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FOUNDATIONS

Soft homogeneity of soft topological sum

Milan Matejdes[1]

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Abstract

The article deals with a soft homogeneity of soft topological space generated by a family of topological spaces and the correspondence between some soft topological notions and topological ones. A solution of an open problem concerning a characterization of soft homogeneity of soft topological space given by a topological sum is presented.

Keywords Relation · Set valued mapping · Soft set · Soft topology · Soft continuity · Soft homogeneity

1 Introduction

Molodtsov (1999) initiated the concept of soft sets as a completely different approach for dealing with uncertainties. In the past few years, the fundamentals of soft set theory have been studied by many authors. For example, different variants of operations on soft sets have been studied in Maji et al. (2003) and a comprehensive overview of the various operations on soft sets is provided in Al-shami and Kocinac (2019) and Al-shami and El-Shafei (2020b).

Since the concept of soft topology was introduced in Shabir and Naz (2011), many terms of general topology have found their analogy in soft topological spaces. Let us mention the soft separation axioms (Al-shami and El-Shafei 2020a; El-Shafei and Al-shami 2020; El-Shafei et al. 2018; Al-shami and Kočinac 2019; Pazar Varol and Aygün 2013), soft compactness (Al-shami 2021; Al-shami et al. 2018; Aygünoğlu and Aygün 2012) and soft sum of soft topological spaces (Al-shami and Kočinac 2019; Al Ghour and Bin-Saadon 2019; Al-shami et al. 2020; Terepeta 2019).

The aim of the article is the investigation of soft homogeneity of soft topological space generated by a family of topological spaces and to solve the open problem set out in Al Ghour and Bin-Saadon (2019) concerning the characterization of homogeneity of soft topological sum. It is worth calling that it is useful to consider only soft sets which are represented by set valued mappings defined on a fixed set of parameters E, which is also our case.

✉ Milan Matejdes
milan.matejdes@truni.sk

1 Department of Mathematics and Computer Science, Faculty of Education, Trnava University in Trnava, Priemyselná 4, 918 43 Trnava, Slovak Republic

2 Relations and set valued mappings

Any subset S of the Cartesian product $E \times U$ is a binary relation from a set E to a set U. Let $S(e) = \{u \in U : (e, u) \in S\}$. By $\mathbf{R}(E, U)$ we denote the set of all binary relations from E to U.

The operations of sum $S \cup T$, $\cup_{t \in T} S_t$, intersection $S \cap T$, $\cap_{t \in T} S_t$, complement S^c and difference $S \setminus T$ of relations are defined in the obvious way as in set theory.

By $F : E \to 2^U$ we denote a set valued mapping (multifunction) from E to power set 2^U of U. The set of all set valued mappings from E to 2^U is denoted by $\mathbf{F}(E, U)$. A set valued mapping F for which $F(e) = \{u\}$ and it is empty valued otherwise is denoted by F_e^u.

If F, G are two set valued mappings, then $F \subset G$ ($F = G$) means $F(e) \subset G(e)$ ($F(e) = G(e)$) for any $e \in E$. The difference $F \setminus G$ of F and G is defined as a set valued mapping given by $(F \setminus G)(e) = F(e) \setminus G(e)$ for any $e \in E$. The intersection (union) of family $\{G_t : t \in T\}$ of set valued mappings is defined as a set valued mapping $H : E \to 2^U$ for which $H(e) = \cap_{t \in T} G_t(e)$ ($H(e) = \cup_{t \in T} G_t(e)$) for any $e \in E$ and it is denoted by $\cap_{t \in T} G_t$ ($\cup_{t \in T} G_t$). The complement F^c of F is defined as a set valued mapping for which $F^c(e) = U \setminus F(e)$ for all $e \in E$.

A graph of $G \in \mathbf{F}(E, U)$ is a set $Gr(G) = \{(e, u) \in E \times U : u \in G(e)\}$ and it is a subset of $E \times U$, hence $Gr(G) \in \mathbf{R}(E, U)$. So, any set valued mapping G determines a relation from $\mathbf{R}(E, U)$ denoted by $\mathbf{R}_G = \{(e, u) \in E \times U : u \in G(e)\} = Gr(G)$.

On the other hand, any relation $S \in \mathbf{R}(E, U)$ determines a set valued mapping $\mathbf{F}_S$ from E to 2^U where $\mathbf{F}_S(e) = S(e)$. So, for $G \in \mathbf{F}(E, U)$, $\mathbf{F}_S = G \Leftrightarrow S = \mathbf{R}_G$.

3 Soft topological space and topological space

In the following we will show that many terms of soft topology can be derived and investigated by means of general topology. In the literature (see references) a definition of a soft set is introduced by a set valued mapping. The next definition introduced the basic operations on the set of all soft sets with respect to a fixed set of parameters E.

Definition 1 (Maji et al. 2003; Shabir and Naz 2011) Let E, U be two nonempty sets.

(1) If $F : E \to 2^U$ is a set valued mapping, then F is called a soft set over U with respect to E. A soft set F for which $F(e) = \emptyset$ ($F(e) = U$) for any $e \in E$ is called the null soft set (the absolute soft set) and F_e^u is called a soft point.

