Soft Computing (2021) 25:6013–6027
https://doi.org/10.1007/s00500-021-05594-8

METHODOLOGIES AND APPLICATION

Fuzzy topological structures via fuzzy graphs and their applications

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Published online: 9 February 2021

Abstract

Fuzzy graphs are an individual of application tools in the area of mathematics, which permit the users to define the relative between concepts because the wildlife of fuzziness is satisfactory for any situation. They are helpful to give more exactness and suppleness to the classification as associated with the traditional models. A topological structure is a set model for graphs. The main purpose of this paper is to introduce a new kind of fuzzy topological structures in terms of fuzzy graphs called fuzzy topological graphs due to a class of fuzzy subsets, and some of their properties are investigated. Also, a new procedure to calculate the number of edges in fuzzy graphs will be defined. Further, we consider the concept of a homeomorphic between fuzzy topological graphs as a fuzzy topological property that can be used to prove the isomorphic between fuzzy graphs. Moreover, an algorithm based on the proposed operations that build some fuzzy topological graphs will be presented. Finally, we give a new method to explain the homeomorphic between some fuzzy topological graphs which will be applied in smart cities.

Keywords Fuzzy sets · Fuzzy graphs · Fuzzy topology · Isomorphic fuzzy graphs · Smart cities

1 Introduction

In general, graphs are mentioned as a style of relative between the constructions and substances in the state achieved according with the material. Graphs are suitable to prompt the structure which delivers us information to operate and appreciate the conduct over the ideas verified on it. Graphs have two significant groups specifically vertices which are linked by a relation said to be edges. When there is a doubt in the choice of each of the vertices and edges, it is necessary to describe it under the situations of fuzzy graphs. A symmetric binary relation on a nonempty set of vertices V in a graph G offers the family of edges E. Similarly, a symmetric binary relation is specified on fuzzy subset which induces to get fuzzy graph model. The notion of fuzzy sets and their relations was initiated by Zadeh (1965) which displayed the significance of the ethics that lies between the binary digits 0 and 1. These ideas presented ascent to discover the wildlife of uncertainty. It may be also denoted as imprecision, uncertainty, etc. The ground-breaking concept of Zadeh has found many applications in many areas, including chemical industry, telecommunication, decision-making, networking, computer science, and smart city (Kozae et al. 2019; Quijano-Sanchez et al. 2020; Ma et al. 2020).

Kaufmann (1973) introduced the concept of fuzzy graphs (FGs, for short), and some of their remarks are discussed by Bhattacharya (1987). Additional progress was made by Rosenfeld (1971, 1975) who measured the fuzzy relation on fuzzy sets and presented a new approach for fuzzy graphs combination of a graph sense in fuzziness. He considered some graph concepts such as a fuzzy tree, a fuzzy cycle, fuzzy bridges, etc., and some properties on fuzzy graphs are presented. Several real-world phenomena gave the inspiration to describe FGs. Ghorai and Pal (2016); Mathew and Mordeson (2017) and Li et al. (2019) developed the structure of fuzzy graphs corresponding to several fuzzy graph concepts. Bhutani (1989) introduced the concept of a weak isomorphism, a co-weak isomorphism and an isomorphism between fuzzy graphs. Mordeson introduced the concept of fuzzy line graphs in Mordeson (1993). Morde-

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son and Nair (2000) defined fuzzy graph complement, and Sunitha and Vijayakumar (2002) further studied it. Recently, they studied some applications of fuzzy graphs in Mordeson and Mathew (2019). Samanta and Pal introduced several types of fuzzy graphs such as fuzzy planar graphs (Samanta and Pal 2015), fuzzy competition graphs Samanta and Pal (2013) and Samanta and Pal (2015); Samanta et al. (2015), fuzzy tolerance graphs (Samanta and Pal 2011a, b), and fuzzy threshold graphs (Samanta and Pal 2011a, b). Akram and Luqman (2020) described fuzzy hypergraphs and relevant extensions. Naz and Ashraf (2018) introduced the notion of Pythagorean fuzzy graphs (PFGs, for short). (Akram et al. 2018a, b) proposed the idea of the graphs under PFGs environment. The idea of complex fuzzy graphs (CFGs, for short) is initiated by Thirunavukarasu et al. (2016). The novel definition of complex neutrosophic hypergraphs was presented by Luqman and Akram (2019). Yaqoob et al. (2019) introduced the concept of a complex intuitionistic fuzzy graphs (CIFGs, for short) with the implementation of cellular network provider enterprises. Furthermore, Akram (2019) gave the notion of complex Pythagorean fuzzy graphs (CPFGs, for short). Abdul-Jabbar et al. (2009) introduced the concept of fuzzy dual graphs (abbr. FDGs) and studied some of its properties. Also, (Alshehri and Akram 2014) preceded the idea of intuitionistic fuzzy planar graphs (abbr. IFPGs). They extended the IFSs notion to PGs. Further, Akram et al. (2018a, b) investigated the concept of Pythagorean Fuzzy Planar Graphs (abbr. PFPGs). Akram et al. (2020). Also, they presented the notion of complex Pythagorean fuzzy planar graph (CPFPG, for short) (Akram and Naz 2019). Many structures can be represented by graphs (El Atik et al. 2020) such as self-similar fractals (El Atik and Nasef 2020) which may be useful to study fractals in physics (El-Naschie 2006).

Topology is a branch of geometry called rubber sheet geometry. It deals with the properties of things that does not depend on the dimension, which means that it allows increases and decreases, but without cutting things. It has many applications on various fields of research such as machine learning, data analysis, data mining, and quantum gravity (Hofer et al. 2017; Lum et al. 2013; Nicolau et al. 2011; Sardiu et al. 2017). Homeomorphisms are the isomorphisms of the category of topological spaces that play an important role in the theory. If there exists a homeomorphism among two topological spaces, then these spaces have the same topological properties. Chang (1968) defined the concept of fuzzy topological space in 1968 and generalized some basic topological notions, such as open set, closed set, continuity, and compactness on fuzzy topological spaces. After that, Lowen investigated some other description of a fuzzy topological space by changing some topological properties (Lowen 1976, 1977). In addition, Coker introduced the notion of intuitionistic fuzzy topological space and studied some analogue versions of certain classical topology concepts, such as continuity and compactness (Coker and Haydar Es 1995; Coker 1997). Some scholars had an interest in fuzzy metric spaces (see, for example, Kramosil and Michlek 1975). Moreover, some researchers studied the notion of fuzzy soft topological space and its applications in decision-making (Riaz and Hashmi 2017, 2018).

Based on this development, the paper is built as follows: the preliminaries and concepts will be awarded in this object in Sect. 2. The main work is studied in Sect. 3 with definitions and representations for a collection of fuzzy sets by fuzzy graphs. Moreover, some types of fuzzy topological structures will be generated in terms of fuzzy graphs and said to be fuzzy topological graphs. The outline of Sect. 4 is to construct some algebraic operations on fuzzy topological graphs, and the number of edges will be calculated by a given algorithm in Sect. 5. Section 6 puts forward a novel idea to construct an isomorphism between graphs, and the homeomorphic between fuzzy topological graphs will be discussed. In Sect. 7, the factors of smart cities will be contrived into two homeomorphic fuzzy topological graphs which may be used in decision making. Section 8 gives the conclusion of our models and points out some possible lines for future research. Finally, the proofs are given separately in "Appendix," to facilitate the reading of this paper.

2 Preliminaries

Throughout this paper, X is a nonempty fuzzy set. A collection τ of subsets of X is called a fuzzy topology on X if 1 and 0 belongs to τ, the finite intersection of any two fuzzy sets in τ belongs to τ, and the union of any number of fuzzy sets in τ belongs to τ.

Definition 1 (Harary 1972) A graph G is an ordered pair (V, E), where V is a set vertices and E is a set edges. Two vertices x and y in the undirected graph G are said to be adjacent if $\{x, y\}$ is an edge. A simple graph is an undirected graph that has no loops between any two different vertices and no more than one vertex.

Definition 2 (Bondy and Murty 1975) Two graphs G_1 and G_2 are homomorphic if $F : G_1 \to G_2$ is available, such that for each $v_1, v_2 \in V(G_1)$, and $\{v_1, v_2\} \in E(G_1)$, then $\{F(v_1), F(v_2)\} \in E(G_2)$ is used. Also, G_1 and G_2 are isomorphic if there exists a bijective function $F : V(G_1) \to V(G_2)$, such that for each two adjacent vertices u and v in G_1, then $F(u)$ and $F(v)$ are also adjacent in G_2.

Definition 3 (Bhattacharya 1987) A fuzzy graph $G = (\sigma, \mu)$ is a pair of functions together with underlying vertex set V and edge set E, where $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$ such that $\mu(uv) \leq \sigma(u) \wedge \sigma(v)$ for all $u, v \in V$. Here, $\sigma(u) \wedge \sigma(v)$ argues the minimum among $\sigma(u)$ and $\sigma(v)$.

Definition 4 (Nagoor Gani and Malarvizhi 2008) Let $G = (\sigma, \mu)$ be a fuzzy graph. The degree of a vertex u is $d_G(u) = \sum_{u \neq v} \mu(uv)$.

Definition 5 (Bhutani 1989) A homomorphism of fuzzy graphs $h : G \to \acute{G}$ is a map $h : S \to \acute{G}$ which satisfies $\sigma(x) \leq \acute{\sigma}(h(x))\ \forall\ x \in S$ and $\mu(x, y) \leq \mu(h(x), h(y))\ \forall\ x, y \in S$.

Definition 6 (Nagoor Gani and Malarvizhi 2008) An isomorphism of fuzzy graphs $h : G \to \acute{G}$ is a map $h : S \to \acute{S}$ which is a bijective that satisfies $\sigma(x) = \acute{\sigma}(h(x))\ \forall\ x \in S$ and $\mu(x, y) = \mu(h(x), h(y))\ \forall\ x, y \in S$.

Theorem 1 (Nagoor Gani and Malarvizhi 2008) *The order and size for any two isomorphic fuzzy graphs are the same.*

Definition 7 (Chang 1968) Let $\widetilde{X}$ be a fuzzy set and τ be a collection of fuzzy subsets of $\widetilde{X}$ such that

(i) $\widetilde{X}, \widetilde{\phi} \in \tau$,
(ii) $\widetilde{T_1}, \widetilde{T_2} \in \tau$ implies $\widetilde{T_1} \cap \widetilde{T_2} \in \tau$,
(iii) $\widetilde{T_i} \in \tau$ implies $\bigcup_{i \in I} \widetilde{T_i} \in \tau$ for $i \in I$.

