Soft Computing (2021) 25:85–108
https://doi.org/10.1007/s00500-020-05460-z

FOUNDATIONS

# Methods for solving *LR*-bipolar fuzzy linear systems

Muhammad Akram[1] · Tofigh Allahviranloo[2] · Witold Pedrycz[3] · Muhammad Ali[1]

Published online: 2 January 2021
© Springer-Verlag GmbH Germany, part of Springer Nature 2021

## Abstract

In this paper, we propose a technique to solve $LR$-bipolar fuzzy linear system(BFLS), $LR$-complex bipolar fuzzy linear (CBFL) system with real coefficients and $LR$-complex bipolar fuzzy linear (CBFL) system with complex coefficients of equations. Initially, we solve the $LR$-BFLS of equations using a pair of positive($*$) and negative($\bullet$) of two $n \times n$ $LR$-real linear systems by using mean values and left-right spread systems. We also provide the necessary and sufficient conditions for the solution of $LR$-BFLS of equations. We illustrate the method by using some numerical examples of symmetric and asymmetric $LR$-BFLS equations and obtain the strong and weak solutions to the systems. Further, we solve the $LR$-CBFL system of equations with real coefficients and $LR$-CBFL system of equations with complex coefficients by pair of positive($*$) and negative($\bullet$) two $n \times n$ real and complex $LR$-bipolar fuzzy linear systems by using mean values and left-right spread systems. Finally, we show the usage of technique to solve the current flow circuit which is represented by $LR$-CBFL system with complex coefficients and obtain the unknown current in term of $LR$-bipolar fuzzy complex number.

**Keywords** $LR$-bipolar fuzzy linear system · $LR$-bipolar fuzzy number · $LR$-complex bipolar fuzzy linear system · $LR$-bipolar fuzzy complex number

## 1 Introduction

The concepts of fuzzy sets were introduced by Zadeh (1965, 1971, 1975). Fuzzy set theory is a useful mathematical model for computing the uncertainty and vagueness. Dubois and Prade (1978) discussed the basic arithmetic operations of fuzzy numbers. Zhang (1998) introduced the idea of the term Yin and Yang which were based on double-sided or bipolar judgmental thinking on a positive and negative side. Zhang (1994) introduced the concept of bipolar fuzzy set in 1994 and bipolar fuzzy number in 1996 which was the extension of fuzzy set. Akram (2011) introduced the concept of bipolar fuzzy graph in 2011. Alghamdi et al. (2018) considered the multi-criteria decision-making techniques in bipolar fuzzy environment. The term simultaneous linear equations play vital role in different kinds of fields including mathematics, physics, networking, circuit analysis, economic model and attribute decision making. Fuzzy linear system (FLS) is a powerful tool to measure the fuzziness and uncertainty. In many places, a bipolar fuzzy linear system (BFLS) is used for linear optimization of system when system is represented in bipolar form. Complex fuzzy linear system is specially used in circuit analysis to measure unknown current in complex form. First the solution of FLS was examined by Friedman et al. (1998), and they used the embedding method to solve the system. They used the technique in which $n \times n$ FLS of equations whose coefficient matrix is a real number matrix and right side column vector is fuzzy number vector containing parameter $r$ is replaced by $2n \times 2n$ real linear system. Abbasbandy and Alavi (2005) also used another method to solve $n \times n$ FLS of equations for which coefficient matrix is a real number matrix and right side column vector is fuzzy number vector containing parameter $r$. They used a

Communicated by A. Di Nola.

✉ Muhammad Akram
m.akram@pucit.edu.pk

Tofigh Allahviranloo
tofigh.allahviranloo@eng.bau.edu.tr

Witold Pedrycz
pedrycz@ee.ualberta.ca

Muhammad Ali
muhammadali6481@gmail.com

1 Department of Mathematics, University of the Punjab, New Campus, Lahore, Pakistan

2 Faculty of Engineering and Natural Sciences, Bahcesehir University, Istanbul, Turkey

3 Department of Electrical and Computer Engineering, University of Alberta, Edmonton, Canada

procedure in which $n \times n$ FLS of equations is replaced by two $n \times n$ real linear systems. Allahviranloo (2019) proposed different methods to solve uncertain linear system of equations. Some numerical techniques, to solve FLS of equations, for example, Jacobi, Gauss-Seidel, SOR iterative and steepest descent method, are given in Allahviranloo (2005a, 2004, 2005b) and Abbasbandy and Jafarian (2006). Some other numerical techniques are proposed by Akram et al. (2019b, c, d) to solve BFLS equations, fully bipolar fuzzy linear system FBFLS of equations and linear system of equations in $m$-polar fuzzy environment. In many cases, general fuzzy linear system having unknown parameter in terms of parametric form of fuzzy number containing parameter $r$ and to solve this system the system was extended into real linear system. It is not an easy task to solve the extended fuzzy linear system containing parameter $r$, $0 \leq r \leq 1$, which makes their calculation inconvenient in some sense. To make the multiplication easy, Dubois and Prade (1978) introduced the concept of LR fuzzy number. Dehghan et al. (2007) proposed a method to solve the fully fuzzy linear system $\widetilde{A}\widetilde{x} = \widetilde{b}$ whose coefficients matrix and right-side vector are $LR$ fuzzy numbers. Allahviranloo et al. (2013) also discussed a method to solve LR fuzzy linear systems based on a 1-level expansion. The concept of fuzzy complex set and fuzzy complex number was introduced by Buckley (1989). Rahgooy et al. (2009) studied the $n \times n$ complex fuzzy linear systems with complex coefficients and proposed a method to solve complex fuzzy linear system; it was applied on problem of circuit analysis ($CA$) to find the current in terms of complex fuzzy number. Jahantigh et al. (2010) studied the $n \times n$ complex fuzzy linear system and discussed the general technique to solve the system. Behera and Chakraverty (2012) also proposed a numerical method to solve FLS of equations and CFLS system with complex coefficients by using fuzzy center and width. Ghanbari et al. (2020) discussed the graph of generalized fuzzy complex number and find algebraic and general solutions of rectangular fuzzy complex linear system.

In this article, we present a technique to solve $n \times n$ $LR$-BFLS of equations, $LR$-CBFL system with real coefficients and $LR$-CBFL system with complex coefficients for which coefficient matrix is a real and complex number and right-hand side column vector is $LR$-bipolar fuzzy number (BFN) and $LR$-bipolar fuzzy complex number (BFCN), respectively. Initially, we propose a method to solve $LR$-BFLS of equations we replace $n \times n$ $LR$-BFLS of equations by pair of positive($*$) and negative($\bullet$) of $n \times n$ $LR$-real linear system and then solve the system by using mean value and left-right spread system which are free from parameters $r$ and $s$. Some numerical examples are also discuss to describe the efficiency of the technique. In the next section, we discuss the solution of two different systems: one is $LR$-CBFL system with real coefficients and the other is $LR$-CBFL system with complex coefficients. First, we propose a method to solve $LR$-CBFL system with real coefficients we replace $n \times n$ $LR$-CBFL system with real coefficients by pair of real and imaginary parts of $n \times n$ $LR$-BFLS of equation and then solve this system by using mean value and left-right spread system. Further, we extent the technique of Guo and Zhang (2016) to solve the $LR$-CBFL system with complex coefficients in which coefficient matrix is complex number and right side column vector is a $LR$-BFCN. In this method, we can replace $n \times n$ $LR$-CBFL system with complex coefficients by pair of positive($*$) and negative($\bullet$) of two $2n \times 2n$ mean value system and $4n \times 4n$ left-right spread system. We utilize the current flow circuit system given in Rahgooy et al. (2009) which is represented by $LR$-CBFL system with complex coefficients to find unknown current in the circuit in terms of $LR$-BFCN. In many real life problems, we deal with uncertainty of two kinds, one for the positivity and other for the negativity. Bipolar fuzzy numbers and more generally bipolar fuzzy system of linear equations are used to handle such problems. In present paper, $LR$-bipolar fuzzy systems of linear equations and bipolar fuzzy complex system are studied. The study of $LR$-bipolar fuzzy systems deal the problems that have uncertainty of two faces: certain property and its counter property. In addition, parametric BFN and BFCN, containing parameters $r$ and $s$ are replaced by $LR$-BFN and $LR$-BFCN for efficient mathematical calculations. For other notions and applications, readers are referred to Akram et al. (2021, 2020a, b), Koam et al. (2020), Saqib et al. (2020a, b), Akram et al. (2019a), Allahviranloo et al. (2008, 2014), Amirfakhrian (2012), Amirfakhrian et al. (2018), Ezzati (2011), Fariborzi Araghi and Fallahzadeh (2013), Moloudzadeh et al. (2013), Zheng and Wang (2006) and Rao et al. (2020a, b).

## 2 Preliminaries

In this section, we describe some basic definitions like $LR$-bipolar fuzzy number (BFN), $LR$-bipolar fuzzy linear system, $LR$-bipolar fuzzy complex number (BFCN) and $LR$-complex bipolar fuzzy linear (CBFL) system of equations.

**Definition 1** Abbasbandy and Alavi (2005) A fuzzy number is a fuzzy set $\nu : \Re \rightarrow [0, 1]$ which satisfies:

1. $\nu$ is upper semi continuous.
2. $\nu(x) = 0$ outside some interval [a,d].
3. There are real numbers $b, c : a \leq b \leq c \leq d$ for which

(a) $\nu(x)$ is monotonic non-decreasing on [a,b],
(b) $\nu$ is monotonic non-increasing on [c,d],
(c) $\nu(x) = 1, b \leq x \leq c$.