(2) A soft set F is a soft subset of G, denoted by $F \subset G$ if $F(e) \subset G(e)$ for any $e \in E$. The operations of sum $F \cup G, \cup_{t \in T} G_t$, intersection $F \cap G, \cap_{t \in T} G_t$, complement F^c and difference $F \setminus G$ of soft sets are defined in the same way as for set valued mappings above. For example the intersection (union) of family $\{G_t : t \in T\}$ of soft sets is defined as a soft set $H : E \to 2^U$ for which $H(e) = \cap_{t \in T} G_t(e)$ ($H(e) = \cup_{t \in T} G_t(e)$) for any $e \in E$.

(3) The family of all soft sets over U with respect to E is denoted by $SS(E, U)$. It is clear $SS(E, U) = \mathbf{F}(E, U)$. The family of all soft points is denoted by $SP(E, U)$.

Definition 2 (Maji et al. 2003; Shabir and Naz 2011) Let E, U be two nonempty sets. A soft topological space over U with respect to E is a triplet (E, U, τ) where $\tau \subset SS(E, U)$ is closed under finite intersection, arbitrary union of soft sets and contains the null soft set and the absolute soft set. A soft set from τ is called a soft open set and its complement is called a soft closed set.

In the following we will denote by (X, τ) $((E, \tau), (U, \tau), (E \times U, \tau))$ a topological space where τ is a topology on X ($E, U, E \times U$, respectively). If (X, τ_1) and (Y, τ_2) are topological spaces, then $(X \times Y, \tau_1 \times \tau_2)$ is a topological space equipped with the product topology (the Cartesian product) of τ_1 and τ_2. A topological space (X, τ) is homogeneous if for any points x, y there is a homeomorphism $f : (X, \tau) \to (X, \tau)$ for which $f(x) = y$. For example the real line, Euclidean space, connected manifolds and the underlying spaces of topological groups are all homogeneous. The closed interval $[0, 1]$ is not homogeneous because there is no homeomorphism sending 0 to any point in the open interval $(0, 1)$. The Sierpinsky space in not homogeneous and a pair of intersecting lines is not homogeneous because every homeomorphism fixes the point of intersection.

Theorem 1 (Matejdes 2016) *There is a one-to-one correspondence between the family of all soft topological spaces over U with respect to E and the family of all topological spaces on $E \times U$ as follows.*

(1) *If (E, U, τ) is a soft topological space, then $(E \times U, \mathbf{R}_\tau)$ is a topological space where $\mathbf{R}_\tau = \{\mathbf{R}_G : G \in \tau\}$, i.e., $G \in \tau \Leftrightarrow \mathbf{R}_G \in \mathbf{R}_\tau$.*

(2) *If $(E \times U, \tau)$ is a topological space, then $(E, U, \mathbf{F}_\tau)$ is a soft topological space where $\mathbf{F}_\tau = \{\mathbf{F}_G : G \in \tau\}$, i.e., $G \in \tau \Leftrightarrow \mathbf{F}_G \in \mathbf{F}_\tau$.*

Definition 3 A topological space $(E \times U, \mathbf{R}_\tau)$ from Theorem 1 item (1) is called the corresponding topological space to (E, U, τ) or $(E \times U, \mathbf{R}_\tau)$ is given by (E, U, τ). A soft topological space $(E, U, \mathbf{F}_\tau)$ from Theorem 1 item (2) is called the corresponding soft topological space to $(E \times U, \tau)$ or $(E, U, \mathbf{F}_\tau)$ is given by $(E \times U, \tau)$. We say (E, U, τ) is given by a topological space $(E \times U, \theta)$ (one is given by the other or they are mutually correspondence) if $\tau = \mathbf{F}_\theta$ and $\theta = \mathbf{R}_\tau$. Similarly we say that a soft set G and a relation A are mutually correspondence if $G = \mathbf{F}_A$ and $\mathbf{R}_G = A$. The reader should keep in mind that a notation $(E \times U, \tau)$ $((E_1 \times U_1, \tau), (E_2 \times U_2, \theta))$ indicates a topological space while (E, U, τ) $((E_1, U_1, \tau), (E_2, U_2, \theta))$ indicates a soft topological space.

Remark 1 Let (E, U, τ) be a soft topological space, G be a soft set and S be a relation. Then

(1) for any $e \in E$ a system $\tau_e = \{F(e) : F \in \tau\}$ defines a topological space denoted by (U, τ_e).

(2) $\mathbf{F}_{\{(e,u)\}} = F_e^u \Leftrightarrow \{(e, u)\} = \mathbf{R}_{F_e^u}$,
$(e, u) \in \mathbf{R}_G \Leftrightarrow F_e^u \subset G$,
$(e, u) \in S \Leftrightarrow F_e^u \subset \mathbf{F}_S$.

4 Soft mapping, soft continuity and soft homogeneity

In soft theory (see for example Al Ghour and Bin-Saadon 2019), a soft mapping f_{pu} between two families of soft sets $SS(E_1, U_1)$ and $SS(E_2, U_2)$ is usually defined by two mappings $u : E_1 \to E_2$ and $p : U_1 \to U_2$. If $A \in SS(E_1, U_1)$, $B \in SS(E_2, U_2)$, then the image $f_{pu}(A)$ of A under f_{pu}, the inverse image $f_{pu}^{-1}(B)$ of B under f_{pu} is defined as a soft set from $SS(E_2, U_2)$, $SS(E_1, U_1)$ given by

$f_{pu}(A) : e_2 \mapsto \cup_{e_1 \in u^{-1}(e_2)} p(A(e_1))$ for $e_2 \in E_2$,
$f_{pu}^{-1}(B) : e_1 \mapsto p^{-1}(B(u(e_1)))$ for $e_1 \in E_1$,
respectively.

