$ holds for all $(x, y) \in U \times U$, then R is said to be symmetric;
(4) If the formula $h_R(x, y) \barwedge h_R(y, z) \preceq h_R(x, z)$ holds for all $x, y, z \in U$, then R is said to be transitive. Alternatively, R is transitive if the following condition is satisfied:

$$h^{\sigma(k)}(x, y) \wedge h^{\sigma(k)}(y, z) \leq h^{\sigma(k)}(x, z), k = 1, 2, \ldots, l,$$

where $l = max\{l(h_R(x, y)), l(h_R(y, z)), l(h_R(x, z))\}$.

Definition 7 (Zhang et al. 2019). Suppose U is a nonempty and finite universe of discourse, R is a hesitant fuzzy relation on U. Then the pair (U, R) is referred to as a hesitant fuzzy approximation space. For any $A \in HF(U)$, the lower and upper approximations of A with respect to (U, R), denoted by $\underline{R}(A)$ and $\overline{R}(A)$, are two hesitant fuzzy sets and are defined respectively by

$$\underline{R}(A) = \{\langle x, h_{\underline{R}(A)}(x)\rangle | x \in U\},$$
$$\overline{R}(A) = \{\langle x, h_{\overline{R}(A)}(x)\rangle | x \in U\},$$

where $h_{\underline{R}(A)}(x) = \underset{y\in U}{\barwedge} \{h_{R^c}(x, y) \veebar h_A(y)\}$, $h_{\overline{R}(A)}(x) = \underset{y\in U}{\veebar} \{h_R(x, y) \barwedge h_A(y)\}$.

The pair $(\underline{R}(A), \overline{R}(A))$ is referred to as the hesitant fuzzy rough set of A with respect to (U, R), and $\underline{R}(A), \overline{R}(A)$: $HF(U) \rightarrow HF(U)$ are known as the lower and upper hesitant fuzzy rough approximation operators, respectively. Obviously, we have that

$$h_{\underline{R}(A)}(x) = \{\wedge_{y\in U} h_{R^c}^{\sigma(k)}(x, y) \vee h_A^{\sigma(k)}(y) | k = 1, 2 \ldots, l_x\},$$
$$h_{\overline{R}(A)}(x) = \{\vee_{y\in U} h_R^{\sigma(k)}(x, y) \wedge h_A^{\sigma(k)}(y) | k = 1, 2 \ldots, l_x\},$$

where $l_x = \max \max_{y\in U} l(h_R(x, y)), l(h_A(y))$.

Property 1 (Zhang et al. 2019). Given a hesitant fuzzy approximation space (U, R), the lower and upper hesitant fuzzy rough approximation operators $\underline{R}, \overline{R}$: $HF(U) \rightarrow HF(U)$ satisfy the following properties: $\forall A, B \in HF(U)$, $\forall a_i \in [0, 1], i = 1, 2, \ldots, m, \forall M \subseteq U, \forall (x, y) \in (U \times U)$,

(1) $\overline{R}(A^c) = (\underline{R}(A))^c$, and $\underline{R}(A^c) = (\overline{R}(A))^c$;
(2) $A \subseteq B$ implies $\overline{R}(A) \subseteq \overline{R}(B)$ and $\underline{R}(A) \subseteq \underline{R}(B)$;
(3) $\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$, and $\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$;
(4) $\overline{R}(A \cap B) \subseteq \overline{R}(A) \cap \overline{R}(B)$, and $\underline{R}(A \cup B) \supseteq \underline{R}(A) \cup \underline{R}(B)$;
(5) $\overline{R}(A \cap \widehat{a_1 \ldots a_m}) = \overline{R}(A) \cap \overline{R}(\widehat{a_1 \ldots a_m})$, and $\underline{R}(A \cup \widehat{a_1 \ldots a_m}) = \underline{R}(A) \cup \underline{R}(\widehat{a_1 \ldots a_m})$;
(6) $\overline{R}(\emptyset) = \emptyset$, and $\underline{R}(U) = U$;
(7) $h_{\overline{R}(1_M)}(x) = \underset{y\in M}{\veebar} h_R(x, y)$, and $h_{\underline{R}(1_M)}(x) = \underset{y\in M}{\barwedge} h_{R^c}(x, y)$;
(8) $h_{\overline{R}(1_y)}(x) = h_R(x, y)$;
(9) $h_{\underline{R}(1_{U-\{y\}})}(x) = h_{R^c}(x, y)$.

Theorem 2 (Yang and Song 2014). *Suppose U is a nonempty and finite universe of discourse, and R is a hesitant fuzzy relation on U. $\underline{R}$, $\overline{R}$: $HF(U) \rightarrow HF(U)$ are the lower and upper hesitant fuzzy rough approximation operators defined in* Definition 7*, respectively, then*

(1) *R is serial* $\Leftrightarrow \underline{R}(\emptyset) = \emptyset$;

$$\Leftrightarrow \overline{R}(U) = U;$$
$$\Leftrightarrow \underline{R}(A) \subseteq \overline{R}(A), \forall A \in HF(U).$$

(2) *R is reflexive* $\Leftrightarrow \underline{R}(A) \subseteq A, \forall A \in HF(U)$;

$$\Leftrightarrow A \subseteq \overline{R}(A), \forall A \in HF(U).$$

(3) *R is symmetric* $\Leftrightarrow h_{\underline{R}(U-\{x\})}(y) = h_{\underline{R}(U-\{y\})}(x), \forall x, y \in U$;

$$\Leftrightarrow h_{\overline{R}(U-\{x\})}(y) = h_{\overline{R}(U-\{y\})}(x), \forall x, y \in U.$$

(4) *R is transitive* $\Leftrightarrow \underline{R}(A) \subseteq \underline{R}(\underline{R}(A)), \forall A \in HF(U)$;

$$\Leftrightarrow \overline{R}(\overline{R}(A)) \subseteq \overline{R}(A), \forall A \in HF(U).$$

Definition 8 (Yang and Song 2014). Given two hesitant fuzzy set-theoretic operators $L, H : HF(U) \rightarrow HF(U)$, we call them dual hesitant fuzzy operators if they satisfy one of the following equivalent conditions:

$$L(A) = (H(A^c))^c; H(A) = (L(A^c))^c.$$

Theorem 3 (Yang and Song 2014). *Let $L, H : HF(U) \rightarrow HF(U)$ be a pair of dual hesitant fuzzy operators, then there exists a hesitant fuzzy relation R on U such that $\forall A \in HF(U), L(A) = \underline{R}(A), H(A) = \overline{R}(A)$ holds, if and only if L satisfies the following axioms:*

(L1) $L(A) \cup \widehat{a_1 \ldots a_m} = L(A \cup \widehat{a_1 \ldots a_m}), \forall A \in HF(U), \forall \{a_1 \ldots a_m\} \in 2^{[0,1]}$;

(L2) $L(A \cap B) = L(A) \cap L(B)$, $\forall A, B \in HF(U)$.

Or equivalently H satisfies the following axioms:

(H1) $H(A) \cap \widehat{a_1 \dots a_m} = H(A \cap \widehat{a_1 \dots a_m})$, $\forall A \in HF(U)$, $\forall \{a_1 \dots a_m\} \in 2^{[0,1]}$;
(H2) $H(A \cup B) = H(A) \cup H(B)$, $\forall A, B \in HF(U)$.

3 Single axiomatic characterization of classical hesitant fuzzy rough approximation operators

In Yang and Song (2014), the axiomatic characterization of hesitant fuzzy rough approximation operators was given for the first time. In this section, novel concepts of inner and outer product operations between two hesitant fuzzy sets are first defined, and their properties are examined. Then, by using the two product operations, the axiom set is simplified to obtain the single axiom characterization of the hesitant fuzzy rough approximation operator proposed in Zhang et al. (2019).

Property 2 Given a hesitant fuzzy approximation space (U, R), the lower and upper hesitant fuzzy rough approximation operators $\underline{R}$ and $\overline{R}$: $HF(U) \to HF(U)$ satisfy the following properties: $\forall A_j \in HF(U)$, $j \in J$, where J is an index set.

(1) $\underline{R}(\bigcap_{j\in J} A_j) = \bigcap_{j\in J} \underline{R}(A_j)$;
(2) $\overline{R}(\bigcup_{j\in J} A_j) = \bigcup_{j\in J} \overline{R}(A_j)$.

Proof (1) $\forall x \in U$, by Definition 7 we have

$$\begin{aligned} h_{\underline{R}\left(\bigcap_{j\in J} A_j\right)}(x) &= \bar{\wedge}_{y\in U} \{h_{R^c}(x,y) \veebar h_{\bigcap_{j\in J} A_j}(y)\} \\ &= \bar{\wedge}_{y\in U} \{h_{R^c}(x,y) \veebar [\bar{\wedge}_{j\in J} h_{A_j}(y)]\} \\ &= \bar{\wedge}_{j\in J} \bar{\wedge}_{y\in U} \{h_{R^c}(x,y) \veebar h_{A_j}(y)\} \\ &= h_{\bigcap_{j\in J} \underline{R}(A_j)}(x). \end{aligned}$$

Thus $\underline{R}(\bigcap_{j\in J} A_j) = \bigcap_{j\in J} \underline{R}(A_j)$ holds.
(2) The proof is similar to (1). □

Theorem 4 . *Suppose L and H : $HF(U) \to HF(U)$ are a pair of dual hesitant fuzzy operators, then there exists a hesitant fuzzy relation R on U such that $L = \underline{R}$, $H = \overline{R}$ if and only if L satisfies the following axioms:*

(L1) $L(A) \cup \widehat{a_1 \dots a_m} = L(A \cup \widehat{a_1 \dots a_m})$, $\forall A \in HF(U)$, $\forall \{a_1 \dots a_m\} \in 2^{[0,1]}$;
(L3) $L(\bigcap_{j\in J} A_j) = \bigcap_{j\in J} L(A_j)$, $\forall A_j \in HF(U)$, $j \in J$.

Or equivalently H satisfies the following axioms:

(H1) $H(A) \cap \widehat{a_1 \dots a_m} = H(A \cap \widehat{a_1 \dots a_m})$, $\forall A \in HF(U)$, $\forall \{a_1 \dots a_m\} \in 2^{[0,1]}$;
(H3) $H(\bigcup_{j\in J} A_j) = \bigcup_{j\in J} H(A_j)$, $\forall A_j \in HF(U)$, $j \in J$.

Where J is an index set, and $\forall \{a_1 \dots a_m\} \in 2^{[0,1]}$.

Proof If there exists a hesitant fuzzy relation R on U such that $L = \underline{R}$ holds, then by Property 2 we have

$$L\left(\bigcap_{j\in J} A_j\right) = \underline{R}\left(\bigcap_{j\in J} A_j\right) = \bigcap_{j\in J} \underline{R}(A_j) = \bigcap_{j\in J} L(A_j).$$

Thus (L3) holds. By (5) in Property 1, we conclude that (L1) holds.