The membership function $\nu(x)$ can be represented as:

$$\nu(x)=\begin{cases}0, & \text{if } x\le a,\\ h(x), & \text{if } a\le x\le b,\\ 1, & \text{if } b\le x\le c,\\ g(x), & \text{if } c\le x\le d,\\ 0, & \text{if } x\ge d,\end{cases}$$

where $h(x)$ is non-decreasing function, called left membership function, and $g(x)$ is non-increasing function, called the right membership function.

The set of all fuzzy numbers on $\Re$ is denoted by $E^1$.

**Definition 2** Guo and Zhang (2016) A fuzzy number $\widetilde{H}$ is said to be a $LR$ fuzzy number if there are real numbers $h,\ \mu>0$ and $\nu>0$ such that

$$\sigma_{\widetilde{H}}(x)=\begin{cases}L\left(\frac{h-x}{\mu}\right), & \text{if } x\le h,\quad \mu>0,\\ R\left(\frac{x-h}{\nu}\right), & \text{if } x\ge h,\quad \nu>0,\end{cases}$$

where $h$, $\mu$ and $\nu$ are called the mean value and left and right spreads of $\widetilde{H}$, respectively.

The LR fuzzy number can also be represented as

$$\widetilde{H}=(h,\mu,\nu)_{LR}.$$

A real number $a$ can be denoted by $a=(a,0,0)_{LR}$.

Let L and R both be decreasing functions from positive real numbers $\Re^+$ to the interval [0,1] such that the following conditions:

1. $L(x)<1$, for $x>0$,
2. $L(x)>0$, for $x<1$,
3. $L(0)=1$, $L(1)=0$.

If the two functions $L$ and $R$ are in the form

$$T(x)=\begin{cases}1-x, & \text{if } 0\le x\le 1,\\ 0, & \text{Otherwise},\end{cases}$$

then the $LR$ fuzzy number is called triangular fuzzy number. $\widetilde{H}=(h,\mu,\nu)_{LR}$ is called symmetric fuzzy number if and only if $\mu=\nu$.

Clearly, two $LR$ fuzzy numbers $\widetilde{H}=(h,\mu,\nu)_{LR}$ and $\widetilde{G}=(g,\acute{\mu},\acute{\nu})_{LR}$ are said to be equal if and only if $h=g$, $\mu=\acute{\mu}$ and $\nu=\acute{\nu}$. Also, $\widetilde{H}=(h,\mu,\nu)_{LR}$ is called positive (negative) if and only if $h-\mu>0(h+\nu<0)$.

**Definition 3** Guo and Zhang (2016) For arbitrary two $LR$ fuzzy numbers $\widetilde{H}=(h,\mu,\nu)_{LR}$ and $\widetilde{G}=(g,\alpha,\beta)_{LR}$ and $c$ is arbitrary scalar number, we define addition, subtraction, product and scalar multiplication by $c$ as follows:

(i) Addition:

$$\begin{aligned}\widetilde{H}\oplus\widetilde{G}&=(h,\mu,\nu)_{LR}\oplus(g,\alpha,\beta)_{LR},\\ &=(h+g,\mu+\alpha,\nu+\beta)_{LR},\end{aligned}$$

(ii) Subtraction:

$$\begin{aligned}\widetilde{H}-\widetilde{G}&=(h,\mu,\nu)_{LR}-(g,\alpha,\beta)_{LR},\\ &=(h-g,\mu-\beta,\nu-\alpha)_{LR},\end{aligned}$$

(iii) Multiplication:
$If\,\widetilde{H}>0$ and $\widetilde{G}>0$, then

$$\begin{aligned}\widetilde{H}\otimes\widetilde{G}&=(h,\mu,\nu)_{LR}\otimes(g,\alpha,\beta)_{LR},\\ &\cong(hg,h\alpha+g\mu,h\beta+g\nu)_{LR},\end{aligned}$$

(iv) Scalar multiplication:

$$\begin{aligned}c\otimes\widetilde{H}&=c\otimes(h,\mu,\nu)_{LR},\\ &\cong\begin{cases}(ch,c\mu,c\nu)_{LR}, & c\ge 0,\\ (ch,-c\nu,-c\mu)_{LR}, & c<0.\end{cases}\end{aligned}$$

**Definition 4** Akram and Arshad (2019) A bipolar fuzzy number (BFN)

$$\xi=\prec M,N\succ=\prec[m_1,m_2,m_3,m_4],[n_1,n_2,n_3,n_4]\succ$$

is a bipolar fuzzy set of the mapping $\sigma:R\longrightarrow[0,1]\times[-1,0]$, with satisfaction degree $\sigma_M$ and dissatisfaction degree $\sigma_N$ such that:

$$\sigma_M(\kappa)=\begin{cases}\sigma_M^L(\kappa), & \text{if } \kappa\in[m_1,m_2],\\ 1, & \text{if } \kappa\in[m_2,m_3],\\ \sigma_M^R(\kappa), & \text{if } \kappa\in[m_3,m_4],\\ 0, & \text{otherwise},\end{cases}$$

$$\sigma_N(\kappa)=\begin{cases}\sigma_N^L(\kappa), & \text{if } \kappa\in[n_1,n_2],\\ -1, & \text{if } \kappa\in[n_2,n_3],\\ \sigma_N^R(\kappa), & \text{if } \kappa\in[n_3,n_4],\\ 0, & \text{otherwise},\end{cases}$$

and

$$\sigma_M^L(\kappa):[m_1,m_2]\to[0,1],\quad \sigma_M^R(\kappa):[m_3,m_4]\to[0,1],$$
$$\sigma_N^L(\kappa):[n_1,n_2]\to[-1,0],\quad \sigma_N^R(\kappa):[n_3,n_4]\to[-1,0],$$

where $\sigma_M^L(\kappa)$ and $\sigma_N^L(\kappa)$ represent the left membership functions for $\sigma_M(\kappa)$ and $\sigma_N(\kappa)$, respectively. Similarly, $\sigma_M^R(\kappa)$ and $\sigma_N^R(\kappa)$ represent right membership functions for $\sigma_M(\kappa)$ and $\sigma_N(\kappa)$, respectively.

A BFN in a parametric form is given as $h = \prec [\underline{h}^*(r), \overline{h}^*(r)], [\underline{h}^\bullet(s), \overline{h}^\bullet(s)] \succ$ of the mappings $\underline{h}^*(r), \overline{h}^*(r), \underline{h}^\bullet(s)$ and $\overline{h}^\bullet(s)$ where $0 \le r \le 1$, $-1 \le s \le 0$. A bipolar fuzzy number in parametric form always contains some parameters named r and s which create some difficulties to solve bipolar fuzzy systems. For the alleviation of this problem, we have introduced new class of bipolar fuzzy number which is called $LR$-bipolar fuzzy number in this form parameters r and s are eliminated.

**Definition 5** A bipolar fuzzy number $\widetilde{H}$ is said to be a $LR$-bipolar fuzzy number of the form $\widetilde{H} = \prec [h^*, \mu_l^*, \nu_r^*]_{LR}, [h^\bullet, \mu_l^\bullet, \nu_r^\bullet]_{LR} \succ$, where $h^*, h^\bullet$ are real numbers and $\mu_l^*, \mu_l^\bullet > 0$ and $\nu_r^*, \nu_r^\bullet > 0$ such that

$$\sigma_{\widetilde{H}^*}(\kappa) = \begin{cases} L\left(\frac{h^*-\kappa}{\mu_l^*}\right), & \text{if } \kappa \le h^*, \quad \mu_l^* > 0, \\ R\left(\frac{\kappa-h^*}{\nu_r^*}\right), & \text{if } \kappa \ge h^*, \quad \nu_r^* > 0, \end{cases}$$

where $h^*$, $\mu_l^*$ and $\nu_r^*$ are called the mean value and left and right spreads of positive side of $\widetilde{H}$, respectively. Also L and R both are decreasing functions from positive real numbers $\Re^+$ to the interval [0,1] such that the following conditions hold:

(1) $L(\kappa) < 1$, for $\kappa > 0$,
(2) $L(\kappa) > 0$, for $\kappa < 1$,
(3) $L(0) = 1, L(1) = 0$.

and,

$$\sigma_{\widetilde{H}^\bullet}(\kappa) = \begin{cases} L\left(\frac{h^\bullet-\kappa}{\mu_l^\bullet}\right), & \text{if } \kappa \le h^\bullet, \quad \mu_l^\bullet > 0, \\ R\left(\frac{\kappa-h^\bullet}{\nu_r^\bullet}\right), & \text{if } \kappa \ge h^\bullet, \quad \nu_r^\bullet > 0, \end{cases}$$

where $h^\bullet$, $\mu_l^\bullet$ and $\nu_r^\bullet$ are called the mean value and left and right spreads of negative side of $\widetilde{H}$, respectively. Also, $L$ and $R$ both are decreasing functions from positive real numbers $\Re^+$ to the interval $[-1, 0]$ such that the following conditions hold:

(1) $L(\kappa) > -1$, for $\kappa < 0$,
(2) $L(\kappa) < 0$, for $\kappa > -1$,
(3) $L(0) = -1, L(1) = 0$.

If the two functions $L$ and $R$ for the positive($*$) part are in the form,

$$T(\kappa) = \begin{cases} 1-\kappa, & \text{if } 0 \le \kappa \le 1, \\ 0, & \text{Otherwise}. \end{cases}$$

for the negative($\bullet$) part

$$T(\kappa) = \begin{cases} -1-\kappa, & \text{if } -1 \le \kappa \le 0, \\ 0, & \text{Otherwise}, \end{cases}$$

then the $LR$-bipolar fuzzy number (BFN) is called triangular $LR$-bipolar fuzzy number.