Definition 4 (Al Ghour and Bin-Saadon 2019) Let (E_1, U_1, τ), (E_2, U_2, θ) be the soft topological spaces. A soft mapping $f_{pu} : (E_1, U_1, \tau) \to (E_2, U_2, \theta)$ is soft continuous (soft open) if the inverse image (the image) of any soft open set under f_{pu} is soft open and f_{pu} is a soft homeomorphism if it is soft continuous and a soft open bijection (bijection as a soft mapping $f_{pu} : SP(E_1, U_1) \to SP(E_2, U_2)$). A soft topological space (E, U, τ) is called soft homogeneous if for any soft points F_e^v, F_f^w there is a soft homeomorphism f_{pu} from (E, U, τ) to (E, U, τ) such that $f_{pu}(F_e^v) = F_f^w$.

Definition 5 (Al Ghour and Bin-Saadon 2019) A soft topological space generated by an indexed family $\{(U, \mathfrak{J}_e) : e \in E\}$ of topological spaces is denoted by $(E, U, \oplus_{e\in E}\mathfrak{J}_e)$ where $\oplus_{e\in E}\mathfrak{J}_e = \{F \in SS(E, U) : F(e) \in \mathfrak{J}_e$ for all $e \in E\}$.

In Al Ghour and Bin-Saadon (2019) (Theorem 5.29) it is proven if a soft topological space $(E, U, \oplus_{e\in E}\mathfrak{J}_e)$ generated by an indexed family $\{(U, \mathfrak{J}_e) : e \in E\}$ of topological spaces is soft homogenous, then all topological spaces $(U, \mathfrak{J}_e)$ are homogeneous and mutually homeomorphic.

The next example shows the opposite implication is not valid (see Question 5.30 in Al Ghour and Bin-Saadon 2019).

Example 1 Let $E = \{e_1, e_2\}$, $\mathbb{R}$ be the real line with two topologies $\mathfrak{J}_r = \{(c, \infty) : c \in \mathbb{R}\}$, $\mathfrak{J}_l = \{(-\infty, c) : c \in \mathbb{R}\}$. Then for any $a, b \in \mathbb{R}$, the value of a function $y = x - a + b$, $y = -x + b + a$ at a is equal to b. That means the topological space $(\mathbb{R}, \mathfrak{J}_r)$, $(\mathbb{R}, \mathfrak{J}_l)$ is homogeneous, respectively. It is obvious $(\mathbb{R}, \mathfrak{J}_r)$ is homeomorphic to $(\mathbb{R}, \mathfrak{J}_l)$ ($y = -x$ is a required homeomorphism) and any homeomorphism $f : (\mathbb{R}, \mathfrak{J}_r), \to (\mathbb{R}, \mathfrak{J}_r)$ ($f : (\mathbb{R}, \mathfrak{J}_l), \to (\mathbb{R}, \mathfrak{J}_l)$) is increasing (decreasing). A soft topological space $(E, U, \mathfrak{J}_r \oplus \mathfrak{J}_l)$ where $U = \mathbb{R}$ and $\mathfrak{J}_r \oplus \mathfrak{J}_l = \{F : F(e_1) \in \mathfrak{J}_r, F(e_2) \in \mathfrak{J}_l\}$ meets all conditions of Question 5.30 in Al Ghour and Bin-Saadon (2019), but there is no soft mapping f_{pu} such that $f_{pu}(F_{e_1}^u) = F_{e_1}^v$ for $u < v$ (p cannot be increasing and decreasing simultaneously).

Recall that f_{pu} is a special soft mapping. Generally we can introduce a one-to-one correspondence between a mapping from $E_1 \times U_1$ to $E_2 \times U_2$ and a soft mapping from $SP(E_1, U_1)$ to $SP(E_2, U_2)$.

Any mapping $h : SP(E_1, U_1) \to SP(E_2, U_2)$ is called a soft maping, i.e., every soft point $F_{e_1}^{u_1}$ in $SP(E_1, U_1)$ is uniquely associated with a soft point $h(F_{e_1}^{u_1})$ in $SP(E_2, U_2)$. For $A \in SS(E_1, U_1)$, $B \in SS(E_2, U_2)$, we define the image of A under h, the inverse image of B under h by

$$h(A) = \cup\{h(F_{e_1}^{u_1}) : F_{e_1}^{u_1} \text{ is a soft subset of } A\},$$
$$h^{-1}(B) = \cup\{F_{e_1}^{u_1} : h(F_{e_1}^{u_1}) \text{ is a soft subset of } B\},$$

respectively.

Definition 6 Let $g : E_1 \times U_1 \to E_2 \times U_2, h : SP(E_1, U_1) \to SP(E_2, U_2)$. Then a soft mapping $\Phi_g : SP(E_1, U_1) \to SP(E_2, U_2)$ is defined by

$$\Phi_g(F_{e_1}^{u_1}) = F_{e_2}^{u_2} \Leftrightarrow g((e_1, u_1)) = (e_2, u_2)$$

and a mapping $\Psi_h : E_1 \times U_1 \to E_2 \times U_2$ is defined by

$$\Psi_h((e_1, u_1)) = (e_2, u_2) \Leftrightarrow h(F_{e_1}^{u_1}) = F_{e_2}^{u_2}.$$

A soft mapping Φ_g, h and a mapping g, Ψ_h are said to be mutually correspondence, respectively. A soft mapping f_1 and a mapping f_2 are mutually correspondence if $\Psi_{f_1} = f_2$ and $\Phi_{f_2} = f_1$.