Conversely, if L satisfies axioms (L1) and (L3). By taking $J = \{1, 2\}$ we have $L(A_1 \cap A_2) = L(A_1) \cap L(A_2)$. By Theorem 3, there exists a hesitant fuzzy relation R on U such that $L = \underline{R}$ holds.

For the hesitant fuzzy operator H, the prove is similar to L. □

Theorem 5 *Suppose L and H : $HF(U) \to HF(U)$ are a pair of dual hesitant fuzzy operators, then there exists a hesitant fuzzy relation R on U such that*

(1) *$L = \underline{R}$ if and only if L satisfies the following axiom: $\forall A_j \in HF(U)$, $\forall j \in J$, where J is an index set, and $\forall \{a_1 \dots a_m\} \in 2^{[0,1]}$,*

$$L\left(\bigcap_{j\in J} (\widehat{a_1^j \dots a_m^j} \cup A_j)\right) = \bigcap_{j\in J} (\widehat{a_1^j \dots a_m^j} \cup L(A_j)). \tag{1}$$

(2) *$H = \overline{R}$ if and only if H satisfies the following axiom: $\forall A_j \in HF(U)$, $\forall j \in J$, where J is an index set, and $\forall \{a_1 \dots a_m\} \in 2^{[0,1]}$,*

$$H\left(\bigcup_{j\in J} \left(\widehat{a_1^j \dots a_m^j} \cap A_j\right)\right) = \bigcup_{j\in J} \left(\widehat{a_1^j \dots a_m^j} \cap H(A_j)\right). \tag{2}$$

Proof (1) Firstly, we prove the necessity. By Theorem 4, $L(\bigcap_{j\in J} (\widehat{a_m^j \dots a_m^j} \cup A_j)) = \bigcap_{j\in J} L(\widehat{a_m^j \dots a_m^j} \cup A_j) = \bigcap_{j\in J} (\widehat{a_m^j \dots a_m^j} \cup L(A_j))$.

Secondly, we prove the sufficiency. Assume that $L(\underset{j\in J}{\cap}(\widehat{a_m^j \dots a_m^j} \cup A_j)) = \underset{j\in J}{\cap}(\widehat{a_m^j \dots a_m^j} \cup L(A_j))$ holds, then, from L we define a hesitant fuzzy relation R on U: $\forall(x, y) \in U \times U$,

$$R(x, y) = 1 - L(1_{U-\{y\}})(x).$$

$\forall A \in HF(U)$, noticing that $A = \underset{y\in U}{\cap}(\widehat{h_A(y)} \cup 1_{U-\{y\}})$, then $\forall x \in U$ we have

$$\begin{aligned}L(A)(x) &= L\left(\underset{y\in U}{\cap}(\widehat{h_A(y)} \cup 1_{U-\{y\}})\right)(x)\\ &= \underset{y\in U}{\cap}(\widehat{h_A(y)} \cup L(1_{U-\{y\}}))(x)\\ &= \underset{y\in U}{\bar{\wedge}}(h_A(y) \underline{\vee} L(1_{U-\{y\}})(x))\\ &= \underset{y\in U}{\bar{\wedge}}(h_A(y) \underline{\vee} 1 - h_R(x, y))\\ &= \underset{y\in U}{\bar{\wedge}}(h_A(y) \underline{\vee} h_{R^c}(x, y))\\ &= \underline{R}(A)(x).\end{aligned}$$

Thus $L(A) = \underline{R}(A)$ holds. □

(2) Necessity. By Theorem 4, $H(\underset{j\in J}{\cup}(\widehat{a_m^j \dots a_m^j} \cap A_j)) = \underset{j\in J}{\cup} H(\widehat{a_m^j \dots a_m^j} \cap A_j) = \underset{j\in J}{\cup}(\widehat{a_m^j \dots a_m^j} \cap H(A_j))$.

Sufficiency. Assume that $H(\underset{j\in J}{\cup}(\widehat{a_m^j \dots a_m^j} \cap A_j)) = \underset{j\in J}{\cup}(\widehat{a_m^j \dots a_m^j} \cap H(A_j))$ holds, then, from H we define a hesitant fuzzy relation R on U: $\forall(x, y) \in U \times U$,

$$R(x, y) = H(1_y)(x).$$

Noticing that $\forall A \in HF(U)$, $A = \underset{y\in U}{\cup}(\widehat{h_A(y)} \cap 1_y)$, then $\forall x \in U$ we have

$$\begin{aligned}H(A)(x) &= H\left(\underset{y\in U}{\cup}(\widehat{h_A(y)} \cap 1_y)\right)(x)\\ &= \underset{y\in U}{\cup}\left(\widehat{h_A(y)} \cap H(1_y)\right)(x)\\ &= \underset{y\in U}{\bar{\wedge}}(h_A(y) \bar{\wedge} h(1_y)(x))\\ &= \underset{y\in U}{\bar{\wedge}}(h_A(y) \bar{\wedge} h_R(x, y))\\ &= \overline{R}(A)(x).\end{aligned}$$

Thus $H(A) = \overline{R}(A)$ holds. □

In Wu and Xu (2016), Wu et al. defined the operations of inner product and outer product of two fuzzy sets based on t-norm and t-conorm, and studied the single axiomatic characterization of fuzzy rough approximation operators. Next, we extend these notions to the hesitant fuzzy environment, and give novel definitions and properties of inner product and outer product between two hesitant fuzzy sets.

Definition 9 $\forall A, B \in HF(U)$, the outer product of A and B, denoted as $[A, B]$, is defined by

$$[A, B] = \underset{x\in U}{\bar{\wedge}}(h_A(x) \underline{\vee} h_B(x)).$$

The inner product of A and B, recorded as (A, B), is defined by

$$(A, B) = \underset{x\in U}{\underline{\vee}}(h_A(x) \bar{\wedge} h_B(x)).$$

Property 3 The outer product between two hesitant fuzzy sets satisfies the following properties: $\forall A, B, A_j \in HF(U)$, $j \in J$, where J is an index set, $\forall\{a_1, \dots, a_m\} \in 2^{[0,1]}$.

(1) $[A, B] = [B, A]$;
(2) $[\emptyset, B] = \underset{x\in U}{\bar{\wedge}} h_B(x)$, $[U, B] = 1$;
(3) $A \subseteq B$ if and only if $[A, C] \preceq [B, C]$, $\forall C \in HF(U)$;
(4) If $[A, C] = [B, C]$, $\forall C \in HF(U)$, then $A = B$;
(5) $[(\widehat{a_1 \dots a_m} \cup A), B] = \{a_1 \dots a_m\} \underline{\vee} [A, B]$;
(6) $[\underset{j\in J}{\cap} A_j, B] = \underset{j\in J}{\bar{\wedge}} [A_j, B]$.

Proof

(1) and (2) are obvious by Definition 9.
(3) The necessity is obvious by Definition 9.
Secondly, we prove the sufficiency. $\forall x \in U$, let $C = 1_{U-\{x\}}$, then

$$\begin{aligned}[A, C] &= \underset{y\in U}{\bar{\wedge}}(h_A(y) \underline{\vee} h_C(y))\\ &= \left(\underset{y\neq x}{\bar{\wedge}}(h_A(y) \underline{\vee} h_C(y)) \bar{\wedge} (h_A(x) \underline{\vee} h_C(x))\right) = h_A(x).\end{aligned}$$

Similarly, we can verify that $[B, C] = h_B(x)$. As $[A, C] \preceq [B, C]$, we have $h_A(x) \preceq h_B(x)$ for all $x \in U$. Consequently $A \subseteq B$.
(4) It can be immediately obtained from (3).

(5) For any $\{a_1 \dots a_m\} \in 2^{[0,1]}$, we have

$$
\begin{aligned}
&[(\widehat{a_1 \dots a_m} \cup A), B] \\
&= \mathop{\bar{\wedge}}_{x\in U} (h_{\widehat{a_1\dots a_m}\cup A}(x) \veebar h_B(x)) \\
&= \mathop{\bar{\wedge}}_{x\in U} (\{\widehat{a_1 \dots a_m}\} \veebar h_A(x) \veebar h_B(x)) \\
&= \{\widehat{a_1 \dots a_m}\} \veebar (\mathop{\bar{\wedge}}_{x\in U} \veebar h_A(x) \veebar h_B(x)) \\
&= \{\widehat{a_1 \dots a_m}\} \veebar [A, B].
\end{aligned}
$$

(6) By Definition 3, it is obvious. □

Property 4 The inner product between two hesitant fuzzy sets satisfies the following properties: $\forall A, B, A_j \in HF(U)$, $j \in J$, where J is an index set, $\forall \{a_1 \dots a_m\} \in 2^{[0,1]}$.

(1) $(A, B) = (B, A)$;
(2) $(\emptyset, B) = \emptyset$, $(U, B) = \mathop{\veebar}_{x\in U} h_B(x)$;
(3) $A \subseteq B$ if and only if $(A, C) \preceq (B, C), \forall C \in HF(U)$;
(4) If $(A, C) = (B, C), \forall C \in HF(U)$, then $A = B$;
(5) $((\widehat{a_1 \dots a_m} \cap A), B) = \{a_1 \dots a_m\} \bar{\wedge} (A, B)$;
(6) $(\bigcup_{j\in J} A_j, B) = \mathop{\veebar}_{j\in J} (A_j, B)$.

Proof The proofs are similar to those of Property 3. □

Definition 10 Suppose U is a nonempty and finite universe of discourse, for a hesitant fuzzy operators $O : HF(U) \to HF(U)$, $\forall A \in HF(U)$, $y \in U$, denote

$$
\begin{aligned}
O_{\veebar}^{-1}(A)(y) &= [O(1_{U-\{y\}}), A] \\
&= \mathop{\bar{\wedge}}_{x\in U} (h_{O(1_{U-\{y\}})}(x) \veebar h_A(x)), \\
O_{\bar{\wedge}}^{-1}(A)(y) &= (O(1_y), A) \\
&= \mathop{\veebar}_{x\in U} (h_{O(1_y)}(x) \bar{\wedge} h_A(x)).
\end{aligned}
$$

Then $O_{\veebar}^{-1}$ and $O_{\bar{\wedge}}^{-1} : HF(U) \to HF(U)$ are called respectively the lower inverse operator and upper inverse operators of O.