$\widetilde{H} = \prec [h^*, \mu_l^*, \nu_r^*]_{LR}, [h^\bullet, \mu_l^\bullet, \nu_r^\bullet]_{LR} \succ$ is called symmetric $LR$-bipolar fuzzy number if and only if $\mu_l^* = \nu_r^*$ and $\mu_l^\bullet = \nu_r^\bullet$.

For example, $\widetilde{H} = \prec [h^*, \mu_l^*, \nu_r^*]_{LR}, [h^\bullet, \mu_l^\bullet, \nu_r^\bullet]_{LR} \succ = \prec [7, 2, 3]_{LR}, [5, 3, 2]_{LR} \succ$ is $LR$-bipolar triangular fuzzy number and $\widetilde{G} = \prec [g^*, \alpha_l^*, \beta_r^*]_{LR}, [g^\bullet, \alpha_l^\bullet, \beta_r^\bullet]_{LR} \succ = \prec [5, 2, 2]_{LR}, [1, 2, 2]_{LR} \succ$ is a symmetric $LR$-bipolar triangular fuzzy number.

**Definition 6** For arbitrary two $LR$-bipolar fuzzy numbers $\widetilde{H} = \prec [h^*, \mu_l^*, \nu_r^*]_{LR}, [h^\bullet, \mu_l^\bullet, \nu_r^\bullet]_{LR} \succ$ and $\widetilde{G} = \prec [g^*, \alpha_l^*, \beta_r^*]_{LR}, [g^\bullet, \alpha_l^\bullet, \beta_r^\bullet]_{LR} \succ$ and c is arbitrary scalar number, we define addition, subtraction, product and scalar multiplication by c as follows:

(i) Addition:

$$\begin{aligned} \widetilde{H} \oplus \widetilde{G} &= \prec [h^*, \mu_l^*, \nu_r^*]_{LR}, [h^\bullet, \mu_l^\bullet, \nu_r^\bullet]_{LR} \succ \\ &\quad \oplus \prec [g^*, \alpha_l^*, \beta_r^*]_{LR}, [g^\bullet, \alpha_l^\bullet, \beta_r^\bullet]_{LR} \succ, \\ &= \prec [h^* + g^*, \mu_l^* + \alpha_l^*, \nu_r^* + \beta_r^*]_{LR}, \\ &\quad [h^\bullet + g^\bullet, \mu_l^\bullet + \alpha_l^\bullet, \nu_r^\bullet + \beta_r^\bullet]_{LR} \succ, \end{aligned}$$

(ii) Subtraction:

$$\begin{aligned} \widetilde{H} - \widetilde{G} &= \prec [h^*, \mu_l^*, \nu_r^*]_{LR}, [h^\bullet, \mu_l^\bullet, \nu_r^\bullet]_{LR} \succ \\ &\quad - \prec [g^*, \alpha_l^*, \beta_r^*]_{LR}, [g^\bullet, \alpha_l^\bullet, \beta_r^\bullet]_{LR} \succ, \\ &= \prec [h^* - g^*, \mu_l^* - \beta_r^*, \nu_r^* - \alpha_l^*]_{LR}, \\ &\quad [h^\bullet - g^\bullet, \mu_l^\bullet - \beta_r^\bullet, \nu_r^\bullet - \alpha_l^\bullet]_{LR} \succ, \end{aligned}$$

(iii) Multiplication:
$If\, \widetilde{H} > 0$ and $\widetilde{G} > 0$, then

$$\begin{aligned} \widetilde{H} \otimes \widetilde{G} &= \prec [h^*, \mu_l^*, \nu_r^*]_{LR}, [h^\bullet, \mu_l^\bullet, \nu_r^\bullet]_{LR} \succ \\ &\quad \otimes \prec [g^*, \alpha_l^*, \beta_r^*]_{LR}, [g^\bullet, \alpha_l^\bullet, \beta_r^\bullet]_{LR} \succ, \\ &\cong \prec [h^* g^*, h^*\alpha_l^* + g^*\mu_l^*, h^*\beta_r^* + g^*\nu_r^*]_{LR}, \\ &\quad [h^\bullet g^\bullet, h^\bullet\alpha_l^\bullet + g^\bullet\mu_l^\bullet, h^\bullet\beta_r^\bullet + g^\bullet\nu_r^\bullet]_{LR} \succ, \end{aligned}$$

(iv) Scalar multiplication:

$$\begin{aligned} c \otimes \widetilde{H} &= c \otimes \prec [h^*, \mu_l^*, \nu_r^*]_{LR}, [h^\bullet, \mu_l^\bullet, \nu_r^\bullet]_{LR} \succ, \\ &\cong \begin{cases} \prec [ch^*, c\mu_l^*, c\nu_r^*]_{LR}, [ch^\bullet, c\mu_l^\bullet, c\nu_r^\bullet]_{LR} \succ, & c \ge 0, \\ \prec [ch^*, -c\nu_r^*, -c\mu_l^*]_{LR}, [ch^\bullet, -c\nu_r^\bullet, -c\mu_l^\bullet]_{LR} \succ, & c < 0. \end{cases} \end{aligned}$$

**Definition 7** Jha (2018) Let $\zeta$ be a complex set. Then, the fuzzy subset $\widetilde{Z}$ is called fuzzy complex set of $\zeta$ is defined by the membership function $\mu_{\widetilde{Z}} : \zeta \to [0, 1]$.

**Definition 8** Buckley (1989) A fuzzy complex number(FCN) $\widetilde{H}$ is a fuzzy complex set with its membership function $£(h|\widetilde{H})$ is a mapping $£ : C \to [0, 1]$, which may be represented as $\widetilde{H} = u + iv$, where $u = (\underline{u}(r), \overline{u}(r))$ and $v = (\underline{v}(r), \overline{v}(r))$, for all $0 \le r \le 1$.

**Definition 9** Let $\zeta$ be a complex set. Then, the bipolar fuzzy subset $\widetilde{Z}$ is called bipolar fuzzy complex set of $\zeta$ is defined by the membership function $\mu_{\widetilde{Z}} : \zeta \to [-1, 0] \times [0, 1]$.

**Definition 10** A bipolar fuzzy complex number (BFCN) $\widetilde{H}$ is bipolar fuzzy complex set with its membership function $£(h|\widetilde{H})$ is a mapping $£ : C \to [-1, 0] \times [0, 1]$, which may be represented as $\widetilde{H} = u + iv$, where $u = \prec [\underline{u}^*(r), \overline{u}^*(r)], [\underline{u}^\bullet(s), \overline{u}^\bullet(s)] \succ$ and

$v = \prec [\underline{v}^*(r), \overline{v}^*(r)], [\underline{v}^\bullet(s), \overline{v}^\bullet(s)] \succ$,

for all $0 \le r \le 1$ and $-1 \le s \le 0$.

**Definition 11** For any two arbitrary BFCN $\widetilde{h}_1 = u_1 + iv_1$ and $\widetilde{h}_2 = u_2 + iv_2$ (where $u_1$, $u_2$, $v_1$ and $v_2$ are BFN), the bipolar fuzzy complex arithmetic for addition and multiplication is given as:

(i) $\widetilde{h}_1 + \widetilde{h}_2 = (u_1 + u_2) + i(v_1 + v_2)$,
(ii) $\widetilde{h}_1 \times \widetilde{h}_2 = \{(u_1 \times u_2) - (v_1 \times v_2)\} + i\{(u_1 \times v_2) + (v_1 \times u_2)\}$.

**Definition 12** An $LR$-BFCN should be represented as $\widetilde{h} = \widetilde{u} + i\widetilde{v}$, where $\widetilde{u} = \prec [u^*, u_l^*, u_r^*]_{LR}, [u^\bullet, u_l^\bullet, u_r^\bullet]_{LR} \succ$ and $\widetilde{v} = \prec [v^*, v_l^*, v_r^*]_{LR}, [v^\bullet, v_l^\bullet, v_r^\bullet]_{LR} \succ$. Further, $\widetilde{h}$ can be written as $\prec [u^*, u_l^*, u_r^*]_{LR}, [u^\bullet, u_l^\bullet, u_r^\bullet]_{LR} \succ + \mathrm{i} \prec [v^*, v_l^*, v_r^*]_{LR}, [v^\bullet, v_l^\bullet, v_r^\bullet]_{LR} \succ$.

**Definition 13** For any two arbitrary $LR$-BFCN $\widetilde{h}_1 = \widetilde{u}_1 + i\widetilde{v}_1$ and $\widetilde{h}_2 = \widetilde{u}_2 + i\widetilde{v}_2$ (where $\widetilde{u}_1$, $\widetilde{u}_2$, $\widetilde{v}_1$ and $\widetilde{v}_2$ are $LR$-BFCN), the LR bipolar fuzzy complex arithmetic for addition and multiplication is given as:

(i) $\widetilde{h}_1 + \widetilde{h}_2 = (\widetilde{u}_1 + \widetilde{u}_2) + i(\widetilde{v}_1 + \widetilde{v}_2)$,
(ii) $c\widetilde{h} = c\widetilde{u} + ic\widetilde{v}, \quad c \in \Re$,
(iii) $\widetilde{h}_1 \times \widetilde{h}_2 = \{(\widetilde{u}_1 \times \widetilde{u}_2) - (\widetilde{v}_1 \times \widetilde{v}_2)\} + i\{(\widetilde{u}_1 \times \widetilde{v}_2) + (\widetilde{v}_1 \times \widetilde{u}_2)\}$.