The following definition represents a more general concept of soft continuity than the soft continuity with respect to f_{pu}. In the next sections a soft homoemorphism is understood in the sense of a definition below unless otherwise stated.

Definition 7 Let (E_1, U_1, τ) and (E_2, U_2, θ) be soft topological spaces. A soft mapping h from $SP(E_1, U_1)$ to $SP(E_2, U_2)$

(1) is soft continuous if the inverse image of any soft open set under h is soft open,
(2) is soft open if the image of any soft open set under h is soft open,
(3) is a soft homeomorphism if it is soft continuous and a soft open bijection.
(4) (E_1, U_1, τ) and (E_2, U_2, θ) are soft homeomorphic if there is a soft homeomorphism h from $SP(E_1, U_1)$ to $SP(E_2, U_2)$.

The following lemma points to a closed link between the soft topological approach and the topological one.

Lemma 1 *Let (E_1, U_1, τ), (E_2, U_2, θ) be soft topological spaces, $g : E_1 \times U_1 \to E_2 \times U_2$, $h : SP(E_1, U_1) \to SP(E_2, U_2)$ and $A \in SS(E_1, U_1)$, $B \in SS(E_2, U_2)$. Then*

(1) $\Psi_{\Phi_g} = g, \quad \Phi_{\Psi_h} = h,$
$g^{-1}(\mathbf{R}_B) = \mathbf{R}_{\Phi_g^{-1}(B)}, \quad g(\mathbf{R}_A) = \mathbf{R}_{\Phi_g(A)},$
(2) *Φ_g, h is soft continuous (soft open) if and only if g, Ψ_h is continuous (open) as a function between the corresponding topological spaces, respectively,*
(3) *Φ_g, h is a soft homeomorphism if and only if g, Ψ_h is a homeomorphism as a function between the corresponding topological spaces, respectively,*
(4) *(E_1, U_1, τ) and (E_2, U_2, θ) are soft homeomorphic if and only if their correspondence topological spaces are homeomorphic.*

Proof (1) Since $\Psi_{\Phi_g}((e_1,u_1)) = (e_2,u_2) \Leftrightarrow \Phi_g(F_{e_1}^{u_1}) = F_{e_2}^{u_2} \Leftrightarrow g((e_1,u_1)) = (e_2,u_2)$, $\Psi_{\Phi_g} = g$.
Since $\Phi_{\Psi_h}(F_{e_1}^{u_1}) = F_{e_2}^{u_2} \Leftrightarrow \Psi_h((e_1,u_1)) = (e_2,u_2) \Leftrightarrow h(F_{e_1}^{u_1}) = F_{e_2}^{u_2}$, $\Phi_{\Psi_h} = h$.
$(e_1,u_1) \in g^{-1}(\mathbf{R}_B) \Leftrightarrow g((e_1,u_1)) = (e_2,u_2) \in \mathbf{R}_B \Leftrightarrow \Phi_g(F_{e_1}^{u_1}) = F_{e_2}^{u_2} \subset B \Leftrightarrow F_{e_1}^{u_1} \subset \Phi_g^{-1}(B) \Leftrightarrow (e_1,u_1) \in \mathbf{R}_{\Phi_g^{-1}(B)}$, that means $g^{-1}(\mathbf{R}_B) = \mathbf{R}_{\Phi_g^{-1}(B)}$.
$(e_2,u_2) \in g(\mathbf{R}_A) \Leftrightarrow \exists (e_1,u_1) \in \mathbf{R}_A$ such that $g((e_1,u_1)) = (e_2,u_2) \Leftrightarrow \exists F_{e_1}^{u_1} \subset A$ such that $\Phi_g(F_{e_1}^{u_1}) = F_{e_2}^{u_2} \Leftrightarrow F_{e_2}^{u_2} \subset \Phi_g(A) \Leftrightarrow (e_2,u_2) \in \mathbf{R}_{\Phi_g(A)}$, that means $g(\mathbf{R}_A) = \mathbf{R}_{\Phi_g(A)}$.
(2) Let Φ_g be soft continuous and $S \in \mathbf{R}_\theta$. Then $B = \mathbf{F}_S \in \theta$ and $g^{-1}(S) = g^{-1}(\mathbf{R}_B) = \mathbf{R}_{\Phi_g^{-1}(B)} \in \mathbf{R}_\tau$. That means g is continuous.
Let g be continuous and $B \in \theta$. Then $S = \mathbf{R}_B \in \mathbf{R}_\theta$ and $\mathbf{R}_{\Phi_g^{-1}(B)} = g^{-1}(\mathbf{R}_B) = g^{-1}(S) \in \mathbf{R}_\tau$. That means $\Phi_g^{-1}(B) \in \tau$, so Φ_g is soft continuous.
According to just proven, Ψ_h is continuous if and only if $\Phi_{\Psi_h} = h$ is soft continuous.
Similarly we can prove the equivalence for the soft openness and the openness.
The item (3) follows from (2) and (4) from (3) and (1). □

5 Main results

For the purposes of this article, we specify a soft function derived from a family of mappings. If $u : E_1 \to E_2$ and $P = \{p_{e_1} : U_1 \to U_2 : e_1 \in E_1\}$ is an indexed family of mappings, then a mappings $Pu : E_1 \times U_1 \to E_2 \times U_2$ is defined by $Pu((e_1,u_1)) = (u(e_1), p_{e_1}(u_1))$ and a soft mapping h_{Pu} is defined by $h_{Pu}(F_{e_1}^{u_1}) = \Phi_{Pu}(F_{e_1}^{u_1})$.