Theorem 6 *Let $L : HF(U) \to HF(U)$ be a hesitant fuzzy set valued operator, then there exists a hesitant fuzzy relation R on U such that $L = \underline{R}$ if and only if L satisfies the following axiom:*

$$[A, L(B)] = [B, L_{\veebar}^{-1}(A)], \ \forall A, B \in HF(U).$$

Proof Assume that $[A, L(B)] = [B, L_{\veebar}^{-1}(A)]$, $\forall A, B \in HF(U)$. By Theorem 5 and Properties (1) and (4) in Property 3, we only need to prove that

$$
\left[C, L\left(\bigcap_{j\in J} (\widehat{a_m^j \dots a_m^j} \cup A_j)\right)\right] = \left[C, \bigcap_{j\in J} (\widehat{a_m^j \dots a_m^j} \cup L(A_j))\right], \ \forall C \in HF(U).
$$

In fact, $\forall C \in HF(U)$, we have

$$
\begin{aligned}
&\left[C, L\left(\bigcap_{j\in J}\left(\widehat{a_1^j \dots a_m^j} \cup A_j\right)\right)\right] \\
&= \left[\bigcap_{j\in J}\left(\widehat{a_1^j \dots a_m^j} \cup A_j\right), L_{\veebar}^{-1}(C)\right] && \text{by the assumption} \\
&= \mathop{\wedge}_{j\in J} \left[\left(\widehat{a_1^j \dots a_m^j} \cup A_j\right), L_{\veebar}^{-1}(C)\right] && \text{by Property 3, (6)} \\
&= \mathop{\wedge}_{j\in J} \left(\{a_1^j \dots a_m^j\} \veebar [A_j, L_{\veebar}^{-1}(C)]\right) && \text{by Property 3, (5)} \\
&= \mathop{\wedge}_{j\in J} \left(\{a_1^j \dots a_m^j\} \veebar [C, L(A_j)]\right) && \text{by the assumption} \\
&= \mathop{\wedge}_{j\in J} \left(\{a_1^j \dots a_m^j\} \veebar [L(A_j), C]\right) && \text{by Property 3, (1)} \\
&= \mathop{\wedge}_{j\in J} \left[\left(\widehat{a_1^j \dots a_m^j} \cup L(A_j)\right), C\right] && \text{by Property 3, (5)} \\
&= \left[\bigcap_{j\in J}\left(\widehat{a_1^j \dots a_m^j} \cup L(A_j)\right), C\right] && \text{by Property 3, (6)} \\
&= \left[C, \bigcap_{j\in J}\left(\widehat{a_1^j \dots a_m^j} \cup L(A_j)\right)\right]. && \text{by Property 3, (1)}
\end{aligned}
$$

Therefore, $L = \underline{R}$.

Conversely, assume that L satisfies $L = \underline{R}$. Noticing that $\forall B \in HF(U)$, $B = \bigcap_{y\in U} (\widehat{h_B(y)} \cup 1_{U-\{y\}})$, then

$$
\begin{aligned}
&[A, L(B)] \\
&= \mathop{\wedge}_{x\in U} (h_A(x) \veebar h_{L(B)}(x)) \\
&= \mathop{\wedge}_{x\in U} (h_A(x) \veebar h_{L\left(\bigcap_{y\in U} (\widehat{h_B(y)}\cup 1_{U-\{y\}})\right)}(x)) \\
&= \mathop{\wedge}_{x\in U} (h_A(x) \veebar h_{\bigcap_{y\in U} (\widehat{h_B(y)}\cup L(1_{U-\{y\}}))}(x)) && \text{by Theorem 5} \\
&= \mathop{\wedge}_{x\in U} (h_A(x) \veebar \left(\mathop{\wedge}_{y\in U} h_B(y) \veebar h_{L(1_{U-\{y\}})}(x)\right)) \\
&= \mathop{\wedge}_{x\in U} \mathop{\wedge}_{y\in U} (h_A(x) \veebar (h_B(y) \veebar h_{L(1_{U-\{y\}})}(x))) \\
&= \mathop{\wedge}_{y\in U} \mathop{\wedge}_{x\in U} (h_B(y) \veebar (h_A(x) \veebar h_{L(1_{U-\{y\}})}(x))) \\
&= \mathop{\wedge}_{y\in U} \left(h_B(y) \veebar \left(\mathop{\wedge}_{x\in U} (h_A(x) \veebar h_{L(1_{U-\{y\}})}(x))\right)\right) \\
&= \mathop{\wedge}_{y\in U} \left(h_B(y) \veebar L_{\veebar}^{-1}(A)(y)\right) && \text{by Definition 10}
\end{aligned}
$$

$$= \bigwedge_{y\in U}(h_B(y)\,\underline{\vee}\,L_{\underline{\vee}}^{-1}(A)(y))$$
$$= [B, L_{\underline{\vee}}^{-1}(A)]. \qquad \text{by Definition 9}$$

Thus L satisfies $[A, L(B)] = [B, L_{\underline{\vee}}^{-1}(A)], \forall A \in HF(U)$. □

Analogous to Theorem 6, by using the inner product and the upper inverse operator of H, we can obtain another single axiom to characterize the upper hesitant fuzzy rough approximation operator.

Theorem 7 *Let $H : HF(U) \to HF(U)$ be a hesitant fuzzy operator, then there exists a hesitant fuzzy relation R on U such that $\forall A \in HF(U), H(A) = \overline{R}(A)$ holds, if and only if H satisfies the following axiom:*

$$(A, H(B)) = (B, H_{\overline{\wedge}}^{-1}(A)),\ \forall A, B \in HF(U).$$

Proof Sufficiency. Assume that $(A, H(B)) = (B, H_{\overline{\wedge}}^{-1}(A))$, $\forall A, B \in HF(U)$. By Theorem 5 and Properties (1) and (4) in Property 4, we only need to prove that

$$\left(C, H\left(\bigcup_{j\in J}\left(\widehat{a_m^j \ldots a_m^j} \cap A_j\right)\right)\right)$$
$$= \left(C, \bigcup_{j\in J}\left(\widehat{a_m^j \ldots a_m^j} \cap H\left(A_j\right)\right)\right).$$

In fact, $\forall C \in HF(U)$, we have

$$\left(C, H\left(\bigcup_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cap A_j\right)\right)\right)$$
$$= \left(\bigcup_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cap A_j\right)\right), H_{\overline{\wedge}}^{-1}(C)) \quad \text{by The assumption}$$
$$= \bigvee_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cap A_j, H_{\overline{\wedge}}^{-1}(C)\right) \quad \text{by Property 4, (6)}$$
$$= \bigvee_{j\in J}\left(a_1^j \ldots a_m^j\,\overline{\wedge}\left(A_j, H_{\overline{\wedge}}^{-1}(C)\right)\right) \quad \text{by Property 4, (5)}$$
$$= \bigvee_{j\in J}\left(a_1^j \ldots a_m^j\,\overline{\wedge}\left(C, H_{\overline{\wedge}}^{-1}(A_j)\right)\right) \quad \text{by The assumption}$$
$$= \bigvee_{j\in J}\left(a_1^j \ldots a_m^j\,\overline{\wedge}\left(H_{\overline{\wedge}}^{-1}(A_j), C\right)\right) \quad \text{by Property 4, (1)}$$
$$= \bigvee_{j\in J}\left(\left(\widehat{a_1^j \ldots a_m^j} \cap H_{\overline{\wedge}}^{-1}(A_j)\right), C\right) \quad \text{by Property 4, (5)}$$
$$= \left(\bigcup_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cap H_{\overline{\wedge}}^{-1}(A_j)\right), C\right) \quad \text{by Property 4, (6)}$$
$$= \left(C, \bigcup_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cap H_{\overline{\wedge}}^{-1}(A_j)\right)\right). \quad \text{by Property 4, (1)}$$

Thus H satisfies $H(A) = \overline{R}(A), \forall A \in HF(U)$.

Necessity. Noticing that for all $B \in HF(U)$, $B = \bigcup_{y\in U}(\widehat{h_B(y)} \cap 1_y)$, then

$$(A, H(B))$$
$$= \bigvee_{x\in U}\left(h_A(x)\,\overline{\wedge}\,h_{H(B)}(x)\right)$$
$$= \bigvee_{x\in U}\left(h_A(x)\,\overline{\wedge}\,h_{H\left(\bigcup_{y\in U}\left(\widehat{h_B(y)}\cap 1_y\right)\right)}(x)\right)$$
$$= \bigvee_{x\in U}\left(h_A(x)\,\overline{\wedge}\,h_{\bigcup_{y\in U}\left(\widehat{h_B(y)}\cap H(1_y)\right)}(x)\right) \quad \text{by Theorem 5}$$
$$= \bigvee_{x\in U}\left(h_A(x)\,\overline{\wedge}\left(\bigvee_{y\in U}\left(h_B(y)\,\overline{\wedge}\,h_{H(1_y)}(x)\right)\right)\right)$$
$$= \bigvee_{x\in U}\bigvee_{y\in U}\left(h_A(x)\,\overline{\wedge}\left(h_B(y)\,\overline{\wedge}\,h_{H(1_y)}(x)\right)\right)$$
$$= \bigvee_{y\in U}\bigvee_{x\in U}\left(h_B(y)\,\overline{\wedge}\,h_A(x)\,\overline{\wedge}\,h_{H(1_y)}(x)\right)$$
$$= \bigvee_{y\in U}\left(h_B(y)\,\overline{\wedge}\left(\bigvee_{x\in U}\left(h_A(x)\,\overline{\wedge}\,h_{H(1_y)}(x)\right)\right)\right)$$
$$= \bigvee_{y\in U}\left(h_B(y)\,\overline{\wedge}\,H_{\overline{\wedge}}^{-1}(A)(y)\right) \quad \text{by Definition 10}$$
$$= \left(B, H_{\overline{\wedge}}^{-1}(A)\right) \quad \text{by Definition 9}$$

Thus H satisfies $(A, H(B)) = (B, H_{\overline{\wedge}}^{-1}(A)), \forall A, B \in HF(U)$. □

4 Single axiomatic characterization of special hesitant fuzzy rough approximation operators

4.1 Single axiom for serial hesitant fuzzy rough approximation operators

In this subsection, we will study how to use a single axiom to characterize the hesitant fuzzy rough approximation operators generated by a serial hesitant fuzzy relation. We give the following results.

Theorem 8 *Let $L : HF(U) \to HF(U)$ be a hesitant fuzzy operator, then there exists a serial hesitant fuzzy relation R on U such that $\forall A \in HF(U), L(A) = \underline{R}(A)$ holds, if and only if L satisfies the following axiom:*

$$(U - L(\emptyset)) \cap L\left(\bigcap_{j\in J}(\widehat{a_1^j \ldots a_m^j} \cup A_j)\right)$$
$$= \bigcap_{j\in J}(\widehat{a_1^j \ldots a_m^j} \cup L(A_j)).$$

Where J is an index set, and $\forall\{a_1^j \ldots a_m^j\} \in 2^{[0,1]}$.

