



Group decision-making framework under linguistic q -rung orthopair fuzzy Einstein models

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Abstract

The q -rung orthopair fuzzy sets dynamically change the range of indication of decision knowledge by adjusting a parameter q from decision makers, where $q \geq 1$, and outperform the conventional intuitionistic fuzzy sets and Pythagorean fuzzy sets. Linguistic q -rung orthopair fuzzy sets (Lq -ROFSs), a qualitative type of q -rung orthopair fuzzy sets, are characterized by a degree of linguistic membership and a degree of linguistic non-membership to reflect the qualitative preferred and non-preferred judgments of decision makers. Einstein operator is a powerful alternative to the algebraic operators and has flexible nature with its operational laws and fuzzy graphs perform well when expressing correlations between attributes via edges between vertices in fuzzy information systems, which makes it possible for addressing correlational multi-attribute decision-making (MADM) problems. Inspired by the idea of Lq -ROFS and taking the advantage of the flexible nature of Einstein operator, in this paper, we aim to introduce a new class of fuzzy graphs, namely, linguistic q -rung orthopair fuzzy graphs (Lq -ROFGs) and further explore efficient approaches to complicated MAGDM situations. Following the above motivation, we propose the new concepts, including product-connectivity energy, generalized product-connectivity energy, Laplacian energy and signless Laplacian energy and discuss several of its desirable properties in the background of Lq -ROFGs based on Einstein operator. Moreover, product-connectivity energy, generalized product-connectivity energy, Laplacian energy and signless Laplacian energy of linguistic q -rung orthopair fuzzy digraphs (Lq -ROFDGs) are presented. In addition, we present a graph-based MAGDM approach with linguistic q -rung orthopair fuzzy information based on Einstein operator. Finally, an illustrative example related to the selection of mobile payment platform is given to show the validity of the proposed decision-making method. For the sake of the novelty of the proposed approach, comparison analysis is conducted and superiorities in contrast with other methodologies are illustrated.

Keywords Linguistic q -rung orthopair fuzzy sets · Linguistic q -rung orthopair fuzzy graphs · Product-connectivity energy · Generalized product-connectivity energy · Einstein operator

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1 Introduction

With the advent of the era of big data, practical problems and actual scenarios in real life have become more complicated. In some cases, Pythagorean fuzzy set (PFS) (Yager and Abbasov 2013) and intuitionistic fuzzy set (IFS) (Atanassov 1986) cannot effectively deal with certain types of data. To solve this issue, Yager (2016) put forward the concept of q -rung orthopair fuzzy sets (q -ROFSs), which generalize

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the classical IFSs and PFSs. q -ROFSs provide more freedom and choice for decision makers by allowing the sum of the q^{th} power of the membership degree (MD) and the q^{th} power of the non-membership degree (NMD) to be less than or equal to 1. The above information can be illustrated as $Q = (0.6, 0.9)$; it is q -rung orthopair fuzzy number, but not an intuitionistic fuzzy number and Pythagorean fuzzy number (because $0.6^q + 0.9^q \leq 1$ and $0.6^2 + 0.9^2 > 1$, $0.6 + 0.9 > 1$). The q -ROFSs reduced to IFSs, PFSs and Fermatean fuzzy sets (Senapati and Yager 2020) when $q = 1$, $q = 2$ and $q = 3$, respectively. When q expands, the q -ROFSs give more freedom to decision makers to provide their assessment data. Recently, many researchers developed several decision-making methods under generalized fuzzy scenario (Akram et al. 2020; Akram and Shahzadi 2020; Akram et al. 2021, 2020; Feng et al. 2021; Abdullah and Aslam 2020; Ashraf et al. 2019a, b, c; Liu et al. 2021; Qiyas et al. 2020; Zhan et al. 2018; Zhang and Li 2019).

For certain qualitative attributes, in some complicated decision-making problems, the decision makers cannot assign the MD and NMD by crisp numbers, and a feasible solution is to describe them by linguistic variables (LVs). Lin et al. (2020) developed Lq -ROFSs and devised the operational laws, such as weighted averaging operator and weighted geometric operator based on Lq -ROF environment to aggregate the Lq -ROF numbers. Obviously, Lq -ROFS can be regarded as the extension of the linguistic intuitionistic fuzzy set (LIFS) (Zhang 2014) and linguistic Pythagorean fuzzy set (LPFS) (Garg 2018; Lin et al. 2018), when $q = 1$ and $q = 2$ the Lq -ROFS reduces to LIFS and LPFS, respectively. Wang et al. (2019) introduced some basic concepts of Lq -ROFS and developed some new operational laws, comparison methods, and distance measure methods with Lq -ROFS. Liu et al. (2019) proposed a new class of fuzzy sets (Zadeh 1965) called q -rung orthopair uncertain linguistic sets based on the q -ROFSs and uncertain linguistic variables and developed uncertain linguistic aggregation operators. Yue et al. (2020) put forward several new operational laws of probabilistic linguistic term sets along with its application in MADM. Amin et al. (2020) developed MADM approach on the basis of triangular cubic linguistic uncertain fuzzy aggregation operators. Table 1 expresses the superiority and flexibility of the Lq -ROFS model compared to some of the existing models in literature.

Graphs can be used to model many real life systems. The graphs demonstrate the connections within these systems between the entities. As in electrical networks and computer networks, the connections may be physical or relationships as in molecules and ecosystems. Graphs are abstractions of these connections. The spectrum is the set of all eigenvalues of a graph. There has been detailed analysis of spectral properties of graphs. Gutman (2001) proposed the concept of the graph energy in chemistry and evaluated lower and upper

limits for the energy of the graph. If a graph is connected, its distance and adjacent energy is computed as the sum of the absolute values of the associated eigenvalues. If graph is not connected, then a graph's energy is the sum of the energy of its connected components. The Laplacian matrix $L(G)$ of a graph G is obtained by subtracting the adjacency matrix $A(G)$ from its degree matrix $D(G)$ whose vertex f_i has degree d_i is defined by $d_G(r_i)$ when $i = j$ and 0 otherwise. Signless Laplacian matrix $L^+(G)$ of a graph G is obtained by adding the adjacency matrix $A(G)$ in its degree matrix $D(G)$. Laplacian energy (Gutman and Zhou 2006) and signless Laplacian energy (Gutman et al. 2010) of a graph is the sum of the absolute values of the differences of the average vertex degree of G to the Laplacian and signless Laplacian eigenvalues of G , respectively. Graphs are representations of binary relations. Similarly, fuzzy binary relations are represented by graphs called fuzzy graphs (FGs). Fuzzy graphs are generalizations of crisp graphs. Rosenfeld (1975) suggested the idea of FGs and established its structure. Anjali and Mathew (2013) evaluated the energy of FGs.

Naz et al. originally introduced some new classes of FGs called PFGs (Naz et al. 2018) and complex PFGs (Akram and Naz 2019) along its pertinent applications in decision making. Akram and Naz (2018) discussed the theory of graph spectra in the context of the generalized fuzzy circumstances which deal MADM information more objectively. A new concept of q -rung orthopair fuzzy graphs was put forward by Habib et al. (2019) along its application in the soil ecosystem.

Koczy et al. (2020) put forward the notion of picture fuzzy graphs and analyze a Wi-Fi network and a social network based on picture fuzzy graphs. Akram et al. (2021) and Akram and Luqman (2020) defined trapezoidal picture fuzzy numbers along with its graphical representation and proposed a new approach of formation of granular structures based on fuzzy soft graphs. Recently, novel concepts of Zagreb and Harmonic energy in the background of interval-valued q -rung orthopair dual hesitant fuzzy Hamacher graphs are incorporated by Naz et al. (2021). Apart from this, some novel concepts of energy based on well-known molecular descriptors geometric-arithmetic and the atom bond connectivity of dual hesitant q -ROFGs have been presented by Akram et al. (2021).

Einstein operator not only a good alternative to the algebraic operator, and gives the same smooth approximations as the algebraic but also has more and more flexibility and robustness than algebraic operator. Due to the uncertainty of information data from some practical decision-making problems with the interrelated criteria, it might be tough for the decision makers to precisely quantify their judgment with a classical number, but can represent them by natural language. Lq -ROFS theory is one of the successful generalizations of linguistic fuzzy set theory which provides greater range for decision makers to express their uncertain, vague and imprecise

Table 1 Comparison of Lq -ROFS model with extant models in literature

Model	Membership	Non-membership	Indeterminacy	Linguistic membership	Linguistic non-membership	Linguistic indeterminacy
FS	✓	×	×	×	×	×
IFS	✓	✓	✓	×	×	×
PFS	✓	✓	✓	×	×	×
q -ROFS ($q \geq 1$)	✓	✓	✓	×	×	×
LFS	✓	×	×	✓	×	×
LIFS	✓	✓	✓	✓	✓	✓
LPFS	✓	✓	✓	✓	✓	✓
Lq -ROFS ($q \geq 1$)	✓	✓	✓	✓	✓	✓

cise information in natural language, and graphs play an essential role in modern world, due to its broadly spread use for representing, modeling and organizing linked data. In this manner, it is very significant to present the idea of Lq -ROFGs that not only accommodates Lq -ROF information but also can capture the relationships among arguments. To achieve this goal, utilizing LFSs into q -ROFGs, we introduce a new class of fuzzy graphs called linguistic q -rung orthopair fuzzy graphs (Lq -ROFGs) on the basis of Einstein operator and discuss its spectra. Due to its simple analysis and computational flexibility, the presented study has a distinct advantage over the others. The main contributions of this research are:

- (1) to incorporate the theory of LFSs into q -ROFGs and propose a new, powerful tool for describing interrelated uncertain phenomena, called Lq -ROFGs based on Einstein operator;
- (2) to propose the new concepts of the product-connectivity energy, generalized product-connectivity energy, Laplacian energy and signless Laplacian energy of Lq -ROFGs, as well as the Lq -ROFDGs;
- (3) to develop a MAGDM method based on Lq -ROFG and Einstein operator;
- (4) to solve decision-making problem related to mobile payment platform selection to illustrate the applicability and effectiveness of our proposed approach.

Compared with many existing generalized fuzzy graph theories, the newly proposed Lq -ROFGs show extraordinary flexibility and effectiveness and can successfully express the decision-making opinions of decision experts. The most beneficial advantage of the Lq -ROFG is that it uses linguistic terminology to describe the qualitative evaluation details provided by decision makers for a specific evaluation object.

Remaining paper is organized as follows: Sect. 2 recalls some basic definitions related to this paper. Section 3 introduces the concept of Lq -ROFG based on Einstein

operator, the spectra of a graph in Lq -ROF environment and briefly discusses its certain properties. Section 4 find out the product-connectivity energy, generalized product-connectivity energy, Laplacian energy, and signless Laplacian energy of Lq -ROFDGs based on Einstein operator. In Sect. 5, a graph-based MAGDM approach with Lq -ROF information is given and illustrative example shows the validity of our proposed method. Finally, we summarize the paper in Sect. 6.

2 Preliminaries

In this section, the basic concepts of LTS and Lq -ROFS are provided to facilitate the next sections.

Definition 1 (Dutta and Guha 2015) Let there exist a finite LTS $S = \{s_\gamma | \gamma = 0, 1, \dots, \tau\}$ with odd cardinality, where s_γ indicates a possible linguistic term. For instance, a LTS S having seven terms can be described as follows:

$$S = \{s_0 = \text{none}, s_1 = \text{very low}, s_2 = \text{low}, s_3 = \text{medium}, s_4 = \text{high}, s_5 = \text{very high}, s_6 = \text{perfect}\}.$$

Generally, the LTS meets the following characteristics:

- (i) The set is ordered: $s_k > s_l$, if and only if $k > l$.
- (ii) Max operator: $\max(s_k, s_l) = s_k$, if and only if $k \geq l$.
- (iii) Min operator: $\min(s_k, s_l) = s_k$, if and only if $k \leq l$.
- (iv) Negative operator: $\text{Neg}(s_k) = s_l$ such that $l = \tau - k$.

Definition 2 (Lin et al. 2020) If there is a finite reference set $U = \{r_1, r_2, \dots, r_n\}$ and a continuous LTS (CLTS) $S = \{s_\gamma | \gamma \in [0, \tau]\}$ with a nonnegative integer τ ; then Lq -ROFS on U is expressed as

$$\mathcal{L} = \{(r, s_a(r), s_b(r)) | r \in U\},$$

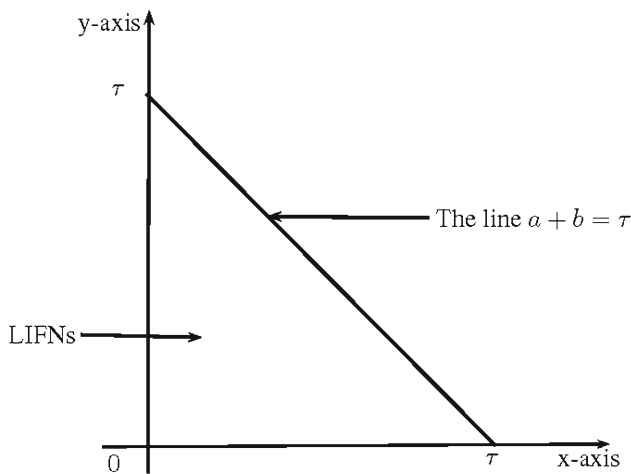


Fig. 1 The distribution area of LIFNs in a LIFS

characterized as the membership function $s_a(r)$ and non-membership function $s_b(r)$ of an element r . Each orthopair $(s_a(r), s_b(r))$ in the L_q -ROFS \mathcal{L} is considered as a L_q -ROFN and is simplified as (s_a, s_b) . In each orthopair (s_a, s_b) , s_a and s_b must fulfill the condition: $a, b \in [0, \tau]$ and $0 \leq a^q + b^q \leq \tau^q$ with $q \geq 1$. Further, $s_\pi(r) = s^{\frac{q}{\tau^q - a^q - b^q}}$ is the hesitant degree of the element r .

Definition 3 (Lin et al. 2020) Consider a L_q -ROFN $p = (s_a, s_b)$ with $s_a, s_b \in S$ and a CLTS $S = \{s_\gamma | \gamma \in [0, \tau]\}$, then the score value of the L_q -ROFN is described as

$$S(p) = s^{\frac{1}{(a^q + b^q)/2}}, \tag{1}$$

and its accuracy value is described as

$$J(p) = s^{\frac{1}{(a^q + b^q)}}. \tag{2}$$

Any two L_q -ROFNs p_1 and p_2 are compared by the following developed method as follows:

- (1) if $S(p_1) > S(p_2)$, then $p_1 \succ p_2$;
- (2) if $S(p_1) = S(p_2)$, then: (a) if $J(p_1) > J(p_2)$, then $p_1 \succ p_2$; (b) if $J(p_1) = J(p_2)$, then $p_1 \sim p_2$.

To illustrate the difference among linguistic intuitionistic fuzzy numbers (LIFNs), linguistic Pythagorean fuzzy numbers (LPFNs) and linguistic q -rung orthopair fuzzy numbers (L_q -ROFNs), we present their space of acceptable membership degrees in Figs. 1, 2 and 3.

Figures 1, 2 and 3 clearly show that as the index of s_a and s_b increases, so does the range of information which can be described by the fuzzy numbers. The L_q -ROFNs can therefore enlarge the information which the attributes can portray and expand the space for decision makers to evaluate

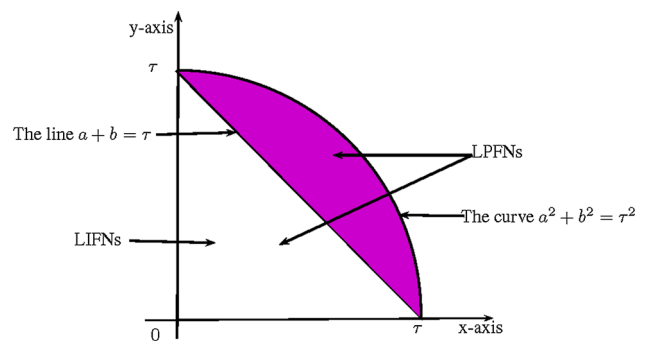


Fig. 2 The distribution areas for LIFNs and LPFNs

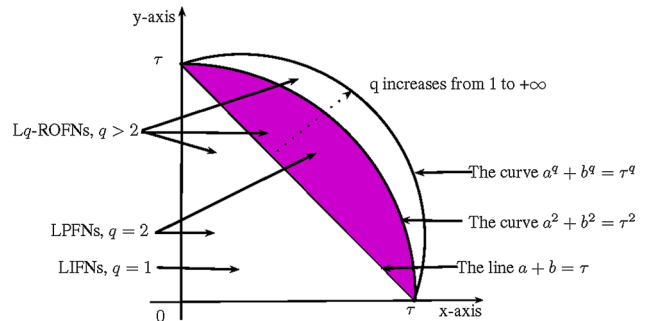


Fig. 3 The distribution areas for L_q -ROFNs

alternatives. It can also be observed that each of LIFS and LPFS is also a L_q -ROFS with $q \geq 1$ and $q \geq 2$, respectively.

Definition 4 (Wang and Liu 2011) For any two real numbers $r, s \in [0, 1]$, the Einstein t-norm and t-conorm are defined as follows:

$$I(r, s) = \frac{rs}{1 + (1-r)(1-s)},$$

$$I^*(r, s) = \frac{r + s}{1 + rs}.$$

Some notations used in this paper are listed in Table 2.

3 Linguistic q -rung orthopair fuzzy Einstein graphs

In this section, we introduce the innovative concept of L_q -ROFGs based on Einstein operator called Linguistic q -rung orthopair fuzzy Einstein graphs or simply L_q -ROFGs. Further, we discuss the spectra of a graph in L_q -ROF environment and discuss its certain properties.