**Definition 14** Consider the $n \times n$ Linear system of equations is given by:

$$\begin{cases} l_{11}h_1 + l_{12}h_2 + \cdots + l_{1n}h_n &= z_1, \\ l_{21}h_1 + l_{22}h_2 + \cdots + l_{2n}h_n &= z_2, \\ . \\ . \\ . \\ l_{n1}h_1 + l_{n2}h_2 + \cdots + l_{nn}h_n &= z_n, \end{cases} \tag{1}$$

where the coefficient elements $(l_{pq})$, $1 \le p, q \le n$ is a real matrix of order $n \times n$ and each $z_q$, $1 \le q \le n$ is a $LR$ form of bipolar fuzzy number and each unknown $h_p$, $1 \le p \le n$ is also $LR$ form of bipolar fuzzy number, is known as the $LR$-bipolar fuzzy linear system of equations ($LR$-BFLS).

The $LR$-BFLS of equations given in (1) can be written in matrix form as:

$$L\widetilde{H} = \widetilde{Z}, \tag{2}$$

we can rewrite the above-mentioned $LR$-BFLS of Eqs. (2) as follows:

$$\begin{aligned} &L \prec [h^*, \mu_l^*, \nu_r^*]_{LR}, [h^\bullet, \mu_l^\bullet, \nu_r^\bullet]_{LR} \succ \\ &\quad = \prec [z^*, \alpha_l^*, \beta_r^*]_{LR}, [z^\bullet, \alpha_l^\bullet, \beta_r^\bullet]_{LR} \succ . \end{aligned} \tag{3}$$

We separate the positive($*$) and negative($\bullet$) parts of Eq. (3) to obtain two $LR$-real linear systems:

$$\begin{aligned} L(h^*, \mu_l^*, \nu_r^*)_{LR} &= (z^*, \alpha_l^*, \beta_r^*)_{LR}, \\ L(h^\bullet, \mu_l^\bullet, \nu_r^\bullet)_{LR} &= (z^\bullet, \alpha_l^\bullet, \beta_r^\bullet)_{LR}. \end{aligned} \tag{4}$$

**Definition 15** A $LR$-BFNs vector $H = (\widetilde{h}_1, \widetilde{h}_2, \ldots . \widetilde{h}_n)^T$ is the form of $H_q = \prec [h_q^*, \mu_{lq}^*, \nu_{rq}^*]_{LR}, [h_q^\bullet, \mu_{lq}^\bullet, \nu_{rq}^\bullet]_{LR} \succ$ where $1 \le q \le n$ is called the solution of $LR$-BFLS of Eq. (1) if:

$$\begin{cases} \sum_{q=1}^{n} l_{pq}h_q^* = z_p^*, \\ \sum_{q=1}^{n} l_{pq}\mu_{lq}^* = \alpha_{lp}^*, \\ \sum_{q=1}^{n} l_{pq}\nu_{rq}^* = \beta_{rp}^*, \\ \sum_{q=1}^{n} l_{pq}h_q^\bullet = z_p^\bullet, \\ \sum_{q=1}^{n} l_{pq}\mu_{lq}^\bullet = \alpha_{lp}^\bullet, \\ \sum_{q=1}^{n} l_{pq}\nu_{rq}^\bullet = \beta_{rp}^\bullet. \end{cases} \tag{5}$$

**Definition 16** For any $LR$-BFLS $L\widetilde{H} = \widetilde{Z}$, we define a matrix $A$ obtained from non-negative entries of $L$ and defined a matrix $B$ obtained from absolute of the negative entries of $L$. Then, $L = A - B$ and we define $L^+ = A + B$.

**Definition 17** Abbasbandy and Alavi (2005) A square matrix is known as permutation matrix in which each row and column contain exactly one unit element and all other entries in each row and column are zero.

**Definition 18** Abbasbandy and Alavi (2005) A matrix $L$ is known as absolutely permutation matrix if absolute of the entries of matrix $L$, i.e., $(L^{+})$ is a permutation matrix.

***Remark 1*** $L^{-1}$ is an absolutely permutation matrix if $L$ is an absolutely permutation matrix and $LL^{T}=I$.

**Theorem 1** Minc (1988) *The inverse of non-negative matrix $L$ is non-negative if $L$ is a permutation matrix.*

## 3 Solution procedure for n × n *LR*-BFLS of equations

To solve the $LR$-BFLS of equations in (1), first we separate the positive($*$) and negative($\bullet$) parts of system of equations to get two $n\times n$ $LR$-real linear systems given as

$$L(h^{*},\mu_{l}^{*},\nu_{r}^{*})_{LR}=(z^{*},\alpha_{l}^{*},\beta_{r}^{*})_{LR}, \tag{6}$$
$$L(h^{\bullet},\mu_{l}^{\bullet},\nu_{r}^{\bullet})_{LR}=(z^{\bullet},\alpha_{l}^{\bullet},\beta_{r}^{\bullet})_{LR}. \tag{7}$$

To solve positive part of $LR$-BFLS of equations $L(h^{*},\mu_{l}^{*},\nu_{r}^{*})_{LR}=(z^{*},\alpha_{l}^{*},\beta_{r}^{*})_{LR}$, we resolve this system into two real linear systems, one is $n\times n$ mean value system and other is $2n\times 2n$ left-right spread system.

The $n\times n$ mean value system for the positive part of (1) is given as:

$$\begin{cases} l_{11}h_1^{*}+l_{12}h_2^{*}+\cdots+l_{1n}h_n^{*} &= z_1^{*},\\ l_{21}h_1^{*}+l_{22}h_2^{*}+\cdots+l_{2n}h_n^{*} &= z_2^{*},\\ . \\ . \\ . \\ l_{n1}h_1^{*}+l_{n2}h_2^{*}+\cdots+l_{nn}h_n^{*} &= z_n^{*}. \end{cases} \tag{8}$$

The matrix notation of system (8) is given as

$$LH_q^{*}=Z_p^{*},$$

where $1\le p,q\le n$.

To solve left-right spread system of Eq. (1), we must solve the $2n\times 2n$ real linear system and where the right side column is function vector $(\alpha_{l1}^{*},\alpha_{l2}^{*},\ldots,\alpha_{ln}^{*},\beta_{r1}^{*},\beta_{r2}^{*},\ldots,\beta_{rn}^{*})^{T}$. Now rearrange the left-right spread system, so the unknown parameters are $\mu_{lq}^{*},\nu_{rq}^{*}$, $1\le q\le n$ and column of right side vector are

$$(\alpha_{l1}^{*},\alpha_{l2}^{*},\ldots,\alpha_{ln}^{*},\beta_{r1}^{*},\beta_{r2}^{*},\ldots,\beta_{rn}^{*})^{T}.$$

We get first $2n\times 2n$ left-right spread system of positive($*$) part of $LR$-BFLS of equations,

$$\begin{cases} t_{11}\mu_{l1}^{*}+t_{12}\mu_{l2}^{*}+\cdots+t_{1n}\mu_{ln}^{*}+t_{1,n+1}(-\nu_{r1}^{*})+t_{1,n+2}(-\nu_{r2}^{*})\\ \quad+\cdots+t_{1,2n}(-\nu_{rn}^{*})=\alpha_{l1}^{*},\\ . \\ . \\ . \\ t_{n1}\mu_{l1}^{*}+t_{n2}\mu_{l2}^{*}+\cdots+t_{nn}\mu_{ln}^{*}+t_{n,n+1}(-\nu_{r1}^{*})+t_{n,n+2}(-\nu_{r2}^{*})\\ \quad+\cdots+t_{n,2n}(-\nu_{rn}^{*})=\alpha_{ln}^{*},\\ t_{n+1,1}\mu_{l1}^{*}+t_{n+1,2}\mu_{l2}^{*}+\cdots+t_{n+1,n}\mu_{ln}^{*}+t_{n+1,n+1}(-\nu_{r1}^{*})\\ \quad+t_{n+1,n+2}(-\nu_{r2}^{*})+\cdots+t_{n+1,2n}(-\nu_{rn}^{*})=\beta_{r1}^{*},\\ . \\ . \\ . \\ t_{2n,1}\mu_{l1}^{*}+t_{2n,2}\mu_{l2}^{*}+\cdots+t_{2n,n}\mu_{ln}^{*}+t_{2n,n+1}(-\nu_{r1}^{*})+t_{2n,n+2}\\ \quad(-\nu_{r2}^{*})+\cdots+t_{2n,2n}(-\nu_{rn}^{*})=\beta_{rn}^{*}, \end{cases}$$

where $l_{pq}$ are determined as follows:

$$\begin{aligned} &l_{pq}\ge 0\Longrightarrow t_{pq}=l_{pq},\, t_{p+n,q+n}=l_{pq},\\ &l_{pq}<0\Longrightarrow t_{p,q+n}=-l_{pq},\, t_{p+n,q}=-l_{pq}, \end{aligned} \tag{9}$$

where $1\le p,q\le 2n$ and any $t_{pq}$ which is not determined from Eq. (9) is zero.

Using the matrix notation, we obtain

$$M(\mu_l^{*},\nu_r^{*})=(\alpha_l^{*},\beta_r^{*}), \tag{10}$$

where $M=l_{pq}$, $1\le p,q\le 2n$

$$(\mu_{l1}^{*},\mu_{l2}^{*},\ldots,\mu_{ln}^{*},\nu_{r1}^{*},\nu_{r2}^{*},\ldots,\nu_{rn}^{*})^{T}\ \textit{and}\\ (\alpha_{l1}^{*},\alpha_{l2}^{*},\ldots,\alpha_{ln}^{*},\beta_{r1}^{*},\beta_{r2}^{*},\ldots,\beta_{rn}^{*})^{T}.$$

We can also write the matrix form of Eq. (10) as:

$$\begin{pmatrix} A & -B\\ -B & A\end{pmatrix}\begin{pmatrix}\mu_l^{*}\\ \nu_r^{*}\end{pmatrix}=\begin{pmatrix}\alpha_l^{*}\\ \beta_r^{*}\end{pmatrix}. \tag{11}$$

In the similar way we can take the negative($\bullet$) part of the $LR$-BFLS of equations given in (1).