Let $u : E_1 \to E_2$ and $z : E_2 \to E_3$ be two functions and $P = \{p_{e_1} : U_1 \to U_2 : e_1 \in E_1\}$ and $Q = \{q_{e_2} : U_2 \to U_3 : e_2 \in E_2\}$ be two families of mappings. Then it is easy to prove: if u is bijective and P is a family of bijections, then the inverse function $(Pu)^{-1} : E_2 \times U_2 \to E_1 \times U_1$ is given by u^{-1} and by a family $\{p^{-1}_{u^{-1}(e_2)} : U_2 \to U_1 : e_2 \in E_2\}$. Moreover, the composition $Qz \circ Pu$ is given by $z \circ u$ and by $\{q_{u(e_1)} \circ p_{e_1} : U_1 \to U_3 : e_1 \in E_1\}$.

Remark 2 The corresponding mapping to the soft mapping h_{Pu} is equal to Pu, i.e., $\Psi_{h_{Pu}} = Pu$. Proof follows from: $\Psi_{h_{Pu}((e_1,u_1))} = (e_2,u_2) \Leftrightarrow h_{Pu}(F_{e_1}^{u_1}) = F_{e_2}^{u_2} \Leftrightarrow \Phi_{Pu}(F_{e_1}^{u_1}) = F_{e_2}^{u_2} \Leftrightarrow Pu((e_1,u_1)) = (e_2,u_2)$.

For the following a special type of soft homeomorphism (soft homogeneity) we have chosen a designation $\oplus$-soft homeomorphism ($\oplus$-soft homogeneity) that corresponds to the topological sum (the sum of the functions), see Engelking (1977).

Definition 8 The soft topological spaces (E_1,U_1,τ) and (E_2,U_2,θ) are $\oplus$-soft homeomorphic if there are a bijection $u : E_1 \to E_2$ and an indexed family $P = \{p_{e_1} : (U_1,\tau_{e_1}) \to (U_2,\theta_{u(e_1)}) : e_1 \in E_1\}$ of homeomorphic mappings where (U_1,τ_{e_1}), $(U_2,\theta_{u(e_1)})$ is a topological space derived from (E_1,U_1,τ), (E_2,U_2,θ), respectively, see Remark 1. The corresponding soft mapping h_{Pu} is called a $\oplus$-soft homeomorphism, Pu is called a $\oplus$-homeomorphism from $(E_1 \times U_1, \mathbf{R}_\tau)$ to $(E_2 \times U_2, \mathbf{R}_\theta)$ and $(E_1 \times U_1, \mathbf{R}_\tau)$ and $(E_2 \times U_2, \mathbf{R}_\theta)$ are called to be $\oplus$-homeomorphic.

Definition 9 A soft topological space (E,U,τ) is called $\oplus$-soft homogeneous if for any soft points F_e^v, F_f^w there are a bijection $u : E \to E$ and an indexed family $P = \{p_e : (U,\tau_e) \to (U,\tau_{u(e)}) : e \in E\}$ of homeomorphic mappings such that $h_{Pu}(F_e^v) = F_f^w$ (i.e., $Pu((e,v)) = (u(e),p_e(v)) = (f,w)$). The corresponding topological space $(E \times U, \mathbf{R}_\tau)$ is said to be $\oplus$-homogeneous.

It is clear from two definitions above that many soft topological terms can be defined by using corresponding topological terms. The following lemma can be considered as a definition of $\oplus$-soft homogeneity and $\oplus$-soft homeomorphism.

Lemma 2 *Let (E_1,U_1,τ), (E_2,U_2,θ) be soft topological space and $(E_1 \times U_1, \mathbf{R}_\tau)$, $(E_2 \times U_2, \mathbf{R}_\theta)$ be the corresponding topological space, respectively. Then*

(1) *a soft mapping h_{Pu} is a $\oplus$-soft homeomorphism from (E_1,U_1,τ) to (E_2,U_2,θ) if and only if Pu is a $\oplus$-homeomorphism from $(E_1 \times U_1, \mathbf{R}_\tau)$ to $(E_2 \times U_2, \mathbf{R}_\theta)$,*
(2) *a soft topological space (E_1,U_1,τ) is $\oplus$-soft homogeneous if and only if the corresponding topological space $(E_1 \times U_1, \mathbf{R}_\tau)$ is $\oplus$-homogeneous.*

Proof The item (1) follows from Definition 8.