Proof Firstly, we prove the necessity. Assume that there exists a serial hesitant fuzzy relation R on U such that $\forall A \in HF(U)$, $L(A) = \underline{R}(A)$. By Theorem 2, $L(\emptyset) = \underline{R}(\emptyset) = \emptyset$. Thus

$$\begin{aligned}(U - L(\emptyset)) \cap L\left(\bigcap_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cup A_j\right)\right) \\ = U \cap L\left(\bigcap_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cup A_j\right)\right) \\ = L\left(\bigcap_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cup A_j\right)\right) \\ = \bigcap_{j\in J}(\widehat{a_1^j \ldots a_m^j} \cup L(A_j)).\end{aligned}$$

Secondly, we prove the sufficiency. Assume that L satisfies $(U - L(\emptyset)) \cap L(\bigcap_{j\in J}(\widehat{a_1^j \ldots a_m^j} \cup A_j)) = \bigcap_{j\in J}(\widehat{a_1^j \ldots a_m^j} \cup L(A_j))$. By taking $J = \{1\}, A_1 = \emptyset, a_1^1 \ldots a_m^1 = 1$ we have $(U - L(\emptyset)) \cap L(\widehat{1} \cup \emptyset) = \widehat{1} \cup L(\emptyset)$, that is, $(U - L(\emptyset)) \cap L(\widehat{1} \cup \emptyset) = U$ thus $L(\emptyset) = \emptyset$, by the assumption, $L(\bigcap_{j\in J}(\widehat{a_1^j \ldots a_m^j} \cup A_j)) = \bigcap_{j\in J}(\widehat{a_1^j \ldots a_m^j} \cup L(A_j))$. According to Theorem 5, there exists a hesitant fuzzy relation R on U such that $\forall A \in HF(U)$, $L(A) = \underline{R}(A)$. Therefore, we conclude by Theorem 2 that R is serial. □

Theorem 9 *Let $H : HF(U) \to HF(U)$ be a hesitant fuzzy operator, then there exists a serial hesitant fuzzy relation R on U such that $H = \overline{R}$ if and only if H satisfies the following axiom:*

$$\begin{aligned}(U - H(U)) \cup H(\bigcup_{j\in J}(\widehat{a_1^j \ldots a_m^j} \cap A_j)) \\ = \bigcup_{j\in J}(\widehat{a_1^j \ldots a_m^j} \cap H(A_j)).\end{aligned}$$

Where J is an index set, and $\forall\{a_1^j \ldots a_m^j\} \in 2^{[0,1]}$.

Proof Assume that there exists a hesitant fuzzy relation R such that $\forall A \in HF(U)$, $H(A) = \overline{R}(A)$. Then by Theorem 2, we have that $H(U) = \overline{R}(U) = U$, thus $U - H(U) = \emptyset$. By Theorem 5, we see that H satisfies:

$$\begin{aligned}(U - H(U)) \cup H(\bigcup_{j\in J}(\widehat{a_1^j \ldots a_m^j} \cap A_j)) \\ = \bigcup_{j\in J}(\widehat{a_1^j \ldots a_m^j} \cap H(A_j)).\end{aligned}$$

On the contrary, assume that L satisfies $(U - H(U)) \cup H(\bigcup_{j\in J}(\widehat{a_1^j \ldots a_m^j} \cap A_j)) = \bigcup_{j\in J}(\widehat{a_1^j \ldots a_m^j} \cap H(A_j))$. By taking $J = \{1\}, A_1 = U, a_1^1 \ldots a_m^1 = 0$ we have $(U - H(U)) \cup H(\widehat{0} \cap U) = \widehat{0} \cap H(U) = \emptyset$, thus $(U - H(U)) = \emptyset \Rightarrow U = H(U)$. By the assumption, we conclude that

$$\begin{aligned}(U - H(U)) \cup H\left(\bigcup_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cap A_j\right)\right) \\ = H\left(\bigcup_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cap A_j\right)\right) \\ = \bigcup_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cap H(A_j)\right).\end{aligned}$$

By Theorem 5, there exists a hesitant fuzzy relation R on U such that $\forall A \in HF(U)$, $H(A) = \overline{R}(A)$. Moreover, by Theorem 2, we conclude that R is serial. □

4.2 Single axiom for reflexive hesitant fuzzy rough approximation operators

In this subsection we will study how to use a single axiom to characterize the hesitant fuzzy rough approximation operators generated by a reflexive hesitant fuzzy relation. The results are summarized as follows.

Theorem 10 *Let $L : HF(U) \to HF(U)$ be a hesitant fuzzy operator, then there exists a reflexive hesitant fuzzy relation R on U such that $L = \underline{R}$ if and only if L satisfies the following axiom:*

$$\begin{aligned}L\left(\bigcap_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cup A_j\right)\right) = \left(\bigcap_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cup A_j\right)\right) \\ \cap\left(\bigcap_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cup L(A_j)\right)\right).\end{aligned}$$

Where $A_j \in HF(U)$, J is an index set, and $\forall\{a_1^j \ldots a_m^j\} \in 2^{[0,1]}$.

Proof Firstly, we consider the necessity. If there exists a reflexive fuzzy relation R on U such that $\forall A \in HF(U)$, $L(A) = \underline{R}(A)$ holds, then we have by Theorem 2 that $\forall A \in HF(U)$, $L(A) = \underline{R}(A) \subseteq A$. Therefore, $(\widehat{a_1^j \ldots a_m^j} \cup L(A)) \subseteq (\widehat{a_1^j \ldots a_m^j} \cup A)$. By Theorem 5, we see that L satisfies

$$\begin{aligned}L\left(\bigcap_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cup A_j\right)\right) \\ = \bigcap_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cup L(A_j)\right) \\ = \left(\bigcap_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cup A_j\right)\right) \cap \left(\bigcap_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cup L(A_j)\right)\right).\end{aligned}$$

Next we consider the sufficiency. Assume that L satisfies $L(\bigcap_{j\in J}(\widehat{a_1^j \ldots a_m^j} \cup A_j)) = (\bigcap_{j\in J}(\widehat{a_1^j \ldots a_m^j} \cup A_j)) \cap (\bigcap_{j\in J}(\widehat{a_1^j \ldots a_m^j} \cup L(A_j)))$. Where $A_j \in HF(U)$, J is an index set, and $\forall\{a_1 \dots a_m\} \in 2^{[0,1]}$. By taking $J = \{1\}, a_1^1 \dots a_m^1 = 0$ and $A_1 = B, B \in HF(U)$ we have that $L(\widehat{0} \cup B) = (\widehat{0} \cup B) \cap (\widehat{0} \cup L(B) \Rightarrow L(B) = B \cap L(B)$. Thus $L(B) \subseteq B$, by the assumption we conclude that

$$L\left(\bigcap_{j\in J}\left(\widehat{a_1^j \dots a_m^j} \cup A_j\right)\right) = \left(\bigcap_{j\in J}\left(\widehat{a_1^j \dots a_m^j} \cup A_j\right)\right) \cap \left(\bigcap_{j\in J}\left(\widehat{a_1^j \dots a_m^j} \cup L(A_j)\right)\right) = \bigcap_{j\in J}\left(\widehat{a_1^j \dots a_m^j} \cup L(A_j)\right).$$

By Theorem 5, there exists a hesitant fuzzy relation R on U such that $\forall A \in HF(U)$, $L(A) = \underline{R}(A)$. Thus, we conclude by Theorem 2 that R is reflexive. □

Theorem 11 *Let $H : HF(U) \to HF(U)$ be a hesitant fuzzy operator, then there exists a reflexive hesitant fuzzy relation R on U such that $\forall A \in HF(U)$, $H(A) = \overline{R}(A)$ holds, if and only if H satisfies the following axiom:*

$$H\left(\bigcup_{j\in J}\left(\widehat{a_1^j \dots a_m^j} \cap A_j\right)\right) = \left(\bigcup_{j\in J}\left(\widehat{a_1^j \dots a_m^j} \cap A_j\right)\right) \cup \left(\bigcup_{j\in J}\left(\widehat{a_1^j \dots a_m^j} \cap H\left(A_j\right)\right)\right).$$

Where $A_j \in HF(U)$, J is an index set, and $\forall\{a_1^j \dots a_m^j\} \in 2^{[0,1]}$.

Proof Necessity. If there exists a reflexive hesitant fuzzy relation R on U such that $\forall A \in HF(U)$, $H(A) = \overline{R}(A)$ holds. Then, by Theorem 2, we have $\forall A \in HF(U)$, $A \subseteq \overline{R}(A) = H(A)$. Thus, we see by Theorem 5 that

$$H\left(\bigcup_{j\in J}\left(\widehat{a_1^j \dots a_m^j} \cap A_j\right)\right) = \bigcup_{j\in J}\left(\widehat{a_1^j \dots a_m^j} \cap H\left(A_j\right)\right) = \left(\bigcup_{j\in J}\left(\widehat{a_1^j \dots a_m^j} \cap A_j\right)\right) \cup \left(\bigcup_{j\in J}\left(\widehat{a_1^j \dots a_m^j} \cap H\left(A_j\right)\right)\right).$$

Sufficiency. Assume that $\forall A_j \in HF(U)$ and $\forall\{a_1 \dots a_m\} \in 2^{[0,1]}$, $H(\bigcup_{j\in J}(\widehat{a_1^j \dots a_m^j} \cap A_j)) = (\bigcup_{j\in J}(\widehat{a_1^j \dots a_m^j} \cap A_j)) \cup (\bigcup_{j\in J}(\widehat{a_1^j \dots a_m^j} \cap H(A_j)))$ holds. By taking $J = \{1\}$, $a_1^1 \dots a_m^1 = 1$ and $A_1 = B$, $B \in HF(U)$ we have that $H(\widehat{1} \cap B) = (\widehat{1} \cap B) \cup (\widehat{1} \cap H(B)) \Rightarrow H(B) = B \cup H(B)$. Hence, from the arbitrariness of B, it is easy to observe that $(\widehat{a_1^j \dots a_m^j} \cap A_j) \subseteq (\widehat{a_1^j \dots a_m^j} \cap H(A_j))$. By the assumption, we conclude that

$$H\left(\bigcup_{j\in J}\left(\widehat{a_1^j \dots a_m^j} \cap A_j\right)\right) = \left(\bigcup_{j\in J}\left(\widehat{a_1^j \dots a_m^j} \cap A_j\right)\right) \cup (\bigcup_{j\in J}\left(\widehat{a_1^j \dots a_m^j} \cap H(A_j)\right)) = \bigcup_{j\in J}(\widehat{a_1^j \dots a_m^j} \cap H(A_j)).$$

By Theorem 5, there exists a hesitant fuzzy relation R on U such that $\forall A \in HF(U)$, $H(A) = \overline{R}(A)$. Moreover, we conclude by Theorem 2 that R is reflexive. □

4.3 Single axiom for symmetric hesitant fuzzy rough approximation operators

In this subsection we study how to use a single axiom to characterize the hesitant fuzzy rough approximation operators generated by a symmetric hesitant fuzzy relation. We first examine some properties of the lower and upper inverse operators.

Theorem 12 *Let $L : HF(U) \to HF(U)$ be a hesitant fuzzy operator. If L satisfies* Eq. (1) *of* Theorem 5*, then the following statements are equivalent:*

(1) $h_{L(1_{U-\{x\}})}(y) = h_{L(1_{U-\{y\}})}(x)$, $\forall x, y \in U$.
(2) $L(A) = L_{\underline{\vee}}^{-1}(A)$, $\forall A \in HF(U)$.