To solve the negative part of $LR$-BFLS of equation $L(h^{\bullet},\mu_l^{\bullet},\nu_r^{\bullet})_{LR}=(z^{\bullet},\alpha_l^{\bullet},\beta_r^{\bullet})_{LR}$, we resolve this system into two real linear systems, one is $n\times n$ mean value system and other is $2n\times 2n$ left-right spread system for negative part.

The $n\times n$ mean value system for the negative part of (1) is given as:

$$\begin{cases} l_{11}h_1^{\bullet}+l_{12}h_2^{\bullet}+\cdots+l_{1n}h_n^{\bullet} &= z_1^{\bullet},\\ l_{21}h_1^{\bullet}+l_{22}h_2^{\bullet}+\cdots+l_{2n}h_n^{\bullet} &= z_2^{\bullet},\\ . \\ . \\ . \\ l_{n1}h_1^{\bullet}+l_{n2}h_2^{\bullet}+\cdots+l_{nn}h_n^{\bullet} &= z_n^{\bullet}. \end{cases} \tag{12}$$

The matrix notation of system (12) is given as

$$LH_q^\bullet = Z_p^\bullet,$$

where $1 \le p, q \le n$.

To solve negative part of left-right spread system of Eq. (1), we must solve the $2n \times 2n$ real linear system and where the right side column is function vector $(\alpha_{l1}^\bullet, \alpha_{l2}^\bullet, \ldots, \alpha_{ln}^\bullet, \beta_{r1}^\bullet, \beta_{r2}^\bullet, \ldots, \beta_{rn}^\bullet)^T$. Now rearrange the left-right spread system, so the unknown parameters are $\mu_{lq}^\bullet, \nu_{rq}^\bullet$, $1 \le q \le n$ and column of right side vector are

$$(\alpha_{l1}^\bullet, \alpha_{l2}^\bullet, \ldots, \alpha_{ln}^\bullet, \beta_{r1}^\bullet, \beta_{r2}^\bullet, \ldots, \beta_{rn}^\bullet)^T.$$

We get first $2n \times 2n$ left-right spread system of negative($\bullet$) part of $LR$-BFLS of equations,

$$\begin{cases} t_{11}\mu_{l1}^\bullet + t_{12}\mu_{l2}^\bullet + \cdots + t_{1n}\mu_{ln}^\bullet + t_{1,n+1}(-\nu_{r1}^\bullet) + t_{1,n+2}(-\nu_{r2}^\bullet) \\ \quad + \cdots + t_{1,2n}(-\nu_{rn}^\bullet) = \alpha_{l1}^\bullet, \\ . \\ . \\ . \\ t_{n1}\mu_{l1}^\bullet + t_{n2}\mu_{l2}^\bullet + \cdots + t_{nn}\mu_{ln}^\bullet + t_{n,n+1}(-\nu_{r1}^\bullet) + t_{n,n+2}(-\nu_{r2}^\bullet) \\ \quad + \cdots + t_{n,2n}(-\nu_{rn}^\bullet) = \alpha_{ln}^\bullet, \\ t_{n+1,1}\mu_{l1}^\bullet + t_{n+1,2}\mu_{l2}^\bullet + \cdots + t_{n+1,n}\mu_{ln}^\bullet + t_{n+1,n+1}(-\nu_{r1}^\bullet) \\ \quad + t_{n+1,n+2}(-\nu_{r2}^\bullet) + \cdots + t_{n+1,2n}(-\nu_{rn}^\bullet) = \beta_{r1}^\bullet, \\ . \\ . \\ . \\ t_{2n,1}\mu_{l1}^\bullet + t_{2n,2}\mu_{l2}^\bullet + \cdots + t_{2n,n}\mu_{ln}^\bullet + t_{2n,n+1}(-\nu_{r1}^\bullet) + t_{2n,n+2} \\ \quad (-\nu_{r2}^\bullet) + \cdots + t_{2n,2n}(-\nu_{rn}^\bullet) = \beta_{rn}^\bullet, \end{cases}$$

where $l_{pq}$ are determined as follows:

$$\begin{aligned} l_{pq} \ge 0 &\Longrightarrow t_{pq} = l_{pq}, t_{p+n,q+n} = l_{pq}, \\ l_{pq} < 0 &\Longrightarrow t_{p,q+n} = -l_{pq}, t_{p+n,q} = -l_{pq}, \end{aligned} \tag{13}$$

where $1 \le p, q \le 2n$ and any $t_{pq}$ which is not determined from Eq. (13) is zero.

Using the matrix notation, we obtain

$$M(\mu_l^\bullet, \nu_r^\bullet) = (\alpha_l^\bullet, \beta_r^\bullet), \tag{14}$$

where $M = l_{pq}$, $1 \le p, q \le 2n$

$$(\mu_{l1}^\bullet, \mu_{l2}^\bullet, \ldots, \mu_{ln}^\bullet, \nu_{r1}^\bullet, \nu_{r2}^\bullet, \ldots, \nu_{rn}^\bullet)^T \textit{and} \\ (\alpha_{l1}^\bullet, \alpha_{l2}^\bullet, \ldots, \alpha_{ln}^\bullet, \beta_{r1}^\bullet, \beta_{r2}^\bullet, \ldots, \beta_{rn}^\bullet)^T.$$

We can also write the matrix form of Eq. (14) as:

$$\begin{pmatrix} A & -B \\ -B & A \end{pmatrix} \begin{pmatrix} \mu_l^\bullet \\ \nu_r^\bullet \end{pmatrix} = \begin{pmatrix} \alpha_l^\bullet \\ \beta_r^\bullet \end{pmatrix}. \tag{15}$$

To find the solution of $LR$-BFLS of Eq. (1), we can find the solution of Eqs. (8), (11), (12) and (15) by using matrix inverse method.

**Theorem 2** *For any arbitrary LR bipolar fuzzy vector Z the unique solution $\widetilde{H}$ is a LR bipolar fuzzy vector of $LR$-BFLS of equations in (1) if and only if the matrix M=* $\begin{pmatrix} A & -B \\ -B & A \end{pmatrix}$ *and matrix L is non-singular, the necessary and sufficient condition that the matrices $(A+B)^{-1}$ and $(A-B)^{-1}$ exist.*

***Example 1*** Consider the $2 \times 2$ non-symmetric $LR$-BFLS of equations

$$\begin{aligned} \widetilde{h}_1 - \widetilde{h}_2 &= \prec [1, 1, 1]_{LR}, [-2, 2, 2]_{LR} \succ, \\ \widetilde{h}_1 + 3\widetilde{h}_2 &= \prec [5, 1, 2]_{LR}, [4, 2, 3]_{LR} \succ . \end{aligned}$$

To solve the above $LR$-BFLS equation first we put $\widetilde{h}_1 = \prec [h_1^*, \mu_{l1}^*, \nu_{r1}^*]_{LR}, [h_1^\bullet, \mu_{l1}^\bullet, \nu_{r1}^\bullet]_{LR} \succ$ and $\widetilde{h}_2 = \prec [h_2^*, \mu_{l2}^*, \nu_{r2}^*]_{LR}, [h_2^\bullet, \mu_{l2}^\bullet, \nu_{r2}^\bullet]_{LR} \succ$ in original system and get,

$$\begin{aligned} &\prec [h_1^*, \mu_{l1}^*, \nu_{r1}^*]_{LR}, [h_1^\bullet, \mu_{l1}^\bullet, \nu_{r1}^\bullet]_{LR} \succ \\ &\quad - \prec [h_2^*, \mu_{l2}^*, \nu_{r2}^*]_{LR}, [h_2^\bullet, \mu_{l2}^\bullet, \nu_{r2}^\bullet]_{LR} \succ \\ &\quad = \prec [1, 1, 1]_{LR}, [-2, 2, 2]_{LR} \succ, \\ &\prec [h_1^*, \mu_{l1}^*, \nu_{r1}^*]_{LR}, [h_1^\bullet, \mu_{l1}^\bullet, \nu_{r1}^\bullet]_{LR} \succ \\ &\quad +3 \prec [h_2^*, \mu_{l2}^*, \nu_{r2}^*]_{LR}, [h_2^\bullet, \mu_{l2}^\bullet, \nu_{r2}^\bullet]_{LR} \succ \\ &\quad = \prec [5, 1, 2]_{LR}, [4, 2, 3]_{LR} \succ . \end{aligned}$$

To solve the $LR$-BFLS of equation in (1), first we take the positive($*$) part of the system (1) given as:

$$\begin{aligned} (h_1^*, \mu_{l1}^*, \nu_{r1}^*)_{LR} - (h_2^*, \mu_{l2}^*, \nu_{r2}^*)_{LR} &= (1, 1, 1)_{LR}, \\ (h_1^*, \mu_{l1}^*, \nu_{r1}^*)_{LR} + 3(h_2^*, \mu_{l2}^*, \nu_{r2}^*)_{LR} &= (5, 1, 2)_{LR}. \end{aligned}$$

To solve this positive part of $LR$-BFLS of equations we can express this system into two systems, one is mean value system and other is left-right spread system.