Then, $(\widetilde{X}, \tau)$ is a fuzzy topological space. The elements of τ are said be open fuzzy sets. A fuzzy subset $\widetilde{A}$ of $\widetilde{X}$ is called fuzzy closed iff a complement of $\widetilde{A}$ with respect to $\widetilde{X}$ is fuzzy open.

Definition 8 (Chang 1968) Let λ be a fuzzy subset of X. A collection τ of fuzzy subsets of λ satisfying:

(i) $k \cap \lambda \in \tau, \forall\ k \in I$,
(ii) $\mu_i \in \tau, \forall\ i \in \Delta$ implies $\bigcup\{\mu_i : i \in \Delta\} \in \tau$,
(iii) $\mu, \nu \in \tau$ implies $\mu \cap \nu \in \tau$.

is called a fuzzy topology on λ. The pair (λ, τ) is called a fuzzy topological space. Members of τ are called fuzzy open sets, and their complements with respect to λ are called fuzzy closed sets of (λ, τ).

3 A class of fuzzy sets and its fuzzy graphs

In this section, we introduce a new concept of parallel classes of fuzzy sets. Each class can be represented to a fuzzy graph, and some algebraic operations on these fuzzy graphs will be established.

Definition 9 Let X be a nonempty universe set, and $\mathcal{F}(X)$ will denote to the set of all fuzzy sets of X. Let $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathcal{F}(X)$, we say $\mathcal{C}_1$ is parallel to $\mathcal{C}_2$ (say, $X \sim Y$), if there exists a bijective fuzzy function $F : X \to Y$ such that for each $c_1 \in C_1$, and $c_2 \in \mathcal{C}_2$ and $\mathcal{F}(c_1) = c_2$. Let $X = Y$. We say that $\mathcal{C}_1$ is parallel with $\mathcal{C}_2$ if there exists a transformation $\mathcal{F} : X \to X$, such that $F(C_1) = C_2$.

Example 1 Let $X = \{\frac{0.7}{x_1}, \frac{0.2}{x_2}, \frac{0.8}{x_3}\}$ be a fuzzy set. Consider

$$\mathcal{C}_1 = \left\{\left\{\frac{0.8}{x_3}\right\}, \left\{\frac{0.7}{x_1}, \frac{0.2}{x_2}\right\}, \left\{\frac{0.8}{x_3}, \frac{0.7}{x_1}\right\}\right\} \quad \text{and}$$
$$\mathcal{C}_2 = \left\{\left\{\frac{0.7}{x_1}\right\}, \left\{\frac{0.2}{x_2}, \frac{0.8}{x_3}\right\}, \left\{\frac{0.7}{x_1}, \frac{0.2}{x_2}\right\}\right\}.$$

Then, $\mathcal{C}_1$ and $\mathcal{C}_2$ are parallel, since there exists a bijective fuzzy function $F : X \to X$, such that

$$F\left(\left\{\frac{0.8}{x_3}\right\}\right) = \left\{\frac{0.7}{x_1}\right\},$$
$$F\left(\left\{\frac{0.7}{x_1}, \frac{0.2}{x_2}\right\}\right) = \left\{\frac{0.2}{x_2}, \frac{0.8}{x_3}\right\} \text{ and}$$
$$F\left(\left\{\frac{0.8}{x_3}, \frac{0.7}{x_1}\right\}\right) = \left\{\frac{0.7}{x_1}, \frac{0.2}{x_2}\right\}.$$

In the following, every class of fuzzy sets can be represented by a general fuzzy graph through a new operation $\wedge$ between classes. Any fuzzy graph $\mathcal{G}$ with only one edge can be represented by a class $\{\mathcal{A}, \mathcal{B}\}$ such that $\mathcal{A} \wedge \mathcal{B}$ is a singleton and so $|\mathcal{A} \wedge \mathcal{B}| = 1$. If $\mathcal{G}$ contains r-edges, then $|E(\mathcal{G})| = |\mathcal{A} \wedge \mathcal{B}| = r$, where $|\mathcal{A}|$ refers to the cardinality of a set $\mathcal{A}$. So, every graph can be illustrated by distinct classes that will be stated in Example 2.

Example 2 The classes of fuzzy sets

$$\mathcal{C}_1 = \left\{B = \left\{\frac{0.3}{x_1}\right\}, C = \left\{\frac{0.5}{x_2}\right\},\right.$$
$$\left. A = \left\{\frac{0.3}{x_1}, \frac{0.5}{x_2}\right\}\right\};$$
$$\mathcal{C}_2 = \left\{E = \left\{\frac{0.3}{x_1}, \frac{0.1}{x_2}, \frac{0.4}{x_3}\right\}, F = \left\{\frac{0.5}{x_4}, \frac{0.3}{x_5}\right\},\right.$$
$$\left. D = \left\{\frac{0.4}{x_3}, \frac{0.5}{x_4}\right\}\right\};$$
$$\mathcal{C}_3 = \left\{H = \left\{\frac{0.4}{x_3}, \frac{0.5}{x_7}\right\}, K = \left\{\frac{0.5}{x_8}, \frac{0.3}{x_9}, \frac{0.8}{x_{10}}\right\},\right.$$
$$\left. J = \left\{\frac{0.4}{x_3}, \frac{0.8}{x_{10}}, \frac{0.5}{x_{12}}\right\}\right\},$$

represent the same fuzzy graphs as shown in Fig. 1.

Definition 10 If $\mathcal{C} = \{\mathcal{C}_i : i \in I\}$ is the set of all classes of a fuzzy set X. Then, $\mathcal{C}_i$ can be represented by the same fuzzy graph $\mathcal{G}$. The graph number of a fuzzy graph $\mathcal{G}$ equals $\{|\bigvee \mathcal{C}_i| : i \in I\}$. In other words, a graph number of $\mathcal{G}$ equals m, meaning there is no number $r < m$ such that $|\bigvee_{i \in I} \mathcal{C}_i| = r$.

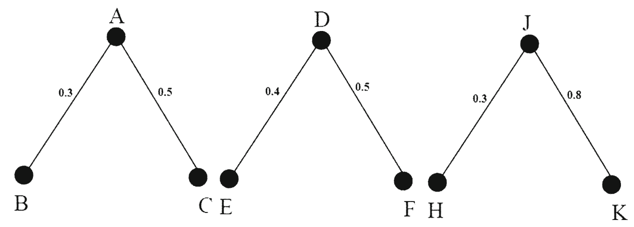


Fig. 1 A simple fuzzy graph $\mathcal{G}$

Example 3 (Continued for Example 2). We have

$$\bigvee \mathcal{C}_1 = \left\{\frac{0.3}{x_1}, \frac{0.5}{x_2}\right\},$$
$$\bigvee \mathcal{C}_2 = \left\{\frac{0.3}{x_1}, \frac{0.1}{x_2}, \frac{0.4}{x_3}, \frac{0.5}{x_4}, \frac{0.3}{x_5}\right\} \quad \text{and}$$
$$\bigvee \mathcal{C}_3 = \left\{\frac{0.4}{x_3}, \frac{0.5}{x_7}, \frac{0.5}{x_8}, \frac{0.3}{x_9}, \frac{0.8}{x_{10}}, \frac{0.5}{x_{12}}\right\}.$$

The number of a fuzzy graph $|\bigvee \mathcal{C}_1|$ (resp. $|\bigvee \mathcal{C}_2|$ and $|\bigvee \mathcal{C}_3|$) for $\mathcal{C}_1$ (resp. $\mathcal{C}_2$ and $\mathcal{C}_3$) equals 2 (resp. 5 and 6).

In the following, we show that every fuzzy graph $\mathcal{G}$ can be illustrated by a class of fuzzy sets. We reformulate the operation $\wedge$ to an operator $\mathcal{N}$ for points of fuzzy graphs.

Definition 11 Let $\mathcal{G}$ be a fuzzy graph and $v_i, v_j \in V(\mathcal{G})$. If the vertex v_i is represented by a fuzzy set $\mathcal{A}$ and v_j is represented by a fuzzy set $\mathcal{B}$ in a fuzzy set graph $\mathcal{G}$, then $\mathcal{N}(v_i, v_j)$ is defined by $|\mathcal{A} \wedge \mathcal{B}|$. It is clear that $\mathcal{N}(v_i, X) = |\mathcal{A}|$ and $\mathcal{N}(v_j, X) = |\mathcal{B}|$, where X is a universe fuzzy set of vertices for $\mathcal{G}$.

Example 4 (Continued for Example 3). If $\mathcal{A}=\left\{\frac{0.3}{x_1}\right\}$ and $\mathcal{B} = \left\{\frac{0.3}{x_1}, \frac{0.5}{x_2}\right\}$, then $\mathcal{N}(\mathcal{A}, \mathcal{B}) = 1$, for $\mathcal{A} \wedge \mathcal{B} = \left\{\frac{0.3}{x_1}\right\}$ and $|\mathcal{A} \wedge \mathcal{B}| = 1$.

Theorem 2 *If $\mathcal{G}_1$ and $\mathcal{G}_2$ are two fuzzy graphs which represented by two parallel classes $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, then $\mathcal{G}_1$ and $\mathcal{G}_2$ are an isomorphic.*

Proof The proof is found in "Appendix." □

The converse of Theorem 2 may not be true, in general. Because every fuzzy graph can be represented by many classes, it may not be parallel as shown in Example 5.

Example 5 Assume that

$$\mathcal{C}_1 = \left\{B = \left\{\frac{0.3}{x_1}\right\}, C = \left\{\frac{0.5}{x_2}\right\},\right.$$
$$\left. A = \left\{\frac{0.3}{x_1}, \frac{0.5}{x_2}\right\}\right\}; \text{and}$$
$$\mathcal{C}_2 = \left\{E = \left\{\frac{0.3}{x_1}, \frac{0.1}{x_2}, \frac{0.4}{x_3}\right\}, F = \left\{\frac{0.5}{x_4}, \frac{0.3}{x_5}\right\},\right.$$

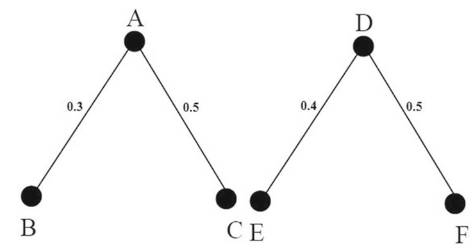


Fig. 2 A simple fuzzy graph $\mathcal{G}$

Fig. 3 Disjoint graph fuzzy sets

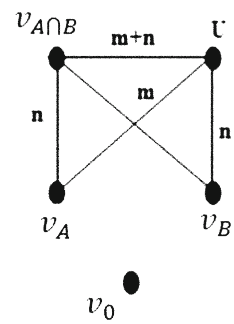
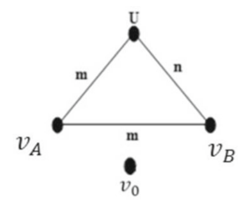


Fig. 4 Inclusion graph fuzzy sets



$$\left. D = \left\{\frac{0.4}{x_3}, \frac{0.5}{x_4}\right\}\right\}.$$

Then, fuzzy graphs corresponding to $\mathcal{C}_1$ and $\mathcal{C}_2$ are isomorphic that are shown in Fig. 2 while fuzzy graphs cannot be represented by two parallel classes.