Proof (1) ⇒ (2) Noticing that for all $A \in HF(U)$, $A = \bigcap_{y\in U}(\widehat{h_A(y)} \cup 1_{U-\{y\}})$, we have $\forall x \in U$ that

$$\begin{aligned} L(A)(x) &= L\left(\bigcap_{y\in U}(h_A(y) \cup 1_{U-\{y\}})\right)(x) \\ &= \left(\bigcap_{y\in U}\left(\widehat{h_A(y)} \cup L(1_{U-\{y\}})\right)\right) && \text{by Theorem 5} \\ &= \bar{\bigwedge_{y\in U}}\,(h_A(y) \underline{\vee}\, h_{L(1_{U-\{y\}})}(x)) \\ &= \bar{\bigwedge_{y\in U}}\,(h_A(y) \underline{\vee}\, h_{L(1_{U-\{x\}})}(y)) && \text{by (1)} \\ &= L_{\underline{\vee}}^{-1}(A)(x). \end{aligned}$$

Therefore, $L(A) = L_{\underline{\vee}}^{-1}(A)$, $\forall A \in HF(U)$.

(2) ⇒ (1) For any $(x, y) \in U \times U$, since $\forall A \in HF(U)$, $L(A) = L_{\underline{\vee}}^{-1}(A)$, we conclude by taking $A =$

$1_{U-\{x\}}$ that

$$\begin{aligned} h_{L(1_{U-\{x\}})}(y) &= h_{L_{\underline{\vee}}^{-1}(1_{U-\{x\}})}(y) \\ &= [1_{U-\{x\}}, L(1_{U-\{y\}})] \\ &= \mathop{\overline{\wedge}}_{z\in U} (h_{1_{U-\{x\}}}(z) \underline{\vee}\, h_{L(1_{U-\{y\}})}(z)) \\ &= \left(\mathop{\overline{\wedge}}_{\{z\in U|z\neq x\}} (\{1\} \underline{\vee}\, h_{L(1_{U-\{y\}})}(z)) \right) \overline{\wedge}\, (\{0\} \\ &\quad \underline{\vee}\, h_{L(1_{U-\{y\}})}(x)) \\ &= h_{L(1_{U-\{y\}})}(x). \end{aligned}$$

□

Theorem 13 *Let $H : HF(U) \to HF(U)$ be a hesitant fuzzy operator. If H satisfies* Eq. (2) *of* Theorem 5, *then the following statements are equivalent:*

(1) $h_{H(1_{\{x\}})}(y) = h_{H(1_{\{y\}})}(x)$, $\forall x, y \in U$.
(2) $H(A) = H_{\overline{\wedge}}^{-1}(A)$, $\forall A \in HF(U)$.

Proof (1) $\Rightarrow$ (2) Since for all $A \in HF(U)$, $A = \bigcup_{y\in U} (\widehat{h_A(y)} \cap 1_y)$, then $\forall x \in U$ we have that

$$\begin{aligned} H(A)(x) &= H(\bigcup_{y\in U} (\widehat{h_A(y)} \cap 1_y))(x) \\ &= (\bigcup_{y\in U} (\widehat{h_A(y)} \cap H(1_y)))(x) \quad \text{by Theorem 5} \\ &= \mathop{\underline{\vee}}_{y\in U} (h_A(y)\, \overline{\wedge}\, h_{H(1_y)}(x)) \\ &= \mathop{\underline{\vee}}_{y\in U} (h_A(y)\, \overline{\wedge}\, h_{H(1_x)}(y)) \quad \text{by (1)} \\ &= H_{\overline{\wedge}}^{-1}(A)(x). \end{aligned}$$

As a result, $H(A) = H_{\overline{\wedge}}^{-1}(A)$, $\forall A \in HF(U)$.

(2) $\Rightarrow$ (1) $\forall (x, y) \in U \times U$, since $\forall A \in HF(U)$, $H(A) = H_{\overline{\wedge}}^{-1}(A)$. By taking $A = 1_x$, we have that

$$\begin{aligned} h_{H(1_x)}(y) &= h_{H_{\overline{\wedge}}^{-1}(1_x)}(y) \\ &= (1_x, H(1_y)) \\ &= \mathop{\underline{\vee}}_{z\in U} (h_{1_x}(z)\, \overline{\wedge}\, h_{H(1_y)}(z)) \\ &= \left(\mathop{\underline{\vee}}_{\{z\in U|z\neq x\}} (\{0\}\, \overline{\wedge}\, h_{H(1_y)}(z)) \right) \\ &\quad \underline{\vee}\, (\{1\}\, \overline{\wedge}\, h_{H(1_y)}(x)) \\ &= h_{H(1_y)}(x). \end{aligned}$$

□

Theorem 14 *Let $L : HF(U) \to HF(U)$ be a hesitant fuzzy operator, then there exists a symmetric hesitant fuzzy relation R on U such that $\forall A \in HF(U), L(A) = \underline{R}(A)$ holds, if and only if L satisfies the following axiom: $\forall \{a_1 \ldots a_m\}, \{a_1^j \ldots a_m^j\} \in 2^{[0,1]}$, $\forall A, A_j \in HF(U)$, $j \in J$, where J is an index set,*

$$\begin{aligned} &(\widehat{a_1 \ldots a_m} \cup L_{\underline{\vee}}^{-1}(A)) \cap L\left(\bigcap_{j\in J} \left(\widehat{a_1^j \ldots a_m^j} \cup A_j\right)\right) \\ &\quad = \left(\widehat{a_1 \ldots a_m} \cup L(A)\right) \cap \left(\bigcap_{j\in J} \left(\widehat{a_1^j \ldots a_m^j} \cup L(A_j)\right)\right). \end{aligned}$$

Proof Firstly, we prove the necessity. If there exists a symmetric hesitant fuzzy relation R on U such that $\forall A \in HF(U)$, $L(A) = \underline{R}(A)$ holds, then, we know that L satisfies Equation (1) of Theorem 5. Since the symmetry of R, we conclude that $h_{\underline{R}(U-\{x\})}(y) = h_{\underline{R}(U-\{y\})}(x)$, $\forall (x, y) \in U \times U$. Thus $h_{L(1_{U-\{x\}})}(y) = h_{L(l_{U-\{y\}})}(x)$. Using Theorem 12 we have $L(A) = L_{\underline{\vee}}^{-1}(A)$, $\forall A \in HF(U)$. Hence, by equation (1) of Theorem 5, we conclude that

$$\begin{aligned} &\left(\widehat{a_1 \ldots a_m} \cup L_{\underline{\vee}}^{-1}(A)\right) \cap L\left(\bigcap_{j\in J} \left(\widehat{a_1^j \ldots a_m^j} \cup A_j\right)\right) \\ &\quad = \left(\widehat{a_1 \ldots a_m} \cup L(A)\right) \cap \left(\bigcap_{j\in J} \left(\widehat{a_1^j \ldots a_m^j} \cup L(A_j)\right)\right). \end{aligned}$$

Secondly, we prove the sufficiency. Assume that $\forall A \in HF(U)$, $A_j \in HF(U)$, $j \in J$, where J is an index set, and $\{a_1 \ldots a_m\}$, $\{a_1^j \ldots a_m^j\} \in 2^{[0,1]}$, $(\widehat{a_1 \ldots a_m} \cup L_{\underline{\vee}}^{-1}(A)) \cap L(\bigcap_{j\in J} (\widehat{a_1^j \ldots a_m^j} \cup A_j)) = (\widehat{a_1 \ldots a_m} \cup L(A)) \cap (\bigcap_{j\in J} (\widehat{a_1^j \ldots a_m^j} \cup L(A_j)))$ holds. By taking $a_1 = \ldots = a_m = 1$ we have $L(\bigcap_{j\in J} (\widehat{a_1^j \ldots a_m^j} \cup A_j)) = (\bigcap_{j\in J} (\widehat{a_1^j \ldots a_m^j} \cup L(A_j)))$. Thus by Theorem 5, there exists a hesitant fuzzy relation R on U such that $\forall A \in HF(U), L(A) = \underline{R}(A)$. Then by taking $J = \{1\}$, $a_1 = \ldots = a_m = a_1^j = \ldots = a_m^j = 1$, $A = A_1$ we have

$$\begin{aligned} &(\widehat{1} \cup L_{\underline{\vee}}^{-1}(A)) \cap L(\widehat{1} \cup A) \\ &\quad = (\widehat{1} \cup L(A)) \cap (\widehat{1} \cup L(A)) \Rightarrow \widehat{1} \cap L(\widehat{1}) = \widehat{1}, \end{aligned}$$

thus $L(\widehat{1}) = L(U) = U$.

On the other hand, by taking $J = \{1\}, a_1 = \ldots = a_m = 0$, $a_1^1 = \ldots = a_m^1 = 1$, $A = A_1$, we have

$$\begin{aligned} &(\widehat{0} \cup L_{\underline{\vee}}^{-1}(A)) \cap L(\widehat{1} \cup A) \\ &\quad = (\widehat{0} \cup L(A)) \cap (\widehat{1} \cup L(A)) \Rightarrow L_{\underline{\vee}}^{-1}(A) \cap L(A) = L(A). \end{aligned}$$

Therefore, we conclude $L_{\underline{\vee}}^{-1}(A) = L(A)$, $\forall A \in HF(U)$. By Theorem 12 we conclude that R is a symmetric hesitant fuzzy relation. □

Theorem 15 *Let $H : HF(U) \to HF(U)$ be a hesitant fuzzy operator, then there exists a symmetric hesitant fuzzy relation R on U such that $\forall A \in HF(U)$, $H(A) = \overline{R}(A)$ holds, if and only if H satisfies the following axiom: $\forall \{a_1 \ldots a_m\}$, $\{a_1^j \ldots a_m^j\} \in 2^{[0,1]}$, $\forall A, A_j \in HF(U)$, $j \in J$, where J is an index set,*

$$(\widehat{a_1 \ldots a_m} \cap H_{\bar{\wedge}}^{-1}(A)) \cup H(\bigcup_{j\in J}(\widehat{a_1^j \ldots a_m^j} \cap A_j))$$
$$= (\widehat{a_1 \ldots a_m} \cap H(A)) \cup (\bigcup_{j\in J}(\widehat{a_1^j \ldots a_m^j} \cap H(A_j))).$$