Definition 5 Let U be the universe of discourse. A L_q -ROFS \mathcal{E} in $U \times U$ is said to be a linguistic q -rung orthopair fuzzy relation (L_q -ROFR) in U , denoted by

$$\mathcal{E} = \{(rs, s_{a_{\mathcal{E}}}(rs), s_{b_{\mathcal{E}}}(rs)) | rs \in U \times U\},$$

Table 2 List of notations

Notation	Description	Notation	Description
U	Classical set	$L^+(\mathcal{M})$	SLM of Lq -ROFG
S	CLTS	ζ_i^+	Eigenvalues of $L^+(s_{a_E}(r_i r_j))$
\mathcal{L}	Lq -ROFS	ν_i^+	Eigenvalues of $L^+(s_{b_E}(r_i r_j))$
\mathcal{E}	Lq -ROFR	\mathcal{D}	Lq -ROFDG
\mathcal{M}	Lq -ROFG	$A(\mathcal{D})$	Adjacency matrix of Lq -ROFDG
s_a	Linguistic membership	$PC_\alpha(\mathcal{D})$	GPC matrix of Lq -ROFDG
s_b	Linguistic non-membership	\tilde{t}_i	Eigenvalues of $PC_\alpha(s_{a_{\bar{E}}}(r_i r_j))$
$A(\mathcal{M})$	Adjacency matrix of Lq -ROFG	\tilde{w}_i	Eigenvalues of $PC_\alpha(s_{b_{\bar{E}}}(r_i r_j))$
$PC_\alpha(\mathcal{M})$	GPC matrix of Lq -ROFG	$D(\mathcal{D})$	Degree matrix of Lq -ROFDG
κ_i	Eigenvalues of $PC_\alpha(s_{a_E}(r_i r_j))$	$L(\mathcal{D})$	LM of Lq -ROFDG
χ_i	Eigenvalues of $PC_\alpha(s_{b_E}(r_i r_j))$	\mathfrak{R}_i	Eigenvalues of $L(s_{a_{\bar{E}}}(r_i r_j))$
$D(\mathcal{M})$	Degree matrix of Lq -ROFG	\mathfrak{h}_i	Eigenvalues of $L(s_{b_{\bar{E}}}(r_i r_j))$
$L(\mathcal{M})$	LM of Lq -ROFG	$L^+(\mathcal{D})$	SLM of Lq -ROFDG
ζ_i	Eigenvalues of $L(s_{a_E}(r_i r_j))$	\mathfrak{R}_i^+	Eigenvalues of $L^+(s_{a_{\bar{E}}}(r_i r_j))$
ν_i	Eigenvalues of $L(s_{b_E}(r_i r_j))$	\mathfrak{h}_i^+	Eigenvalues of $L^+(s_{b_{\bar{E}}}(r_i r_j))$

where $s_{a_E} : U \times U \rightarrow [0, \tau]$ and $s_{b_E} : U \times U \rightarrow [0, \tau]$ represent the membership and non-membership function of \mathcal{E} , respectively, such that $0 \leq a_E^q(rs) + b_E^q(rs) \leq \tau^q$ for all $rs \in U \times U$.

Definition 6 A Lq -ROFG based on Einstein operator under CLTS $S = \{s_\gamma | \gamma \in [0, \tau]\}$ with a positive integer τ is a pair $\mathcal{M} = (\mathcal{L}, \mathcal{E})$, where \mathcal{L} is a Lq -ROFS on U and \mathcal{E} is a Lq -ROFR on U such that:

$$s_{a_E}(rs) \leq s \frac{\left(\frac{a}{\tau}\right)_L(r) \left(\frac{a}{\tau}\right)_L(s)}{1 + \left(1 - \left(\frac{a}{\tau}\right)_L(r)\right) \left(1 - \left(\frac{a}{\tau}\right)_L(s)\right)},$$

$$s_{b_E}(rs) \leq s \frac{\left(\frac{b}{\tau}\right)_L(r) + \left(\frac{b}{\tau}\right)_L(s)}{1 + \left(\frac{b}{\tau}\right)_L(r) \left(\frac{b}{\tau}\right)_L(s)}$$

and $0 \leq a_E^q(rs) + b_E^q(rs) \leq \tau^q$ for all $r, s \in U$. We call \mathcal{L} and \mathcal{E} the Lq -ROFS of vertices and the Lq -ROFS of edges of \mathcal{M} , respectively. Here, \mathcal{E} is a symmetric linguistic q -rung orthopair fuzzy relation on \mathcal{L} . If \mathcal{E} is not symmetric on \mathcal{L} , then $\mathcal{D} = (\mathcal{L}, \overrightarrow{\mathcal{E}})$ is called Lq -ROFDG.

The adjacency matrix $A(\mathcal{M}) = (A(s_{a_E}(r_i r_j)), A(s_{b_E}(r_i r_j)))$ of a Lq -ROFG $\mathcal{M} = (\mathcal{L}, \mathcal{E})$ is a square matrix $A(\mathcal{M}) = [a_{ij}]$, $a_{ij} = (s_{a_E}(r_i r_j), s_{b_E}(r_i r_j))$, where $s_{a_E}(r_i r_j)$ and $s_{b_E}(r_i r_j)$ indicate the relationship and non-relationship strength between r_i and r_j , respectively.

Example 1 Consider a $L3$ -ROFG $\mathcal{M} = (\mathcal{L}, \mathcal{E})$, shown in Fig. 4, which models collaboration between researchers. The researchers are represented by vertices $V = \{f_1, f_2, f_3, f_4, f_5, f_6\}$ and the collaborations between any two researchers are represented by edges $E = \{f_1 f_2, f_1 f_4, f_1 f_5, f_1 f_6, f_2 f_3, f_3 f_4, f_4 f_5\}$. Let $S = \{s_0 = \text{worst}, s_1 = \text{very poor}, s_2 =$

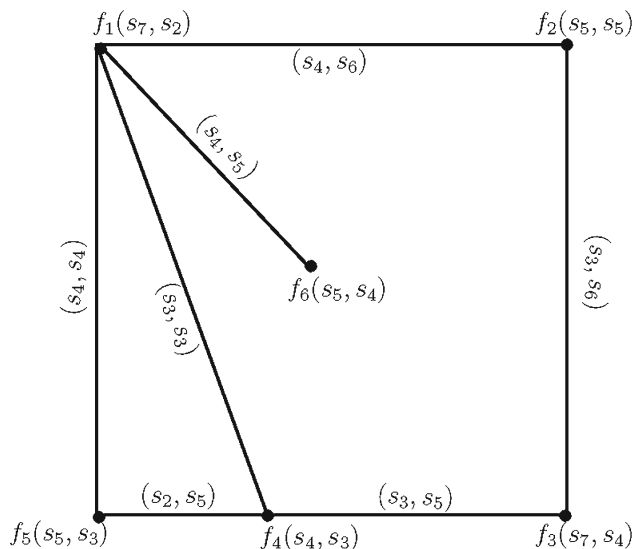


Fig. 4 Linguistic 3-rung orthopair fuzzy Einstein graph with CLTS $S = \{s_\gamma | \gamma \in [0, 8]\}$

poor, $s_3 = \text{slightly poor}, s_4 = \text{medium}, s_5 = \text{slightly good}, s_6 = \text{good}, s_7 = \text{very good}, s_8 = \text{outstanding}\}$ be a CLTS. Then,

$$\mathcal{L} = \left\langle \left(\frac{f_1}{s_7}, \frac{f_2}{s_5}, \frac{f_3}{s_7}, \frac{f_4}{s_4}, \frac{f_5}{s_5}, \frac{f_6}{s_5} \right), \left(\frac{f_1}{s_2}, \frac{f_2}{s_5}, \frac{f_3}{s_4}, \frac{f_4}{s_3}, \frac{f_5}{s_3}, \frac{f_6}{s_4} \right) \right\rangle,$$

$$\mathcal{E} = \left\langle \left(\frac{f_1 f_2}{s_4}, \frac{f_1 f_4}{s_3}, \frac{f_1 f_5}{s_4}, \frac{f_1 f_6}{s_4}, \frac{f_2 f_3}{s_3}, \frac{f_3 f_4}{s_3}, \frac{f_4 f_5}{s_2} \right), \right\rangle,$$

$$\left(\frac{f_1 f_2}{s_6}, \frac{f_1 f_4}{s_3}, \frac{f_1 f_5}{s_4}, \frac{f_1 f_6}{s_5}, \frac{f_2 f_3}{s_6}, \frac{f_3 f_4}{s_5}, \frac{f_4 f_5}{s_5} \right)$$

Clearly, $\mathcal{M} = (\mathcal{L}, \mathcal{E})$ shown in Fig. 4, is a L3-ROFG. The adjacency matrix of the above Lq-ROFG $A(\mathcal{M})$ is as follows:

$$A(\mathcal{M}) = \begin{bmatrix} (s_0, s_0) & (s_4, s_6) & (s_0, s_0) & (s_3, s_3) & (s_4, s_4) & (s_4, s_5) \\ (s_4, s_6) & (s_0, s_0) & (s_3, s_6) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_3, s_6) & (s_0, s_0) & (s_3, s_5) & (s_0, s_0) & (s_0, s_0) \\ (s_3, s_3) & (s_0, s_0) & (s_3, s_5) & (s_0, s_0) & (s_2, s_5) & (s_0, s_0) \\ (s_4, s_4) & (s_0, s_0) & (s_0, s_0) & (s_2, s_5) & (s_0, s_0) & (s_0, s_0) \\ (s_4, s_5) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \end{bmatrix}.$$

$$d_{ij} = \begin{cases} s_0 & \text{if } i = j, \\ s_{(d_{\mathcal{M}}(r_i)d_{\mathcal{M}}(r_j))^\alpha} & \text{if the vertices } r_i \text{ and } r_j \text{ of the Lq-ROFG } \mathcal{M} \text{ are adjacent,} \\ s_0 & \text{if the vertices } r_i \text{ and } r_j \text{ of the Lq-ROFG } \mathcal{M} \text{ are non-adjacent.} \end{cases}$$

Now we put forward some new concepts of spectral graph theory including product-connectivity energy, generalized product-connectivity energy, Laplacian energy and signless Laplacian energy in Lq-ROF environment based on Einstein operator.

3.1 Generalized product-connectivity energy of Lq-ROFGs

This subsection defines and investigates the product-connectivity energy and generalized product-connectivity energy of a Lq-ROFG and provides its properties in detail.

Definition 7 Let $\mathcal{M} = (\mathcal{L}, \mathcal{E})$ be a Lq-ROFG on n vertices. The product-connectivity matrix, $PC(\mathcal{M}) = (PC(s_{a_{\mathcal{E}}}(r_i r_j)), PC(s_{b_{\mathcal{E}}}(r_i r_j))) = [d_{ij}]$, of \mathcal{M} is a $n \times n$ matrix defined as:

$$d_{ij} = \begin{cases} s_0 & \text{if } i = j, \\ s_{\frac{1}{\sqrt{d_{\mathcal{M}}(r_i)d_{\mathcal{M}}(r_j)}}} & \text{if the vertices } r_i \text{ and } r_j \text{ of the Lq-ROFG } \mathcal{M} \text{ are adjacent,} \\ s_0 & \text{if the vertices } r_i \text{ and } r_j \text{ of the Lq-ROFG } \mathcal{M} \text{ are non-adjacent.} \end{cases}$$

where N_{PC} and O_{PC} are the sets of product-connectivity eigenvalues of $PC(s_{a_{\mathcal{E}}}(r_i r_j))$ and $PC(s_{b_{\mathcal{E}}}(r_i r_j))$, respectively.

Now, we define generalized product-connectivity energy of the Lq-ROFG and provide its properties in detail.

Definition 9 Let $\mathcal{M} = (\mathcal{L}, \mathcal{E})$ be a Lq-ROFG on n vertices. The general product-connectivity matrix, $PC_\alpha(\mathcal{M}) = (PC_\alpha(s_{a_{\mathcal{E}}}(r_i r_j)), (PC_\alpha(s_{b_{\mathcal{E}}}(r_i r_j)))) = [d_{ij}]$, of \mathcal{M} is a $n \times n$ matrix defined as:

Definition 10 The general product-connectivity energy of a Lq-ROFG $\mathcal{M} = (\mathcal{L}, \mathcal{E})$ is defined as:

$$PCE_\alpha(\mathcal{M}) = (PCE_\alpha(s_{a_{\mathcal{E}}}(r_i r_j)), (PCE_\alpha)(s_{b_{\mathcal{E}}}(r_i r_j))) = \left(s \sum_{\substack{i=1 \\ \kappa_i \in \mathcal{N}_{PC}}}^n |\kappa_i|, s \sum_{\substack{i=1 \\ \chi_i \in \mathcal{O}_{PC}}}^n |\chi_i| \right).$$

where \mathcal{N}_{PC} and \mathcal{O}_{PC} are the sets of general product-connectivity eigenvalues of $PC_\alpha(s_{a_{\mathcal{E}}}(r_i r_j))$ and $PC_\alpha(s_{b_{\mathcal{E}}}(r_i r_j))$, respectively.

Example 2 Let $\mathcal{M} = (\mathcal{L}, \mathcal{E})$ be a L3-ROFG on $V = \{f_1, f_2, f_3, f_4, f_5, f_6\}$ and $E = \{f_1 f_2, f_1 f_4, f_1 f_5, f_1 f_6, f_2 f_3, f_3 f_4, f_3 f_5, f_3 f_6, f_5 f_6\}$ with CLTS as $S = \{s_0 = \text{very poor}, s_1 = \text{poor}, s_2 = \text{slightly poor}, s_3 = \text{medium}, s_4 = \text{slightly good}, s_5 = \text{good}, s_6 = \text{very good}\}$, as shown in Fig. 5.

Definition 8 The product-connectivity energy of a Lq-ROFG $\mathcal{M} = (\mathcal{L}, \mathcal{E})$ is defined as:

$$PCE(\mathcal{M}) = (PCE(s_{a_{\mathcal{E}}}(r_i r_j)), PCE(s_{b_{\mathcal{E}}}(r_i r_j))) = \left(s \sum_{\substack{i=1 \\ \kappa_i \in \mathcal{N}_{PC}}}^n |\kappa_i|, s \sum_{\substack{i=1 \\ \chi_i \in \mathcal{O}_{PC}}}^n |\chi_i| \right),$$

The adjacency matrix of a Lq-ROFG $A(\mathcal{M})$, shown in Fig. 5, is:

$$\begin{aligned} sPC_{\alpha}^2(s_{a_{\mathcal{E}}}(r_i r_j)) &= s \sum_{k=1}^n (PC_{\alpha}(s_{a_{\mathcal{E}}}(r_i r_k)))(PC_{\alpha}(s_{a_{\mathcal{E}}}(r_k r_j))) \\ &= s \sum_{\substack{k \sim i \\ k \sim j}} (PC_{\alpha}(s_{a_{\mathcal{E}}}(r_i r_k)))(PC_{\alpha}(s_{a_{\mathcal{E}}}(r_k r_j))) \\ &= S(d_{s_a}(r_i)d_{s_a}(r_j))^{\alpha} \sum_{\substack{k \sim i \\ k \sim j}} d_{s_a}^{2\alpha}(r_k). \end{aligned}$$

Therefore, $s_{Tr}(PC_{\alpha}^2(s_{a_{\mathcal{E}}})) = S \sum_{i=1}^n PC_{\alpha}^2(s_{a_{\mathcal{E}}}(r_i r_i))$
 $= S \sum_{i=1}^n \sum_{i \sim j} (d_{s_a}(r_i)d_{s_a}(r_j))^{2\alpha} = S^2 \sum_{i \sim j} (d_{s_a}(r_i)d_{s_a}(r_j))^{2\alpha}$.
 Similarly, we can show that

$$s_{Tr}(PC_{\alpha}^2(s_{b_{\mathcal{E}}}(r_i r_i))) = S^2 \sum_{i \sim j} (d_{s_b}(r_i)d_{s_b}(r_j))^{2\alpha}.$$

Hence, $s_{Tr}(PC_{\alpha}^2(\mathcal{M})) = S^2 \sum_{i \sim j} (d_{\mathcal{M}}(r_i)d_{\mathcal{M}}(r_j))^{2\alpha}$.

3. The diagonal elements of $PC_{\alpha}^3(\mathcal{M})$ are

$$\begin{aligned} sPC_{\alpha}^3(s_{a_{\mathcal{E}}}(r_i r_i)) &= s \sum_{j=1}^n (PC_{\alpha}(s_{a_{\mathcal{E}}}(r_i r_j)))(PC_{\alpha}^2(s_{a_{\mathcal{E}}}(r_j r_i))) \\ &= s \sum_{j \sim i} (d_{s_a}(r_i)d_{s_a}(r_j))^{\alpha} (PC_{\alpha}^2(s_{a_{\mathcal{E}}}(r_i r_j))) \\ &= s \sum_{i \sim j} (d_{s_a}(r_i)d_{s_a}(r_j))^{2\alpha} \left(\sum_{\substack{k \sim i \\ k \sim j}} d_{s_a}^{2\alpha}(r_k) \right). \end{aligned}$$

Therefore,

$$\begin{aligned} s_{Tr}(PC_{\alpha}^3(s_{a_{\mathcal{E}}}(r_i r_i))) &= s \sum_{i=1}^n PC_{\alpha}^3(s_{a_{\mathcal{E}}}(r_i r_i)) \\ &= s \sum_{i=1}^n \left(\sum_{i \sim j} (d_{s_a}(r_i)d_{s_a}(r_j))^{2\alpha} \sum_{\substack{k \sim i \\ k \sim j}} d_{s_a}^{2\alpha}(r_k) \right) \\ &= s \sum_{i \sim j} (d_{s_a}(r_i)d_{s_a}(r_j))^{2\alpha} \left(\sum_{\substack{k \sim i \\ k \sim j}} d_{s_a}^{2\alpha}(r_k) \right). \end{aligned}$$

Similarly, we can show that

$$\begin{aligned} s_{Tr}(PC_{\alpha}^3(s_{b_{\mathcal{E}}}(r_i r_i))) &= s \sum_{i=1}^n PC_{\alpha}^3(s_{b_{\mathcal{E}}}(r_i r_i)) \\ &= s \sum_{i \sim j} (d_{s_b}(r_i)d_{s_b}(r_j))^{2\alpha} \left(\sum_{\substack{k \sim i \\ k \sim j}} d_{s_b}^{2\alpha}(r_k) \right). \end{aligned}$$

Hence, $s_{Tr}(PC^3(\mathcal{M})) = S \sum_{i \sim j} (d_{\mathcal{M}}(r_i)d_{\mathcal{M}}(r_j))^{2\alpha} \left(\sum_{\substack{k \sim i \\ k \sim j}} d_{\mathcal{M}}^{2\alpha}(r_k) \right)$.