In the following, we describe how to generate a fuzzy class from a simple fuzzy graph $\mathcal{G}$.

Definition 12 In a fuzzy graph $\mathcal{G}$, we have

(i) the universe fuzzy set X can be represented by a fuzzy subset of natural number $\mathbb{N}$. The class of a universe fuzzy vertex set X is adjacent with each vertex set in its fuzzy set graph.
(ii) any isolated point can be represented by 0 in a new definite class $\mathcal{C}$.

Now, we explain that the class of fuzzy sets in Definition 12 which represent a fuzzy graph.

Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty fuzzy sets and $|\mathcal{A}| = m$, $|\mathcal{B}| = n$. The fuzzy graphs will be represented by a fuzzy class $\{\mathcal{A}, \mathcal{B}, \mathcal{A} \wedge \mathcal{B}, \mathcal{A} \vee \mathcal{B}\}$ that are given by three cases:

Case 1: if $\mathcal{A} \wedge \mathcal{B} = 0$, then the fuzzy graph $\mathcal{G}$ is shown in Fig. 3.

Case 2: If $\mathcal{A} \leq \mathcal{B}$, then $|\mathcal{A} \wedge \mathcal{B}| = |\mathcal{A}|$ and $|\mathcal{A} \vee \mathcal{B}| = |B|$. The fuzzy graph $\mathcal{G}$ is shown in Fig. 4.

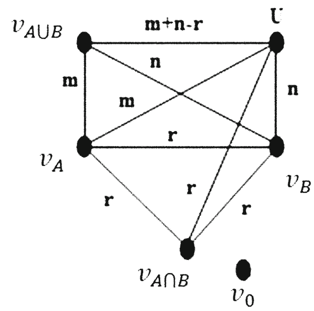


Fig. 5 Intersection graph fuzzy sets

Case 3: If $\mathcal{A}\wedge\mathcal{B}\neq 0$,$|\mathcal{A}|\neq|\mathcal{B}|$ and $|\mathcal{A}\wedge\mathcal{B}|=r$, where $r\in N$, then the fuzzy graph $\mathcal{G}$ is shown in Fig. 5.

Remark 1 Every graph fuzzy set for a subgraph $\mathcal{H}$ of $\mathcal{G}$ is a fuzzy subset from a graph fuzzy set of $\mathcal{G}$. Also, by Definition 12, each graph fuzzy set can also be represented by a class $\mathcal{P}(\mathbb{N})$, where $\mathcal{P}(\mathbb{N})$ denotes to the power fuzzy set of $\mathbb{N}$.

4 Some algebraic operations on fuzzy topological graphs

In this section, we use fuzzy set graphs to generate fuzzy topological spaces. We can also say it fuzzy topological graphs or fuzzy topological structures. The fuzzy set graph will be represented by $\mathcal{P}(\mathbb{N})$. Some algebraic operations on fuzzy topological graphs such as $\vee$, $\wedge$ and $\leq$ will be defined on vertices by

(i) $v_{\mathcal{A}_1}\vee v_{\mathcal{A}_2}\vee v_{\mathcal{A}_3}\vee\cdots=v_{\mathcal{A}_1\vee\mathcal{A}_2\vee\mathcal{A}_3\vee\cdots}$.
(ii) $v_{\mathcal{A}}\wedge v_{\mathcal{B}}=v_{\mathcal{A}\wedge\mathcal{B}}$.
(iii) $v_{\mathcal{A}}\leq v_{\mathcal{B}}$ iff $\mathcal{A}\leq\mathcal{B}$.

Definition 13 A fuzzy topology on a fuzzy set $X=\left\{\frac{a}{x_1},\frac{b}{x_2},\frac{c}{x_3},\ldots,\frac{d}{x_n}\right\}$ can be established from a fuzzy graph $\mathcal{G}$ by each class in X being a vertex in $\mathcal{G}$ and number of edges of $\mathcal{G}$ being the number of elements which are determined by the intersection of classes of X and the degree of edges is the degree of each vertex in its intersection. In a fuzzy simple graph, if the intersection between two classes is more than one element, then we choose a maximum degree between them.

Remark 2 If there are no loops in a fuzzy pseudographs $\mathcal{P}$, we have a discrete fuzzy topological graph, say $\mathcal{D}$. Also, in simple fuzzy graphs, we draw only one edge between each of the adjacent vertices in a discrete fuzzy topological graph.

Theorem 3 declared some fundamental properties for Definition 13. Its proof is found in "Appendix" section.

Theorem 3 (i) *The number of edges $|\mathcal{E}_p(\mathcal{G})|$ of a fuzzy pseudograph $\mathcal{G}$ of the fuzzy set $X=\left\{\frac{a_1}{x_1},\frac{a_2}{x_2},\frac{a_3}{x_3},\ldots,\frac{a_4}{x_n}\right\}$ equals $n2^{n-2}(2^{n-1}-1)+n2^{n-1}$.*
(ii) *The number of edges $|\mathcal{E}_d(\mathcal{G})|$ of a discrete fuzzy graph arising from a fuzzy pseudograph, i.e., by deleting loops in $\mathcal{P}$ equals $n2^{n-2}(2^{n-1}-1)$.*
(iii) *The number of edges $|\mathcal{E}_s(\mathcal{G})|$ of a fuzzy simple graph arising from the fuzzy discrete graph, i.e., by drawing only one edge joint two adjacent vertices in $\mathcal{D}$ equals $\frac{1}{2}(2^{2n}-2^n-3^n+1)$.*

We apply Theorem 3 in Examples 6 and 7.

Example 6 The number of edges of a pseudograph, a discrete graph, say $\mathcal{D}$, and a simple graph, say $\mathcal{S}$, of a fuzzy set $X=\left\{\frac{0.5}{x_1},\frac{0.4}{x_2},\frac{0.2}{x_3}\right\}$ for $\mathcal{G}$ is given by

$$
\begin{aligned}
|E_{\mathcal{P}}(\mathcal{G})| &= 3.2^{3-2}\left(2^{3-1}-1\right)\\
&\quad +3.2^{3-1}=3.2.3+16=18+12=30,\\
|E_{\mathcal{D}}(\mathcal{G})| &= 3.2^{3-2}(2^{3-1}-1)=3.2.3=18,\\
|E_{\mathcal{S}}(\mathcal{G})| &= \frac{1}{2}(2^6-2^3-3^3+1)=15,
\end{aligned}
$$

respectively.

Example 7 From Theorem 3 and Example 6, we have $|\mathcal{E}_{\mathcal{P}}(\mathcal{G})|=14.9$; $|\mathcal{E}_{\mathcal{D}}(\mathcal{G})|=5.5$ and $|\mathcal{E}_{\mathcal{S}}(\mathcal{G}))|=4.7$.

A fuzzy topological graph from a fuzzy graph will be formed via Theorem 4.

Theorem 4 *Let $\mathcal{G}$ be a fuzzy graph that satisfies the conditions*

(i) *$\mathcal{G}$ contains only one isolated vertex v_0.*
(ii) *$\mathcal{G}$ contains a vertex v adjacent with each vertex in $\mathcal{G}\backslash\{v_0\}$ and $\mu_R(v_i,v)\leq\mu_R(v_i,X)\leq\mu_R(v,X)$, for every $v_i\in\mathcal{G}\backslash\{v_0\}$.*
(iii) *For every two distinct vertices v_i ,$v_j\in V(\mathcal{G})$, $v_i\vee v_j\in V(\mathcal{G})$ and $v_i\wedge v_j\in V(\mathcal{G})$. Then, the class τ of vertices is a fuzzy topological graph.*

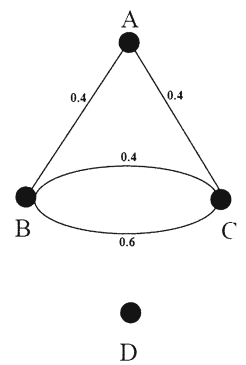
Proof The proof is found in appendix. □

Now, we use the fuzzy topological graph τ to calculate the number of edges in $\mathcal{G}$ which can be shown in Theorem 5.

Theorem 5 *The number of edges of a fuzzy topological graph is represented by a fuzzy topology*

$$
\tau=\left\{0,\left\{\frac{a}{x_1}\right\},\left\{\frac{a}{x_1},\frac{b}{x_2}\right\},\ldots,1\right.
$$

Fig. 6 A fuzzy topological graph for 3 points



$$= \left\{\frac{a}{x_1}, \frac{b}{x_2}, \ldots, \frac{d}{x_n}\right\}\bigg\}.$$

Proof The proof is found in "Appendix." □

To explain results of Theorems 4 and 5, we present some fuzzy topological structures in Examples 8 and 9.

Example 8 Let $X = \left\{\frac{0.4}{x_1}, \frac{0.6}{x_2}, \frac{0.2}{x_3}\right\}$ with a fuzzy topological space

$$\tau = \left\{D = 0, A = \left\{\frac{0.4}{x_1}\right\},\right.$$
$$B = \left\{\frac{0.4}{x_1}, \frac{0.6}{x_2}\right\},$$
$$C = \left.\left\{\frac{0.4}{x_1}, \frac{0.6}{x_2}, \frac{0.2}{x_3}\right\}\right\}.$$

The total degree of edges of a fuzzy topological graph with 3 vertices equals 1.8. (see Fig. 6).

Example 9 Let $Y = \left\{\frac{0.4}{x_1}, \frac{0.6}{x_2}, \frac{0.2}{x_3}, \frac{0.5}{x_4}\right\}$ with a fuzzy topological space

$\tau = \Big\{E = 0,\ A = \{\frac{0.4}{x_1}\},\ B = \{\frac{0.4}{x_1}, \frac{0.6}{x_2}\},\ C = \{\frac{0.4}{x_1}, \frac{0.6}{x_2}, \frac{0.2}{x_3}\},\ D = \{\frac{0.4}{x_1}, \frac{0.6}{x_2}, \frac{0.2}{x_3}, \frac{0.5}{x_4}\}\Big\}.$

The total degree of edges of a fuzzy topological graph with 4 vertices equals 4.4. (see Fig. 7).