Proof If there exists a symmetric hesitant fuzzy relation R on U such that $\forall A \in HF(U)$, $H(A) = \overline{R}(A)$ holds, then, we know that H satisfies Equation (2) of Theorem 5. Since the symmetry of R, we have $h_{\overline{R}(1_x)}(y) = h_{\overline{R}(1_x)}(y)$, $\forall (x, y) \in U \times U$, thus, $h_{H(1_x)}(y) = h_{H(1_x)}(y)$. Then by Theorem 13, we have $H(A) = H_{\bar{\wedge}}^{-1}(A)$, $\forall A \in HF(U)$ holds. Hence, by Equation (2) of Theorem 5, we conclude that

$$(\widehat{a_1 \ldots a_m} \cap H_{\bar{\wedge}}^{-1}(A)) \cup H(\bigcup_{j\in J}(\widehat{a_1^j \ldots a_m^j} \cap A_j))$$
$$= (\widehat{a_1 \ldots a_m} \cap H(A)) \cup (\bigcup_{j\in J}(\widehat{a_1^j \ldots a_m^j} \cap H(A_j))).$$

Conversely, assume that $\forall A \in HF(U)$, $A_j \in HF(U)$, $j \in J$, and $\{a_1 \ldots a_m\}$, $\{a_1^j \ldots a_m^j\} \in 2^{[0,1]}$, $(\widehat{a_1 \ldots a_m} \cap H_{\bar{\wedge}}^{-1}(A)) \cup H(\bigcup_{j\in J}(\widehat{a_1^j \ldots a_m^j} \cap A_j)) = (\widehat{a_1 \ldots a_m} \cap H(A)) \cup (\bigcup_{j\in J}(\widehat{a_1^j \ldots a_m^j} \cap H(A_j)))$. holds. By taking $a_1 = \ldots = a_m = 0$ we have $H(\bigcup_{j\in J}(\widehat{a_1^j \ldots a_m^j} \cap A_j)) = (\bigcup_{j\in J}(\widehat{a_1^j \ldots a_m^j} \cap H(A_j)))$. Thus, by Theorem 5, there exists a hesitant fuzzy relation R on U such that $\forall A \in HF(U)$, $H(A) = \overline{R}(A)$ holds. Then by taking $J = \{1\}$, $a_1 = \ldots = a_m = a_1^j = \ldots = a_m^j = 0$, $A = A_1$, we have

$$(\widehat{0} \cap H_{\bar{\wedge}}^{-1}(A)) \cup H(\widehat{0} \cap A)$$
$$= (\widehat{0} \cap H(A)) \cup (\widehat{0} \cap H(A)) \Rightarrow H(\widehat{0}) = \emptyset.$$

On the other hand, by taking $J = \{1\}$, $a_1 = \ldots = a_m = 1$, $a_1^1 = \ldots = a_m^1 = 0$, $A = A_1$, we have

$$(\widehat{1} \cap H_{\bar{\wedge}}^{-1}(A)) \cup H(\widehat{0} \cap A)$$
$$= (\widehat{1} \cap H(A)) \cup (\widehat{0} \cap H(A)) \Rightarrow H_{\bar{\wedge}}^{-1}(A) \cup H(\widehat{0}) = H(A).$$

Therefore, we have that $H_{\bar{\wedge}}^{-1}(A) = H(A)$, $\forall A \in HF(U)$. By Theorem 13, we conclude that hesitant fuzzy relation R is symmetric. □

Theorem 16 *Let $L : HF(U) \to HF(U)$ be a hesitant fuzzy operator, then there exists a symmetric hesitant fuzzy relation R on U such that $\forall A \in HF(U)$, $L(A) = \underline{R}(A)$ holds, if and only if L satisfies the following axiom:*

$$[A, L(B)] = [L(A), B], \ \forall A, B \in HF(U).$$

Proof Necessity. If there exists a symmetric hesitant fuzzy relation R on U such that $L = \underline{R}$, then by Theorem 6 we have that $[A, L(B)] = [B, L_{\underline{\vee}}^{-1}(A)], \forall A, B \in HF(U)$. From Theorems 2 and 12 we know that $L = L_{\underline{\vee}}^{-1}$. Thus we conclude that $[A, L(B)] = [L(A), B], \forall A, B \in HF(U)$.

Sufficiency. Assume that $[A, L(B)] = [L(A), B], \forall A, B \in HF(U)$ holds. For any $A \in HF(U)$ and $x \in U$, by taking $B = 1_{U-\{x\}}$. Then, by Definition 10 and Property 3, we see that $[A, L(B)] = [A, L(1_{U-\{x\}})] = [L(1_{U-\{x\}}), A] = L_{\underline{\vee}}^{-1}(A)(x)$. On the other hand, $[L(A), B] = [L(A), 1_{U-\{x\}}] = \bar{\wedge}_{y\in U} (h_{L(A)}(y) \underline{\vee} h_{1_{U-\{x\}}}(y)) = \bar{\wedge}_{\{y\in U | y\neq x\}} (h_{L(A)}(y) \underline{\vee} 1) \wedge (h_{L(A)}(x) \underline{\vee} 0) = L(A)(x)$.

In summary $L(A)(x) = L_{\underline{\vee}}^{-1}(A)(x)$, thus $[A, L(B)] = [L(A), B] = [B, L(A)] = [B, L_{\underline{\vee}}^{-1}(A)]$. By Theorem 6, there exists a hesitant fuzzy relation R on U such that $L = \underline{R}$ holds. Then, by Theorems 2 and 12 we know that hesitant fuzzy relation R is a symmetric hesitant fuzzy relation. □

Theorem 17 *Let $H : HF(U) \to HF(U)$ be a hesitant fuzzy operator, then there exists a symmetric hesitant fuzzy relation R on U such that $\forall A \in HF(U)$, $H(A) = \overline{R}(A)$ holds, if and only if H satisfies the following axiom:*

$$(A, H(B)) = (H(A), B), \ \forall A, B \in HF(U).$$

Proof Necessity. If there exists a symmetric hesitant fuzzy relation R on U such that $H = \overline{R}$, then by Theorem 7 we have that $(A, H(B)) = (B, H_{\bar{\wedge}}^{-1}(A))$, $\forall A, B \in HF(U)$. From Theorems 2 and 13 we know that $H = H_{\bar{\wedge}}^{-1}$. Thus we conclude that $(A, H(B)) = (H(A), B), \forall A, B \in HF(U)$.

Sufficiency. Assume that $(A, H(B)) = (H(A), B)$, $\forall A, B \in HF(U)$ holds. For any $A \in HF(U)$ and $x \in U$, by taking $B = 1_x$. Then, by Definition 10 and Property 4, we see that $(A, H(B)) = (A, H(1_x)) = (H(1_x), A) = H_{\bar{\wedge}}^{-1}(A)(x)$. On the other hand, $(H(A), B) = (H(A), 1_x) = \underline{\vee}_{y\in U} (h_{H(A)}(y) \bar{\wedge} h_{1_x}(y)) = \underline{\vee}_{\{y\in U | y\neq x\}} (h_{H(A)}(y) \bar{\wedge} 0) \vee (h_{L(A)}(x) \bar{\wedge} 1) = H(A)(x)$.

In summary $H(A)(x) = H_{\bar{\wedge}}^{-1}(A)(x)$, thus $(A, H(B)) = (H(A), B) = (B, H(A)) = (B, H_{\bar{\wedge}}^{-1}(A))$. By Theorem 7, there exists a hesitant fuzzy relation R on U such that $H = \overline{R}$ holds. Then, by Theorems 2 and 13 we know that hesitant fuzzy relation R is a symmetric hesitant fuzzy relation. □

4.4 Single axiom for transitive hesitant fuzzy rough approximation operators

In this subsection we will study single axioms to characterize hesitant fuzzy rough approximation operators generated by a transitive hesitant fuzzy relation. Axiomatic characterizations of transitive hesitant fuzzy rough approximation operators are summarized as follows.

Theorem 18 *Let $L : HF(U) \to HF(U)$ be a hesitant fuzzy operator, then there exists a transitive hesitant fuzzy relation R on U such that $\forall A \in HF(U)$, $L(A) = \underline{R}(A)$ holds, if and only if L satisfies the following axiom: $\forall A, A_j \in HF(U)$, and $\forall \{a_1^j \ldots a_m^j\} \in 2^{[0,1]}$, $j \in J$, J is an index set,*

$$L\left(\bigcap_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cup A_j\right)\right) = \left(\bigcap_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cup L(A_j)\right)\right) \cap \left(\bigcap_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cup L(L(A_j))\right)\right).$$

Proof Firstly, we consider the necessity. If there exists a transitive hesitant fuzzy relation R on U such that $\forall A \in HF(U)$, $L(A) = \underline{R}(A)$ holds, by Theorem 2, we have $\forall A \in HF(U)$, $\underline{R}(A) \subseteq \underline{R}(\underline{R}(A))$ i.e. $L(A) \subseteq L(L(A))$. Thus, we conclude that $(\widehat{a_1 \ldots a_m} \cup L(A)) \subseteq (\widehat{a_1 \ldots a_m} \cup L(L(A)))$, $\forall \{a_1 \ldots a_m\} \in 2^{[0,1]}$. Then by equation (1) of Theorem 5, we conclude that

$$L\left(\bigcap_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cup A_j\right)\right) = \left(\bigcap_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cup L(A_j)\right)\right) \cap \left(\bigcap_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cup L(L(A_j))\right)\right).$$

Then we consider the sufficiency. Assume that L satisfies

$$L\left(\bigcap_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cup A_j\right)\right) = \left(\bigcap_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cup L(A_j)\right)\right) \cap \left(\bigcap_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cup L(L(A_j))\right)\right),$$

where $\forall A, A_j \in HF(U)$, and $\forall \{a_1^j \ldots a_m^j\} \in 2^{[0,1]}$, $j \in J$, J is an index set. By taking $J = \{1\}$, $a_1^1 = \ldots = a_m^1 = 0$, $A_1 = A$, $\forall A \in HF(U)$ we have

$$L(\widehat{0} \cup A) = (\widehat{0} \cup L(A)) \cap (\widehat{0} \cup L(L(A))),$$

That is, $L(A) = L(A) \cap L(L(A))$. Therefore, we conclude that

$$\left(\widehat{a_1^j \ldots a_m^j} \cup L(A)\right) \subseteq \left(\widehat{a_1^j \ldots a_m^j} \cup L(L(A))\right),$$

thus equation $(\widehat{a_1^j \ldots a_m^j} \cup L(A)) = (\widehat{a_1^j \ldots a_m^j} \cup L(A)) \cap (\widehat{a_1^j \ldots a_m^j} \cup L(L(A)))$ holds. Then we conclude that

$$\begin{aligned} L\left(\bigcap_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cup A_j\right)\right) &= \left(\bigcap_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cup L(A_j)\right)\right) \\ &\quad \cap \left(\bigcap_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cup L(L(A_j))\right)\right) \\ &= \left(\bigcap_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cup L(A_j)\right)\right). \end{aligned}$$

Thus, by Theorem 5, there exists a hesitant fuzzy relation R on U such that $\forall A \in HF(U)$, $L(A) = \underline{R}(A)$ holds, hence we have $\underline{R}(A) \subseteq \underline{R}(\underline{R}(A)), \forall A \in HF(U)$. Then by Theorem 2, that hesitant fuzzy relation R is transitive. □