We now calculate $s_{Tr}(PC_{\alpha}^4(s_{a_{\mathcal{E}}}(r_i r_j)))$. Because $tr(PC_{\alpha}^4)(s_{a_{\mathcal{E}}}(r_i r_j)) = \|(PC_{\alpha}^2)(s_{a_{\mathcal{E}}}(r_i r_j))\|_F^2$, where $\|(PC_{\alpha}^2)(s_{a_{\mathcal{E}}}(r_i r_j))\|_F$ denotes the Frobenius norm of $(PC_{\alpha}^2)(s_{a_{\mathcal{E}}}(r_i r_j))$, we obtain

$$\begin{aligned} s_{Tr}(PC_{\alpha}^4(s_{a_{\mathcal{E}}}(r_i r_j))) &= s \sum_{i,j=1}^n |(PC_{\alpha}^2)(s_{a_{\mathcal{E}}}(r_i r_j))|^2 \\ &= s \sum_{i=j} |(PC_{\alpha}^2)(s_{a_{\mathcal{E}}}(r_i r_j))|^2 + \sum_{i \neq j} |(PC_{\alpha}^2)(s_{a_{\mathcal{E}}}(r_i r_j))|^2 \\ &= s \sum_{i=1}^n \left(\sum_{i \sim j} (d_{s_a}(r_i)d_{s_a}(r_j))^{2\alpha} \right)^2 + \sum_{i \neq j} (d_{s_a}(r_i)d_{s_a}(r_j))^{2\alpha} \left(\sum_{\substack{k \sim i \\ k \sim j}} d_{s_a}^{2\alpha}(r_k) \right)^2. \end{aligned}$$

Similarly, we can show that

$$\begin{aligned} s_{Tr}(PC_{\alpha}^4(s_{b_{\mathcal{E}}}(r_i r_i))) &= s \sum_{i=1}^n \left(\sum_{i \sim j} (d_{s_b}(r_i)d_{s_b}(r_j))^{2\alpha} \right)^2 + \sum_{i \neq j} (d_{s_b}(r_i)d_{s_b}(r_j))^{2\alpha} \left(\sum_{\substack{k \sim i \\ k \sim j}} d_{s_b}^{2\alpha}(r_k) \right)^2. \end{aligned}$$

Hence, $s_{Tr}(PC_{\alpha}^4(\mathcal{M})) = S \sum_{i=1}^n \left(\sum_{i \sim j} (d_{\mathcal{M}}(r_i)d_{\mathcal{M}}(r_j))^{2\alpha} \right)^2 + \sum_{i \neq j} (d_{\mathcal{M}}(r_i)d_{\mathcal{M}}(r_j))^{2\alpha} \left(\sum_{\substack{k \sim i \\ k \sim j}} d_{\mathcal{M}}^{2\alpha}(r_k) \right)^2$. \square

Theorem 1 Let $\mathcal{M} = (\mathcal{L}, \mathcal{E})$ be a Lq-ROFG on n vertices. Then,

$$sPCE_{\alpha}(\mathcal{M}) \leq S \sqrt{2n \sum_{i \sim j} (d_{\mathcal{M}}(r_i)d_{\mathcal{M}}(r_j))^{\alpha}}.$$

Furthermore, $sPCE_{\alpha}(\mathcal{M}) = S \sqrt{2n \sum_{i \sim j} (d_{\mathcal{M}}(r_i)d_{\mathcal{M}}(r_j))^{\alpha}}$ if and only if \mathcal{M} is a graph with only isolated vertices, or end vertices.

Proof The variance of the numbers $|\kappa_i^{(\alpha)}| = \frac{1}{n} \sum_{i=1}^n |\kappa_i^{(\alpha)}|^2 - \left(\frac{1}{n} \sum_{i=1}^n |\kappa_i^{(\alpha)}| \right)^2 \geq 0$, $i = 1, 2, \dots, n$, therefore

$$\begin{aligned} s_{Tr}^{1/n}(PC_{\alpha}^2(s_{a_{\mathcal{E}}}(r_i r_j))) - S \left(\frac{1}{n} (PCE_{\alpha}(s_{a_{\mathcal{E}}}(r_i r_j))) \right)^2 &\geq 0 \\ \iff s_{ntr}((PC_{\alpha})^2(s_{a_{\mathcal{E}}}(r_i r_j))) &\geq S((PCE_{\alpha})(s_{a_{\mathcal{E}}}(r_i r_j)))^2 \\ \iff S(PCE_{\alpha}(s_{a_{\mathcal{E}}}(r_i r_j))) &\leq S \sqrt{ntr((PC_{\alpha}^2)(s_{a_{\mathcal{E}}}(r_i r_j)))} \\ \iff S(PCE_{\alpha}(s_{a_{\mathcal{E}}}(r_i r_j))) &\leq S \sqrt{2n \sum_{i \sim j} (d_{s_a}(r_i)d_{s_a}(r_j))^{\alpha}}. \end{aligned}$$

If \mathcal{M} is a graph with only isolated vertices, i.e., without edges, then $\kappa_i^{(\alpha)} = 0$ for all $i = 1, 2, \dots, n$, and therefore $PCE(s_{a_{\mathcal{E}}}(r_i r_j)) = 0$. Since no vertices are adjacent, $\sum_{i \sim j} (d_{s_a}(r_i)d_{s_a}(r_j))^{\alpha} = 0$. If \mathcal{M} is a graph with only

end vertices, i.e., having degree one, then $\kappa_i^{(\alpha)} = \pm d_{s_a}^\alpha(r_i)$, so the variance of $|\kappa_i^{(\alpha)}| = 0, i = 1, 2, \dots, n$. Thus, $s_{PCE_\alpha}(s_{a_E}(r_i r_j)) = s \sqrt{2n \sum_{i \sim j} (d_{s_a}(r_i) d_{s_a}(r_j))^\alpha}$.

Analogously, we can show that $s_{PCE_\alpha}(s_{b_E}(r_i r_j)) \leq s \sqrt{2n \sum_{i \sim j} (d_{s_b}(r_i) d_{s_b}(r_j))^\alpha}$. Hence, $s_{PCE_\alpha}(\mathcal{M}) \leq s \sqrt{2n \sum_{i \sim j} (d_{\mathcal{M}}(r_i) d_{\mathcal{M}}(r_j))^\alpha}$. \square

Theorem 2 Let $\mathcal{M} = (\mathcal{L}, \mathcal{E})$ be a Lq -ROFG on n vertices and at least one edge. Then,

If the Lq -ROFG \mathcal{M} has at least one edge, then all $\kappa_i^{(\alpha)}$'s are not equal to zero. Then, $\sum_{i=1}^n |\kappa_i^{(\alpha)}|^4 \neq 0$ and

$$s_{PCE_\alpha}(s_{a_E}(r_i r_j)) = s \sum_{i=1}^n |\kappa_i^{(\alpha)}| \geq s \sqrt{\frac{\left(\sum_{i=1}^n |\kappa_i^{(\alpha)}|^2\right)^3}{\sum_{i=1}^n |\kappa_i^{(\alpha)}|^4}} = s \sqrt{\frac{tr(PC_\alpha^2(s_{a_E}(r_i r_j)))^3}{tr((PC_\alpha^4(s_{a_E}(r_i r_j))))}}$$

$$s_{PCE_\alpha}(\mathcal{M}) \geq s \sqrt{\frac{2 \sum_{i \sim j} (d_{\mathcal{M}}(r_i) d_{\mathcal{M}}(r_j))^{2\alpha}}{\sum_{i=1}^n \left(\sum_{i \sim j} (d_{\mathcal{M}}(r_i) d_{\mathcal{M}}(r_j))^{2\alpha}\right)^2 + \sum_{i \neq j} (d_{\mathcal{M}}(r_i) d_{\mathcal{M}}(r_j))^{2\alpha} \left(\sum_{\substack{k \sim i \\ k \sim j}} d_{\mathcal{M}}^{2\alpha}(r_k)\right)}}$$

By using Lemma 1, we get

$$s_{PCE_\alpha}(s_{a_E}(r_i r_j)) \geq s \sqrt{\frac{2 \sum_{i \sim j} (d_{s_a}(r_i) d_{s_a}(r_j))^{2\alpha}}{\sum_{i=1}^n \left(\sum_{i \sim j} (d_{s_a}(r_i) d_{s_a}(r_j))^{2\alpha}\right)^2 + \sum_{i \neq j} (d_{s_a}(r_i) d_{s_a}(r_j))^{2\alpha} \left(\sum_{\substack{k \sim i \\ k \sim j}} d_{s_a}^{2\alpha}(r_k)\right)}}$$

$$s_{PCE_\alpha}(s_{b_E}(r_i r_j)) \geq s \sqrt{\frac{2 \sum_{i \sim j} (d_{s_b}(r_i) d_{s_b}(r_j))^{2\alpha}}{\sum_{i=1}^n \left(\sum_{i \sim j} (d_{s_b}(r_i) d_{s_b}(r_j))^{2\alpha}\right)^2 + \sum_{i \neq j} (d_{s_b}(r_i) d_{s_b}(r_j))^{2\alpha} \left(\sum_{\substack{k \sim i \\ k \sim j}} d_{s_b}^{2\alpha}(r_k)\right)}}$$

$$s_{PCE_\alpha}(\mathcal{M}) \geq s \sqrt{\frac{2 \sum_{i \sim j} (d_{\mathcal{M}}(r_i) d_{\mathcal{M}}(r_j))^{2\alpha}}{\sum_{i=1}^n \left(\sum_{i \sim j} (d_{\mathcal{M}}(r_i) d_{\mathcal{M}}(r_j))^{2\alpha}\right)^2 + \sum_{i \neq j} (d_{\mathcal{M}}(r_i) d_{\mathcal{M}}(r_j))^{2\alpha} \left(\sum_{\substack{k \sim i \\ k \sim j}} d_{\mathcal{M}}^{2\alpha}(r_k)\right)}}$$

Proof According to the Hölder inequality \square

$$\sum_{i=1}^n l_i m_i \leq \left(\sum_{i=1}^n l_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n m_i^q\right)^{\frac{1}{q}},$$

which holds for any real numbers $l_i, m_i \geq 0, i = 1, 2, \dots, n$. Setting $l_i = |\kappa_i^{(\alpha)}|^{\frac{2}{3}}, m_i = |\kappa_i^{(\alpha)}|^{\frac{4}{3}}, p = \frac{3}{2}$, and $q = 3$, we obtain

$$\sum_{i=1}^n |\kappa_i^{(\alpha)}|^2 = \sum_{i=1}^n |\kappa_i^{(\alpha)}|^{\frac{2}{3}} |\kappa_i^{(\alpha)}|^{\frac{4}{3}} \leq \left(\sum_{i=1}^n |\kappa_i^{(\alpha)}|\right)^{\frac{2}{3}} \left(\sum_{i=1}^n |\kappa_i^{(\alpha)}|^4\right)^{\frac{1}{3}}.$$

Theorem 3 Let $\mathcal{M} = (\mathcal{L}, \mathcal{E})$ be a Lq -ROFG on n vertices. If \mathcal{M} is regular of degree $(p, q), p, q > 0$, then

$$s_{PCE_\alpha}(\mathcal{M}) = s_{(p^{(2\alpha)}, q^{(2\alpha)})E}(\mathcal{M}).$$

Proof Suppose that \mathcal{M} is a regular Lq -ROFG of degree $(p^{(2\alpha)}, q^{(2\alpha)})(p, q > 0)$, i.e., $d_{s_a}(r_1) = d_{s_a}(r_2) = \dots = d_{s_a}(r_n) = p^{2\alpha}$. Then, all nonzero entries of $(PC_\alpha)(s_{a_E}(r_i r_i))$ are equal to $p^{(2\alpha)}$, implying that $(PC_\alpha)(s_{a_E}(r_i r_i)) = p^{(2\alpha)} A(s_{a_E}(r_i r_i))$. Therefore, for all $i = 1, 2, \dots, n$

$$\begin{aligned}
 S_{\kappa_i^{(\alpha)}} &= S_{p^{(2\alpha)}\kappa_i} \\
 S_{\sum_{i=1}^n \kappa_i^{(\alpha)}} &= S_{p^{(2\alpha)} \sum_{i=1}^n \kappa_i} \\
 S_{(PCE_\alpha)(s_{a_E}(r_i r_j))} &= S_{p^{(2\alpha)} E(s_{a_E}(r_i r_j))}.
 \end{aligned}$$

Similarly, we can show that $S_{PCE_\alpha(s_{b_E}(r_i r_j))} = S_{q^{(2\alpha)} E(s_{b_E}(r_i r_j))}$.

Hence

$$PCE_\alpha(\mathcal{M}) = S_{(p^{(2\alpha)}, q^{(2\alpha)})E(\mathcal{M})}.$$

□

3.2 Laplacian energy of Lq-ROFGs

This subsection describes and determines Laplacian energy of Lq-ROFGs and provides descriptions of its properties.

Definition 11 The degree matrix, $D(\mathcal{M}) = (D(s_{a_E}(r_i r_j)), D(s_{b_E}(r_i r_j))) = [d_{ij}]$, of a Lq-ROFG $\mathcal{M} = (\mathcal{L}, \mathcal{E})$ on n vertices, is a $n \times n$ diagonal matrix defined as:

$$d_{ij} = \begin{cases} s_{d_{\mathcal{M}}(r_i)} & \text{if } i = j, \\ s_0 & \text{otherwise.} \end{cases}$$

Definition 12 The Laplacian matrix (LM) of a Lq-ROFG $\mathcal{M} = (\mathcal{L}, \mathcal{E})$ is defined as:

$$L(\mathcal{M}) = (L(s_{a_E}(r_i r_j)), L(s_{b_E}(r_i r_j))) = D(\mathcal{M}) - A(\mathcal{M}),$$

where $A(\mathcal{M})$ and $D(\mathcal{M})$ are the adjacency matrix and the degree matrix of a Lq-ROFG \mathcal{M} , respectively.

Definition 13 The Laplacian energy of a Lq-ROFG $\mathcal{M} = (\mathcal{L}, \mathcal{E})$ is defined as:

$$\begin{aligned}
 LE(\mathcal{M}) &= (LE(s_{a_E}(r_i r_j)), LE(s_{b_E}(r_i r_j))) \\
 &= \left(\begin{matrix} s \sum_{\substack{i=1 \\ \omega_i \in N_L}}^n |\omega_i|, s \sum_{\substack{i=1 \\ \eta_i \in O_L}}^n |\eta_i| \end{matrix} \right),
 \end{aligned}$$

where N_L and O_L are the sets of Laplacian eigenvalues of $L(s_{a_E}(r_i r_j))$ and $L(s_{b_E}(r_i r_j))$, respectively, and

$$\begin{aligned}
 \omega_i &= \zeta_i - \frac{2 \sum_{1 \leq i < j \leq n} s_{a_E}(r_i r_j)}{n}, \\
 \eta_i &= \nu_i - \frac{2 \sum_{1 \leq i < j \leq n} s_{b_E}(r_i r_j)}{n},
 \end{aligned}$$

where ζ_i and ν_i represent eigenvalues of LM.

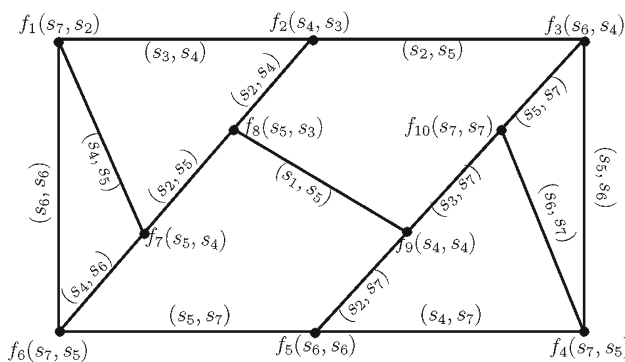


Fig. 6 Linguistic 6-rung orthopair fuzzy graph with CLTS $S = \{s_\gamma | \gamma \in [0, 8]\}$

Theorem 4 Let $\mathcal{M} = (\mathcal{L}, \mathcal{E})$ be a Lq-ROFG with CLTS $S = \{s_\gamma | \gamma = 0, 1, \dots, \tau\}$, and let $L(\mathcal{M})$ be the LM of \mathcal{M} . If $\varsigma_1 \geq \varsigma_2 \geq \dots \geq \varsigma_n$ and $\nu_1 \geq \nu_2 \geq \dots \geq \nu_n$ are the eigenvalues of $L(s_{a_E}(r_i r_j))$ and $L(s_{b_E}(r_i r_j))$, respectively, and $\omega_i = \zeta_i - \frac{2 \sum_{1 \leq i < j \leq n} s_{a_E}(r_i r_j)}{n}$, $\eta_i = \nu_i - \frac{2 \sum_{1 \leq i < j \leq n} s_{b_E}(r_i r_j)}{n}$, then

$$\begin{aligned}
 \sum_{i=1}^n \omega_i &= s_0, & \sum_{i=1}^n \eta_i &= s_0, \\
 \sum_{i=1}^n \omega_i^2 &= s_{\mathcal{M}_N}, & \sum_{i=1}^n \eta_i^2 &= s_{\mathcal{M}_O},
 \end{aligned}$$

where

$$\begin{aligned}
 \mathcal{M}_N &= \sum_{1 \leq i < j \leq n} (s_{a_E}(r_i r_j))^2 \\
 &+ \frac{1}{2} \sum_{i=1}^n \left(d_{s_{a_E}(r_i r_j)}(r_i) - \frac{2 \sum_{1 \leq i < j \leq n} s_{a_E}(r_i r_j)}{n} \right)^2, \\
 \mathcal{M}_O &= \sum_{1 \leq i < j \leq n} (s_{b_E}(r_i r_j))^2 \\
 &+ \frac{1}{2} \sum_{i=1}^n \left(d_{s_{b_E}(r_i r_j)}(r_i) - \frac{2 \sum_{1 \leq i < j \leq n} s_{b_E}(r_i r_j)}{n} \right)^2.
 \end{aligned}$$

Example 3 Let $\mathcal{M} = (\mathcal{L}, \mathcal{E})$ be a L6-ROFG on $V = \{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}\}$ and $E = \{f_1 f_2, f_1 f_6, f_1 f_7, f_2 f_3, f_2 f_8, f_3 f_4, f_3 f_{10}, f_4 f_5, f_4 f_{10}, f_5 f_6, f_5 f_9, f_6 f_7, f_7 f_8, f_8 f_9, f_9 f_{10}\}$ with CLTS as $S = \{s_0 = \text{worst}, s_1 = \text{very bad}, s_2 = \text{bad}, s_3 = \text{below average}, s_4 = \text{average}, s_5 = \text{above average}, s_6 = \text{good}, s_7 = \text{very good}, s_8 = \text{exceptional}\}$, as shown in Fig. 6.