The mean value system for positive part is given as:

$$\begin{aligned} h_1^* - h_2^* &= 1, \\ h_1^* + 3h_2^* &= 5, \end{aligned}$$

by solving above equations, we have $h_1^* = 2$ and $h_2^* = 1$. Now the extended $4 \times 4$ form of positive part of left-right spread is given as:

$$\begin{aligned} 1(\mu_{l1}^*) + 0(\mu_{l2}^*) + 0(\nu_{r1}^*) + 1(\nu_{r2}^*) &= 1, \\ 1(\mu_{l1}^*) + 3(\mu_{l2}^*) + 0(\nu_{r1}^*) + 0(\nu_{r2}^*) &= 1, \\ 0(\mu_{l1}^*) + 1(\mu_{l2}^*) + 1(\nu_{r1}^*) + 0(\nu_{r2}^*) &= 1, \\ 0(\mu_{l1}^*) + 0(\mu_{l2}^*) + 1(\nu_{r1}^*) + 3(\nu_{r2}^*) &= 2. \end{aligned}$$

The matrix form of the above-mentioned system is

$$\begin{pmatrix}1&0&0&1\\1&3&0&0\\0&1&1&0\\0&0&1&3\end{pmatrix}\begin{pmatrix}\mu^*_{l1}\\\mu^*_{l2}\\\nu^*_{r1}\\\nu^*_{r2}\end{pmatrix}=\begin{pmatrix}1\\1\\1\\2\end{pmatrix},$$

this can also be written as:

$$M(\mu^*_l,\nu^*_r)=(\alpha^*_l,\beta^*_r),$$

and the solution of above-mentioned system is,

$$\begin{pmatrix}\mu^*_{l1}\\\mu^*_{l2}\\\nu^*_{r1}\\\nu^*_{r2}\end{pmatrix}=\begin{pmatrix}1.1250&-0.1250&0.3750&-0.3750\\-0.3750&0.3750&-0.1250&0.1250\\0.3750&-0.3750&1.1250&-0.1250\\-0.1250&0.1250&-0.3750&0.3750\end{pmatrix}\begin{pmatrix}1\\1\\1\\2\end{pmatrix}$$

$$\mu^*_{l1}=0.6250,\quad \mu^*_{l2}=0.1250,$$
$$\nu^*_{r1}=0.8750,\quad \nu^*_{r2}=0.3750.$$

Similarly, we can take the negative($\bullet$) part of the system (1) given as:

$$(h^\bullet_1,\mu^\bullet_{l1},\nu^\bullet_{r1})_{LR}-(h^\bullet_2,\mu^\bullet_{l2},\nu^\bullet_{r2})_{LR}=(-2,2,2)_{LR},$$
$$(h^\bullet_1,\mu^\bullet_{l1},\nu^\bullet_{r1})_{LR}+3(h^*_2,\mu^\bullet_{l2},\nu^\bullet_{r2})_{LR}=(4,2,3)_{LR}.$$

To solve this negative part of $LR$-BFLS of equations we can express this system into two systems, one is mean value system and other is left-right spread system.

The mean value system for negative part is given as:

$$h^\bullet_1-h^\bullet_2=-2,$$
$$h^\bullet_1+3h^\bullet_2=4,$$

by solving above equations, we have $h^\bullet_1=\frac{-1}{2}$ and $h^\bullet_2=\frac{3}{2}$. Now the extended $4\times 4$ form of negative part of left-right spread is given as:

$$1(\mu^\bullet_{l1})+0(\mu^\bullet_{l2})+0(\nu^\bullet_{r1})+1(\nu^\bullet_{r2})=2,$$
$$1(\mu^\bullet_{l1})+3(\mu^\bullet_{l2})+0(\nu^\bullet_{r1})+0(\nu^\bullet_{r2})=2,$$
$$0(\mu^\bullet_{l1})+1(\mu^\bullet_{l2})+1(\nu^\bullet_{r1})+0(\nu^\bullet_{r2})=2,$$
$$0(\mu^\bullet_{l1})+0(\mu^\bullet_{l2})+1(\nu^\bullet_{r1})+3(\nu^\bullet_{r2})=3.$$

The matrix form of the above-mentioned system is

$$\begin{pmatrix}1&0&0&1\\1&3&0&0\\0&1&1&0\\0&0&1&3\end{pmatrix}\begin{pmatrix}\mu^\bullet_{l1}\\\mu^\bullet_{l2}\\\nu^\bullet_{r1}\\\nu^\bullet_{r2}\end{pmatrix}=\begin{pmatrix}2\\2\\2\\3\end{pmatrix},$$

this can also be written as:

$$M(\mu^\bullet_l,\nu^\bullet_r)=(\alpha^\bullet_l,\beta^\bullet_r),$$

and the solution of above system is,

$$\begin{pmatrix}\mu^\bullet_{l1}\\\mu^\bullet_{l2}\\\nu^\bullet_{r1}\\\nu^\bullet_{r2}\end{pmatrix}=\begin{pmatrix}1.1250&-0.1250&0.3750&-0.3750\\-0.3750&0.3750&-0.1250&0.1250\\0.3750&-0.3750&1.1250&-0.1250\\-0.1250&0.1250&-0.3750&0.3750\end{pmatrix}\begin{pmatrix}2\\2\\2\\3\end{pmatrix}$$

$$\mu^\bullet_{l1}=1.6250,\quad \mu^\bullet_{l2}=0.1250,$$
$$\nu^\bullet_{r1}=1.8750,\quad \nu^\bullet_{r2}=0.3750.$$

Hence, the solution of $LR$-BFLS of equation in (1) is given as:

$$\widetilde{h}_1=\prec[h^*_1,\mu^*_{l1},\nu^*_{r1}]_{LR},[h^\bullet_1,\mu^\bullet_{l1},\nu^\bullet_{r1}]_{LR}\succ$$
$$=\prec[2,0.625,0.875]_{LR},\left[\frac{-1}{2},1.6250,1.8750\right]_{LR}\succ,$$
$$\widetilde{h}_2=\prec[h^*_2,\mu^*_{l2},\nu^*_{r2}]_{LR},[h^\bullet_2,\mu^\bullet_{l2},\nu^\bullet_{r2}]_{LR}\succ$$
$$=\prec[1,0.1250,0.3750]_{LR},\left[\frac{3}{2},0.1250,0.3750\right]_{LR}\succ.$$

***Example 2*** Consider the $2\times 2$ symmetric $LR$-BFLS of equations

$$\widetilde{h}_1-\widetilde{h}_2=\prec[1,1,1]_{LR},[2,3,3]_{LR}\succ,$$
$$\widetilde{h}_1+3\widetilde{h}_2=\prec[6,2,2]_{LR},[-2,7,7]_{LR}\succ.$$

To solve the above $LR$-BFLS of equations first we put $\widetilde{h}_1=\prec[h^*_1,\mu^*_{l1},\nu^*_{r1}]_{LR},[h^\bullet_1,\mu^\bullet_{l1},\nu^\bullet_{r1}]_{LR}\succ$ and $\widetilde{h}_2=\prec[h^*_2,\mu^*_{l2},\nu^*_{r2}]_{LR},[h^\bullet_2,\mu^\bullet_{l2},\nu^\bullet_{r2}]_{LR}\succ$ in original system and get,

$$\prec[h^*_1,\mu^*_{l1},\nu^*_{r1}]_{LR},[h^\bullet_1,\mu^\bullet_{l1},\nu^\bullet_{r1}]_{LR}\succ$$
$$-\prec[h^*_2,\mu^*_{l2},\nu^*_{r2}]_{LR},[h^\bullet_2,\mu^\bullet_{l2},\nu^\bullet_{r2}]_{LR}\succ$$
$$=\prec[1,1,1]_{LR},[2,3,3]_{LR}\succ,$$
$$\prec[h^*_1,\mu^*_{l1},\nu^*_{r1}]_{LR},[h^\bullet_1,\mu^\bullet_{l1},\nu^\bullet_{r1}]_{LR}\succ$$
$$+3\prec[h^*_2,\mu^*_{l2},\nu^*_{r2}]_{LR},[h^\bullet_2,\mu^\bullet_{l2},\nu^\bullet_{r2}]_{LR}\succ$$
$$=\prec[6,2,2]_{LR},[-2,7,7]_{LR}\succ.$$

To solve the $LR$-BFLS of equations in (2), first we take the positive($*$) part of the system (2) given as:

$$(h_1^*, \mu_{l1}^*, \nu_{r1}^*)_{LR} - (h_2^*, \mu_{l2}^*, \nu_{r2}^*)_{LR} = (1, 1, 1)_{LR},$$
$$(h_1^*, \mu_{l1}^*, \nu_{r1}^*)_{LR} + 3(h_2^*, \mu_{l2}^*, \nu_{r2}^*)_{LR} = (6, 2, 2)_{LR}.$$

To solve this positive part of $LR$-BFLS of equation we can restructure this system into two systems, one is mean value system and other is left-right spread system.
The mean value system for positive part is given as:

$$h_1^* - h_2^* = 1,$$
$$h_1^* + 3h_2^* = 6,$$

by solving above equations, we have $h_1^* = \frac{9}{4}$ and $h_2^* = \frac{5}{4}$.
Now the extended $4 \times 4$ form of positive part of left-right spread is given as:

$$1(\mu_{l1}^*) + 0(\mu_{l2}^*) + 0(\nu_{r1}^*) + 1(\nu_{r2}^*) = 1,$$
$$1(\mu_{l1}^*) + 3(\mu_{l2}^*) + 0(\nu_{r1}^*) + 0(\nu_{r2}^*) = 2,$$
$$0(\mu_{l1}^*) + 1(\mu_{l2}^*) + 1(\nu_{r1}^*) + 0(\nu_{r2}^*) = 1,$$
$$0(\mu_{l1}^*) + 0(\mu_{l2}^*) + 1(\nu_{r1}^*) + 3(\nu_{r2}^*) = 2.$$