Remark 3 Every fuzzy topological graph can be illustrated by a fuzzy graph, but the converse may not be true, in general. So, Examples 10 and 11 are given.

Example 10 Let $\mathcal{G}$ be a fuzzy graph in Fig. 8. We construct a fuzzy topological graph τ by the following procedures:

An isolated vertex v_f is represented by 0; the vertex v_a and v_b or one of them can be considered as a vertex adjacent with all vertices except v_f. Take v_a as a vertex V represented by a set $1 = \left\{\frac{0.4}{x_1}, \frac{0.3}{x_2}, \frac{0.5}{x_3}, \frac{0.7}{x_4}\right\}$. Since $v_e \wedge V = 1$, then v_e is represented by $e = \left\{\frac{0.4}{x_1}\right\}$. At $v_d \wedge V = 2$, v_d will be represented by $d = \left\{\frac{0.4}{x_1}, \frac{0.3}{x_2}\right\}$. Also, at $v_c \wedge V = 2$, v_c will be represented by $c = \left\{\frac{0.4}{x_1}, \frac{0.5}{x_3}\right\}$. Continuing in the same manner, v_b will be represented by $b = \left\{\frac{0.4}{x_1}, \frac{0.3}{x_2}, \frac{0.5}{x_3}\right\}$ and v_a will be formed by $\left\{\frac{0.4}{x_1}, \frac{0.3}{x_2}, \frac{0.5}{x_3}, \frac{0.7}{x_4}\right\}$, for $v_b \wedge V < v \wedge U$, for all v in $\mathcal{G}$.

Therefore, $\tau = \Big\{0, \{\frac{0.4}{x_1}\}, \{\frac{0.4}{x_1}, \frac{0.3}{x_2}\}, \{\frac{0.4}{x_1}, \frac{0.5}{x_3}\}, \{\frac{0.4}{x_1}, \frac{0.3}{x_2}, \frac{0.5}{x_3}\}, 1 = \{\frac{0.4}{x_1}, \frac{0.3}{x_2}, \frac{0.5}{x_3}, \frac{0.7}{x_4}\}\Big\}$ is a fuzzy topology on X. We also call τ is a fuzzy topological graph of $\mathcal{G}$.

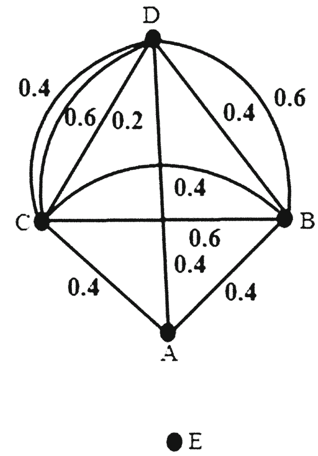


Fig. 7 A fuzzy topological graph for 4 points

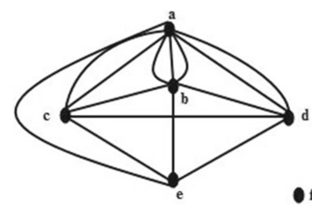
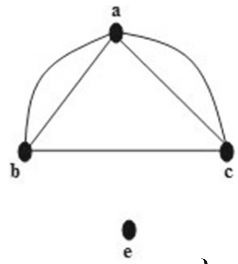


Fig. 8 A fuzzy graph which is a fuzzy topological graph

Example 11 The graph $\mathcal{G}$ in Fig. 9 is not fuzzy topological graph. Because of $deg(v_a) = 4$, where deg denotes the degree of vertex v_a, $deg(v_b) = deg(v_c) = 3$. Now, v_a will be represented by $X = \left\{\frac{0.4}{x_1}, \frac{0.3}{x_2}, \frac{0.5}{x_3}, \frac{0.7}{x_4}\right\}$, v_b will be formed by a set, say $b = \left\{\frac{0.3}{x_2}\right\}$, and v_c is given by a set, say $c = \left\{\frac{0.5}{x_3}\right\}$. Although $v_b \wedge v_c = 1$, we have $\left\{\frac{0.4}{x_2}\right\} \wedge \left\{\frac{0.5}{x_3}\right\} = 0$.

Fig. 9 A fuzzy graph which is not a fuzzy topological graph



Therefore, $\tau = \left\{ , 1 = \{\frac{0.4}{x_1}, \frac{0.3}{x_2}, \frac{0.5}{x_3}, \frac{0.7}{x_4}\}, \{\frac{0.3}{x_2}\}, \{\frac{0.5}{x_3}\} \right\}$ is not a fuzzy topology, for $\left\{\frac{0.3}{x_2}\right\} \vee \left\{\frac{0.5}{x_3}\right\} = \left\{\frac{0.3}{x_2}, \frac{0.5}{x_3}\right\} \notin \tau$.

5 Fuzzy topological graphs algorithm

In this section, we give an algorithm that describes fuzzy graphs and their fuzzy topological graphs. In this way, we calculate the dimension which is used to get a number of edges for fuzzy graphs. In this algorithm, d denotes to dimension, $|V(G)| = n$ and $|\mathcal{A}| = m$, for any $\mathcal{A} \subseteq V(G)$.

Algorithm:
Input: d, n and m.
Output: Degree of each vertex.
1. Insert n,m.
2. **for** $i \in n$
 $\mathcal{A}_i$ =insert d_i .
 end
3. **for** $i \in n$
 fuzzy topology $+= \mathcal{A}_i$.
 end
4. **Function** (For a discrete fuzzy graph)
 $degV_{A\cap B} = m(2^{m-1} - 1)$.
 $degV\, A \cap E = m(2^{n-m} - 1)$.
 $degV_{A\cap L} = m(2^{n-m} - 1)(2^{m-1} - 1)$.
 $degV_A = m(2^{n-1} - 1)$.
 $|Edges| = n2^{n-2}(2^{n-1} - 1)$.
 for $i \le |V(G)|$.
 $deg(V_i) = \sum_{i=1}^{|B|} \mu_R(xy) + \sum_{i=1}^{|A\cap E|} \mu_R(yw) + \sum_{i=1}^{|A\cap L|} \mu_R(yz)$.
 $V(G) = V(G)/V_i$.
 end
 $\mathcal{E}_d(\mathcal{G}) = \sum deg(V_i)$.
end
5. **Function** (For a fuzzy pseudograph)
 $degV_A = m(2^{n-1} - 1) + 2m$.
 $|Edges| = n2^{n-2}(2^{n-1} - 1) + n2^{n-1}$.
 for $i \le |V(G)|$.
 $deg(V_i) = \sum_{i=1}^{|B|} \mu_R(xy) + 2\sum_{i=1}^{|A|} \mu_R(yy) + \sum_{i=1}^{|A\cap E|} \mu_R(yw) + \sum_{i=1}^{|A\cap L|} \mu_R(yz)$.
 $V(G) = V(G)/V_i$.
 end
 $\mathcal{E}_p(\mathcal{G}) = \sum deg(V_i)$.
end
6. **Function** (For a simple fuzzy graph)
 $degV_A = 2^n - 2^{n-m} - 1$.
 $|Edges| = \frac{1}{2}(2^{2n} - 2^n - 3^n + 1)$.
 for $i \le |V(G)|$.
 $deg(V_i) = \sum_{i=1}^{|B|} \mu_R(V_B) + \sum_{i=1}^{|A\cap E|} \mu_R(V_{\mathcal{A}}) + \sum_{i=1}^{|A\cap l|} \mu_R(V_{\mathcal{A}\cap\mathcal{L}})$.
 $V(G) = V(G)/V_i$.
 end
 $\mathcal{E}_s(\mathcal{G}) = \sum deg(V_i)$.
 end
end

This algorithm can be illustrated through the following flowchart in Fig. 10.

6 Isomorphic between fuzzy topological structures

In this section, we study the isomorphic between fuzzy graphs, which is a transformation in graph theory. Through this isomorphic, we study the homeomorphic between fuzzy topological graphs which is stated in Section 2. The following examples discuss special types of fuzzy graphs such as a fuzzy Petersen graph in Fig. 11 and others in Fig. 12.

Example 12 The isomorphic between fuzzy topological graphs is shown in Fig. 11. From Table 1, we have a class $\mathcal{C} = \left\{ u_1{=}\{\frac{\sigma(x_1)}{x_1}, \frac{\sigma(x_2)}{x_2}, \frac{\sigma(x_3)}{x_3}\}, u_2 = \{\frac{\sigma(x_1)}{x_1}, \frac{\sigma(x_4)}{x_4}, \frac{\sigma(x_5)}{x_5}\}, u_3 = \{\frac{\sigma(x_4)}{x_4}, \frac{\sigma(x_6)}{x_6}, \frac{\sigma(x_7)}{x_7}\}, u_4 = \{\frac{\sigma(x_6)}{x_6}, \frac{\sigma(x_8)}{x_8}, \frac{\sigma(x_9)}{x_9}\}, u_5 = \{\frac{\sigma(x_3)}{x_3}, \frac{\sigma(x_9)}{x_9}, \frac{\sigma(x_{10})}{x_{10}}\}, u_6 = \{\frac{\sigma(x_2)}{x_2}, \frac{\sigma(x_{12})}{x_{12}}, \frac{\sigma(x_{13})}{x_{13}}\}, u_7 = \{\frac{\sigma(x_5)}{x_5}, \frac{\sigma(x_{14})}{x_{14}}, \frac{\sigma(x_{15})}{x_{15}}\}, = u_8 \{\frac{\sigma(x_7)}{x_7}, \frac{\sigma(x_{11})}{x_{11}}, \frac{\sigma(x_{12})}{x_{12}}\}, u_9{=}\{\frac{\sigma(x_8)}{x_8}, \frac{\sigma(x_{13})}{x_{13}}, \frac{\sigma(x_{14})}{x_{14}}\}, u_{10}{=}\{\frac{\sigma(x_{10})}{x_{10}}, \frac{\sigma(x_{11})}{x_{11}}, \frac{\sigma(x_{15})}{x_{15}}\} \right\}$,

which represents a fuzzy graph $\mathcal{G}_1$. By the same manner, we have the same class by Table 2, which represents a fuzzy graph $\mathcal{G}_2$. Table 3 proves the homeomorphic between fuzzy topological graphs of $\mathcal{G}_1$ and $\mathcal{G}_2$.

Since there are a homeomorphic between fuzzy topological graphs in Table 3, $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic in Fig. 11.