The adjacency matrix, degree matrix and LM of the L6-ROFG given in Fig. 6 are as follows:

$$\begin{aligned}
 A(\mathcal{M}) &= \begin{bmatrix} (s_0, s_0) & (s_3, s_4) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_6, s_6) & (s_4, s_5) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \\ (s_3, s_4) & (s_0, s_0) & (s_2, s_5) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_2, s_4) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_2, s_5) & (s_0, s_0) & (s_5, s_6) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_5, s_7) \\ (s_0, s_0) & (s_0, s_0) & (s_5, s_6) & (s_0, s_0) & (s_4, s_7) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_6, s_7) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_4, s_7) & (s_0, s_0) & (s_5, s_7) & (s_0, s_0) & (s_0, s_0) & (s_2, s_7) & (s_0, s_0) \\ (s_6, s_6) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_5, s_7) & (s_0, s_0) & (s_4, s_6) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \\ (s_4, s_5) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_4, s_6) & (s_0, s_0) & (s_2, s_5) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_2, s_4) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_2, s_5) & (s_0, s_0) & (s_1, s_5) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_2, s_7) & (s_0, s_0) & (s_0, s_0) & (s_1, s_5) & (s_0, s_0) & (s_3, s_7) \\ (s_0, s_0) & (s_0, s_0) & (s_5, s_7) & (s_6, s_7) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_3, s_7) & (s_0, s_0) \end{bmatrix} \\
 D(\mathcal{M}) &= \begin{bmatrix} (s_{13}, s_{15}) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_7, s_{13}) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_{12}, s_{18}) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_{15}, s_{20}) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_{11}, s_{21}) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_{15}, s_{19}) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_{10}, s_{16}) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_5, s_{14}) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_6, s_{19}) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_{14}, s_{21}) \end{bmatrix} \\
 L(\mathcal{M}) &= \begin{bmatrix} (s_{13}, s_{15}) & (-s_3, -s_4) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_6, -s_6) & (-s_4, -s_5) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) \\ (-s_3, -s_4) & (s_7, s_{13}) & (-s_2, -s_5) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_2, -s_4) & (-s_0, -s_0) & (-s_0, -s_0) \\ (-s_0, -s_0) & (-s_2, -s_5) & (s_{12}, s_{18}) & (-s_5, -s_6) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_5, -s_7) \\ (-s_0, -s_0) & (-s_0, -s_0) & (-s_5, -s_6) & (s_{15}, s_{20}) & (-s_4, -s_7) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_6, -s_7) \\ (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_4, -s_7) & (s_{11}, s_{21}) & (-s_5, -s_7) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_2, -s_7) & (-s_0, -s_0) \\ (-s_6, -s_6) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_5, -s_7) & (s_{15}, s_{19}) & (-s_4, -s_6) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) \\ (-s_4, -s_5) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_4, -s_6) & (s_{10}, s_{16}) & (-s_2, -s_5) & (-s_0, -s_0) & (-s_0, -s_0) \\ (-s_0, -s_0) & (-s_2, -s_4) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_2, -s_5) & (s_5, s_{14}) & (-s_1, -s_5) \\ (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_2, -s_7) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (s_6, s_{19}) & (-s_3, -s_7) \\ (-s_0, -s_0) & (-s_0, -s_0) & (-s_5, -s_7) & (-s_6, -s_7) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_{14}, s_{21}) \end{bmatrix}
 \end{aligned}$$

The Laplacian spectrum and the Laplacian energy of a L6-ROFG \mathcal{M} , given in Fig. 6, are as follows:

Laplacian Spec(\mathcal{M})

$$\begin{aligned}
 &= \{(s_0.0000, s_{-0.0000}), (s_2.2714, s_4.8435), \\
 &\quad (s_4.5784, s_9.3620), (s_5.6234, s_{11.1148}), \\
 &\quad (s_8.0286, s_{16.8117}), (s_{10.1418}, s_{19.7963}), \\
 &\quad (s_{15.8350}, s_{23.7238}), (s_{17.9427}, s_{25.9566}), \\
 &\quad (s_{20.6269}, s_{28.7939}), (s_{22.9518}, s_{35.5974})\},
 \end{aligned}$$

and $LE(\mathcal{M}) = (s_{68.3128}, s_{91.7360})$.

Furthermore, we have

$$\begin{aligned}
 s_{10} &= 0, \quad s_{10} = s_0. \\
 \sum_{i=1}^{s_{10}} \varpi_i &= \sum_{i=1}^{s_{10}} \eta_i \\
 s_{10} &= s_{583.6} = s_2(291.8) = s_2 \mathcal{M}_N, \\
 \sum_{i=1}^{s_{10}} \varpi_i^2 &= s_{1144.4} = s_2(572.2) = s_2 \mathcal{M}_O. \\
 \sum_{i=1}^{s_{10}} \eta_i^2 &
 \end{aligned}$$

Theorem 5 Let $\mathcal{M} = (L, E)$ be a Lq-ROFG, and let $L(\mathcal{M})$ be the LM of \mathcal{M} with CLTS $S = \{s_\gamma | \gamma = 0, 1, \dots, \tau\}$. If $\varsigma_1 \geq \varsigma_2 \geq \dots \geq \varsigma_n$ and $v_1 \geq v_2 \geq \dots \geq v_n$ are the eigenvalues of $L(s_{a_E}(r_i r_j))$ and $L(s_{b_E}(r_i r_j))$, respectively, then

$$\begin{aligned}
 \text{(i)} \quad s \sum_{\substack{i=1 \\ \varsigma_i \in N_L}}^n \varsigma_i &= s_2 \sum_{1 \leq i < j \leq n} s_{a_E}(r_i r_j), \quad s \sum_{\substack{i=1 \\ v_i \in O_L}}^n v_i = s_2 \sum_{1 \leq i < j \leq n} s_{b_E}(r_i r_j). \\
 \text{(ii)} \quad s \sum_{\substack{i=1 \\ \varsigma_i \in N_L}}^n \varsigma_i^2 &= s \sum_{1 \leq i < j \leq n} (s_{a_E}(r_i r_j))^2 + \sum_{i=1}^n d_{s_{a_E}(r_i r_j)}^2(r_i), \quad s \sum_{\substack{i=1 \\ v_i \in O_L}}^n v_i^2 \\
 &= s \sum_{1 \leq i < j \leq n} (s_{b_E}(r_i r_j))^2 + \sum_{i=1}^n d_{s_{b_E}(r_i r_j)}^2(r_i).
 \end{aligned}$$

Proof (i) Since LM of Lq-ROFG $L(\mathcal{M})$ is symmetric and its eigenvalues are nonnegative, we have

$$\begin{aligned}
 s \sum_{\substack{i=1 \\ \varsigma_i \in N_L}}^n \varsigma_i &= s_{tr}(L(\mathcal{M})) \\
 &= s \sum_{i=1}^n d_{s_{a_E}(r_i r_j)}(r_i) = s_2 \sum_{1 \leq i < j \leq n} s_{a_E}(r_i r_j).
 \end{aligned}$$

Therefore, $s \sum_{\substack{i=1 \\ \zeta_i \in N_L}}^n \zeta_i = s_2 \sum_{1 \leq i < j \leq n} s_{a_{\mathcal{E}}}(r_i r_j)$.

Analogously, $s \sum_{\substack{i=1 \\ \nu_i \in O_L}}^n \nu_i = s_2 \sum_{1 \leq i < j \leq n} s_{b_{\mathcal{E}}}(r_i r_j)$.

(ii) Utilizing Definition 12, we get:

$$L(s_{a_{\mathcal{E}}}(r_i r_j)) = \begin{bmatrix} d_{s_{a_{\mathcal{E}}}(r_i r_j)}(r_1) & -s_{a_{\mathcal{E}}}(r_1 r_2) & \dots & -s_{a_{\mathcal{E}}}(r_1 r_n) \\ -s_{a_{\mathcal{E}}}(r_2 r_1) & d_{s_{a_{\mathcal{E}}}(r_i r_j)}(r_2) & \dots & -s_{a_{\mathcal{E}}}(r_2 r_n) \\ \vdots & \vdots & \ddots & \vdots \\ -s_{a_{\mathcal{E}}}(r_n r_1) & -s_{a_{\mathcal{E}}}(r_n r_2) & \dots & d_{s_{a_{\mathcal{E}}}(r_i r_j)}(r_n) \end{bmatrix}.$$

Based on matrix trace properties, we have

$$s_{tr}((L(s_{a_{\mathcal{E}}}(r_i r_j)))^2) = s \sum_{\substack{i=1 \\ \zeta_i \in N_L}}^n \zeta_i^2$$

where

$$\begin{aligned} s_{tr}((L(s_{a_{\mathcal{E}}}(r_i r_j)))^2) &= (d_{s_{a_{\mathcal{E}}}(r_i r_j)}^2(r_1) + s_{a_{\mathcal{E}}}^2(r_1 r_2) + \dots + s_{a_{\mathcal{E}}}^2(r_1 r_n)) \\ &+ (s_{a_{\mathcal{E}}}^2(r_2 r_1) + d_{s_{a_{\mathcal{E}}}(r_i r_j)}^2(r_2) + \dots + s_{a_{\mathcal{E}}}^2(r_2 r_n)) \\ &+ \dots + (s_{a_{\mathcal{E}}}^2(r_n r_1) + s_{a_{\mathcal{E}}}^2(r_n r_2) + \dots + d_{s_{a_{\mathcal{E}}}(r_i r_j)}^2(r_n)) \\ &= s_2 \sum_{1 \leq i < j \leq n} (s_{a_{\mathcal{E}}}(r_i r_j))^2 + \sum_{i=1}^n d_{s_{a_{\mathcal{E}}}(r_i r_j)}^2(r_i). \end{aligned}$$

So,

$$s \sum_{\substack{i=1 \\ \zeta_i \in N_L}}^n \zeta_i^2 = s_2 \sum_{1 \leq i < j \leq n} (s_{a_{\mathcal{E}}}(r_i r_j))^2 + \sum_{i=1}^n d_{s_{a_{\mathcal{E}}}(r_i r_j)}^2(r_i).$$

Similarly, we can easily show that

$$s \sum_{\substack{i=1 \\ \nu_i \in O_L}}^n \nu_i^2 = s_2 \sum_{1 \leq i < j \leq n} (s_{b_{\mathcal{E}}}(r_i r_j))^2 + \sum_{i=1}^n d_{s_{b_{\mathcal{E}}}(r_i r_j)}^2(r_i).$$

□

Theorem 6 *If the Lq-ROFG $\mathcal{M} = (\mathcal{L}, \mathcal{E})$ with CLTS $S = \{s_\gamma | \gamma = 0, 1, \dots, \tau\}$ is regular, then*

- (i) $s_{LE}(s_{a_{\mathcal{E}}}(r_i r_j)) \leq s \frac{|\varpi_1| + \sqrt{(n-1) \left(2 \sum_{1 \leq i < j \leq n} (s_{a_{\mathcal{E}}}(r_i r_j))^2 - \varpi_1^2 \right)}}{|\varpi_1|}$;
- (ii) $s_{LE}(s_{b_{\mathcal{E}}}(r_i r_j)) \leq s \frac{|\eta_1| + \sqrt{(n-1) \left(2 \sum_{1 \leq i < j \leq n} (s_{b_{\mathcal{E}}}(r_i r_j))^2 - \eta_1^2 \right)}}{|\eta_1|}$.

3.3 Signless Laplacian energy of Lq-ROFGs

In this subsection, we introduce a novel concept of signless Laplacian energy of Lq-ROFGs and provide descriptions of its properties.

Definition 14 The signless Laplacian matrix (SLM) of a Lq-ROFG $\mathcal{M} = (\mathcal{L}, \mathcal{E})$ is defined as $L^+(\mathcal{M}) = \langle L^+(s_a)(r_i r_j) \rangle, L^+(s_b(r_i r_j)) \rangle = D(\mathcal{M}) + A(\mathcal{M})$, where $D(\mathcal{M})$ is the degree matrix and $A(\mathcal{M})$ is the adjacency matrix of \mathcal{M} .

Definition 15 The signless Laplacian energy of Lq-ROFG $\mathcal{M} = (\mathcal{L}, \mathcal{E})$ is defined as

$$\begin{aligned} LE^+(\mathcal{M}) &= (LE^+(s_a(r_i r_j)), LE^+(s_b(r_i r_j))) \\ &= \left(s \sum_{\substack{i=1 \\ \varpi_i^+ \in N_{L^+}}}^n |\varpi_i^+|, s \sum_{\substack{i=1 \\ \eta_i^+ \in O_{L^+}}}^n |\eta_i^+| \right) \end{aligned}$$

where N_{L^+} and O_{L^+} are the sets of signless Laplacian eigenvalues of $L^+(s_a(r_i r_j)), L^+(s_b(r_i r_j))$, respectively, and

$$\begin{aligned} \varpi_i^+ &= \zeta_i^+ - \frac{2 \sum_{1 \leq i < j \leq n} s_a(r_i r_j)}{n}, \\ \eta_i^+ &= \nu_i^+ - \frac{2 \sum_{1 \leq i < j \leq n} s_b(r_i r_j)}{n}, \end{aligned}$$

where ζ_i^+ and ν_i^+ are the eigenvalues of SLM.

Theorem 7 *Let $\mathcal{M} = (\mathcal{L}, \mathcal{E})$ be a Lq-ROFG with CLTS $S = \{s_\gamma | \gamma = 0, 1, \dots, \tau\}$ and let $L^+(\mathcal{M})$ be the SLM of \mathcal{M} . If $\zeta_1^+ \geq \zeta_2^+ \geq \dots \geq \zeta_n^+$ and $\nu_1^+ \geq \nu_2^+ \geq \dots \geq \nu_n^+$ are the eigenvalues of $L^+(s_a(r_i r_j))$ and $L^+(s_b(r_i r_j))$, respectively, and $\varpi_i^+ = \zeta_i^+ - \frac{2 \sum_{1 \leq i < j \leq n} s_a(r_i r_j)}{n}, \eta_i^+ = \nu_i^+ - \frac{2 \sum_{1 \leq i < j \leq n} s_b(r_i r_j)}{n}$. Then*

$$\begin{aligned} s \sum_{i=1}^n \varpi_i^+ &= s_0, \quad s \sum_{i=1}^n \eta_i^+ = s_0, \\ s \sum_{i=1}^n (\varpi_i^+)^2 &= s_2 \mathcal{M}_N^+, \quad s \sum_{i=1}^n (\eta_i^+)^2 = s_2 \mathcal{M}_O^+, \end{aligned}$$

where

$$\begin{aligned} \mathcal{M}_N^+ &= \sum_{1 \leq i < j \leq n} (s_a(r_i r_j))^2 \\ &+ \frac{1}{2} \sum_{i=1}^n \left(d_{s_a(r_i r_j)}(r_i) - \frac{2 \sum_{1 \leq i < j \leq n} s_a(r_i r_j)}{n} \right)^2, \end{aligned}$$

$$\mathcal{M}_O^+ = \sum_{1 \leq i < j \leq n} (s_b(r_i r_j))^2 + \frac{1}{2} \sum_{i=1}^n \left(d_{s_b(r_i r_j)}(r_i) - \frac{2 \sum_{1 \leq i < j \leq n} s_b(r_i r_j)}{n} \right)^2.$$

Example 4 Let $\mathcal{M} = (L, \mathcal{E})$ be a L6-ROFG on $V = \{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8\}$ and $E = \{f_1 f_2, f_1 f_6, f_1 f_7, f_2 f_3, f_2 f_8, f_3 f_4, f_3 f_8, f_4 f_5, f_4 f_8, f_5 f_6, f_5 f_7, f_6 f_7\}$ with CLTS as $S = \{s_0 = \text{worst}, s_1 = \text{very bad}, s_2 = \text{bad}, s_3 = \text{below average}, s_4 = \text{average}, s_5 = \text{above average}, s_6 = \text{good}, s_7 = \text{very good}, s_8 = \text{exceptional}\}$, as shown in Fig. 7.

The adjacency matrix, degree matrix and SLM of the L6-ROFG given in Fig. 7 are as follows:

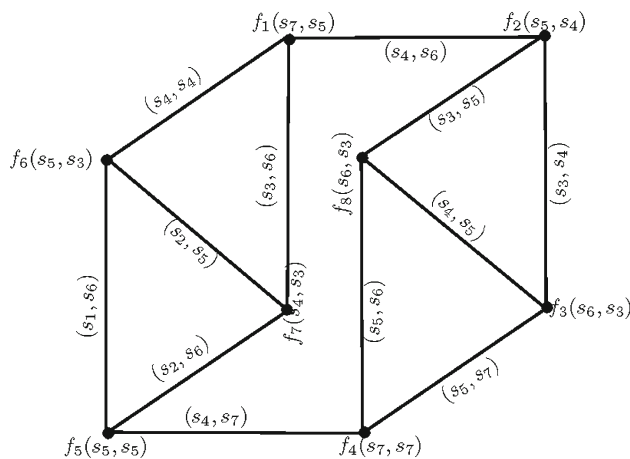


Fig. 7 Linguistic 6-rung orthopair fuzzy graph with CLTS $S = \{s_\gamma | \gamma \in [0, 8]\}$

$$A(\mathcal{M}) = \begin{bmatrix} (s_0, s_0) & (s_4, s_6) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_4, s_4) & (s_3, s_6) & (s_0, s_0) \\ (s_4, s_6) & (s_0, s_0) & (s_3, s_4) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_3, s_5) \\ (s_0, s_0) & (s_3, s_4) & (s_0, s_0) & (s_5, s_7) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_4, s_5) \\ (s_0, s_0) & (s_0, s_0) & (s_5, s_7) & (s_0, s_0) & (s_4, s_7) & (s_0, s_0) & (s_0, s_0) & (s_5, s_6) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_4, s_7) & (s_0, s_0) & (s_1, s_6) & (s_2, s_6) & (s_0, s_0) \\ (s_4, s_4) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_1, s_6) & (s_0, s_0) & (s_2, s_5) & (s_0, s_0) \\ (s_3, s_6) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_2, s_6) & (s_2, s_5) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_3, s_5) & (s_4, s_5) & (s_5, s_6) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \end{bmatrix}.$$

$$D(\mathcal{M}) = \begin{bmatrix} (s_{11}, s_{16}) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_{10}, s_{15}) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_{12}, s_{16}) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_{14}, s_{20}) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_7, s_{19}) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_7, s_{15}) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_7, s_{17}) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_{12}, s_{16}) \end{bmatrix}.$$

$$L^+(\mathcal{M}) = \begin{bmatrix} (s_{11}, s_{16}) & (s_4, s_6) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_4, s_4) & (s_3, s_6) & (s_0, s_0) \\ (s_4, s_6) & (s_{10}, s_{15}) & (s_3, s_4) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_3, s_5) \\ (s_0, s_0) & (s_3, s_4) & (s_{12}, s_{16}) & (s_5, s_7) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_4, s_5) \\ (s_0, s_0) & (s_0, s_0) & (s_5, s_7) & (s_{14}, s_{20}) & (s_4, s_7) & (s_0, s_0) & (s_0, s_0) & (s_5, s_6) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_4, s_7) & (s_7, s_{19}) & (s_1, s_6) & (s_2, s_6) & (s_0, s_0) \\ (s_4, s_4) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_7, s_{15}) & (s_7, s_{15}) & (s_2, s_5) & (s_0, s_0) \\ (s_3, s_6) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_2, s_6) & (s_2, s_5) & (s_7, s_{17}) & (s_0, s_0) \\ (s_0, s_0) & (s_3, s_5) & (s_4, s_5) & (s_5, s_6) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_{12}, s_{16}) \end{bmatrix}.$$