(2): By Definition 9, (E_1,U_1,τ) is $\oplus$-soft homogeneous if and only if for any soft points F_e^v, F_f^w $(e, f \in E_1, v, w \in U_1)$ there are a bijection $u : E_1 \to E_1$ and an indexed family $P = \{p_e : (U_1,\tau_e) \to (U_1,\tau_{u(e)}) : e \in E_1\}$ of homeomorphic mappings such that $h_{Pu}(F_e^v) = F_f^w$ (i.e., $Pu((e,v)) = (u(e),p_e(v)) = (f,w)$) if and only is the corresponding mapping (see Remark 2) Pu is a $\oplus$-homeomorphism from $(E_1 \times U_1, \mathbf{R}_\tau)$ to $(E_1 \times U_1, \mathbf{R}_\tau)$ (by Definition 8) and $Pu((e,v)) = (u(e),p_e(v)) = (f,w)$ if and only if $(E_1 \times U_1, \mathbf{R}_\tau)$ is $\oplus$-homogeneous. □

Lemma 3 *Let (E_1,U_1,τ) be a $\oplus$-soft homogeneous soft topological space. If (E_2,U_2,θ) is $\oplus$-soft homeomorphic to (E_1,U_1,τ), then (E_2,U_2,θ) is $\oplus$-soft homogeneous.*

Proof Let (E_1,U_1,τ) and (E_2,U_2,θ) be $\oplus$-soft homeomorphic. Then there are a bijection $u : E_1 \to E_2$ and an indexed family $P = \{p_{e_1} : (U_1,\tau_{e_1}) \to (U_2,\theta_{u(e_1)}) : e_1 \in E_1\}$ of homeomorphic mappings. Let $F_{e_2}^{v_2}, F_{f_2}^{w_2}$ be soft points from $SP(E_2, U_2)$. Since (E_1, U_1, τ) is $\oplus$-soft homogeneous, for the points

$$(Pu)^{-1}((e_2, v_2)) = (u^{-1}(e_2), p^{-1}_{u^{-1}(e_2)}(v_2))$$
$$(Pu)^{-1}((f_2, w_2)) = (u^{-1}(f_2), p^{-1}_{u^{-1}(f_2)}(w_2))$$

there are a bijection $z : E_1 \to E_1$ and an indexed family $Q = \{q_{e_1} : (U_1, \tau_{e_1}) \to (U_1, \tau_{z(e_1)}) : e_1 \in E_1\}$ of homeomorphic mappings such that

$$Qz((u^{-1}(e_2), p^{-1}_{u^{-1}(e_2)}(v_2))) = (u^{-1}(f_2), p^{-1}_{u^{-1}(f_2)}(w_2)).$$

Then $h_{(Pu)\circ Qz\circ (Pu)^{-1}}$ (given by $u \circ z \circ u^{-1}$ and by a family $\{p_{z(u^{-1}(e_2))} \circ q_{u^{-1}(e_2)} \circ p^{-1}_{u^{-1}(e_2)} : e_2 \in E_2\}$ of homeomorphic mappings from (U_2, θ_{e_2}) to $(U_2, \theta_{u(z(u^{-1}(e_2)))})$) is a $\oplus$-soft homeomorphism from (E_2, U_2, θ) to (E_2, U_2, θ) which transforms $F_{e_2}^{v_2}$ to $F_{f_2}^{w_2}$.

□

By $(E \times U, \oplus_{e\in E}\tau_e)$ we denote a topological sum of indexed family $\{(U, \tau_e) : e \in E\}$ of topological spaces. Note, $\oplus_{e\in E}\tau_e$ is a topology defined as the finest topology on $\oplus_{e\in E}U = \cup_{e\in E}\{e\} \times U = E \times U$ for which all canonical injections $\varphi_e : (U, \tau_e) \to (E \times U, \oplus_{e\in E}\tau_e)$ defined by $\varphi_e(u) = (e, u)$ for $u \in U$ are continuous.

The following lemma will be useful for further investigation, see Engelking (1977).

Lemma 4 *Let $\{(U, \tau_e) : e \in E\}$ be an indexed family of topological spaces. Then*

1. *a canonical injection φ_e is a continuous, open and closed map for any $e \in E$, so it is a homeomorphic embedding,*
2. *a map $Pu : (E_1 \times U_1, \oplus_{e_1\in E_1}\tau_1) \to (E_2 \times U_2, \oplus_{e_2\in E_2}\theta_{e_2})$ is continuous (a $\oplus$-homeomorphism) if and only if for any $e_1 \in E_1$ $p_{e_1} : (U_1, \tau_{e_1}) \to (U_2, \theta_{u(e_1)})$ is continuous (a homeomorphism),*
3. *if each (U, τ_e) is homeomorphic to a fixed topological space (Z, θ), then there is a homeomorphic mapping of the form Pu from $(E \times U, \oplus_{e\in E}\tau_e)$ to a product space $(E \times Z, \tau_{dis} \times \theta)$ where τ_{dis} is the discrete topology on E (Pu is given by $u(e) = e$ and by a family of homeomorphic mappings $P = \{p_e : (U, \tau_e) \to (Z, \theta) : e \in E\}$).*

Remark 3 Let $\{(U_1, \tau_{e_1}) : e_1 \in E_1\}$, $\{(U_2, \theta_{e_2}) : e_2 \in E_2\}$ be the indexed families of topological spaces.