Example 13 Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two fuzzy graphs which are shown in Fig. 12. From Tables 4 and 5, we have a fuzzy topological graph $\tau = \left\{ u_0 = 0, u_1 = \{\frac{0.4}{x_1}\}, u_2 = \{\frac{0.3}{x_2}\}, u_3 = \{\frac{0.6}{x_3}\}, u_4 = \{\frac{0.4}{x_1}, \frac{0.3}{x_2}\}, u_5 = \{\frac{0.4}{x_1}, \frac{0.6}{x_3}\}, u_6 = \{\frac{0.3}{x_2}, \frac{0.6}{x_3}\}, u_7 = 1 = \{\frac{0.4}{x_1}, \frac{0.3}{x_2}, \frac{0.6}{x_3}\} \right\}$, for fuzzy graphs $\mathcal{G}_1$ and $\mathcal{G}_2$. We prove that $\mathcal{G}_1$ is an isomorphic with $\mathcal{G}_2$, in Table 6.

Since there are a homeomorphic between fuzzy topological graphs in Table 6, $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic in Fig. 12.

7 Application on smart city

Smart cities are visualized as patterns of tools through several rulers that are linked through several networks which deliver continuous information viewing the activities of societies and materials via the stream of conclusions about physical and social formula of the city, and this is shown in Fig. 13

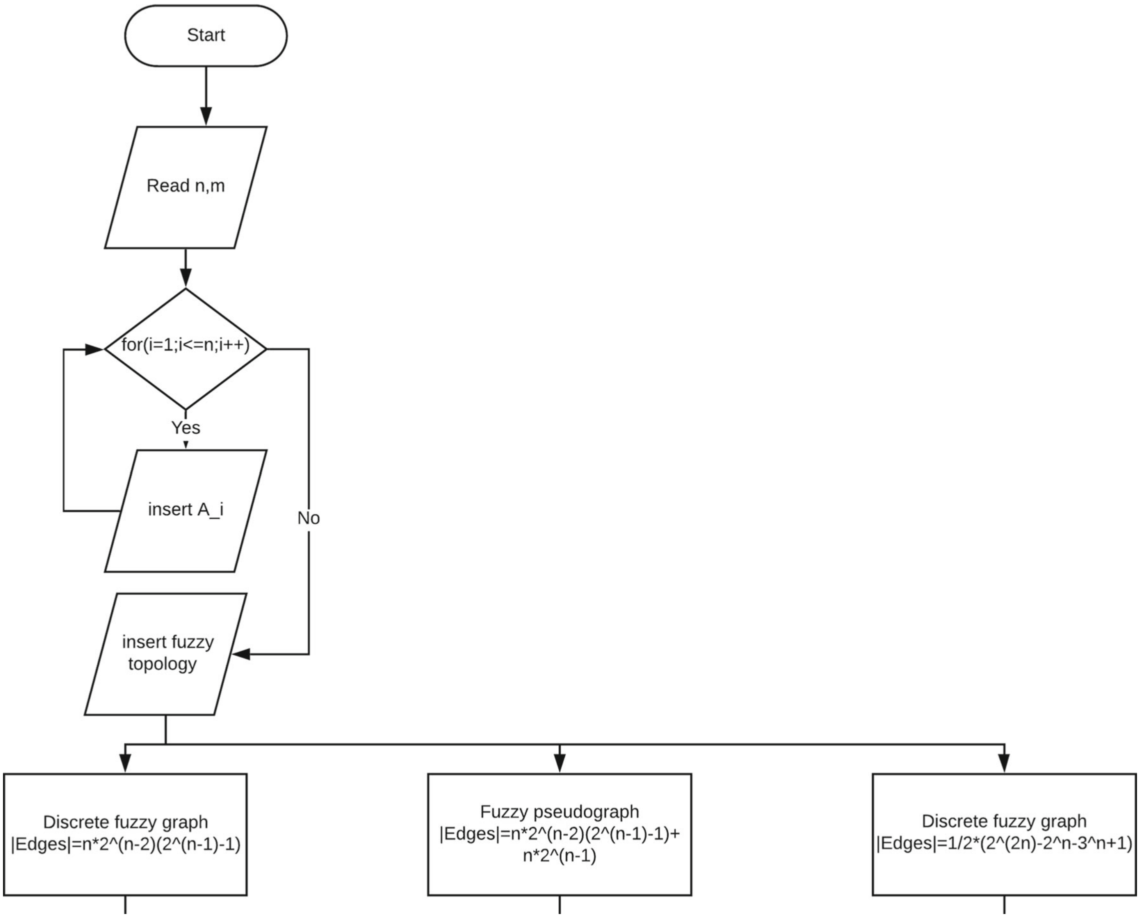


Fig. 10 A flowchart for the given algorithm

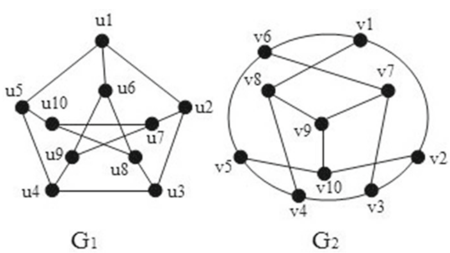


Fig. 11 Two isomorphic fuzzy graphs $\mathcal{G}_1$ and $\mathcal{G}_2$

(see Kumar et al. 2020; Vandercruysse et al. 2020; Westraadt and Calitz 2020; Lara Sánchez et al. 2020; Wang et al. 2020). De Santis et al. (2014) initiated the process of city transformation depending on the commune combination of governance, technological, transitional components, and institutional. Moreover, a greatest important element for building smart cities are organization, policy, governance, technology, people communities, economy, built infrastructure, and natural environment as shown in Fig. 14.

In this section, we make a restructuring for the factors which build smart cites in terms of a connected graph, say $\mathcal{G}_1$. The link between factors in Fig. 14 can be substituted with a non-closed connected path as shown in Fig. 15. A graph $\mathcal{G}_1$ consists of a set of vertices $V(\mathcal{G}_1) = \{v_0, v_1, \ldots, v_8\}$ and 15 edges which link the vertices with each other. The vertices of $\mathcal{G}_1$ refer to Smart city (resp., technology, organization, policy, built infrastructure, economy, people communities, governance, natural environment). $\mathcal{G}_1$ is a connected fuzzy topological structure. We apply Examples 12 and 13 to have an isomorphic fuzzy topological structure, say $\mathcal{G}_2$, in Fig. 16. $\mathcal{G}_2$ consists of 8 vertices $V(\mathcal{G}_2) = \{u_0, u_1, \ldots, u_8\}$, and 15 edges. Topologically, $\mathcal{G}_1$ and $\mathcal{G}_2$ are fuzzy topological homeomorphic (see Sect. 6). $\mathcal{G}_2$ gives some other building of smart cites which may be more useful from the initiated smart city in Fig. 14. In this case, the expert can select a justest choice in decision making.

8 Conclusion and future work

The field of mathematical science which goes under the name of the fuzzy topology is concerned with all questions directly or indirectly related to fuzzy topological graphs.

Table 1 A fuzzy topological graph of $\mathcal{G}_1$

Step/Vertex	1	2	3	4	5	6	7	8
u_1	$\{\frac{\sigma_{x_1}}{x_1}, \frac{\sigma_{x_2}}{x_2}, \frac{\sigma_{x_3}}{x_3}\}$							
u_2	$\{\frac{\sigma_{x_1}}{x_1},$	$\{\frac{\sigma_{x_1}}{x_1}, \frac{\sigma_{x_4}}{x_4}, \frac{\sigma_{x_5}}{x_5}\}$						
u_3		$\{\frac{\sigma_{x_4}}{x_4},$	$\{\frac{\sigma_{x_4}}{x_4}, \frac{\sigma_{x_6}}{x_6}, \frac{\sigma_{x_7}}{x_7}\}$					
u_4			$\{\frac{\sigma_{x_6}}{x_6},$	$\{\frac{\sigma_{x_6}}{x_6}, \frac{\sigma_{x_8}}{x_8}, \frac{\sigma_{x_9}}{x_9}\}$				
u_5	$\{\frac{\sigma_{x_3}}{x_3},$			$\{\frac{\sigma_{x_3}}{x_3}, \frac{\sigma_{x_9}}{x_9},$	$\{\frac{\sigma_{x_3}}{x_3}, \frac{\sigma_{x_9}}{x_9}, \frac{\sigma_{x_{10}}}{x_{10}}\}$			
u_6	$\{\frac{\sigma_{x_2}}{x_2},$					$\{\frac{\sigma_{x_2}}{x_2}, \frac{\sigma_{x_{12}}}{x_{12}}, \frac{\sigma_{x_{13}}}{x_{13}}\}$		
u_7		$\{\frac{\sigma_{x_5}}{x_5},$					$\{\frac{\sigma_{x_5}}{x_5}, \frac{\sigma_{x_{14}}}{x_{14}}, \frac{\sigma_{x_{15}}}{x_{15}}\}$	
u_8			$\{\frac{\sigma_{x_7}}{x_7},$			$\{\frac{\sigma_{x_7}}{x_7}, \frac{\sigma_{x_{12}}}{x_{12}},$		$\{\frac{\sigma_{x_7}}{x_7}, \frac{\sigma_{x_{11}}}{x_{11}}, \frac{\sigma_{x_{12}}}{x_{12}}\}$
u_9				$\{\frac{\sigma_{x_8}}{x_8},$		$\{\frac{\sigma_{x_8}}{x_8}, \frac{\sigma_{x_{13}}}{x_{13}},$	$\{\frac{\sigma_{x_8}}{x_8}, \frac{\sigma_{x_{13}}}{x_{13}}, \frac{\sigma_{x_{14}}}{x_{14}}\}$	
u_{10}					$\{\frac{\sigma_{x_{10}}}{x_{10}},$		$\{\frac{\sigma_{x_{10}}}{x_{10}}, \frac{\sigma_{x_{15}}}{x_{15}},$	$\{\frac{\sigma_{x_{10}}}{x_{10}}, \frac{\sigma_{x_{11}}}{x_{11}}, \frac{\sigma_{x_{15}}}{x_{15}}\}$