The signless Laplacian spectrum and the signless Laplacian energy of a L6-ROFG \mathcal{M} , shown in Fig. 7, are as follows:

Signless Laplacian $\text{Spec}(\mathcal{M})$

$$= \{(s_2.1844, s_3.8768), (s_4.3714, s_9.8139), (s_5.9818, s_{10}.8682), (s_6.9401, s_{11}.3698), (s_8.0000, s_{11}.9107), (s_{12}.5861, s_{23}.2483), (s_{16}.4815, s_{28}.4615), (s_{23}.4547, s_{34}.4509)\},$$

and $LE^+(\mathcal{M}) = (s_{45}.0446, s_{71}.8214).$

Furthermore,

$$\sum_{i=1}^8 \varpi_i^+ = s_0, \quad \sum_{i=1}^8 \eta_i^+ = s_0, \\ \sum_{i=1}^8 (\varpi_i^+)^2 = s_{352} = s_2(176) = s_2 \mathcal{M}_N^+,$$

$$s_8 \sum_{i=1}^8 (\eta_i^+)^2 = s_7 93.5 = s_2(396.75) = s_2 \mathcal{M}_O^+$$

Theorem 8 Let $\mathcal{M} = (\mathcal{L}, \mathcal{E})$ be a graph with CLTS $S = \{s_\gamma | \gamma = 0, 1, \dots, \tau\}$ and let $L^+(\mathcal{M})$ be the SLM of \mathcal{M} . If $s_1^+ \geq s_2^+ \geq \dots \geq s_n^+$ and $v_1^+ \geq v_2^+ \geq \dots \geq v_n^+$ are the eigenvalues of $L^+(s_a(r_i r_j))$ and $L^+(s_b(r_i r_j))$, respectively. Then

$$\begin{aligned} \text{(i)} \quad & s \sum_{i=1}^n s_i^+ = s_2 \sum_{1 \leq i < j \leq n} s_a(r_i r_j), \\ & s \sum_{s_i^+ \in N_{L^+}} v_i^+ = s_2 \sum_{1 \leq i < j \leq n} s_b(r_i r_j) \\ \text{(ii)} \quad & s \sum_{i=1}^n (s_i^+)^2 = s_2 \sum_{1 \leq i < j \leq n} (s_a(r_i r_j))^2 + \sum_{i=1}^n d_{s_a(r_i r_j)}^2(r_i), \\ & s \sum_{s_i^+ \in N_{L^+}} (v_i^+)^2 = s_2 \sum_{1 \leq i < j \leq n} (s_b(r_i r_j))^2 + \sum_{i=1}^n d_{s_b(r_i r_j)}^2(r_i). \end{aligned}$$

Proof Proof of this theorem follows Theorem 5. □

4 Spectra of Lq-ROFDGs

Concepts of product-connectivity energy, generalized product-connectivity energy, Laplacian energy and signless Laplacian energy of Lq-ROFDGs to Lq-ROFDGs are generalized in this section. As adjacency matrix of Lq-ROFDG is not necessarily symmetric, its eigenvalues might be complex numbers.

4.1 Generalized product-connectivity energy of Lq-ROFDGs

Definition 16 Let $\mathcal{D} = (\mathcal{L}, \vec{\mathcal{E}})$ be a Lq-ROFDG on n vertices. The general product-connectivity matrix, $PC_\alpha(\mathcal{D}) = (PC_\alpha(s_{a_{\vec{\mathcal{E}}}}(r_i r_j)), PC_\alpha(s_{b_{\vec{\mathcal{E}}}}(r_i r_j))) = [d_{ij}]$, of \mathcal{D} is a $n \times n$ matrix defined as:

$$d_{ij} = \begin{cases} s_0 & \text{if } i = j, \\ s_{(d_M^{\text{out}}(r_i) d_M^{\text{out}}(r_j))^\alpha} & \text{if the vertices } r_i \text{ and } r_j \text{ of the Lq-ROFDG } \mathcal{D} \text{ are adjacent,} \\ s_0 & \text{if the vertices } r_i \text{ and } r_j \text{ of the Lq-ROFDG } \mathcal{D} \text{ are non-adjacent.} \end{cases}$$

Definition 17 Let $\mathcal{D} = (\mathcal{L}, \vec{\mathcal{E}})$ be a Lq-ROFDG on n vertices. The general product-connectivity energy of \mathcal{D} is defined as:

$$PCE_\alpha(\mathcal{D}) = (PCE_\alpha(s_{a_{\vec{\mathcal{E}}}}(r_i r_j)), PCE_\alpha(s_{b_{\vec{\mathcal{E}}}}(r_i r_j)))$$

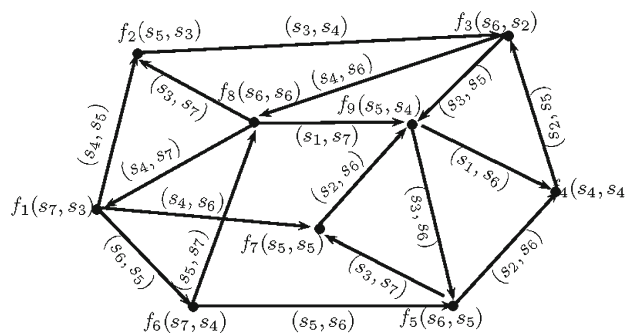


Fig. 8 L5-ROFDG with CLTS $S = \{s_\gamma | \gamma \in [0, 8]\}$

$$= \left(\begin{matrix} s \sum_{\substack{i=1 \\ \tilde{t}_i \in \mathcal{N}}}^n |\text{Re}(\tilde{t}_i)|, & s \sum_{\substack{i=1 \\ \tilde{w}_i \in \mathcal{O}}}^n |\text{Re}(\tilde{w}_i)| \end{matrix} \right)$$

where \mathcal{N} and \mathcal{O} are the sets of eigenvalues of $PC_\alpha(s_{a_{\vec{\mathcal{E}}}}(r_i r_j))$ and $PC_\alpha(s_{b_{\vec{\mathcal{E}}}}(r_i r_j))$, respectively, and $\text{Re}(\tilde{t}_i)$ and $\text{Re}(\tilde{w}_i)$ represent the real part of eigenvalues \tilde{t}_i and \tilde{w}_i , respectively.

Theorem 9 Let $\mathcal{D} = (\mathcal{L}, \vec{\mathcal{E}})$ be a Lq-ROFDG with CLTS $S = \{s_\gamma | \gamma = 0, 1, \dots, \tau\}$ and $PC_\alpha(\mathcal{D})$ be its adjacency matrix. If $\tilde{t}_1 \geq \tilde{t}_2 \geq \dots \geq \tilde{t}_n$ and $\tilde{w}_1 \geq \tilde{w}_2 \geq \dots \geq \tilde{w}_n$ are the eigenvalues of $PC_\alpha(s_{a_{\vec{\mathcal{E}}}}(r_i r_j))$ and $PC_\alpha(s_{b_{\vec{\mathcal{E}}}}(r_i r_j))$, respectively, then

$$s \sum_{\substack{i=1 \\ \tilde{t}_i \in \mathcal{N}}}^n \text{Re}(\tilde{t}_i) = s_0 \quad \text{and} \quad s \sum_{\substack{i=1 \\ \tilde{w}_i \in \mathcal{O}}}^n \text{Re}(\tilde{w}_i) = s_0.$$

Example 5 Consider a digraph $\mathcal{D} = (V, \vec{\mathcal{E}})$, where $V = \{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9\}$ and $\vec{\mathcal{E}} = \{f_1 f_2, f_1 f_6, f_1 f_7, f_2 f_3, f_3 f_8, f_3 f_9, f_4 f_3, f_5 f_4, f_5 f_7, f_6 f_5, f_6 f_8, f_7 f_9, f_8 f_1, f_8 f_2, f_8 f_9, f_9 f_4, f_9 f_5\}$ with CLTS as $S = \{s_0 = \text{worst}, s_1 = \text{very bad}, s_2 = \text{bad}, s_3 = \text{below average}, s_4 = \text{average}, s_5 = \text{above average}, s_6 = \text{good}, s_7 = \text{very good}, s_8 = \text{exceptional}\}$. Let $\mathcal{D} = (\mathcal{L}, \vec{\mathcal{E}})$ be a L5-ROFDG on V , as shown in Fig. 8.

The adjacency matrix of a L5-ROFDG given in Fig. 8 is:

$$A(\mathcal{D}) = \begin{bmatrix} (s_0, s_0) & (s_4, s_5) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_6, s_5) & (s_4, s_6) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_3, s_4) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_4, s_6) & (s_3, s_5) \\ (s_0, s_0) & (s_0, s_0) & (s_2, s_5) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_2, s_6) & (s_0, s_0) & (s_0, s_0) & (s_3, s_7) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_5, s_6) & (s_0, s_0) & (s_0, s_0) & (s_5, s_7) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_2, s_6) \\ (s_4, s_7) & (s_3, s_7) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_1, s_7) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_1, s_6) & (s_3, s_6) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \end{bmatrix}.$$

The generalized product-connectivity matrix with $\alpha = -1/2$ of a L5-ROFDG $PC(\mathcal{D})$, shown in Fig. 8, is:

$$PC_{-1/2}(\mathcal{D}) = \begin{bmatrix} (s_0, s_0) & (s_0.1543, s_0.1250) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0.2182, s_0.1508) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0.2673, s_0.1348) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0.3162, s_0.1240) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0.1414, s_0.0769) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \\ (s_0.0945, s_0.0546) & (s_0.2041, s_0.1091) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0.3536, s_0.1291) & (s_0.2236, s_0.0801) \\ (s_0.0845, s_0.0693) & (s_0.1890, s_0.1021) & (s_0, s_0) & (s_0, s_0) & \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & \\ (s_0, s_0) & (s_0, s_0) & (s_0.1336, s_0.0658) & (s_0.1890, s_0.0870) & \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & \\ (s_0, s_0) & (s_0.3162, s_0.1132) & (s_0, s_0) & (s_0, s_0) & \\ (s_0, s_0) & (s_0, s_0) & (s_0.1118, s_0.0605) & (s_0, s_0) & \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0.3536, s_0.1179) & \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0.1768, s_0.0630) & \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & \end{bmatrix}.$$

Therefore,

The spectrum and the energy values of a L q -ROFDG \mathcal{D} , given in Fig. 8, are as follows:

$$\begin{aligned} \text{Spec}(s_{a_E}(r_i r_j)) &= \{s_0.4053, s_{-0.1328+0.3164i}, s_{-0.1328-0.3164i}, \\ & s_{-0.1027+0.1367i}, s_{-0.1027-0.1367i}, \\ & s_0.0843+0.0631i, s_0.0843-0.0631i, s_{-0.1029}, s_0\}, \end{aligned}$$

$$\begin{aligned} \text{Spec}(s_{b_E}(r_i r_j)) &= \{s_0.1754, s_{-0.0524+0.1372i}, s_{-0.0524-0.1372i}, \\ & s_{-0.0732}, s_{-0.0412+0.0696i}, \\ & s_{-0.0412-0.0696i}, s_0.0425+0.0254i, s_0.0425-0.0254i, s_0\}. \end{aligned}$$

$$\begin{aligned} \text{Spec}(\mathcal{D}) &= \{(s_0.4053, s_0.1754), \\ & (s_{-0.1328+0.3164i}, s_{-0.0524+0.1372i}), \\ & (s_{-0.1328-0.3164i}, s_{-0.0524-0.1372i}), \\ & (s_{-0.1027+0.1367i}, s_{-0.0732}), \\ & (s_{-0.1027-0.1367i}, s_{-0.0412+0.0696i}), \\ & (s_0.0843+0.0631i, \\ & s_{-0.0412-0.0696i}), (s_0.0843-0.0631i, s_0.0425+0.0254i), \\ & (s_{-0.1029}, s_0.0425-0.0254i), (s_0, s_0)\}. \end{aligned}$$

Now, $PCE_{-1/2}(s_{a_E}(r_i r_j)) = s_{1.1481}$ and $PCE_{-1/2}(s_{b_E}(r_i r_j)) = s_0.5209$.

Therefore, $PCE_{-1/2}(\mathcal{D}) = (s_{1.1481}, s_0.5209)$.

Furthermore,

$$s \sum_{\substack{i=1 \\ \tilde{r}_i \in \mathcal{N}}}^9 \operatorname{Re}(\tilde{r}_i) = s_0 \quad \text{and} \quad s \sum_{\substack{i=1 \\ \tilde{w}_i \in \mathcal{O}}}^9 \operatorname{Re}(\tilde{w}_i) = s_0.$$

4.2 Laplacian energy of Lq-ROFDGs

This subsection discusses the Laplacian energy of Lq-ROFDGs.

Definition 18 Let $\mathcal{D} = (\mathcal{L}, \vec{\mathcal{E}})$ be a Lq-ROFDG on n vertices. The degree matrix, $D(\mathcal{D}) = (D(s_{a_{\vec{\mathcal{E}}}}(r_i r_j)), D(s_{b_{\vec{\mathcal{E}}}}(r_i r_j))) = [d_{ij}]$, of \mathcal{D} is a $n \times n$ diagonal matrix defined as:

$$d_{ij} = \begin{cases} s_{d_{\mathcal{D}}^{\text{out}}(r_i)} & \text{if } i = j, \\ s_0 & \text{otherwise.} \end{cases}$$

Definition 19 The Laplacian matrix (LM) of a Lq-ROFDG $\mathcal{D} = (\mathcal{L}, \vec{\mathcal{E}})$ is defined as $L(\mathcal{D}) = (L(s_{a_{\vec{\mathcal{E}}}}(r_i r_j)), L(s_{b_{\vec{\mathcal{E}}}}(r_i r_j))) = D^{\text{out}}(\mathcal{D}) - A(\mathcal{D})$, where $A(\mathcal{D})$ is an adjacency matrix and $D^{\text{out}}(\mathcal{D})$ is an out-degree matrix of \mathcal{D} .

Definition 20 The Laplacian energy of a Lq-ROFDG $\mathcal{D} = (\mathcal{L}, \vec{\mathcal{E}})$ is defined as:

$$\begin{aligned} \operatorname{LE}(\mathcal{D}) &= \left(\operatorname{LE}(s_{a_{\vec{\mathcal{E}}}}(r_i r_j)), \operatorname{LE}(s_{b_{\vec{\mathcal{E}}}}(r_i r_j)) \right) \\ &= \left(\begin{matrix} s \sum_{\substack{i=1 \\ \wp_i \in \mathcal{N}_L}}^n |\wp_i|, s \sum_{\substack{i=1 \\ \mathfrak{s}_i \in \mathcal{O}_L}}^n |\mathfrak{s}_i| \end{matrix} \right) \end{aligned}$$

where \mathcal{N}_L and \mathcal{O}_L are the sets of Laplacian eigenvalues of $L(s_{a_{\vec{\mathcal{E}}}}(r_i r_j))$ and $L(s_{b_{\vec{\mathcal{E}}}}(r_i r_j))$, respectively, and

$$\begin{aligned} \wp_i &= \operatorname{Re}(\mathfrak{R}_i) - \frac{\sum_{1 \leq i < j \leq n} s_{a_{\vec{\mathcal{E}}}}(r_i r_j)}{n}, \\ \mathfrak{s}_i &= \operatorname{Re}(\mathfrak{H}_i) - \frac{\sum_{1 \leq i < j \leq n} s_{b_{\vec{\mathcal{E}}}}(r_i r_j)}{n}, \end{aligned}$$

where $\operatorname{Re}(\mathfrak{R}_i)$ and $\operatorname{Re}(\mathfrak{H}_i)$ are the real parts of the eigenvalues of LM.

Theorem 10 Let $\mathcal{D} = (\mathcal{L}, \vec{\mathcal{E}})$ be a Lq-ROFDG with CLTS $S = \{s_\gamma | \gamma = 0, 1, \dots, \tau\}$, and let $L(\mathcal{D})$ be the LM of \mathcal{D} . If $\mathfrak{R}_1 \geq \mathfrak{R}_2 \geq \dots \geq \mathfrak{R}_n$ and $\mathfrak{H}_1 \geq \mathfrak{H}_2 \geq \dots \geq \mathfrak{H}_n$ are the eigenvalues of $L(s_{a_{\vec{\mathcal{E}}}}(r_i r_j))$ and $L(s_{b_{\vec{\mathcal{E}}}}(r_i r_j))$, respectively, then

$$\begin{aligned} s \sum_{\substack{i=1 \\ \mathfrak{R}_i \in \mathcal{N}_L}}^n \operatorname{Re}(\mathfrak{R}_i) &= s \sum_{1 \leq i < j \leq n} s_{a_{\vec{\mathcal{E}}}}(r_i r_j), \\ s \sum_{\substack{i=1 \\ \mathfrak{H}_i \in \mathcal{O}_L}}^n \operatorname{Re}(\mathfrak{H}_i) &= s \sum_{1 \leq i < j \leq n} s_{b_{\vec{\mathcal{E}}}}(r_i r_j). \end{aligned}$$

Theorem 11 Let $\mathcal{D} = (\mathcal{L}, \vec{\mathcal{E}})$ be a Lq-ROFDG with CLTS $S = \{s_\gamma | \gamma = 0, 1, \dots, \tau\}$, and let $L(\mathcal{D})$ be the LM of \mathcal{D} . If $\mathfrak{R}_1 \geq \mathfrak{R}_2 \geq \dots \geq \mathfrak{R}_n$ and $\mathfrak{H}_1 \geq \mathfrak{H}_2 \geq \dots \geq \mathfrak{H}_n$ are the eigenvalues of $L(s_{a_{\vec{\mathcal{E}}}}(r_i r_j))$ and $L(s_{b_{\vec{\mathcal{E}}}}(r_i r_j))$, respectively, and

$$\wp_i = \operatorname{Re}(\mathfrak{R}_i) - \frac{\sum_{1 \leq i < j \leq n} s_{a_{\vec{\mathcal{E}}}}(r_i r_j)}{n}, \quad \mathfrak{s}_i = \operatorname{Re}(\mathfrak{H}_i) - \frac{\sum_{1 \leq i < j \leq n} s_{b_{\vec{\mathcal{E}}}}(r_i r_j)}{n},$$

then $s \sum_{i=1}^n \wp_i = s_0$, $s \sum_{i=1}^n \mathfrak{s}_i = s_0$.