Theorem 19 *Let $H : HF(U) \to HF(U)$ be a hesitant fuzzy operator, then there exists a transitive hesitant fuzzy relation R on U such that $\forall A \in HF(U)$, $H(A) = \overline{R}(A)$ holds, if and only if L satisfies the following axiom: $\forall A, A_j \in HF(U)$, and $\forall \{a_1^j \ldots a_m^j\} \in 2^{[0,1]}$, $j \in J$, J is an index set,*

$$H\left(\bigcup_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cap A_j\right)\right) = \bigcup_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cap H(A_j)\right) \cup \left(\bigcup_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cap H(H(A_j))\right)\right).$$

Proof Necessity. If there exists a transitive hesitant fuzzy relation R on U such that $\forall A \in HF(U)$, $H(A) = \overline{R}(A)$ holds. By Theorem 2, we have $\forall A \in HF(U)$, $\overline{R}(\overline{R}(A)) \subseteq \overline{R}(A)$ i.e. $H(H(A)) \subseteq H(A)$. Thus, we conclude that $(\widehat{a_1 \ldots a_m} \cap H(H(A))) \subseteq (\widehat{a_1 \ldots a_m} \cap H(A)), \forall \{a_1 \ldots a_m\} \in 2^{[0,1]}$. Then by Equation (2) of Theorem 5, we conclude that

$$H\left(\bigcup_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cap A_j\right)\right) = \bigcup_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cap H(A_j)\right) \cup \left(\bigcup_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cap H(H(A_j))\right)\right).$$

Sufficiency. Assume that H satisfies

$$H\left(\bigcup_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cap A_j\right)\right) = \bigcup_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cap H(A_j)\right) \cup \left(\bigcup_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cap H(H(A_j))\right)\right).$$

By taking $J = \{1\}$, $a_1^1 = \ldots = a_m^1 = 1$, $A_1 = A$, $\forall A \in HF(U)$ we have

$$H(\widehat{1} \cap A) = (\widehat{1} \cap H(A)) \cup (\widehat{1} \cap H(H(A))),$$

i.e. $H(A) = H(A) \cup H(H(A))$. Therefore, we conclude that

$$\left(\widehat{a_1^j \ldots a_m^j} \cap H(H(A))\right) \subseteq (\widehat{a_1^j \ldots a_m^j} \cap H(A)).$$

Therefore, $(\widehat{a_1^j \ldots a_m^j} \cap H(A)) = (\widehat{a_1^j \ldots a_m^j} \cap H(A)) \cup (\widehat{a_1^j \ldots a_m^j} \cap H(H(A)))$ holds. Then we conclude that

$$H\left(\bigcup_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cap A_j\right)\right) = \bigcup_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cap H\left(A_j\right)\right) \cup \left(\bigcup_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cap H\left(H\left(A_j\right)\right)\right)\right) = \bigcup_{j\in J}\left(\widehat{a_1^j \ldots a_m^j} \cap H\left(A_j\right)\right).$$

Thus, by Theorem 5, there exists a hesitant fuzzy relation R on U such that $\forall A \in HF(U), H(A) = \overline{R}(A)$ holds. Hence we have $\overline{R}(\overline{R}(A)) \subseteq \overline{R}(A), \forall A \in HF(U)$. Thus by Theorem 2, R is a transitive hesitant fuzzy relation. □

5 Comparative analysis

In this section, we will compare and analyze the advantages and disadvantages of fuzzy rough set, hesitant fuzzy set and hesitant fuzzy rough set by some cases. Fuzzy rough set, hesitant fuzzy set and hesitant fuzzy rough set are extensively used in multi-criteria decision making, multi-attribute decision making(MADM), attribute reduction, clustering analysis and other fields. In solving the problem of MADM, we often encounter the situation of hesitation in the determination of attribute value i.e. the attribute value is in the form of hesitant fuzzy element. In view of the above situation, we redefine the similarity between two hesitant fuzzy sets according to the reference Xu and Zhang (2013), and obtain a measure-based hesitant fuzzy MADM method. The approach involves the following steps:

Step 1 Let A_i be the i-th alternative, c_j be the i-th attribute. Determining hesitant fuzzy decision matrix.

Step 2 According to the decision matrix, the positive ideal solution A^+ and the negative ideal solution A^- are calculated.

$$A^+ = \left\{ \bigvee_{i=1,2,\ldots,n} h_{A_i}(c_j) | j = 1,2,\ldots,m \right\} = \left\{ \left\{ \bigvee_{i=1,2,\ldots,n} h_{A_i}^{\sigma(k)}(c_j) | k = 1,2,\ldots,l(h_{A_i}(c_j)) \right\} | j = 1,2,\ldots,m \right\}, \quad (3)$$

$$A^- = \left\{ \bigwedge_{i=1,2,\ldots,n} h_{A_i}(c_j) | j = 1,2,\ldots,m \right\} = \left\{ \left\{ \bigwedge_{i=1,2,\ldots,n} h_{A_i}^{\sigma(k)}(c_j) | k = 1,2,\ldots,l(h_{A_i}(c_j)) \right\} | j = 1,2,\ldots,m \right\}, \quad (4)$$

where n is the number of alternatives and m is the number of attributes.

Step 3 Calculate the separation measures d_i^+ and d_i^- of each alternative A_i from the positive ideal solution A^+ and the negative ideal solution A^-, respectively.

$$d_i^+(A_i, A^+) = \frac{\sum_{j=1,2,\ldots,m} \frac{\sum_k |h_{A_i}^{\sigma(k)}(c_j) - h_{A^+}^{\sigma(k)}(c_j)|}{l^+}}{m}, k = 1,2,\ldots,l^+, \quad (5)$$

$$d_i^-(A_i, A^-) = \frac{\sum_{j=1,2,\ldots,m} \frac{\sum_k |h_{A_i}^{\sigma(k)}(c_j) - h_{A^-}^{\sigma(k)}(c_j)|}{l^-}}{m}, k = 1,2,\ldots,l^-, \quad (6)$$

where $l^+ = max\{l(h_{A_i}(c_j)), l(h_{A^+}(c_j))\}$, $l^- = max\{l(h_{A_i}(c_j)), l(h_{A^-}(c_j))\}$.

Step 4 Calculate the relative closeness and then select the most desirable one. Obviously, the smaller the value of C_i the better corresponding alternative A_i.

$$C_i = \frac{d_i^+(A_i)}{d_i^+(A_i) + d_i^-(A_i)}. \quad (7)$$

Next, the method is applied to an example in LI (2019) to illustrate its effectiveness.

Example 1 This example analyzes the influencing factors of the evaluation index of "postgraduate training". Suppose that there are four alternatives: A_1: doctoral student scale; A_2: scale of master degree students; A_3: outstanding achievements; A_4: scientific research level and four attributes: c_1: academic output power; c_2: academic influence power; c_3: academic innovation power; c_4: academic growth power. The hesitant fuzzy decision matrix for the above information is given in Table 1.

Step 2 Utilize formulas (3) and (4) to determine the positive ideal solution A^+ and the negative ideal solution A^-, respectively:

$$A^+ = \{\{0.6, 0.5, 0.3\}, \{0.8, 0.6, 0.3\}, \{0.8, 0.7, 0.5, 0.4\}, \{0.8, 0.7, 0.6\}\},$$
$$A^- = \{\{0.4, 0.3, 0.2, 0.1\}, \{0.5, 0.4, 0.2\}, \{0.6, 0.5, 0.4, 0.3, 0.2\}, \{0.6, 0.5, 0.3, 0.2\}\}.$$

Step 3 Utilize formulas (5) and (6) to calculate the separation measures d_i^+ and d_i^- of each alternative A_i from the positive ideal solution A^+ and the negative ideal solution A^-,

Table 1 The hesitant fuzzy decision matrix

	c_1	c_2	c_3	c_4
A_1	{0.5, 0.3, 0.2}	{0.8, 0.6, 0.3}	{0.8, 0.7, 0.5, 0.3, 0.2}	{0.6, 0.5, 0.4, 0.3, 0.2}
A_2	{0.5, 0.4, 0.2, 0.1}	{0.5, 0.4, 0.3, 0.2}	{0.6, 0.5, 0.4, 0.3}	{0.8, 0.6, 0.5, 0.4}
A_3	{0.6, 0.5, 0.3}	{0.6, 0.4, 0.2}	{0.7, 0.6, 0.5, 0.3}	{0.6, 0.5, 0.3, 0.2}
A_4	{0.4, 0.3, 0.2}	{0.6, 0.4, 0.3}	{0.6, 0.5, 0.4}	{0.8, 0.7, 0.6}

Table 2 The fuzzy decision system

	c_1	c_2	c_3	c_4	d
x_1	0.8	0.1	0.1	0.5	1
x_2	0.3	0.5	0.2	0.8	1
x_3	0.2	0.2	0.6	0.7	0
x_4	0.6	0.3	0.1	0.2	1
x_5	0.3	0.4	0.3	0.3	0

respectively:

$$d_1^+ = 0.08958,\ d_2^+ = 0.13125,$$
$$d_3^+ = 0.12917,\ d_4^+ = 0.10625,$$
$$d_1^- = 0.13125,\ d_2^- = 0.0675,$$
$$d_3^- = 0.0708,\ d_4^- = 0.10667.$$

Step 4 The relative closenesses are calculated by formula (7):

$$C_1 = 0.537549,\ C_2 = 0.660377,$$
$$C_3 = 0.641822,\ C_4 = 0.499022.$$

Thus, $C_4 < C_1 < C_3 < C_2$. Obviously, A_4 is the best alternative. The results are consistent with those in reference LI (2019).

We find that hesitant fuzzy membership function has a certain subjective apriority. However, in rough set theory, the upper and lower approximation operators are obtained by objective calculation, so fuzzy rough set has a certain objectivity in dealing with uncertain information. This objectivity can be reflected by attribute reduction method based on fuzzy rough set in reference Chen et al. (2019).

Example 2 Table 2 is a fuzzy decision system $(U, C \cup \{d\})$, in which object set $U = \{x_1, x_2, \ldots, x_5\}$, attribute set $C = \{c_1, c_2, \ldots, c_4\}$.