Table 2 A fuzzy topological graph of $\mathcal{G}_2$

Step	1	2	3	4	5	6	7	8
v_1	$\{\frac{\sigma_{x_1}}{x_1}, \frac{\sigma_{x_2}}{x_2}, \frac{\sigma_{x_3}}{x_3}\}$							
v_2	$\{\frac{\sigma_{x_1}}{x_1},$	$\{\frac{\sigma_{x_1}}{x_1}, \frac{\sigma_{x_4}}{x_4}, \frac{\sigma_{x_5}}{x_5}\}$						
v_3		$\{\frac{\sigma_{x_4}}{x_4},$	$\{\frac{\sigma_{x_4}}{x_4}, \frac{\sigma_{x_6}}{x_6}, \frac{\sigma_{x_7}}{x_7}\}$					
v_4			$\{\frac{\sigma_{x_7}}{x_7},$	$\{\frac{\sigma_{x_7}}{x_7}, \frac{\sigma_{x_{11}}}{x_{11}}, \frac{\sigma_{x_{12}}}{x_{12}}\}$				
v_5				$\{\frac{\sigma_{x_{11}}}{x_{11}},$	$\{\frac{\sigma_{x_{10}}}{x_{10}}, \frac{\sigma_{x_{11}}}{x_{11}}, \frac{\sigma_{x_{15}}}{x_{15}}\}$			
v_6	$\{\frac{\sigma_{x_3}}{x_3},$				$\{\frac{\sigma(x_3)}{x_3}, \frac{\sigma(x_{10})}{x_{10}}$	$\{\frac{\sigma(x_3)}{x_3}, \frac{\sigma(x_9)}{x_9}, \frac{\sigma(x_{10})}{x_{10}}\}$		
v_7			$\{\frac{\sigma(x_6)}{x_6},$			$\{\frac{\sigma(x_6)}{x_6}, \frac{\sigma(x_9)}{x_9},$	$\{\frac{\sigma(x_6)}{x_6}, \frac{\sigma(x_8)}{x_8}, \frac{\sigma(x_9)}{x_9}\}$	
v_8							$\{\frac{\sigma(x_8)}{x_8},$	$\{\frac{\sigma(x_8)}{x_8}, \frac{\sigma(x_{13})}{x_{13}}, \frac{\sigma(x_{14})}{x_{14}}\}$
v_9	$\{\frac{\sigma(x_2)}{x_2},$				$\{\frac{\sigma(x_2)}{x_2}, \frac{\sigma(x_{12})}{x_{12}},$			$\{\frac{\sigma(x_2)}{x_2}, \frac{\sigma(x_{12})}{x_{12}}, \frac{\sigma(x_{13})}{x_{13}}\}$
v_{10}		$\{\frac{\sigma(x_5)}{x_5},$			$\{\frac{\sigma(x_5)}{x_5}, \frac{\sigma(x_{15})}{x_{15}}$			$\{\frac{\sigma(x_5)}{x_5}, \frac{\sigma(x_{14})}{x_{14}}, \frac{\sigma(x_{15})}{x_{15}}\}$

Table 3 Homeomorphic between fuzzy topological graphs of $\mathcal{G}_1$ and $\mathcal{G}_2$

Set	G_1	G_2
$\{\frac{\sigma(x_1)}{x_1}, \frac{\sigma(x_2)}{x_2}, \frac{\sigma(x_3)}{x_3}\}$	u_1	v_1
$\{\frac{\sigma(x_1)}{x_1}, \frac{\sigma(x_4)}{x_4}, \frac{\sigma(x_5)}{x_5}\}$	u_2	v_2
$\{\frac{\sigma(x_4)}{x_4}, \frac{\sigma(x_6)}{x_6}, \frac{\sigma(x_7)}{x_7}\}$	u_3	v_3
$\{\frac{\sigma(x_6)}{x_6}, \frac{\sigma(x_8)}{x_8}, \frac{\sigma(x_9)}{x_9}\}$	u_4	v_7
$\{\frac{\sigma(x_3)}{x_3}, \frac{\sigma(x_9)}{x_9}, \frac{\sigma(x_{10})}{x_{10}}\}$	u_5	v_6
$\{\frac{\sigma(x_2)}{x_2}, \frac{\sigma(x_{12})}{x_{12}}, \frac{\sigma(x_{13})}{x_{13}}\}$	u_6	v_9
$\{\frac{\sigma(x_5)}{x_5}, \frac{\sigma(x_{14})}{x_{14}}, \frac{\sigma(x_{15})}{x_{15}}\}$	u_7	v_{10}
$\{\frac{\sigma(x_7)}{x_7}, \frac{\sigma(x_{11})}{x_{11}}, \frac{\sigma(x_{12})}{x_{12}}\}$	u_8	v_4
$\{\frac{\sigma(x_8)}{x_8}, \frac{\sigma(x_{13})}{x_{13}}, \frac{\sigma(x_{14})}{x_{14}}\}$	u_9	v_8
$\{\frac{\sigma(x_{10})}{x_{10}}, \frac{\sigma(x_{11})}{x_{11}}, \frac{\sigma(x_{15})}{x_{15}}\}$	u_{10}	v_5

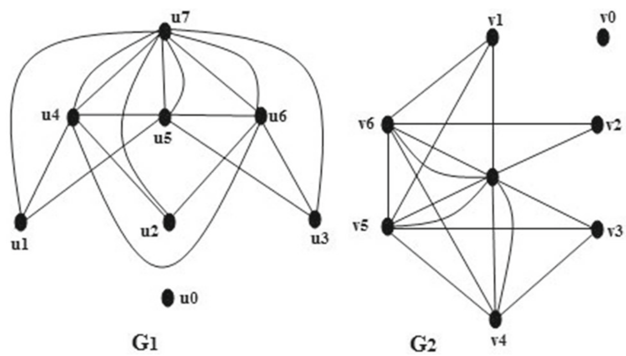


Fig. 12 Two isomorphic fuzzy graphs $\mathcal{G}_1$ and $\mathcal{G}_2$

Table 4 A fuzzy topological graph of $\mathcal{G}_1$

Step/Vertex	1	2	3	4
u_0	0			
u_1		$\{\frac{0.4}{x_1}\}$		
u_2			$\{\frac{0.3}{x_2}\}$	
u_3				$\{\frac{0.6}{x_3}\}$
u_4		$\{\frac{0.4}{x_1},$	$\{\frac{0.4}{x_1}, \frac{0.3}{x_2}\}$	
u_5		$\{\frac{0.4}{x_1},$		$\{\frac{0.4}{x_1}, \frac{0.6}{x_3}\}$
u_6			$\{\frac{0.3}{x_2},$	$\{\frac{0.3}{x_2}, \frac{0.6}{x_3}\}$
u_7		$\{\frac{0.4}{x_1},$	$\{\frac{0.4}{x_1}, \frac{0.3}{x_2},$	$\{\frac{0.4}{x_1}, \frac{0.3}{x_2}, \frac{0.6}{x_3}\}$

Fuzzy topological structures are an important base for knowledge extraction and processing. Therefore, an interesting and a natural research topic in a fuzzy set theory is to study the relationship between fuzzy sets and fuzzy topological spaces. In the last few years, fuzzy topology (Liu and Luo 1998), as an important research field in fuzzy set theory, has been developed into a quite mature discipline. In contrast with classical topology, fuzzy topology is endowed with richer structure,

Table 5 A fuzzy topological graph of $\mathcal{G}_2$

Step/Vertex	1	2	3	4
v_0	0			
v_1		$\{\frac{0.4}{x_1}\}$		
v_2			$\{\frac{0.3}{x_2}\}$	
v_3				$\{\frac{0.6}{x_3}\}$
v_4			$\{\frac{0.3}{x_2},$	$\{\frac{0.3}{x_2}, \frac{0.6}{x_3}\}$
v_5		$\{\frac{0.4}{x_1},$		$\{\frac{0.4}{x_1}, \frac{0.6}{x_3}\}$
v_6		$\{\frac{0.4}{x_1},$	$\{\frac{0.4}{x_1}, \frac{0.3}{x_2}\}$	
v_7		$\{\frac{0.4}{x_1},$	$\{\frac{0.4}{x_1}, \frac{0.3}{x_2},$	$\{\frac{0.4}{x_1}, \frac{0.3}{x_2}, \frac{0.6}{x_3}\}$

Table 6 Homeomorphic between two fuzzy topological graphs of $\mathcal{G}_1$ and $\mathcal{G}_2$

Sets	G_1	G_2
0	u_0	v_0
$\{\frac{0.4}{x_1}\}$	u_1	v_1
$\{\frac{0.3}{x_2}\}$	u_2	v_2
$\{\frac{0.6}{x_3}\}$	u_3	v_3
$\{\frac{0.4}{x_1}, \frac{0.3}{x_2}\}$	u_4	v_6
$\{\frac{0.4}{x_1}, \frac{0.6}{x_3}\}$	u_5	v_5
$\{\frac{0.3}{x_2}, \frac{0.6}{x_3}\}$	u_6	v_4
$\{\frac{0.4}{x_1}, \frac{0.3}{x_2}, \frac{0.6}{x_3}\}$	u_7	v_2

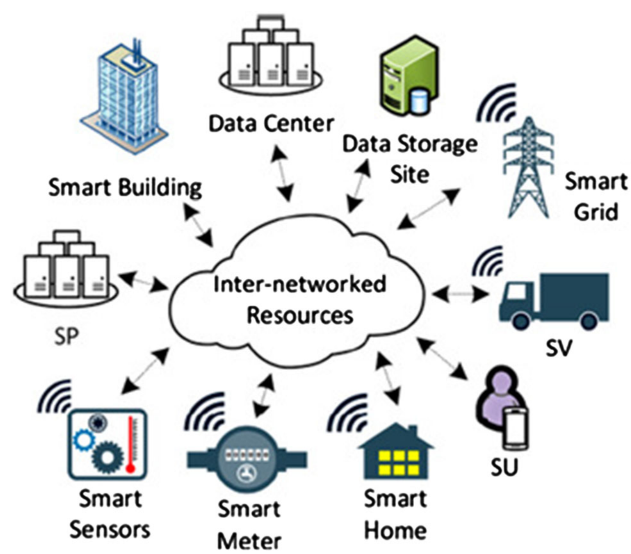


Fig. 13 Smart city ecosystem (Jararweh et al. 2020)

to a certain extent, which is manifested with different ways to generalize certain classical concepts. Therefore, the theory of fuzzy graphs and fuzzy topological spaces became the most important subjects in mathematics which can be applied in smart city. On the other hand, topology plays a

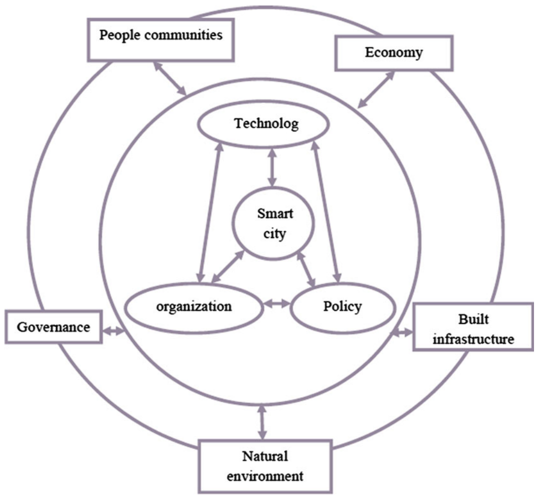


Fig. 14 Factors for building smart city (**Chourabi 2012**)