Example 6 The adjacency matrix is given in Example 5, but out-degree matrix and LM of the Lq-ROFDG shown in Fig. 8 are as follows:

$$D^{\text{out}}(\mathcal{D}) = \begin{bmatrix} (s_{14}, s_{16}) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_3, s_4) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_7, s_{11}) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_2, s_5) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_5, s_{13}) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_{10}, s_{13}) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_2, s_6) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_8, s_{21}) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_4, s_{12}) & (s_0, s_0) \end{bmatrix}.$$

$$L(\mathcal{D}) = \begin{bmatrix} (s_{14}, s_{16}) & (-s_4, -s_5) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_6, -s_5) & (-s_4, -s_6) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) \\ (-s_0, -s_0) & (s_3, s_4) & (-s_3, -s_4) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) \\ (-s_0, -s_0) & (-s_0, -s_0) & (s_7, s_{11}) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_4, -s_6) & (-s_3, -s_5) & (-s_0, -s_0) \\ (-s_0, -s_0) & (-s_0, -s_0) & (-s_2, -s_5) & (s_2, s_5) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) \\ (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_2, -s_6) & (s_5, s_{13}) & (-s_0, -s_0) & (-s_3, -s_7) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) \\ (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_5, -s_6) & (s_{10}, s_{13}) & (-s_0, -s_0) & (-s_5, -s_7) & (-s_0, -s_0) & (-s_0, -s_0) \\ (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (s_2, s_6) & (-s_0, -s_0) & (-s_2, -s_6) & (-s_0, -s_0) \\ (-s_4, -s_7) & (-s_3, -s_7) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (s_8, s_{21}) & (-s_1, -s_7) & (-s_0, -s_0) \\ (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_1, -s_6) & (-s_3, -s_6) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (-s_0, -s_0) & (s_4, s_{12}) \end{bmatrix}.$$

The Laplacian spectrum and the Laplacian energy of a Lq -ROFDG \mathcal{D} , given in Fig. 8, are as follows:

Laplacian Spec(\mathcal{D})

$$= \{(s_{13.1598+3.5023i}, s_{19.6057+4.5379i}), (s_{13.1598-3.5023i}, s_{19.6057-4.5379i}), (s_{6.8303+1.3240i}, s_{13.0597+6.6373i}), (s_{6.8303-1.3240i}, s_{13.0597-6.6373i}), (s_{5.4910+2.5200i}, s_{15.7977}), (s_{5.4910-2.5200i}, s_0), (s_0, s_{9.3916}), (s_{2.0190+0.4356i}, s_{6.6055}), (s_{2.0190-0.4356i}, s_{3.8743})\},$$

and $LE(\mathcal{D}) = (s_{31.0714}, s_{50.0349})$.

Furthermore, we have

$$\sum_{i=1}^9 \wp_i = s_0, \quad \sum_{i=1}^9 \mathfrak{s}_i = s_0.$$

4.3 Signless Laplacian energy of Lq -ROFDGs

We now discuss the signless Laplacian energy of Lq -ROFDGs and its properties.

Definition 21 The SLM of a Lq -ROFDG $\mathcal{D} = (L, \vec{E})$ is defined as $L^+(\mathcal{D}) = (L^+(s_{a_{\vec{E}}}(r_i r_j)), L^+(s_{b_{\vec{E}}}(r_i r_j))) = D^{\text{out}}(\mathcal{D}) + A(\mathcal{D})$, where $A(\mathcal{D})$ and $D^{\text{out}}(\mathcal{D})$ are the adjacency matrix and out-degree matrix of \mathcal{D} , respectively.

Definition 22 The spectrum of the SLM of a Lq -ROFDG $L^+(\mathcal{D})$ is defined as $(\mathcal{N}_{L^+}, O_{L^+})$, where \mathcal{N}_{L^+} and O_{L^+} are the sets of signless Laplacian eigenvalues of $L^+(s_{a_{\vec{E}}}(r_i r_j))$ and $L^+(s_{b_{\vec{E}}}(r_i r_j))$, respectively.

Definition 23 The energy of signless Laplacian of a Lq -ROFDG $\mathcal{D} = (L^+, \vec{E})$ is defined as:

$$LE^+(\mathcal{D}) = (LE^+(s_{a_{\vec{E}}}(r_i r_j)), LE^+(s_{b_{\vec{E}}}(r_i r_j))) = \left(\sum_{\wp_i^+ \in \mathcal{N}_{L^+}} \wp_i^+, \sum_{\mathfrak{s}_i^+ \in O_{L^+}} \mathfrak{s}_i^+ \right)$$

where \mathcal{N}_{L^+} and O_{L^+} are the sets of signless Laplacian eigenvalues of $L^+(s_{a_{\vec{E}}}(r_i r_j))$ and $L^+(s_{b_{\vec{E}}}(r_i r_j))$, respectively, and

$$\wp_i^+ = \text{Re}(\mathfrak{R}_i^+) - \frac{\sum_{1 \leq i < j \leq n} s_{a_{\vec{E}}}(r_i r_j)}{n},$$

$$\mathfrak{s}_i^+ = \text{Re}(\mathfrak{h}_i^+) - \frac{\sum_{1 \leq i < j \leq n} s_{b_{\vec{E}}}(r_i r_j)}{n}$$

where $\text{Re}(\mathfrak{R}_i^+)$ and $\text{Re}(\mathfrak{h}_i^+)$ are the real parts of eigenvalues of SLM.

Theorem 12 Let $\mathcal{D} = (L, \vec{E})$ be a Lq -ROFDG with CLTS $S = \{s_\gamma | \gamma = 0, 1, \dots, \tau\}$, and let $L^+(\mathcal{D})$ be the SLM of

\mathcal{D} . If $\Re_1^+ \geq \Re_2^+ \geq \dots \geq \Re_n^+$ and $\hbar_1^+ \geq \hbar_2^+ \geq \dots \geq \hbar_n^+$ are the eigenvalues of $L^+(s_{a_{\vec{E}}}(r_i r_j))$ and $L^+(s_{b_{\vec{E}}}(r_i r_j))$, respectively, then

$$\sum_{\substack{i=1 \\ \Re_i^+ \in \mathcal{R}_L^+}}^n Re(\Re_i^+) = s \sum_{1 \leq i < j \leq n} s_{a_{\vec{E}}}(r_i r_j),$$

$$s \sum_{\substack{i=1 \\ \hbar_i^+ \in \mathcal{O}_L^+}}^n Re(\hbar_i^+) = s \sum_{1 \leq i < j \leq n} s_{b_{\vec{E}}}(r_i r_j).$$

Theorem 13 Let $\mathcal{D} = (L, \vec{E})$ be a Lq -ROFDG with CLTS $S = \{s_\gamma | \gamma = 0, 1, \dots, \tau\}$, and let $L^+(\mathcal{D})$ be the SLM of \mathcal{D} . If $\Re_1^+ \geq \Re_2^+ \geq \dots \geq \Re_n^+$ and $\hbar_1^+ \geq \hbar_2^+ \geq \dots \geq \hbar_n^+$ are the eigenvalues of $L^+(s_{a_{\vec{E}}}(r_i^+ r_j))$ and $L^+(s_{b_{\vec{E}}}(r_i^+ r_j))$,

respectively, and $\wp_i^+ = Re(\Re_i^+) - \frac{\sum_{1 \leq i < j \leq n} s_{a_{\vec{E}}}(r_i^+ r_j)}{n}$, $\mathfrak{s}_i^+ = Re(\hbar_i^+) - \frac{\sum_{1 \leq i < j \leq n} s_{b_{\vec{E}}}(r_i^+ r_j)}{n}$, then $s \sum_{i=1}^n \wp_i^+ = s_0$, $s \sum_{i=1}^n \mathfrak{s}_i^+ = s_0$.

Example 7 The adjacency matrix and out-degree matrix are shown in Examples 5 and 6, respectively, but SLM of the Lq -ROFDG shown in Fig. 8 is as follows:

$$L^+(\mathcal{D}) = \begin{bmatrix} (s_{14}, s_{16}) & (s_4, s_5) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_6, s_5) & (s_4, s_6) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_3, s_4) & (s_3, s_4) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_7, s_{11}) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_4, s_6) & (s_3, s_5) \\ (s_0, s_0) & (s_0, s_0) & (s_2, s_5) & (s_2, s_5) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_2, s_6) & (s_5, s_{13}) & (s_0, s_0) & (s_3, s_7) & (s_0, s_0) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_5, s_6) & (s_{10}, s_{13}) & (s_0, s_0) & (s_5, s_7) & (s_0, s_0) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_2, s_6) & (s_0, s_0) & (s_2, s_6) \\ (s_4, s_7) & (s_3, s_7) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_8, s_{21}) & (s_1, s_7) \\ (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_1, s_6) & (s_3, s_6) & (s_0, s_0) & (s_0, s_0) & (s_0, s_0) & (s_4, s_{12}) \end{bmatrix}.$$

The signless Laplacian spectrum and the signless Laplacian energy of a Lq -ROFDG \mathcal{D} , given in Fig. 8, are as follows:

Signless Laplacian Spec(\mathcal{D})

$$= \{(s_{16.4749}, s_{25.0539}), (s_{7.7464+3.3079i}, s_{16.6484}), (s_{7.7464-3.3079i}, s_{12.2291+4.3870i}), (s_{6.8156}, s_{12.2291-4.3870i}), (s_{4.6676+1.1182i}, s_{7.1516+5.6459i}), (s_{4.6676-1.1182i}, s_{7.1516-5.6459i}), (s_{2.4384+1.7303i}, s_{4.4982+0.5709i}), (s_{2.4384-1.7303i}, s_{4.4982-0.5709i}), (s_{2.0047}, s_{11.5399})\},$$

and $LE^+(\mathcal{D}) = (s_{28.6778}, s_{43.1785})$.

Furthermore, we have

$$s \sum_{i=1}^9 r_i = s_0, \quad s \sum_{i=1}^9 s_i = s_0.$$

5 A graph-based MAGDM approach with linguistic q -rung orthopair fuzzy information

In this section, the Lq -ROF Einstein averaging (Lq -ROFEA) operator is utilized to develop a novel graph-based MAGDM model.

In the MAGDM problems, let $\ell = \{\ell_1, \ell_2, \dots, \ell_n\}$ be a discrete set of alternatives, $e = \{e_1, e_2, \dots, e_m\}$ be the set of organization composed of many experts, and $\omega = (\omega_1, \omega_2, \dots, \omega_m)^T$ be the weight vector satisfying $\sum_{k=1}^m \omega_k = 1$ and $\omega_k \geq 0, k = 1, 2, \dots, m$. Each expert in an organization provides preference relation for each pair of alternatives, so for each organization its linguistic q -rung orthopair fuzzy preference relation (Lq -ROFPR) is constructed with $H^{(k)} = (p_{ij}^{(k)})_{n \times n} = (s_{a_{ij}}^{(k)}, s_{b_{ij}}^{(k)})_{n \times n}$.

Step 1. We utilize the Lq -ROFEA operator to aggregate all $p_{ij}^{(k)} (j = 1, 2, \dots, n)$ that correspond to the alternative ℓ_i . We get the Lq -ROFN $p_i^{(k)}$ of the alternative ℓ_i over all the other alternatives for the organization e_k .

$$p_i^{(k)} = Lq\text{-ROFEA}(p_{i1}^{(k)}, p_{i2}^{(k)}, \dots, p_{in}^{(k)})$$

$$= \left(s \left(\frac{\prod_{i=1}^n (1 + (\frac{a_i}{\tau})^q)^{\omega_i} - \prod_{i=1}^n (1 - (\frac{a_i}{\tau})^q)^{\omega_i}}{\prod_{i=1}^n (1 + (\frac{a_i}{\tau})^q)^{\omega_i} + \prod_{i=1}^n (1 - (\frac{a_i}{\tau})^q)^{\omega_i}} \right)^{1/q}, \right.$$

$$\left. s \left(\frac{2^{1/q} \prod_{i=1}^n (\frac{b_i}{\tau})^{\omega_i}}{\left(\prod_{i=1}^n (2 - (\frac{b_i}{\tau})^q)^{\omega_i} + \prod_{i=1}^n (\frac{b_i}{\tau})^{q\omega_i} \right)^{1/q}} \right) \right).$$

Step 2. If the weight vector $\omega = (\omega_1, \omega_2, \dots, \omega_m)^T$ of all organizations is known, then go to Step 4; otherwise, the weight of each organization needs to be accounted for. To calculate the weights of decision organization, it proceeds as follows:

- (i) We first use the linguistic q -rung orthopair fuzzy Hamming distance (L q -ROFHD) between two L q -ROFNs to find out the weights of the experts

$$D(p_1, p_2) = \frac{\tau}{2} \left(\left| \left(\frac{a_{p_1}}{\tau} \right)^q - \left(\frac{a_{p_2}}{\tau} \right)^q \right| + \left| \left(\frac{b_{p_1}}{\tau} \right)^q - \left(\frac{b_{p_2}}{\tau} \right)^q \right| + \left| \left(\frac{s_{\pi p_1}}{\tau} \right)^q - \left(\frac{s_{\pi p_2}}{\tau} \right)^q \right| \right)$$

where $s_{\pi p_1} = \sqrt[q]{\tau^q - a_{p_1}^q - b_{p_1}^q}$,
 $s_{\pi p_2} = \sqrt[q]{\tau^q - a_{p_2}^q - b_{p_2}^q}$ (3)

to obtain the difference matrices,

$$D_{lk} = (d_{ij}^{(lk)})_{n \times n} = d(p_{ij}^{(l)}, p_{ij}^{(k)})_{n \times n},$$

$l, k = 1, 2, \dots, m$

where

- (a) $d_{ij}^{(lk)} \geq 0$, especially, if $l = k$, then $d_{ij}^{(lk)} = 0, i, j = 1, 2, \dots, n$;
- (b) $d_{ij}^{(lk)} = 0, i = j$;
- (c) $d_{ij}^{(lk)} = d_{ji}^{(lk)}$.

D_{lk} is a symmetric matrix and the values of diagonal elements are zero.

- (ii) Determine the average value of the matrix D_{lk} from

$$\bar{d}_{lk} = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n d_{ij}^{(lk)} \quad (4)$$

- (iii) Afterward, let $\bar{d} = \sum_{k=1, k \neq l}^m \bar{d}_{lk}$, then

$$\bar{d}_l = \frac{1}{n^2} \sum_{k=1, k \neq l}^m \sum_{i=1}^n \sum_{j=1}^n d_{ij}^{(lk)}, \quad (5)$$

which denotes the deviation of the organization e_l from the rest organizations.

- (iv) The weight vector ω_l of the organizations e_l can be calculated as

$$\omega_l = \frac{(\bar{d}_l)^{-1}}{\sum_{l=1}^m (\bar{d}_l)^{-1}}, \quad l = 1, 2, \dots, m \quad (6)$$

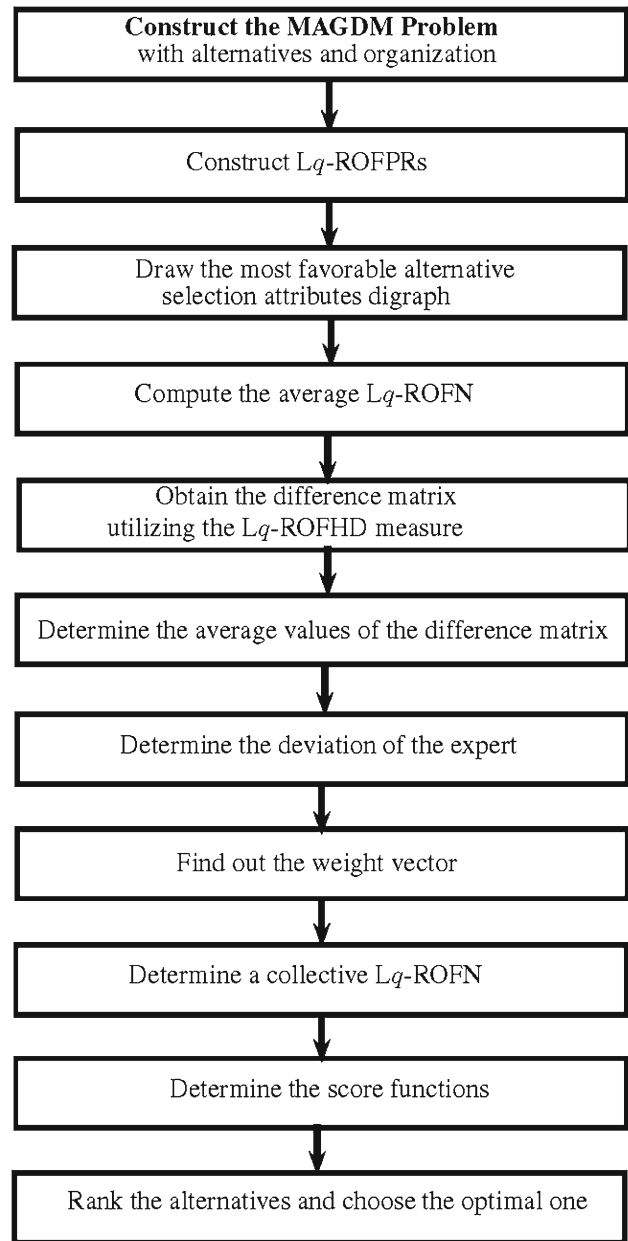


Fig. 9 The specific steps of the proposed approach to MAGDM

Step 3. To aggregate all $p_i^{(k)} (k = 1, 2, \dots, m)$ into a collective L q -ROFN $p_i^{(k)}$ for the alternative ℓ_i , use L q -ROF Einstein weighted averaging (L q -ROFEWA) operator.

Step 4. Compute the score functions of $p_i (i = 1, 2, \dots, n)$ by Eq. (1) to rank all the alternatives $\ell_i (i = 1, 2, \dots, n)$ according to $S(\ell_i) (i = 1, 2, \dots, n)$.

The specific steps of this proposed approach to MAGDM are depicted in Fig. 9.