Next, use each attribute c_j to define a fuzzy equivalence relation $\widetilde{R}_j$,

$$\widetilde{R}_j(x_i, x_{i'}) = \begin{cases} min\{c_j(x_i), c_j(x_{i'})\}, i \neq i' \\ 1, i = i' \end{cases}$$

The calculation is as follows.

$$\widetilde{R}_1 = \begin{pmatrix} 1 & 0.3 & 0.2 & 0.6 & 0.3 \\ & 1 & 0.2 & 0.3 & 0.3 \\ & & 1 & 0.2 & 0.2 \\ & & & 1 & 0.3 \\ & & & & 1 \end{pmatrix}$$

$$\widetilde{R}_2 = \begin{pmatrix} 1 & 0.1 & 0.1 & 0.1 & 0.1 \\ & 1 & 0.2 & 0.3 & 0.4 \\ & & 1 & 0.2 & 0.2 \\ & & & 1 & 0.3 \\ & & & & 1 \end{pmatrix}$$

$$\widetilde{R}_3 = \begin{pmatrix} 1 & 0.1 & 0.1 & 0.1 & 0.1 \\ & 1 & 0.2 & 0.1 & 0.2 \\ & & 1 & 0.1 & 0.3 \\ & & & 1 & 0.1 \\ & & & & 1 \end{pmatrix}$$

$$\widetilde{R}_4 = \begin{pmatrix} 1 & 0.5 & 0.5 & 0.2 & 0.3 \\ & 1 & 0.7 & 0.2 & 0.3 \\ & & 1 & 0.2 & 0.3 \\ & & & 1 & 0.2 \\ & & & & 1 \end{pmatrix}$$

Then construct fuzzy equivalence relation $\widetilde{R} = \underset{j=1,2,\ldots,4}{\cap} \widetilde{R}_j$.

$$\widetilde{R} = \begin{pmatrix} 1 & 0.1 & 0.1 & 0.1 & 0.1 \\ & 1 & 0.2 & 0.1 & 0.2 \\ & & 1 & 0.1 & 0.2 \\ & & & 1 & 0.1 \\ & & & & 1 \end{pmatrix}$$

Obviously, the decision attribute d divides U into two classes $A = \{x_1, x_2, x_4\}$, $B = \{x_3, x_5\}$. From the definition of fuzzy rough lower approximation defined by Chen et al. (2019), the lower approximations of A and B with respect to equivalence relation $\widetilde{R}$ can be calculated: $\widetilde{R}_*A = \frac{0.9}{x_1} + \frac{0.8}{x_2} + \frac{0.9}{x_4}$, $\widetilde{R}_*B = \frac{0.8}{x_3} + \frac{0.8}{x_5}$. Next, the fuzzy rough discernibility matrix defined by Chen et al. (2019)$M_C(D)$ can be calculated as

$$M_C(D) = \begin{pmatrix} & & 23 & & 23 \\ & & 123 & & 3 \\ 234 & 123 & & 1234 & \\ & & 3 & & 3 \\ 23 & 3 & & 34 & \end{pmatrix}$$

It is easy to get the unique relative reduction: $\{c_3\}$.

In the process of attribute reduction, the lower approximation is obtained by objective calculation, so this method has a certain objectivity. Next, the above fuzzy rough attribute reduction method is extended to hesitant fuzzy environment. The steps of hesitant fuzzy rough attribute reduction are formulated as follows.

Step 1 Determining hesitant fuzzy decision system $(U, C \cup D)$.

Step 2 Calculating fuzzy equivalence relation $R = \bigcap_{j=1,2,\ldots,4} R_j$, $j = 1, 2, \ldots, m$, m is the number of attributes.

$$h_{R_j}(x_i, x_{i'}) = \begin{cases} h_{c_j}(x_i) \,\overline{\wedge}\, h_{c_j}(x_{i'}) = \{h_{c_j}^{\sigma(k)}(x_i) \wedge h_{c_j}^{\sigma(k)}(x_{i'}) \\ \qquad |k = 1, 2, \ldots, l\}, i \neq i' \\ 1, i = i' \end{cases} \tag{8}$$

where $l = max\{l(h_{c_j}(x_i)), l(h_{c_j}(x_{i'}))\}$.

Step 3 Calculating the decision partition by $(U, C \cup D)$ and the hesitant fuzzy rough lower approximations for each decision class, by Definition 7.

Step 4 Determining the hesitant fuzzy rough discernibility matrix $HM_c(D) = (c_{ii'})$.

$$c_{ii'} = \begin{cases} \{c \in C | \overline{h_{R_j^c}(x_i, x_{i'})} \geq \overline{\lambda(x_i)}\}, D(x_i) \neq D(x_{i'}) \\ \emptyset, D(x_i) = D(x_{i'}) \end{cases} \tag{9}$$

where $\overline{h_{R_j^c}(x_i, x_{i'})} = \frac{\sum_{k=1,2,\ldots,l_1} h_{R^c}^{\sigma(k)}(x_i, x_{i'})}{l_1}$, $k = 1, 2, \ldots, l_1$, $l_1 = l(h_{R^c}(x_i, x_{i'}))$, $\overline{\lambda(x_i)} = \frac{\sum_{k=1,2,\ldots,l_2} h_{R_*([x_i]_D)}^{\sigma(k)}(x_i)}{l_2}$, $l_2 = l(h_{R_*([x_i]_D)}(x_i))$, $[x_i]_D$ is an equivalent class of decision attributes.

If the membership degree of an object is given by several experts, the fuzzy decision system will be transformed into a hesitant fuzzy decision system. Obviously, fuzzy rough set can't solve the problem of attribute reduction in hesitant fuzzy decision system, thus it is meaningful to extend fuzzy rough set to hesitant fuzzy rough set.

Example 3 Several experts give the membership of the object in Example 2, and the membership degree will be a hesitant fuzzy element. The following Table 3 is a hesitant fuzzy decision system $(U, C \cup D)$.

Step 2 Utilize formula (8) to calculate fuzzy equivalence relation R_j and R.

$$R_1 = \begin{pmatrix} \{1\} & \{0.3, 0.2\} & \{0.3, 0.2\} & \{0.6, 0.5\} & \{0.4, 0.3, \} \\ & \{1\} & \{0.3, 0.2\} & \{0.3, 0.2\} & \{0.5, 0.4, 0.3\} \\ & & \{1\} & \{0.3, 0.2\} & \{0.3, 0.2\} \\ & & & \{1\} & \{0.3, 0.2\} \\ & & & & \{1\} \end{pmatrix}$$

$$R_2 = \begin{pmatrix} \{1\} & \{0.2, 0.1\} & \{0.2, 0.1\} & \{0.2, 0.1\} & \{0.2, 0.1\} \\ & \{1\} & \{0.3, 0.2\} & \{0.3, 0.2\} & \{0, 5, 0.4, 0.3\} \\ & & \{1\} & \{0.3, 0.2\} & \{0.3, 0.2\} \\ & & & \{1\} & \{0.3, 0.2\} \\ & & & & \{1\} \end{pmatrix}$$

$$R_3 = \begin{pmatrix} \{1\} & \{0.2, 0.1\} & \{0.2, 0.1\} & \{0.2, 0.1\} & \{0.2, 0.1\} \\ & \{1\} & \{0.3, 0.2, 0.1\} & \{0.2, 0.1\} & \{0.3, 0.2, 0.1\} \\ & & \{1\} & \{0.2, 0.1\} & \{0.3, 0.2\} \\ & & & \{1\} & \{0.2, 0.1\} \\ & & & & \{1\} \end{pmatrix}$$

$$R_4 = \begin{pmatrix} \{1\} & \{0.6, 0.5, 0.4\} & \{0.6, 0.5, 0.4\} & \{0.3, 0.2\} & \{0.3, 0.2, 0.1\} \\ & \{1\} & \{0.8, 0.7\} & \{0.3, 0.2\} & \{0.3, 0.2, 0.1\} \\ & & \{1\} & \{0.3, 0.2\} & \{0.3, 0.2, 0.1\} \\ & & & \{1\} & \{0.3, 0.2, 0.1\} \\ & & & & \{1\} \end{pmatrix}$$

$$R = \begin{pmatrix} \{1\} & \{0.2, 0.1\} & \{0.2, 0.1\} & \{0.2, 0.1\} & \{0.2, 0.1\} \\ & \{1\} & \{0.3, 0.2, 0.1\} & \{0.2, 0.1\} & \{0.3, 0.2, 0.1\} \\ & & \{1\} & \{0.2, 0.1\} & \{0.3, 0.2, 0.1\} \\ & & & \{1\} & \{0.2, 0.1\} \\ & & & & \{1\} \end{pmatrix}$$

Step 3 Obviously, the decision attribute d divides U into two classes $A = \{x_1, x_2, x_4\}$, $B = \{x_3, x_5\}$. By Definition 7, the lower approximations of A and B with respect to can be calculated $R_*A = \frac{\{0.9,0.8\}}{x_1} + \frac{\{0.9,0.8,0.7\}}{x_2} + \frac{\{0.9,0.8\}}{x_4}$, $R_*B = \frac{\{0.9,0.8,0.7\}}{x_3} + \frac{\{0.9,0.8,0.7\}}{x_5}$. Next, calculate the fuzzy rough discernibility $HM_C(D)$ by formula (9):

$$HM_C(D) = \begin{pmatrix} & & 23 & & 23 \\ & & 3 & & 34 \\ 23 & 3 & & 3 & \\ & & 3 & & 3 \\ 234 & 34 & & 34 & \end{pmatrix}$$

Obviously, the unique relative reduction is $\{c_3\}$.

To sum up, when solving the problem of attribute reduction in hesitant fuzzy environment, integrating the objectivity of fuzzy rough set into hesitant fuzzy rough set can make the final result objective.

6 Conclusion

In the development of hesitant fuzzy rough set theory, the axiomatization of approximation operator is a significant direction to research the mathematical structure of hesitant

Table 3 The hesitant fuzzy decision system

	c_1	c_2	c_3	c_4	d
x_1	{0.8, 0.7, 0.6}	{0.2, 0.1}	{0.2, 0.1}	{0.6, 0.5, 0.4}	1
x_2	{0.3, 0.2}	{0.6, 0.5, 0.4}	{0.3, 0.2, 0.1}	{0.9, 0.8}	1
x_3	{0.3, 0.2}	{0.3, 0.2}	{0.8, 0.7, 0.6}	{0.8, 0.7}	0
x_4	{0.6, 0.5}	{0.3, 0.2}	{0.2, 0.1}	{0.3, 0.2}	1
x_5	{0.4, 0.3, 0.2}	{0.5, 0.4, 0.3}	{0.3, 0.2}	{0.3, 0.2, 0.1}	0

fuzzy rough set. The preponderance of axiomatic characterization is that it focuses on the algebraic properties of approximation operators, and it also lays a foundation for further research on uncertainty theory.

The axiomatic characterization of hesitant fuzzy rough approximation operator was first studied by Yang and Song (2014). Furthermore, Zhang et al. (2019) improved Yang's model of hesitant fuzzy rough sets such that any two hesitant fuzzy sets are antisymmetric. In this paper, based on the new hesitant fuzzy rough set model proposed by Zhang et al. (2019), by defining the inner product and outer product operations between two hesitant fuzzy sets, the single axiomatic characterization of the classical hesitant fuzzy approximation operators is obtained. Besides, we study the single axiomatic characterization that the upper and lower approximation operators generated by fuzzy preference relation satisfy sequence, reflexivity, symmetry and transitivity respectively. Finally, we compare and analyze the advantages and disadvantages of hesitant fuzzy set, fuzzy rough set and hesitant fuzzy rough set by some cases.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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