1. The topological space $(E_1 \times U_1, \oplus_{e_1\in E_1}\tau_{e_1})$, $(E_2 \times U_2, \oplus_{e_2\in E_2}\theta_{e_2})$ and the soft topological space $(E_1, U_1, \mathbf{F}_{\oplus_{e_1\in E_1}\tau_{e_1}})$, $(E_2, U_2, \mathbf{F}_{\oplus_{e_2\in E_2}\theta_{e_2}})$ are mutually correspondence, respectively.
2. $h_{Pu} : (E_1, U_1, \mathbf{F}_{\oplus_{e_1\in E_1}\tau_{e_1}}) \to (E_2, U_2, \mathbf{F}_{\oplus_{e_2\in E_2}\theta_{e_2}})$ is a $\oplus$-soft homeomorphism $\Leftrightarrow$ for any $e_1 \in E_1$, $p_{e_1} : (U_1, \tau_{e_1}) \to (U_2, \theta_{u(e_1)})$ a homeomorphism, by Lemma 2 item (1) and Lemma 4 item (2).

The following theorem answers Question 5.30 of Al Ghour and Bin-Saadon (2019). As we can see, the equivalence holds if we replace a soft homeomorphism (soft homogeneity) under f_{pu} by a soft homeomorphism (soft homogeneity) under h_{Pu}.

Theorem 2 *Let $\{(U, \tau_e) : e \in E\}$ be an indexed family of topological spaces. Then the following conditions are equivalent.*

1. *$(E, U, \mathbf{F}_{\oplus_{e\in E}\tau_e})$ is $\oplus$-soft homogenous,*
2. *each (U, τ_e) is homeomorphic to a fixed homogeneous topological space (U, τ).*

Proof (1) $\Rightarrow$ (2): First we will prove that for any $e_1, e_2 \in E$, the spaces (U, τ_{e_1}) and (U, τ_{e_2}) are homeomorphic. Since $(E, U, \mathbf{F}_{\oplus_{e\in E}\tau_e})$ is $\oplus$-soft homogenous, for $v \in U$ there is a $\oplus$-soft homeomorphism h_{Pu} such that $(e_2, v) = Pu((e_1, v)) = (u(e_1), p_{e_1}(v))$ where $p_{e_1} : (U, \tau_{e_1}) \to (U, \tau_{e_2})$ is a requested homeomorphism.

Since (U, τ_{e_1}) and (U, τ_{e_2}) are homeomorphic, we can put $\tau = \tau_{e_0}$ for a fixed $e_0 \in E$. So each (U, τ_e) is homeomorphic to (U, τ).

We prove (U, τ) is homogeneous. $(E, U, \mathbf{F}_{\oplus_{e\in E}\tau_e})$ is $\oplus$-soft homogenous, so for $e_0 \in E$ and any $u_1, u_2 \in U$ there are a bijection $u : E \to E$ and a family $P = \{p_e : (U, \tau_e) \to (U, \tau_{u(e)}) : e \in E\}$ of homeomorphic mappings such that $u(e_0) = e_0$, $p_{e_0}(u_1) = u_2$. Since $p_{e_0} : (U, \tau_{e_0}) \to (U, \tau_{e_0})$ is a homeomorphism, (U, τ_{e_0}) is homogenous. Since (U, τ_{e_0}) and (U, τ) are homeomorphic, (U, τ) is homogenous.

(2) $\Rightarrow$ (1): Let $F_{e_1}^v, F_{e_2}^w$ be soft points. By assumptions, there is a homeomorphism $f : (U, \tau_{e_1}) \to (U, \tau_{e_2})$. Since each (U, τ_e) is homeomorphic to the homogeneous topological space (U, τ), (U, τ_{e_2}) is homogeneous. Then there is a homeomorphism $g : (U, \tau_{e_2}) \to (U, \tau_{e_2})$ such that $g(f(v)) = w$. Put $u(e_1) = e_2$, $u(e_2) = e_1$, $u(e) = e$ otherwise and $p_{e_1} = g \circ f : (U, \tau_{e_1}) \to (U, \tau_{e_2})$, $p_{e_2} = f^{-1} : (U, \tau_{e_2}) \to (U, \tau_{e_1})$ and $p_e(u) = u$ for $e \neq e_1, e_2$. Then $P = \{p_e : (U, \tau_e) \to (U, \tau_{u(e)}), e \in E\}$ is a family of homeomorphic mappings, Pu is a $\oplus$-homeomorphism from $(E \times U, \oplus_{e\in E}\tau_e)$ to $(E \times U, \oplus_{e\in E}\tau_e)$ (by Lemma 4 (2)), the corresponding soft mapping h_{Pu} is $\oplus$-soft homeomorphism from $(E, U, \mathbf{F}_{\oplus_{e\in E}\tau_e})$ to $(E, U, \mathbf{F}_{\oplus_{e\in E}\tau_e})$ (by Lamma 2 (1)) and $h_{Pu}(F_{e_1}^v) = F_{u(e_1)}^{p_{e_1}(v)} = F_{e_2}^{(g\circ f)(v)} = F_{e_2}^w$. So $(E, U, \mathbf{F}_{\oplus_{e\in E}\tau_e})$ is $\oplus$-soft homogenous. □

The previous theorem can be reformulated into a topological characterization (Theorem 3 below) and a soft topological characterization (Theorem 4 below) of $\oplus$-soft homogeneity of $(E, U, \mathbf{F}_{\oplus_{e\in E}\tau_e})$.