The matrix form of the above-mentioned system is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} \mu_{l1}^* \\ \mu_{l2}^* \\ \nu_{r1}^* \\ \nu_{r2}^* \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix},$$

this can also be written as:

$$M(\mu_l^*, \nu_r^*) = (\alpha_l^*, \beta_r^*),$$

and the solution of above system is,

$$\begin{pmatrix} \mu_{l1}^* \\ \mu_{l2}^* \\ \nu_{r1}^* \\ \nu_{r2}^* \end{pmatrix} = \begin{pmatrix} 1.1250 & -0.1250 & 0.3750 & -0.3750 \\ -0.3750 & 0.3750 & -0.1250 & 0.1250 \\ 0.3750 & -0.3750 & 1.1250 & -0.1250 \\ -0.1250 & 0.1250 & -0.3750 & 0.3750 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}$$

$$\mu_{l1}^* = 0.5, \quad \mu_{l2}^* = 0.5,$$
$$\nu_{r1}^* = 0.5, \quad \nu_{r2}^* = 0.5.$$

Similarly, we can take the negative($\bullet$) part of the system (2) given as:

$$(h_1^\bullet, \mu_{l1}^\bullet, \nu_{r1}^\bullet)_{LR} - (h_2^\bullet, \mu_{l2}^\bullet, \nu_{r2}^\bullet)_{LR} = (2, 3, 3)_{LR},$$
$$(h_1^\bullet, \mu_{l1}^\bullet, \nu_{r1}^\bullet)_{LR} + 3(h_2^\bullet, \mu_{l2}^\bullet, \nu_{r2}^\bullet)_{LR} = (-2, 7, 7)_{LR}.$$

To solve this negative part of $LR$-BFLS of equations given in (2) we can restructure this system into two systems, one is mean value system and other is left-right spread system.
The mean value system for negative part is given as:

$$h_1^\bullet - h_2^\bullet = 2,$$
$$h_1^\bullet + 3h_2^\bullet = -2,$$

by solving above equations, we have $h_1^\bullet = 1$ and $h_2^\bullet = -1$.
Now the extended $4 \times 4$ form of negative part of left-right spread is given as:

$$1(\mu_{l1}^\bullet) + 0(\mu_{l2}^\bullet) + 0(\nu_{r1}^\bullet) + 1(\nu_{r2}^\bullet) = 3,$$
$$1(\mu_{l1}^\bullet) + 3(\mu_{l2}^\bullet) + 0(\nu_{r1}^\bullet) + 0(\nu_{r2}^\bullet) = 7,$$
$$0(\mu_{l1}^\bullet) + 1(\mu_{l2}^\bullet) + 1(\nu_{r1}^\bullet) + 0(\nu_{r2}^\bullet) = 3,$$
$$0(\mu_{l1}^\bullet) + 0(\mu_{l2}^\bullet) + 1(\nu_{r1}^\bullet) + 3(\nu_{r2}^\bullet) = 7.$$

The matrix form of the above-mentioned system is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} \mu_{l1}^\bullet \\ \mu_{l2}^\bullet \\ \nu_{r1}^\bullet \\ \nu_{r2}^\bullet \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 3 \\ 7 \end{pmatrix},$$

this can also be written as:

$$M(\mu_l^\bullet, \nu_r^\bullet) = (\alpha_l^\bullet, \beta_r^\bullet),$$

and the solution of above system is,

$$\begin{pmatrix} \mu_{l1}^\bullet \\ \mu_{l2}^\bullet \\ \nu_{r1}^\bullet \\ \nu_{r2}^\bullet \end{pmatrix} = \begin{pmatrix} 1.1250 & -0.1250 & 0.3750 & -0.3750 \\ -0.3750 & 0.3750 & -0.1250 & 0.1250 \\ 0.3750 & -0.3750 & 1.1250 & -0.1250 \\ -0.1250 & 0.1250 & -0.3750 & 0.3750 \end{pmatrix} \begin{pmatrix} 3 \\ 7 \\ 3 \\ 7 \end{pmatrix}$$

$$\mu_{l1}^\bullet = 1, \quad \mu_{l2}^\bullet = 2,$$
$$\nu_{r1}^\bullet = 1, \quad \nu_{r2}^\bullet = 2.$$

Hence, the solution of $LR$-BFLS of equation in (2) is symmetric given as:

$$\widetilde{h}_1 = \prec [h_1^*, \mu_{l1}^*, \nu_{r1}^*]_{LR}, [h_1^\bullet, \mu_{l1}^\bullet, \nu_{r1}^\bullet]_{LR} \succ$$

$$= \prec \left[\frac{9}{4}, 0.5, 0.5\right]_{LR}, [1, 1, 1]_{LR} \succ,$$

$$\widetilde{h}_2 = \prec [h_2^*, \mu_{l2}^*, \nu_{r2}^*]_{LR}, [h_2^\bullet, \mu_{l2}^\bullet, \nu_{r2}^\bullet]_{LR} \succ$$

$$= \prec \left[\frac{5}{4}, 0.5, 0.5\right]_{LR}, [-1, 2, 2]_{LR} \succ .$$

### 3.1 Weak *LR*-Bipolar Fuzzy Solution

We restrict our discussion to $LR$-bipolar fuzzy triangular number, i.e., if $\widetilde{h}_p = \prec [h_p^*, \mu_{lp}^*, \nu_{rp}^*]_{LR}, [h_p^\bullet, \mu_{lp}^\bullet, \nu_{rp}^\bullet]_{LR} \succ$ is the unique solution of (1) such that $\mu_{lp}^* > 0$, $\nu_{rp}^* > 0$, $\mu_{lp}^\bullet > 0$ and $\nu_{rp}^\bullet > 0$ then $\widetilde{h}_p = \prec [h_p^*, \mu_{lp}^*, \nu_{rp}^*]_{LR}, [h_p^\bullet, \mu_{lp}^\bullet, \nu_{rp}^\bullet]_{LR} \succ$ is called the strong $LR$-bipolar fuzzy solution of (1). Meanwhile, if one of them from $\mu_{lp}^*, \nu_{rp}^*, \mu_{lp}^\bullet, \nu_{rp}^\bullet$ be negative, then is said to be a weak $LR$-bipolar fuzzy solution of system (1) so the strong solution is given as:

$$\widetilde{h}_p^* = \begin{cases} (h_p^*, \mu_{lp}^*, \nu_{rp}^*)_{LR}, & \mu_{lp}^* > 0, \quad \nu_{rp}^* > 0, \\ (h_p^*, 0, \max\{-\mu_{lp}^*, \nu_{rp}^*\})_{LR}, & \mu_{lp}^* < 0, \quad \nu_{rp}^* > 0, \\ (h_p^*, \max\{\mu_{lp}^*, -\nu_{rp}^*\}, 0)_{LR}, & \mu_{lp}^* > 0, \quad \nu_{rp}^* < 0, \\ (h_p^*, -\nu_{rp}^*, -\mu_{lp}^*)_{LR}, & \mu_{lp}^* < 0, \quad \nu_{rp}^* < 0, \end{cases}$$

$$\widetilde{h}_p^\bullet = \begin{cases} (h_p^\bullet, \mu_{lp}^\bullet, \nu_{rp}^\bullet)_{LR}, & \mu_{lp}^\bullet > 0, \quad \nu_{rp}^\bullet > 0, \\ (h_p^\bullet, 0, \max\{-\mu_{lp}^\bullet, \nu_{rp}^\bullet\})_{LR}, & \mu_{lp}^\bullet < 0, \quad \nu_{rp}^\bullet > 0, \\ (h_p^\bullet, \max\{\mu_{lp}^\bullet, -\nu_{rp}^\bullet\}, 0)_{LR}, & \mu_{lp}^\bullet > 0, \quad \nu_{rp}^\bullet < 0, \\ (h_p^\bullet, -\nu_{rp}^\bullet, -\mu_{lp}^\bullet)_{LR}, & \mu_{lp}^\bullet < 0, \quad \nu_{rp}^\bullet < 0, \end{cases}$$

where $p = 1, 2, 3, \ldots, n$.