5.1 Illustrative example and comparative analysis

In the following, a numerical example related to “mobile payment platform selection” is given to describe the validity of our approach:

5.1.1 Mobile payment platform selection

The advancement of disruptive technologies has made mobile devices acquire new functionalities that support multiple mobile financial services including bill payments, account transfers, person-to-person proximity payments, exchanges, remote payments, and other forms of services such as mobile ticketing and discounts (Liana-Cabanillas et al. 2018). There is a development, called “mobile payment technology,” which is undergoing rapid growth among the different mobile applications offered today. Mobile payment can be seen as an application between mobile devices and a payment mechanism that helps the user to pay for transaction through mobile devices. In sub-Saharan Africa, the adoption of mobile payment technology has been impressive, with various operators offering diverse services. As the beneficial impact mobile payment has on organizations, numerous mobile payment platforms have proliferated around the world. Assessment and selection of numerous mobile payment platforms is often difficult, as they require detailed identification. Also, because of uncertainty and vagueness in real-life situations, decision makers can find it difficult to examine the mobile payment platforms accurately. However, this problem can be solved using L_q -ROF MAGDM. An example related to selected mobile payment platform is used to illustrate proposed decision-making approach. A group of five decision makers $\{e_1, e_2, e_3, e_4, e_5\}$, are invited to evaluate six mobile payment platforms MTN Mobile Money (MTN MM), M-Pesa (MP), Tigo Pesa (TP), Vodafone Cash (VC), Orange Money (OM), and Airtel Money (AM), i.e., alternatives $\ell_i (i = 1, 2, \dots, 6)$. Here, we suppose that the decision makers have equal weights. In order to completely communicate the experts’ appreciation, linguistic variables can be added within the predefined LTS $S = \{s_0 = \text{extremely bad}, s_1 = \text{very bad}, s_2 = \text{bad}, s_3 = \text{relatively bad}, s_4 = \text{normal}, s_5 = \text{relatively good}, s_6 = \text{good}, s_7 = \text{very good}, s_8 = \text{extraordinary}\}$. The five decision makers $e_k (k = 1, 2, \dots, 5)$ presented their assessments in L4-ROFPRs as follows:

The L4-ROFDGs \mathcal{D}_i corresponding to L4-ROFPRs $R_k (k = 1, 2, \dots, 5)$ given in Tables 3, 4, 5, 6 and 7 are shown in Fig. 10.

Compute the averaged L_q -ROFN $p_i^{(k)}$ of an alternative ℓ_i over all the other alternatives for the experts $e_k (k = 1, 2, \dots, 5)$ using L_q -ROFEA operator, as in Table 8.

We first utilize Eq. (3) between two L_q -ROFNs to find out the weights of the experts to compute $d(p_{ij}^{(l)}, p_{ij}^{(k)})$, $i, j =$

Table 3 L4-ROFPR given by decision organization (e_1)

R_1	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6
ℓ_1	(s_4, s_4)	(s_2, s_7)	(s_2, s_6)	(s_5, s_3)	(s_3, s_4)	(s_2, s_5)
ℓ_2	(s_7, s_2)	(s_4, s_4)	(s_4, s_2)	(s_1, s_5)	(s_5, s_4)	(s_3, s_2)
ℓ_3	(s_6, s_2)	(s_2, s_4)	(s_4, s_4)	(s_3, s_5)	(s_3, s_3)	(s_6, s_2)
ℓ_4	(s_3, s_5)	(s_5, s_1)	(s_5, s_3)	(s_4, s_4)	(s_1, s_6)	(s_7, s_1)
ℓ_5	(s_4, s_3)	(s_4, s_5)	(s_3, s_3)	(s_6, s_1)	(s_4, s_4)	(s_3, s_4)
ℓ_6	(s_5, s_2)	(s_2, s_3)	(s_2, s_6)	(s_1, s_7)	(s_4, s_3)	(s_4, s_4)

Table 4 L4-ROFPR given by decision organization (e_2)

R_2	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6
ℓ_1	(s_4, s_4)	(s_5, s_2)	(s_4, s_3)	(s_3, s_6)	(s_6, s_2)	(s_7, s_1)
ℓ_2	(s_2, s_5)	(s_4, s_4)	(s_2, s_4)	(s_2, s_3)	(s_5, s_4)	(s_4, s_2)
ℓ_3	(s_3, s_4)	(s_4, s_2)	(s_4, s_4)	(s_5, s_1)	(s_3, s_3)	(s_2, s_3)
ℓ_4	(s_6, s_3)	(s_3, s_2)	(s_1, s_5)	(s_4, s_4)	(s_1, s_7)	(s_6, s_1)
ℓ_5	(s_2, s_6)	(s_4, s_5)	(s_3, s_3)	(s_7, s_1)	(s_4, s_4)	(s_3, s_5)
ℓ_6	(s_1, s_7)	(s_2, s_4)	(s_3, s_2)	(s_1, s_6)	(s_5, s_3)	(s_4, s_4)

Table 5 L4-ROFPR given by decision organization (e_3)

R_3	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6
ℓ_1	(s_4, s_4)	(s_4, s_1)	(s_1, s_5)	(s_2, s_4)	(s_7, s_2)	(s_3, s_1)
ℓ_2	(s_1, s_4)	(s_4, s_4)	(s_1, s_6)	(s_2, s_2)	(s_3, s_2)	(s_5, s_2)
ℓ_3	(s_5, s_1)	(s_6, s_1)	(s_4, s_4)	(s_1, s_3)	(s_6, s_3)	(s_3, s_7)
ℓ_4	(s_4, s_2)	(s_2, s_2)	(s_3, s_1)	(s_4, s_4)	(s_3, s_5)	(s_4, s_3)
ℓ_5	(s_2, s_7)	(s_2, s_3)	(s_3, s_6)	(s_5, s_3)	(s_4, s_4)	(s_5, s_5)
ℓ_6	(s_1, s_3)	(s_2, s_5)	(s_7, s_3)	(s_3, s_4)	(s_5, s_5)	(s_4, s_4)

Table 6 L4-ROFPR given by decision organization (e_4)

R_4	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6
ℓ_1	(s_4, s_4)	(s_5, s_2)	(s_3, s_7)	(s_4, s_3)	(s_5, s_5)	(s_3, s_3)
ℓ_2	(s_2, s_5)	(s_4, s_4)	(s_4, s_3)	(s_1, s_6)	(s_7, s_2)	(s_2, s_4)
ℓ_3	(s_7, s_3)	(s_3, s_4)	(s_4, s_4)	(s_4, s_1)	(s_1, s_5)	(s_7, s_1)
ℓ_4	(s_3, s_4)	(s_6, s_1)	(s_1, s_4)	(s_4, s_4)	(s_3, s_6)	(s_5, s_3)
ℓ_5	(s_5, s_5)	(s_2, s_7)	(s_5, s_1)	(s_6, s_3)	(s_4, s_4)	(s_5, s_4)
ℓ_6	(s_3, s_3)	(s_4, s_2)	(s_1, s_7)	(s_3, s_5)	(s_4, s_5)	(s_4, s_4)

Table 7 L4-ROFPR given by decision organization (e_5)

R_5	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6
ℓ_1	(s_4, s_4)	(s_3, s_4)	(s_4, s_5)	(s_6, s_4)	(s_3, s_5)	(s_6, s_3)
ℓ_2	(s_4, s_3)	(s_4, s_4)	(s_7, s_3)	(s_2, s_7)	(s_5, s_6)	(s_3, s_2)
ℓ_3	(s_5, s_4)	(s_3, s_7)	(s_4, s_4)	(s_3, s_3)	(s_1, s_7)	(s_6, s_1)
ℓ_4	(s_4, s_6)	(s_7, s_2)	(s_3, s_3)	(s_4, s_4)	(s_5, s_1)	(s_2, s_4)
ℓ_5	(s_5, s_3)	(s_6, s_5)	(s_7, s_1)	(s_1, s_5)	(s_4, s_4)	(s_4, s_5)
ℓ_6	(s_3, s_6)	(s_2, s_3)	(s_1, s_6)	(s_4, s_2)	(s_5, s_4)	(s_4, s_4)

$1, 2, \dots, 6; l, k = 1, 2, \dots, 5$ and obtained the difference matrices as follows:

$$D_{12} = D_{21} = \begin{bmatrix} s_0 & s_{4.6582} & s_{2.3730} & s_{2.3730} & s_{2.3730} & s_{4.6582} \\ s_{4.6582} & s_0 & s_{0.4688} & s_{1.0625} & s_0 & s_{0.3418} \\ s_{2.3730} & s_{0.4688} & s_0 & s_{1.2187} & s_0 & s_{2.5000} \\ s_{2.3730} & s_{1.0625} & s_{1.2187} & s_0 & s_{2.1582} & s_{2.1582} \\ s_{2.3730} & s_0 & s_0 & s_{2.1582} & s_0 & s_{0.7207} \\ s_{4.6582} & s_{0.3418} & s_{2.5000} & s_{2.1582} & s_{0.7207} & s_0 \end{bmatrix},$$

$$D_{13} = D_{31} = \begin{bmatrix} s_0 & s_{4.6875} & s_{1.3398} & s_{1.1895} & s_{4.5313} & s_{1.2188} \\ s_{4.6875} & s_0 & s_{2.5000} & s_{1.1895} & s_{1.5312} & s_{1.0625} \\ s_{1.3398} & s_{2.5000} & s_0 & s_{1.2187} & s_{2.3730} & s_{4.6582} \\ s_{1.1895} & s_{1.1895} & s_{1.2187} & s_0 & s_{1.3105} & s_{4.1895} \\ s_{4.5313} & s_{1.5312} & s_{2.3730} & s_{1.3105} & s_0 & s_{1.7832} \\ s_{1.2188} & s_{1.0625} & s_{4.6582} & s_{4.1895} & s_{1.7832} & s_0 \end{bmatrix},$$

$$D_{14} = D_{41} = \begin{bmatrix} s_0 & s_{4.6582} & s_{2.2852} & s_{0.7207} & s_{1.7832} & s_{1.0625} \\ s_{4.6582} & s_0 & s_{0.1270} & s_{1.3105} & s_{3.4688} & s_{0.4687} \\ s_{2.2852} & s_{0.1270} & s_0 & s_{1.2187} & s_{1.0625} & s_{2.1582} \\ s_{0.7207} & s_{1.3105} & s_{1.2187} & s_0 & s_{0.1562} & s_{3.4688} \\ s_{1.7832} & s_{3.4688} & s_{1.0625} & s_{0.1562} & s_0 & s_{1.0625} \\ s_{1.0625} & s_{0.4687} & s_{2.1582} & s_{3.4688} & s_{1.0625} & s_0 \end{bmatrix},$$

$$D_{15} = D_{51} = \begin{bmatrix} s_0 & s_{4.1895} & s_{1.3105} & s_{1.6523} & s_{0.7207} & s_{2.5000} \\ s_{4.1895} & s_0 & s_{4.3164} & s_{3.4980} & s_{2.0313} & s_0 \\ s_{1.3105} & s_{4.3164} & s_0 & s_{1.0625} & s_{4.5313} & s_{0.0293} \\ s_{1.6523} & s_{3.4980} & s_{1.0625} & s_0 & s_{2.5293} & s_{4.6582} \\ s_{0.7207} & s_{2.0313} & s_{4.5313} & s_{2.5293} & s_0 & s_{1.0625} \\ s_{2.5000} & s_0 & s_{0.0293} & s_{4.6582} & s_{1.0625} & s_0 \end{bmatrix},$$

$$D_{23} = D_{32} = \begin{bmatrix} s_0 & s_{0.7500} & s_{1.0625} & s_{2.1582} & s_{2.1582} & s_{4.5313} \\ s_{0.7500} & s_0 & s_{2.0313} & s_{0.1270} & s_{1.5312} & s_{0.7207} \\ s_{1.0625} & s_{2.0313} & s_0 & s_{1.2188} & s_{2.3730} & s_{4.6582} \\ s_{2.1582} & s_{0.1270} & s_{1.2188} & s_0 & s_{3.4688} & s_{2.0313} \\ s_{2.1582} & s_{1.5312} & s_{2.3730} & s_{3.4688} & s_0 & s_{1.0625} \\ s_{4.5313} & s_{0.7207} & s_{4.6582} & s_{2.0313} & s_{1.0625} & s_0 \end{bmatrix},$$

$$D_{24} = D_{42} = \begin{bmatrix} s_0 & s_0 & s_{4.5313} & s_{2.3730} & s_{1.3105} & s_{4.5313} \\ s_0 & s_0 & s_{0.4688} & s_{2.3730} & s_{3.4688} & s_{0.4688} \\ s_{4.5313} & s_{0.4688} & s_0 & s_{0.7207} & s_{1.0625} & s_{4.6582} \\ s_{2.3730} & s_{2.3730} & s_{0.7207} & s_0 & s_{2.1582} & s_{1.3105} \\ s_{1.3105} & s_{3.4688} & s_{1.0625} & s_{2.1582} & s_0 & s_{1.0625} \\ s_{4.5313} & s_{0.4688} & s_{4.6582} & s_{1.3105} & s_{1.0625} & s_0 \end{bmatrix},$$

$$D_{25} = D_{52} = \begin{bmatrix} s_0 & s_{1.0625} & s_{1.0625} & s_{2.3730} & s_{2.3730} & s_{2.1582} \\ s_{1.0625} & s_0 & s_{4.6582} & s_{4.5313} & s_{2.0313} & s_{0.3418} \\ s_{1.0625} & s_{4.6582} & s_0 & s_{1.0625} & s_{4.5313} & s_{2.5000} \\ s_{2.3730} & s_{4.5313} & s_{1.0625} & s_0 & s_{4.6875} & s_{2.5000} \\ s_{2.3730} & s_{2.0313} & s_{4.5313} & s_{4.6875} & s_0 & s_{0.3418} \\ s_{2.1582} & s_{0.3418} & s_{2.5000} & s_{2.5000} & s_{0.3418} & s_0 \end{bmatrix},$$

$$D_{34} = D_{43} = \begin{bmatrix} s_0 & s_{0.7500} & s_{3.6250} & s_{0.4688} & s_{3.4688} & s_{0.1562} \\ s_{0.7500} & s_0 & s_{2.3730} & s_{2.5000} & s_{4.5313} & s_{1.1895} \\ s_{3.6250} & s_{2.3730} & s_0 & s_{0.4980} & s_{2.5293} & s_{4.6875} \\ s_{0.4688} & s_{2.5000} & s_{0.4980} & s_0 & s_{1.3105} & s_{0.7207} \\ s_{3.4688} & s_{4.5313} & s_{2.5293} & s_{1.3105} & s_0 & s_{0.7207} \\ s_{0.1562} & s_{1.1895} & s_{4.6875} & s_{0.7207} & s_{0.7207} & s_0 \end{bmatrix},$$

$$D_{35} = D_{53} = \begin{bmatrix} s_0 & s_{0.4980} & s_{0.4980} & s_{2.5000} & s_{4.5313} & s_{2.5293} \\ s_{0.4980} & s_0 & s_{4.6875} & s_{4.6582} & s_{3.5625} & s_{1.0625} \\ s_{0.4980} & s_{4.6875} & s_0 & s_{0.1562} & s_{4.5312} & s_{4.6875} \\ s_{2.5000} & s_{4.6582} & s_{0.1562} & s_0 & s_{1.2187} & s_{0.4688} \\ s_{4.5313} & s_{3.5625} & s_{4.5312} & s_{1.2187} & s_0 & s_{0.7207} \\ s_{2.5293} & s_{1.0625} & s_{4.6875} & s_{0.4688} & s_{0.7207} & s_0 \end{bmatrix},$$

$$D_{45} = D_{54} = \begin{bmatrix} s_0 & s_{1.0625} & s_{3.4688} & s_{2.3730} & s_{1.0625} & s_{2.3730} \\ s_{1.0625} & s_0 & s_{4.1895} & s_{2.1875} & s_{3.4688} & s_{0.4687} \\ s_{3.4688} & s_{4.1895} & s_0 & s_{0.3418} & s_{3.4687} & s_{2.1582} \\ s_{2.3730} & s_{2.1875} & s_{0.3418} & s_0 & s_{2.5293} & s_{1.1895} \\ s_{1.0625} & s_{3.4688} & s_{3.4687} & s_{2.5293} & s_0 & s_{0.7207} \\ s_{2.3730} & s_{0.4687} & s_{2.1582} & s_{1.1895} & s_{0.7207} & s_0 \end{bmatrix},$$

$$D_{11} = D_{22} = D_{33} = D_{44} = D_{55} = \begin{bmatrix} s_0 & s_0 & s_0 & s_0 & s_0 & s_0 \\ s_0 & s_0 & s_0 & s_0 & s_0 & s_0 \\ s_0 & s_0 & s_0 & s_0 & s_0 & s_0 \\ s_0 & s_0 & s_0 & s_0 & s_0 & s_0 \\ s_0 & s_0 & s_0 & s_0 & s_0 & s_0 \end{bmatrix}.$$

Utilize Eq. (4) to determine the average values of the difference matrices, as in Table 9.

Use Eq. (5) to determine the deviation of the expert e_1 from the remaining experts as follows:

$$d_1 = s_{6.7195}, \quad d_2 = s_{6.8700}, \quad d_3 = s_{7.2503}, \\ d_4 = s_{6.4501}, \quad d_5 = s_{7.6489}.$$

Utilizing Eq. (6), we determine the weights of the experts.

$$\omega_1 = s_{0.1923}, \quad \omega_2 = s_{0.1966}, \quad \omega_3 = s_{0.2075}, \\ \omega_4 = s_{0.1846}, \quad \omega_5 = s_{0.2189}.$$

Compute a collective Lq -ROFN p_i ($i = 1, 2, \dots, 6$) of the alternative ℓ_i over all the other alternatives using Lq -ROFEWA operator.

$$p_1 = (s_{4.7348}, s_{3.3786}), \quad p_2 = (s_{4.5951}, s_{3.4232}), \\ p_3 = (s_{4.8233}, s_{2.8875}), \\ p_4 = (s_{4.7003}, s_{2.8788}), \quad p_5 = (s_{4.8156}, s_{3.5688}), \\ p_6 = (s_{4.0637}, s_{3.9677}).$$

Compute the score function of Lq -ROFN by Eq. (1) of p_i ($i = 1, 2, \dots, 6$), and rank all the alternatives ℓ_i ($i = 1, 2, \dots, 6$) according to the values of $S(p_i)$ ($i = 1, 2, \dots, 6$).

$$S(p_1) = s_{6.8751}, \quad S(p_2) = s_{6.8504}, \quad S(p_3) = s_{6.9130}, \\ S(p_4) = s_{6.8931}, \quad S(p_5) = s_{6.8763}, \quad S(p_6) = s_{6.7374}.$$

Then, $\ell_3 \succ \ell_4 \succ \ell_5 \succ \ell_1 \succ \ell_2 \succ \ell_6$. Thus, the optimal choice is ℓ_3 . The curiosity of this decision-making approach is that by comparing graphical structures with Lq -ROF data, we have built up a MAGDM model with the interrelated organizations and depicted various connections among the organizations.