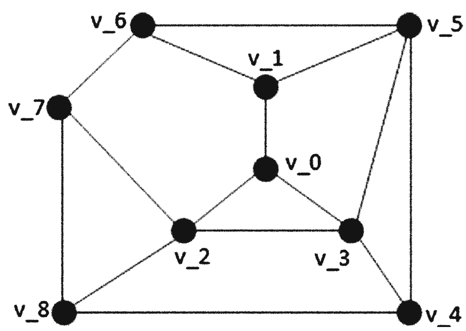


Fig. 15 Representation of smart city as a graph $\mathcal{G}_1$

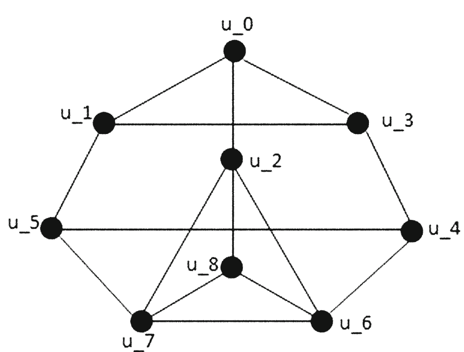


Fig. 16 Representation of smart city as a graph $\mathcal{G}_2$

significant rule in quantum physics, high-energy physics, and superstring theory **(ElNaschie 2006)** and in fractals (El Atik and Nasef 2020). Thus, we study the topological structure of graphs and calculate the degree of vertices and the number of edges of graphs which may have possible applications in quantum physics and superstring theory. In the future, the present work can be extended in fuzzy topological structures as in Akram (2019), and thus, one can get a more affirmative solution in decision making problems (Jiang et al. 2018; Zhan et al. 2019; Zhang et al. 2019a, b) in real-life solutions.

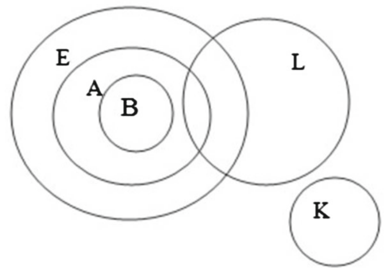


Fig. 17 Relationship between fuzzy sets

We can also study in future two essential projects of research as applications on our study. The homeomorphic between fuzzy topological structures enables us to have isomorphic fuzzy graphs.

(1) **In Chemistry:** Application of these indices are very useful in chemistry. Each object taken as electron for element and edges achieves the link between electrons.
(2) **In Internet Routing:** In Internet routing, when a greater number of internet links are achieved in one area, it can be determined by adding more number of routers in the area.

Acknowledgements The authors are grateful to the editor and referees for their valuable comments and suggestions.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Human and animal rights This article does not contain any studies with human participants or animals performed by any of the authors.

Informed consent Informed consent was obtained from all individual participants included in the study.

Appendix: Proofs

In this section, we prove the results declared in previous sections.

Proof (Continue proof Theorem 2) If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two parallel classes, then there exists a bijective fuzzy function F from $\mathcal{C}_1$ to $\mathcal{C}_2$, i.e., $F : \mathcal{C}_1 \to \mathcal{C}_2$, such that $F(\mathcal{C}_1) = \mathcal{C}_2$, where $\mathcal{C}_1 = \{\mathcal{A}_i : i \in I\}$ and $\mathcal{C}_2 = \{F(\mathcal{A}_i) : i \in I\}$. Then, every vertex $v_{\mathcal{A}_i}$ in $\mathcal{G}_1$, the graph generated by $\mathcal{C}_1$, we have a corresponding vertex $v_{F(\mathcal{A}_i)}$ in $\mathcal{G}_2$, the fuzzy graph generated by C_2. So $\mu_R(v_{\mathcal{A}_i}, v_{\mathcal{A}_j}) = \mu_R(\mathcal{A}_i \cap \mathcal{A}_j) = \mu_R(F(\mathcal{A}_i \cap \mathcal{A}_j)) = \mu_R(F(\mathcal{A}_i) \cap F(\mathcal{A}_j)) = \mu_R(v_{F(\mathcal{A}_i)}, v_{F(\mathcal{A}_j)})$. Also, $\sigma_{v_{\mathcal{A}_i}} = \sigma_{v_{F(\mathcal{A}_i)}}$ and $\sigma_{v_{\mathcal{A}_j}} = \sigma_{v_{F(\mathcal{A}_j)}}$. Thus, F is an isomorphism. □

Proof (Continue proof Theorem 3) The relation between fuzzy sets is shown in Fig. 17.

(i) Let $A \subseteq X$, $|A| = m$ and $|A^c| = n - m$

$$
\begin{aligned}
deg\,(V_A) &= \sum R\,(V_A, V_B) \\
&\quad + \sum R\,(V_A, V_E) \\
&\quad + \sum R\,(V_A, V_L) \\
&\quad + \sum R\,(V_A, V_K) \\
&= \sum |A \cap B| + \sum |A \cap E| \\
&\quad + \sum |(A \cap L| + \sum |A \cap K| \\
&= \sum |B| + \sum |A| + \sum |A \cap L|.
\end{aligned}
$$

$$
\begin{aligned}
\sum |B| &= \sum_{i=1}^{m-1} i\,{}^{m}C_i \\
&= \sum_{i=1}^{m-1} i \frac{m!}{i!(m-i)!} \\
&= m \sum_{i=1}^{m-1} \frac{m-1)!}{(i-1)!(m-i)!} \\
&= m \sum_{i=1}^{m-1} {}^{m-1}C_{i-1} \\
&= m \sum_{j=0}^{m-2} \left({}^{m-1}C_j - 1\right) \\
&= m(2^{m-1} - 1),
\end{aligned}
$$

since $B \subseteq A$, then$\sum |A| = m(2^{m-1} - 1)$.
But $A \subseteq E$, then by complement $\sum |A|$=$m(2^{n-m} - 1)$ and $\sum |A \cap L|$=$(2^{n-m} - 1) \sum_{i=1}^{m-1} i\;{}^{m}C_i$
$= m(2^{n-m} - 1)(2^{m-1} - 1)$.
So $deg(V_A) = m(2^{n-1} - 1) + 2m$.
It follows that the number of edges of pseudographs G equals

$$
\begin{aligned}
|E_p(G)| &= \frac{1}{2} \sum_{m=1}^{n} {}^{n}C_m deg(V_A) \\
&= \frac{1}{2} \sum_{m=1}^{n} {}^{n}C_m \left(m\left(2^{n-1} - 1\right) + 2m\right) \\
&= \frac{1}{2} \sum_{m=1}^{n} {}^{n}C_m m \left(2^{n-1} - 1\right) \\
&\quad + \frac{1}{2} \sum_{m=1}^{n} {}^{n}C_m 2m \\
&= n2^{n-2}\left(2^{n-1} - 1\right) \\
&\quad + \sum_{m=1}^{n} m\,{}^{n}C_m \\
&= n2^{n-2}\left(2^{n-1} - 1\right) \\
&\quad + n \sum_{m=1}^{n} {}^{n-1}C_{m-1} \\
&= n2^{n-2}\left(2^{n-1} - 1\right) \\
&\quad + n \sum_{j=0}^{n-1} {}^{n-1}C_j \\
&= n2^{n-2}\left(2^{n-1} - 1\right) + n2^{n-1}.
\end{aligned}
$$

(ii) $deg(V_A) = m(2^{n-1} - 1)$.

It follows that the number of edges of discrete graphs equals

$$
\begin{aligned}
|E_d(G)| &= \frac{1}{2} \sum_{A} deg(V_A) \\
&= \frac{1}{2} \sum_{A} {}^{n}C_m m.\left(2^{n-1} - 1\right) \\
&= \frac{1}{2}\left(2^{n-1} - 1\right) \sum_{m=1}^{n} m\,{}^{n}C_m \\
&= \frac{1}{2}\left(2^{n-1} - 1\right) \sum_{m=1}^{n} m\,{}^{n}C_m \\
&= \frac{1}{2}\left(2^{n-1} - 1\right) \sum_{m=1}^{n} m \frac{n!}{m!(n-m)!} \\
&= \frac{n}{2}\left(2^{n-1} - 1\right) \sum_{m=1}^{n} \frac{(n-1)!}{(m-1)!(n-m)!)} \\
&= \frac{n}{2}\left(2^{n-1} - 1\right) \sum_{m=1}^{n} {}^{n-1}C_{m-1} \\
&= \frac{n}{2}\left(2^{n-1} - 1\right) \sum_{j=0}^{n-1} {}^{n-1}C_j \\
&= \frac{n}{2}\left(2^{n-1} - 1\right) 2^{n-1} \\
&= n2^{n-2}\left(2^{n-1} - 1\right).
\end{aligned}
$$

(iii) $deg(V_A) = 2^n - 2^{n-m} - 1$.

It follows that the number of edges of simple graphs G equals

$$
\begin{aligned}
|E_s(G)| &= \frac{n}{2}\sum_{m=1}^{n} {}^{n}C_m deg\,(V_A) \\
&= \frac{1}{2}\sum_{m=1}^{n} {}^{n}C_m\left(2^n - 2^{n-m} - 1\right) \\
&= \frac{1}{2}\left(2^n - 1\right)\sum_{m=1}^{n} {}^{n}C_m \\
&\quad -\frac{1}{2}2^n \sum_{m=1}^{n} {}^{n}C_m 2^{-m} \\
&= \frac{1}{2}\left(2^n - 1\right)\left(2^{n-1} - 1\right) \\
&\quad -2^{n-1}\sum_{m=1}^{n}\left(\frac{1}{2}\right)^m {}^{n}C_m \\
&= \frac{1}{2}\left(2^{2n} - 2^{n+1} + 1\right) \\
&\quad -2^{n-1}\left(\left(1+\frac{1}{2}\right)^n - 1\right) \\
&= \frac{1}{2}\left(\left(2^{2n} - 2.2^n + 1\right) - \left(3^n - 2^n\right)\right) \\
&= \frac{1}{2}(2^{2n} - 2^n - 3^n + 1).
\end{aligned}
$$

The total degree of edges of fuzzy topological graphs can be calculated for the three different types of fuzzy graphs as follows:

[For fuzzy pseudographs:] Let $\mathcal{B} \le \mathcal{A} \le \mathcal{E} \le X$, $|\mathcal{A}| = m$ and $|\mathcal{A}^c| = n - m$

$$
\begin{aligned}
deg\,(V_{\mathcal{A}}) &= \sum \mu_R\,(V_{\mathcal{A}}, V_{\mathcal{B}}) \\
&\quad +2\sum \mu_R\,(V_{\mathcal{A}}, V_{\mathcal{A}}) \\
&\quad +\sum \mu_R\,(V_{\mathcal{A}}, V_{\mathcal{E}}) \\
&\quad +\sum \mu_R\,(V_{\mathcal{A}}, V_{\mathcal{L}}) \\
&\quad +\sum \mu_R\,(V_{\mathcal{A}}, V_{\mathcal{K}}) \\
&= \sum \mu_R\,(V_{\mathcal{A}} \cap V_{\mathcal{B}}) \\
&\quad +2\sum \mu_R\,(V_{\mathcal{A}} \cap V_{\mathcal{A}}) \\
&\quad +\sum \mu_R\,(V_{\mathcal{A}} \cap V_{\mathcal{E}}) \\
&\quad +\mu_R\,(V_{\mathcal{A}} \cap V_{\mathcal{L}}) \\
&\quad +\sum \mu_R\,(V_{\mathcal{A}} \cap V_{\mathcal{K}}) \\
&= \sum \mu_R\,(V_{\mathcal{B}}) \\
&\quad +2\sum \mu_R\,(V_{\mathcal{A}}) \\
&\quad +\sum \mu_R\,(V_{\mathcal{A}}) \\
&\quad +\sum \mu_R\,(V_{\mathcal{A}} \cap V_{\mathcal{L}}) \\
&= \sum_{i=1}^{|B|} \mu_R\,(V_{\mathcal{B}}) \\
&\quad +2\sum_{i=1}^{|A|} \mu_R\,(V_{\mathcal{A}}) \\
&\quad +\sum_{i=1}^{|A\cap E|} \mu_R\,(V_{\mathcal{A}}) \\
&\quad +\sum_{i=1}^{|A\cap L|} \mu_R\,(V_{\mathcal{A}\cap\mathcal{L}})\,.
\end{aligned}
$$

$\forall x \in \mathcal{B}$, $y \in \mathcal{A}$, $w \in \mathcal{E}$ and $z \in \mathcal{L}$ we have,

$$
\begin{aligned}
deg\,(V_{\mathcal{A}}) &= \sum_{i=1}^{|B|} \mu_R(xy) + 2\sum_{i=1}^{|A|} \mu_R(yy) \\
&\quad +\sum_{i=1}^{|A\cap E|} \mu_R(yw) + \sum_{i=1}^{|A\cap L|} \mu_R(yz).
\end{aligned}
$$

It follows that the total degree of edges of fuzzy pseudographs $\mathcal{G}$ equals

$$
\sum \mathcal{E}_p(\mathcal{G}) = \sum_{i=1}^{|V(\mathcal{G})|} deg(V_i).
$$

[For fuzzy discrete graphs:]

$$
\begin{aligned}
deg\,(V_{\mathcal{A}}) &= \sum \mu_R\,(V_{\mathcal{A}}, V_{\mathcal{B}}) \\
&\quad +\sum \mu_R\,(V_{\mathcal{A}}, V_{\mathcal{E}}) \\
&\quad +\sum \mu_R\,(V_{\mathcal{A}}, V_{\mathcal{L}}) \\
&\quad +\sum \mu_R\,(V_{\mathcal{A}}, V_{\mathcal{K}}) \\
&= \sum \mu_R\,(V_{\mathcal{A}} \cap V_{\mathcal{B}}) \\
&\quad +\sum \mu_R\,(V_{\mathcal{A}} \cap V_{\mathcal{E}}) \\
&\quad +\mu_R\,(V_{\mathcal{A}} \cap V_{\mathcal{L}}) \\
&\quad +\sum \mu_R\,(V_{\mathcal{A}} \cap V_{\mathcal{K}}) \\
&= \sum \mu_R\,(V_{\mathcal{B}}) \\
&\quad +\sum \mu_R\,(V_{\mathcal{A}}) \\
&\quad +\sum \mu_R\,(V_{\mathcal{A}} \cap V_{\mathcal{L}}) \\
&= \sum_{i=1}^{|B|} \mu_R\,(V_{\mathcal{B}}) \\
&\quad +\sum_{i=1}^{|A\cap E|} \mu_R\,(V_{\mathcal{A}})
\end{aligned}
$$

$$+\sum_{i=1}^{|A\cap L|}\mu_R\left(V_{\mathcal{A}\cap\mathcal{L}}\right).$$

$\forall x\in\mathcal{B}$, $y\in\mathcal{A}$, $w\in\mathcal{E}$ and $z\in\mathcal{L}$ we have,

$$\begin{aligned}deg\left(V_{\mathcal{A}}\right)=&\sum_{i=1}^{|B|}\mu_R(xy)+\sum_{i=1}^{|A\cap E|}\mu_R(yw)\\&+\sum_{i=1}^{|A\cap L|}\mu_R(yz)\end{aligned}$$

It follows that the summation of edges of fuzzy discrete graphs $\mathcal{G}$ equals

$$\sum\mathcal{E}_d\left(\mathcal{G}\right)=\sum_{i=1}^{|V(\mathcal{G})|}deg(V_i)$$

[For fuzzy simple graphs:] Let the number of the fuzzy subset of length $|B|=k$, $|A\cap E|=l$, and $|A\cap L|=r$. Then, we have

$$\begin{aligned}deg\left(V_{\mathcal{A}}\right)=&\sum\mu_R\left(V_{\mathcal{A}},V_{\mathcal{B}}\right)+\sum\mu_R\left(V_{\mathcal{A}},V_{\mathcal{E}}\right)\\&+\sum\mu_R\left(V_{\mathcal{A}},V_{\mathcal{L}}\right)+\sum\mu_R\left(V_{\mathcal{A}},V_{\mathcal{K}}\right)\\=&\sum\mu_R\left(V_{\mathcal{A}}\cap V_{\mathcal{B}}\right)+\sum\mu_R\left(V_{\mathcal{A}}\cap V_{\mathcal{E}}\right)\\&+\mu_R(V_{\mathcal{A}}\cap V_{\mathcal{L}})+\sum\mu_R\left(V_{\mathcal{A}}\cap V_{\mathcal{K}}\right)\\=&\sum\mu_R\left(V_{\mathcal{B}}\right)+\sum\mu_R\left(V_{\mathcal{A}}\right)\\&+\sum\mu_R\left(V_{\mathcal{A}}\cap V_{\mathcal{L}}\right)\\=&\sum_{i=1}^{k}\mu_R\left(V_{\mathcal{B}}\right)+\sum_{i=1}^{l}\mu_R\left(V_{\mathcal{A}}\right)\\&+\sum_{i=1}^{r}\mu_R\left(V_{\mathcal{A}\cap\mathcal{L}}\right).\end{aligned}$$

□

Proof (Continue proof Theorem 4) Let $\mathcal{G}$ be a fuzzy graph. To complete the proof, it is sufficient to prove three conditions of a fuzzy topology on τ.

(i) By Definitions 3 and 7, $\mathcal{G}$ can be represented by different classes. Suppose that the graph number of $\mathcal{G}$ is m. Then, there exists a class, say τ, such that $\tau\leq P(\left\{\frac{a}{x_1},\frac{b}{x_2},\frac{c}{x_3},\ldots,\frac{d}{x_m}\right\})$ and represents $\mathcal{G}$. Each $\mathcal{A}_i\in\tau$ represents v_i and $X\in\tau$ represents the set of vertices $V(\mathcal{G})$. So, by Definition 6, $\mu_R(v_i,V(\mathcal{G}))=\mu_R(v_i,X)$. If $N=\left\{\frac{a}{x_1},\frac{b}{x_2},\frac{c}{x_3},\ldots,\frac{d}{x_m}\right\}$, then $\mu_R(\mathcal{A}\cap X)=\mu_R(\mathcal{A}\cap\mathbb{N})=\mu_R(\mathcal{A})$. This means that each $\mathcal{A}\in\tau$ satisfies that $\mathcal{A}\leq X$. Since every vertex in $\mathcal{G}$ is a fuzzy graph subset of $V(\mathcal{G})$, i.e., for every $v_i\in V(\mathcal{G})$ the singleton $\{v_i\}\subset V(\mathcal{G})$ and each element in τ is a fuzzy subset from $X\in\tau$, then $X=\left\{\frac{a}{x_1},\frac{b}{x_2},\frac{c}{x_3},\ldots,\frac{d}{x_m}\right\}$. Also, the isolated vertex v_0 can be represented by $0\in\tau$.

(ii) Let $v_1,v_2,\ldots$ be an arbitrary different vertices in $V(\mathcal{G})$ represented by $\mathcal{A}_{v_1},\mathcal{A}_{v_2},\ldots$. Since $v_1\vee v_2\vee\cdots=v_{(\mathcal{A}_{v_1}\vee\mathcal{A}_{v_2}\vee\cdots)}$, then $\mathcal{A}_{v_1}\vee\mathcal{A}_{v_2}\vee\cdots\in\tau$.

(iii) If v_i and v_j are two different vertices in $V(\mathcal{G})$ and represented by $\mathcal{A}_i$ and $\mathcal{A}_j$, respectively. Since $v_i\wedge v_j=v_{\mathcal{A}_i\wedge\mathcal{A}_j}$, then $\mathcal{A}_i\wedge\mathcal{A}_j\in\tau$. Therefore, τ is a fuzzy topology. □

Proof (Continue proof Theorem 5) Let $\mathcal{A}_m=\{\frac{a}{x_1},\frac{b}{x_2},\ldots,\frac{c}{x_m}\}$, $1\leq k\leq m$, $k<m<n$ and $B_k\subseteq A_m\subseteq C_n$

$$\begin{aligned}degV_{A_m}&=\sum\mu_R\left(B_k,A_m\right)+\sum\mu_R\left(A_m,C_n\right)\\&=\sum\mu_R\left(B_k\cap A_m\right)+\sum\mu_R\left(A_m\cap C_n\right)\\&=\sum_{i=1}^{|B_k\cap A_m|}\mu_R\left(B_k\right)+\sum_{i=1}^{|A_m\cap C_n|}\mu_R\left(A_m,C_n\right)\\&=\sum_{i=1}^{|B_k\cap A_m|}\mu_R(xy)+\sum_{i=1}^{|A_m\cap C_n|}\mu_R(yz).\end{aligned}$$

$\forall x\in B_k$, $y\in A_m$ and $z\in C_n$. The total degree of edges is $\sum E(\mathcal{G})=\frac{1}{2}\sum_{i=1}^{n}degV_{\mathcal{A}_i}$. □

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