Theorem 3 *Let $\{(U, \tau_e) : e \in E\}$ be an indexed family of topological spaces. Then the following conditions are equivalent.*

(1) $(E, U, \mathbf{F}_{\oplus_{e\in E}\tau_e})$ *is $\oplus$-soft homogenous,*
(2) $(E\times U, \oplus_{e\in E}\tau_e)$ *is $\oplus$-homeomorphic to $(E\times U, \tau_{dis}\times\tau)$ where (U, τ) is a homogenous topological space and τ_{dis} is the discrete topology on E.*

Proof (1) $\Rightarrow$ (2) By Theorem 2, each (U, τ_e) is homeomorphic to a fixed homogeneous topological space (U, τ). By Lemma 4 item (3), there is a $\oplus$-homeomorphism Pu from $(E \times U, \oplus_{e\in E}\tau_e)$ to a product space $(E \times U, \tau_{dis} \times \tau)$ where τ_{dis} is the discrete topology on E.

(2) $\Leftarrow$ (1) We will prove $(E, U, \mathbf{F}_{\tau_{dis}\times\tau})$ is $\oplus$-soft homogeneous. Let $F^v_{e_1}, F^w_{e_2}$ be two soft points. Since (U, τ) is homogeneous, there is a homeomorphism $g : (U, \tau) \to (U, \tau)$ such that $g(v) = w$.

Put $u(e_1) = e_2$, $u(e_2) = e_1$, $u(e) = e$ otherwise and $p_{e_1} = g$, $p_{e_2} = g^{-1}$ and $p_e(u) = u$ for $e \neq e_1, e_2$. Then $P = \{p_e : (U, \tau) \to (U, \tau) : e \in E\}$ is a family of homeomorphic mappings, Pu is a $\oplus$-homeomorphism from $(E \times U, \tau_{dis} \times \tau)$ to $(E \times U, \tau_{dis} \times \tau)$ and $Pu((e_1, v) = (u(e_1), p_{e_1}(v))) = (e_2, g(v)) = (e_2, w)$. That means h_{Pu} is a $\oplus$-soft homeomorphism from $(E, U, \mathbf{F}_{\tau_{dis}\times\tau})$ to $(E, U, \mathbf{F}_{\tau_{dis}\times\tau})$ for which $h_{Pu}(F^v_{e_1}) = F^w_{e_2}$, so $(E, U, \mathbf{F}_{\tau_{dis}\times\tau})$ is $\oplus$-soft homogeneous. By the assumption and Lemma 2, $(E, U, \mathbf{F}_{\oplus_{e\in E}\tau_e})$ is $\oplus$-soft homeomorphic to $(E, U, \mathbf{F}_{\tau_{dis}\times\tau})$ and the proof follows from Lemma 3.

□

As a consequence we have a soft topological characterization of $\oplus$-soft homogeneity.

Theorem 4 *Let $\{(U, \tau_e) : e \in E\}$ be an indexed family of topological spaces. Then $(E, U, \mathbf{F}_{\oplus_{e\in E}\tau_e})$ is $\oplus$-soft homogenous if and only if it is $\oplus$-soft homeomorphic to $(E, U, \mathbf{F}_{\tau_{dis}\times\tau})$ where (U, τ) is a homogenous topological space and τ_{dis} is the discrete topology on E.*

In the next theorem we use the original notation $(E, U, \oplus_{e\in E}\mathfrak{J}_e)$ for a soft topological space generated by $\{(U, \mathfrak{J}_e) : e \in E\}$ (see Al Ghour and Bin-Saadon 2019). Recall the soft topological space $(E, U, \oplus_{e\in E}\mathfrak{J}_e)$ from Al Ghour and Bin-Saadon (2019) is in fact a soft topological space $(E, U, \mathbf{F}_{\oplus_{e\in E}\mathfrak{J}_e})$ and a soft topological space $(E, U, \tau(\mathfrak{J}))$ from Al Ghour and Bin-Saadon (2019) (where $\mathfrak{J}$ is a topology on U and $\tau(\mathfrak{J}) = \{F \in SS(E, U) : F(e) \in \mathfrak{J} \text{ for all } e \in E\}$) is in fact a soft topological space $(E, U, \mathbf{F}_{\oplus_{e\in E}\mathfrak{J}})$ which is $\oplus$-soft homeomorphic to $(E, U, \mathbf{F}_{\tau_{dis}\times\mathfrak{J}})$, by Lemma 2 item (1) and Lemma 4 item (3). Also for a topological space (U, Σ), a soft topological space $(E, U, \mathcal{T}(\Sigma))$ (see Terepeta 2019) is in fact a soft topological space $(E, U, \mathbf{F}_{\oplus_{e\in E}\Sigma})$.

Theorem 5 *Let $\{(U, \mathfrak{J}_e) : e \in E\}$ be an indexed family of topological spaces. Then $(E, U, \oplus_{e\in E}\mathfrak{J}_e)$ is $\oplus$-soft homogenous if and only if it is $\oplus$-soft homeomorphic to $(E, U, \tau(\mathfrak{J}))$ where $(E, \mathfrak{J})$ is a homogenous topological space.*

6 Conclusion

For a further research of soft topological spaces we propose to focus on general topology and to use the correspondence between soft topology and general topology. Topologies such as product topologies $(E \times U, \tau_{dis} \times \tau)$, $(E \times U, \tau_{ind} \times \tau)$, $(E\times U, \tau_1\times\tau_2)$, a wide range of generalized concepts of open sets (semi-open, pre-open, α-open, semi pre-open sets) and many generalized notions of continuity of functions (quasi continuity, quasi pre-open continuity, separate continuity) can be very useful tools for exploring soft topological spaces.

Declarations

Conflict of interest The author declares that he has no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by the author.

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