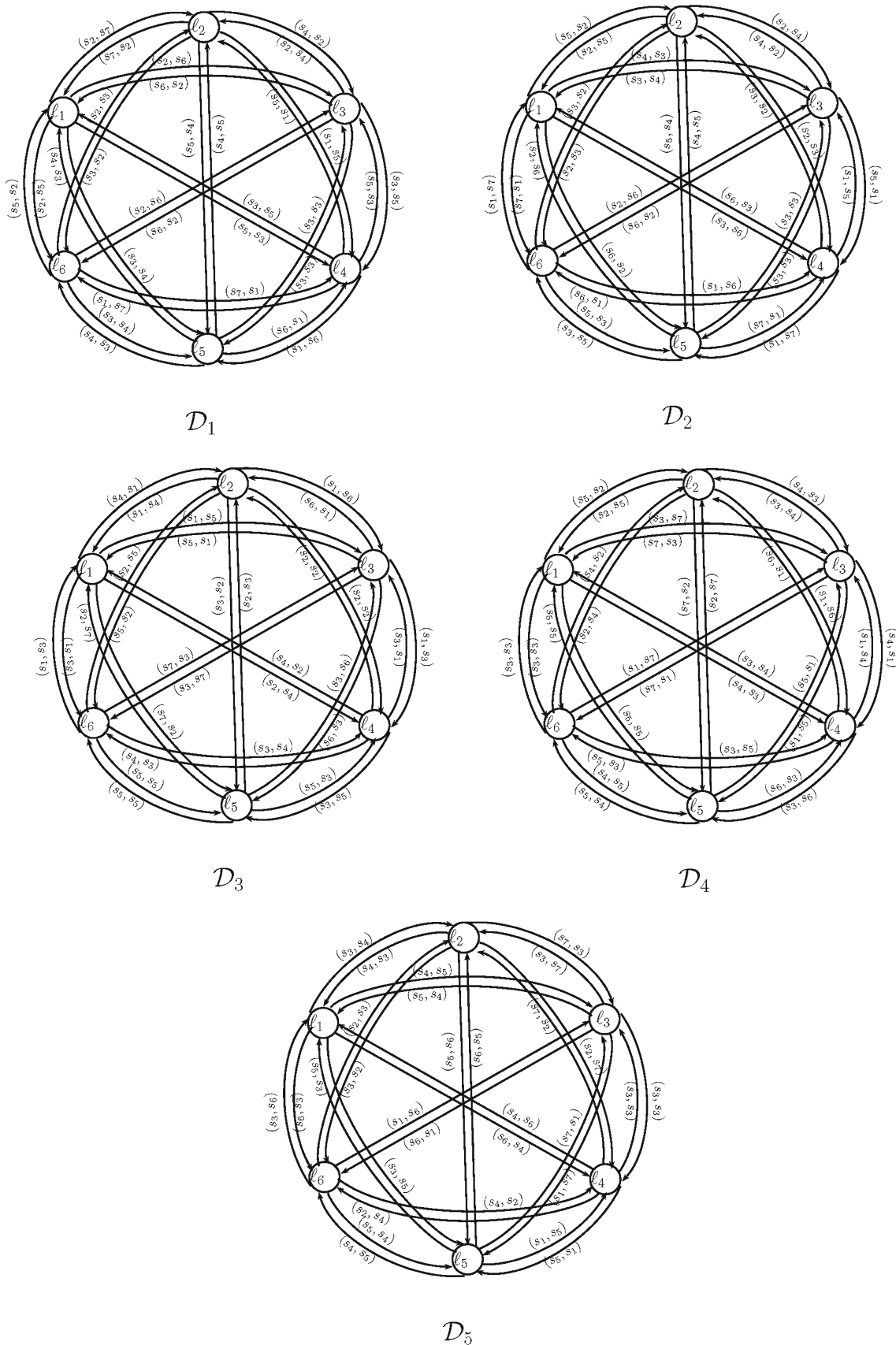


Fig. 10 L4-ROFDG with CLTS $S = \{s_\gamma | \gamma \in [0, 8]\}$

Table 8 Overall assessment values of the experts e_k ($k = 1, 2, \dots, 5$)

Experts	Assessment values	Experts	Assessment values		
e_1	$p_1^{(1)}$	$(s_3.6062, s_4.7065)$	e_4	$p_1^{(4)}$	$(s_4.2353, s_3.7407)$
	$p_2^{(1)}$	$(s_5.0628, s_2.9470)$		$p_2^{(4)}$	$(s_4.8346, s_3.7991)$
	$p_3^{(1)}$	$(s_4.7683, s_3.1518)$		$p_3^{(4)}$	$(s_5.6178, s_2.5058)$
	$p_4^{(1)}$	$(s_5.1756, s_2.6956)$		$p_4^{(4)}$	$(s_4.4646, s_3.2640)$
	$p_5^{(1)}$	$(s_4.4087, s_3.0066)$		$p_5^{(4)}$	$(s_4.9077, s_3.5041)$
	$p_6^{(1)}$	$(s_3.7406, s_3.8677)$		$p_6^{(4)}$	$(s_3.5299, s_4.0747)$
e_2	$p_1^{(2)}$	$(s_5.4352, s_2.5897)$	e_5	$p_1^{(5)}$	$(s_4.8566, s_4.1157)$
	$p_2^{(2)}$	$(s_3.7525, s_3.5359)$		$p_2^{(5)}$	$(s_5.0675, s_3.8677)$
	$p_3^{(2)}$	$(s_3.8508, s_2.5755)$		$p_3^{(5)}$	$(s_4.4646, s_3.7425)$
	$p_4^{(2)}$	$(s_4.7320, s_3.1233)$		$p_4^{(5)}$	$(s_5.0675, s_2.9080)$
	$p_5^{(2)}$	$(s_4.8862, s_3.5255)$		$p_5^{(5)}$	$(s_5.4152, s_3.4089)$
	$p_6^{(2)}$	$(s_3.5793, s_4.0554)$		$p_6^{(5)}$	$(s_3.7912, s_3.9279)$
e_3	$p_1^{(3)}$	$(s_4.8579, s_2.3420)$			
	$p_2^{(3)}$	$(s_3.5793, s_3.0487)$			
	$p_3^{(3)}$	$(s_4.9580, s_2.5520)$			
	$p_4^{(3)}$	$(s_3.5440, s_2.5029)$			
	$p_5^{(3)}$	$(s_4.0578, s_4.4941)$			
	$p_6^{(3)}$	$(s_4.9858, s_3.9256)$			

Table 9 Average values of the difference matrices

$\bar{d}_{12} = \bar{d}_{21} = \frac{s_{54.1286}}{s_{36}} = s_{1.5036}$
$\bar{d}_{13} = \bar{d}_{31} = \frac{s_{69.5664}}{s_{36}} = s_{1.9324}$
$\bar{d}_{14} = \bar{d}_{41} = \frac{s_{50.0234}}{s_{36}} = s_{1.3895}$
$\bar{d}_{15} = \bar{d}_{51} = \frac{s_{68.1836}}{s_{36}} = s_{1.8940}$
$\bar{d}_{23} = \bar{d}_{32} = \frac{s_{59.7660}}{s_{36}} = s_{1.6602}$
$\bar{d}_{24} = \bar{d}_{42} = \frac{s_{60.9962}}{s_{36}} = s_{1.6943}$
$\bar{d}_{25} = \bar{d}_{52} = \frac{s_{72.4298}}{s_{36}} = s_{2.0119}$
$\bar{d}_{34} = \bar{d}_{43} = \frac{s_{59.0586}}{s_{36}} = s_{1.6405}$
$\bar{d}_{35} = \bar{d}_{53} = \frac{s_{72.6208}}{s_{36}} = s_{2.0172}$
$\bar{d}_{45} = \bar{d}_{54} = \frac{s_{62.1250}}{s_{36}} = s_{1.7257}$

5.2 Validity analysis of the developed approach

In this subsection, to show the effectiveness and superiority of our developed approach, we utilize other existing techniques to handle with this example and compare the results with the developed approach.

From this analysis, it is easy to see that our developed method based on graph theory and Lq -ROF Einstein weighted averaging operator is more versatile and reasonable to solve MAGDM problems.

Following are the figures which best describe the above evaluations.

5.2.1 Further discussion

It can be seen from the above analysis that the ranking results of the above methods are same and the best alternative is ℓ_3 . This verifies that the developed approach is reasonable and useful for MAGDM problems with Lq -ROFNs. Lq -ROFS is more and more general and contains more information in the MAGDM process. Therefore, our developed method provides more general and powerful information in MAGDM.

1. Comparing with the Lq -ROF weighted averaging (Lq -ROFWA) operator (i.e., taking algebraic t-norm and t-conorm) proposed by Lin et al. (2020): According to Lq -ROF weighted averaging operator, the ranking results are obtained as: $\ell_3 > \ell_4 > \ell_5 > \ell_1 > \ell_2 > \ell_6$. Clearly, the approach in the literature has the same ranking result as ours. However, our approach adopted the Einstein t-norm and t-conorm, and the literature utilized the algebraic t-norm and t-conorm. As the algebraic t-norm and t-conorm are just a special case of the Einstein t-norm and t-conorm, the approach developed in this paper is more general than the approach given in Lin et al. (2020).
2. Comparing with the Lq -ROF weighted geometric (Lq -ROFWG) operator proposed by Lin et al. (2020): According to Lq -ROF weighted geometric operator, the ranking results are obtained as: $\ell_3 > \ell_4 > \ell_5 > \ell_1 > \ell_2 > \ell_6$. The results corresponding to these approaches are

Table 10 Comparative analysis

Methods	Score of alternatives						Ranking of alternatives
Lin et al. (2020) Lq -ROFWA operator	$S_{6.8967}$	$S_{6.8762}$	$S_{6.9339}$	$S_{6.9138}$	$S_{6.8981}$	$S_{6.7533}$	$l_3 > l_4 > l_5 > l_1 > l_2 > l_6$
Lin et al. (2020) Lq -ROFWG operator	$S_{6.8751}$	$S_{6.8504}$	$S_{6.9130}$	$S_{6.8931}$	$S_{6.8763}$	$S_{6.7374}$	$l_3 > l_4 > l_5 > l_1 > l_2 > l_6$
Proposed (Lq -ROFEWA operator)	$S_{6.6152}$	$S_{6.6074}$	$S_{6.6489}$	$S_{6.6475}$	$S_{6.6310}$	$S_{6.4775}$	$l_3 > l_4 > l_5 > l_1 > l_2 > l_6$
Proposed (Lq -ROFEWG operator)	$S_{6.6256}$	$S_{6.6218}$	$S_{6.6604}$	$S_{6.6600}$	$S_{6.6438}$	$S_{6.4992}$	$l_3 > l_4 > l_5 > l_1 > l_2 > l_6$

summarized in Table 10. These results are graphically represented in Fig. 11, 12.

- Comparing with the LIF Einstein weighted averaging operator (i.e., taking $q = 1$): According to LIF Einstein weighted averaging operator, the ranking results are obtained as: $l_3 > l_4 > l_5 > l_1 > l_2 > l_6$. The LIF Einstein weighted averaging operator just aggregates the LIFNs, and the LIFN must satisfy the conditional limit of $0 \leq a + b \leq \tau$, $a, b \in [0, \tau]$. Therefore, the application range of the LIF Einstein weighted averaging operator is limited. Obviously, most of the evaluation values do not satisfy the conditional limit of $0 \leq a + b \leq \tau$ in this example, so the LIF Einstein weighted averaging operator is not suitable in this example.
- Comparing with the LPF Einstein weighted averaging operator (i.e., taking $q = 2$) in the literature proposed by Rong et al. (2020): According to LPF Einstein weighted averaging operator, the ranking results are obtained as: $l_3 > l_4 > l_5 > l_1 > l_2 > l_6$. The LPF Einstein weighted averaging operator aggregates only the LPFNs and the LPFN must satisfy the conditional limit of $0 \leq a^2 + b^2 \leq \tau^2$, $a, b \in [0, \tau]$. Therefore, the application range of the LPF Einstein weighted averaging operator is wider than the LIF Einstein weighted averaging operator but limited than the Lq -ROFEWA operator in our proposed method. Most of the evaluation values in this decision-making problem meet the conditional limit of $0 \leq a^2 + b^2 \leq \tau^2$. So in this example, the LPF Einstein weighted averaging operator presented in Rong et al. (2020) cannot cope entirely with the decision-making problem.

In summary, under Lq -ROF setting, the approach built in this paper blends the Einstein t -norm and t -conorm with Einstein operator. This not only takes into account the inter-relationship of aggregation arguments but also provides a broad and robust method for dealing with MAGDM problems in Lq -ROF environment.

5.2.2 Advantages of the proposed work

The merits of our developed approach are summarized as follows:

- Apparently, proposed approach is clear and has less loss of data and can be easily utilized to other MAGDM problems in Lq -ROF setting.
- One of the superiorities of the created approach is utilizing graph theory.
- The Lq -ROFSs of developed technique can accessibly depict more general decision-making circumstances.
- The Einstein operator is more valid to deal MAGDM problems under Lq -ROF circumstances.

6 Conclusions

To qualitatively describe the evaluation values and provide greater freedom for the decision makers in MAGDM problems, the concept of Lq -ROFS has been developed under Lq -ROF environment. In fuzzy information systems, correlations between attributes via edges between vertices are well expressed by fuzzy graphs. A q -ROFG model is capable of illustrating the problems with ambiguity, imprecision and inconsistent information in contrast to the classical fuzzy graph models. However, the simplicity and flexibility which arise because of linguistic degree have not been accomplished by q -ROFG. In this paper, we have introduced the concept of Lq -ROFSs concentrating on the theory of graphs and put forward the innovative concept of the Lq -ROFGs on the basis of Einstein operator which is an effective and flexible operator to the algebraic system. Lq -ROFG model gives the framework with more precision, versatility and consistency than others. The Lq -ROFG can well define the complexity of networks of any kind. Further, the theory of graph spectra has been researched in the context of Lq -ROFSs and the product-connectivity energy, generalized product-connectivity energy, Laplacian energy and signless Laplacian energy of Lq -ROFGs as well as Lq -ROFDGs have been proposed. Moreover, the proposed concept of Lq -ROFGs is applied to solve the MAGDM problems with Lq -ROF information. We have represented a graph-based MAGDM approach with Lq -ROF information. Finally, an illustrative example regarding the selection of mobile payment platform is given to demonstrate its validity and effectiveness. Comparison analysis has been conducted, and the superiorities have been illustrated.

Fig. 11 The graph depicts the results of Lq -ROFWA operator and Lq -ROFEWA operator, plotted between the alternatives and ranking index

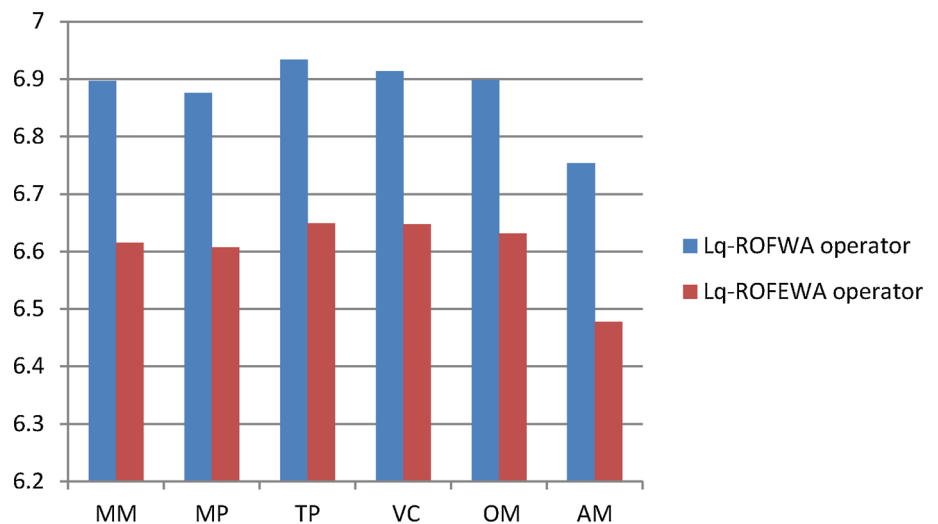
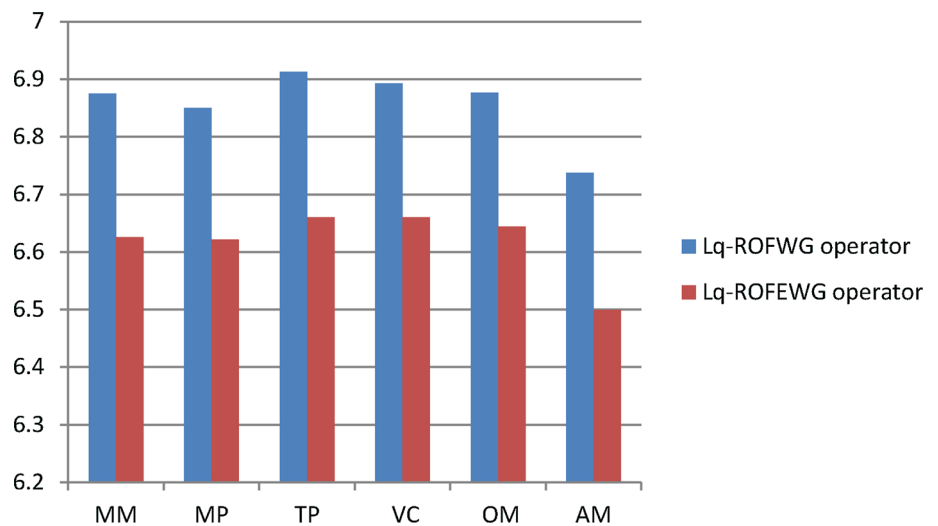


Fig. 12 The graph depicts the results of Lq -ROFWG operator and Lq -ROFEWG operator, plotted between the alternatives and ranking index



It is observed that some important topics still remain in terms of possible future works that are good enough to justify. In future, our research work will be extended to: (1) linguistic interval-valued q -rung orthopair fuzzy graphs; (2) linguistic complex q -rung orthopair fuzzy graphs; and (3) complex 2-tuple linguistic q -rung orthopair fuzzy graphs.

Declarations

Conflict of interest The authors declare no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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