***Example 3*** Consider the $2 \times 2$ symmetric $LR$-BFLS of equations

$$\widetilde{h}_1 - \widetilde{h}_2 = \prec [-3, 2, 2]_{LR}, [-5, 1, 1]_{LR} \succ,$$
$$\widetilde{h}_1 - 2\widetilde{h}_2 = \prec [7, 5, 5]_{LR}, [12, 5, 5]_{LR} \succ .$$

To solve the above $LR$-BFLS of equations first we put $\widetilde{h}_1 = \prec [h_1^*, \mu_{l1}^*, \nu_{r1}^*]_{LR}, [h_1^\bullet, \mu_{l1}^\bullet, \nu_{r1}^\bullet]_{LR} \succ$ and $\widetilde{h}_2 = \prec [h_2^*, \mu_{l2}^*, \nu_{r2}^*]_{LR}, [h_2^\bullet, \mu_{l2}^\bullet, \nu_{r2}^\bullet]_{LR} \succ$ in original system and get,

$$\begin{aligned} &\prec [h_1^*, \mu_{l1}^*, \nu_{r1}^*]_{LR}, [h_1^\bullet, \mu_{l1}^\bullet, \nu_{r1}^\bullet]_{LR} \succ \\ &\quad - \prec [h_2^*, \mu_{l2}^*, \nu_{r2}^*]_{LR}, [h_2^\bullet, \mu_{l2}^\bullet, \nu_{r2}^\bullet]_{LR} \succ \\ &\quad = \prec [-3, 2, 2]_{LR}, [-5, 1, 1]_{LR} \succ, \\ &\prec [h_1^*, \mu_{l1}^*, \nu_{r1}^*]_{LR}, [h_1^\bullet, \mu_{l1}^\bullet, \nu_{r1}^\bullet]_{LR} \succ \\ &\quad -2 \prec [h_2^*, \mu_{l2}^*, \nu_{r2}^*]_{LR}, [h_2^\bullet, \mu_{l2}^\bullet, \nu_{r2}^\bullet]_{LR} \succ \\ &\quad = \prec [7, 5, 5]_{LR}, [12, 5, 5]_{LR} \succ . \end{aligned}$$

To solve the $LR$-BFLS of equation in (3), first we take the positive($*$) part of the system (3) given as:

$$(h_1^*, \mu_{l1}^*, \nu_{r1}^*)_{LR} - (h_2^*, \mu_{l2}^*, \nu_{r2}^*)_{LR} = (-3, 2, 2)_{LR},$$
$$(h_1^*, \mu_{l1}^*, \nu_{r1}^*)_{LR} - 2(h_2^*, \mu_{l2}^*, \nu_{r2}^*)_{LR} = (7, 5, 5)_{LR}.$$

To solve this positive part of $LR$-BFLS of equations we can divide this system into two systems, one is mean value system and other is left-right spread system.
The mean value system for positive part is given as:

$$h_1^* - h_2^* = -3,$$
$$h_1^* - 2h_2^* = 7,$$

by solving above mentioned equations, we have $h_1^* = -13$ and $h_2^* = -10$. Now the extended $4 \times 4$ form of positive part of left-right spread is given as:

$$\begin{aligned} 1(\mu_{l1}^*) + 0(\mu_{l2}^*) + 0(\nu_{r1}^*) + 1(\nu_{r2}^*) &= 2, \\ 1(\mu_{l1}^*) + 0(\mu_{l2}^*) + 0(\nu_{r1}^*) + 2(\nu_{r2}^*) &= 5, \\ 0(\mu_{l1}^*) + 1(\mu_{l2}^*) + 1(\nu_{r1}^*) + 0(\nu_{r2}^*) &= 2, \\ 0(\mu_{l1}^*) + 2(\mu_{l2}^*) + 1(\nu_{r1}^*) + 0(\nu_{r2}^*) &= 5. \end{aligned}$$

The matrix form of the above-mentioned system is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mu_{l1}^* \\ \mu_{l2}^* \\ \nu_{r1}^* \\ \nu_{r2}^* \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 2 \\ 5 \end{pmatrix},$$

this can also be written as:

$$M(\mu_l^*, \nu_r^*) = (\alpha_l^*, \beta_r^*),$$

and the solution of above system is,

$$\begin{pmatrix} \mu_{l1}^* \\ \mu_{l2}^* \\ \nu_{r1}^* \\ \nu_{r2}^* \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & -1 \\ -1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ 2 \\ 5 \end{pmatrix}$$

$$\mu_{l1}^* = -1, \quad \mu_{l2}^* = 3,$$
$$\nu_{r1}^* = -1, \quad \nu_{r2}^* = 3.$$

Similarly, we can take the negative($\bullet$) part of the system (3) given as:

$$(h_1^\bullet, \mu_{l1}^\bullet, \nu_{r1}^\bullet)_{LR} - (h_2^\bullet, \mu_{l2}^\bullet, \nu_{r2}^\bullet)_{LR} = (-5, 1, 1)_{LR},$$
$$(h_1^\bullet, \mu_{l1}^\bullet, \nu_{r1}^\bullet)_{LR} - 2(h_2^*, \mu_{l2}^\bullet, \nu_{r2}^\bullet)_{LR} = (12, 5, 5)_{LR}.$$

To solve this negative part we can divide this system into two systems, one is mean value system and other is left-right spread system.
The mean value system for negative part is given as:

$$h_1^\bullet - h_2^\bullet = -5,$$

$$h_1^\bullet - 2h_2^\bullet = 12,$$

by solving above equations, we have $h_1^\bullet = -22$ and $h_2^\bullet = -17$. Now the extended $4 \times 4$ form of negative part of left-right spread is given as:

$$1(\mu_{l1}^\bullet) + 0(\mu_{l2}^\bullet) + 0(\nu_{r1}^\bullet) + 1(\nu_{r2}^\bullet) = 1,$$
$$1(\mu_{l1}^\bullet) + 0(\mu_{l2}^\bullet) + 0(\nu_{r1}^\bullet) + 2(\nu_{r2}^\bullet) = 5,$$
$$0(\mu_{l1}^\bullet) + 1(\mu_{l2}^\bullet) + 1(\nu_{r1}^\bullet) + 0(\nu_{r2}^\bullet) = 1,$$
$$0(\mu_{l1}^\bullet) + 2(\mu_{l2}^\bullet) + 1(\nu_{r1}^\bullet) + 0(\nu_{r2}^\bullet) = 5.$$

The matrix form of the above-mentioned system is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mu_{l1}^\bullet \\ \mu_{l2}^\bullet \\ \nu_{r1}^\bullet \\ \nu_{r2}^\bullet \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 1 \\ 5 \end{pmatrix},$$

this can also be written as:

$$M(\mu_l^\bullet, \nu_r^\bullet) = (\alpha_l^\bullet, \beta_r^\bullet),$$

and the solution of above system is,

$$\begin{pmatrix} \mu_{l1}^\bullet \\ \mu_{l2}^\bullet \\ \nu_{r1}^\bullet \\ \nu_{r2}^\bullet \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & -1 \\ -1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \\ 1 \\ 5 \end{pmatrix}$$

$$\mu_{l1}^\bullet = -3, \quad \mu_{l2}^\bullet = 4,$$
$$\nu_{r1}^\bullet = -3, \quad \nu_{r2}^\bullet = 4.$$

Hence, the solution of $LR$-BFLS of equations in (3) is symmetric given as:

$$\begin{aligned}\widetilde{h}_1 &= \prec [h_1^*, \mu_{l1}^*, \nu_{r1}^*]_{LR}, [h_1^\bullet, \mu_{l1}^\bullet, \nu_{r1}^\bullet]_{LR} \succ \\ &= \prec [-13, -1, -1]_{LR}, [-22, -3, -3]_{LR} \succ, \\ \widetilde{h}_2 &= \prec [h_2^*, \mu_{l2}^*, \nu_{r2}^*]_{LR}, [h_2^\bullet, \mu_{l2}^\bullet, \nu_{r2}^\bullet]_{LR} \succ \\ &= \prec [-10, 3, 3]_{LR}, [-17, 4, 4]_{LR} \succ .\end{aligned}$$

It is clear to see the $LR$-bipolar fuzzy solution that $\mu_{l1}^* < 0$, $\nu_{r1}^* < 0$, $\mu_{l1}^\bullet < 0$ and $\nu_{r1}^\bullet < 0$, so that $LR$-bipolar fuzzy solution is a weak solution and now solution is given as:

$$\begin{aligned}\widetilde{h}_1 &= \prec [h_1^*, \mu_{l1}^*, \nu_{r1}^*]_{LR}, [h_1^\bullet, \mu_{l1}^\bullet, \nu_{r1}^\bullet]_{LR} \succ \\ &= \prec [-13, 1, 1]_{LR}, [-22, 3, 3]_{LR} \succ, \\ \widetilde{h}_2 &= \prec [h_2^*, \mu_{l2}^*, \nu_{r2}^*]_{LR}, [h_2^\bullet, \mu_{l2}^\bullet, \nu_{r2}^\bullet]_{LR} \succ \\ &= \prec [-10, 3, 3]_{LR}, [-17, 4, 4]_{LR} \succ .\end{aligned}$$

## 4 Solution to *LR* complex bipolar fuzzy linear system

In this section, we develop two different methods to solve $LR$-CBFL system with real coefficients and $LR$-CBFL system with complex coefficients in which unknown parameter, right side vector are $LR$-BFCN and coefficients of unknown are real and complex number, respectively. In the first method, we suggest a technique to solve $LR$-CBFL system with real coefficients. First we compare real and imaginary parts of equation to get two $LR$-BFLS of equations and then solve these $LR$-BFLS of equations by using mean value and left-right spread linear systems. Guo and Zhang (2016) suggested a method to solve $n \times n$ $LR$-complex fuzzy linear system with complex coefficients in which coefficient matrix is a complex number matrix and unknown and right side vector are $LR$-complex fuzzy number. In the next method we extent the Xiaobin and Zhang technique to solve $LR$-CBFL system with complex coefficients in bipolar fuzzy environment. To solve the system we can replace $n \times n$ $LR$-CBFL system with complex coefficients by pair of positive($*$) and negative($\bullet$) of two $2n \times 2n$ mean value system and $4n \times 4n$ left-right spread system. We use this technique to solve the example of circuit flow and find the current which is calculated in terms of $LR$-BFCN.

**Definition 19** Consider the $n \times n$ system of linear equations,

$$\begin{cases} l_{11}h_1 + l_{12}h_2 + \cdots + l_{1n}h_n &= z_1, \\ l_{21}h_1 + l_{22}h_2 + \cdots + l_{2n}h_n &= z_2, \\ . \\ . \\ . \\ l_{n1}h_1 + l_{n2}h_2 + \cdots + l_{nn}h_n &= z_n, \end{cases} \tag{16}$$

is the $LR$-complex bipolar fuzzy linear (CBFL) system of equation with real coefficients, if all the coefficient elements $l_{pq}$, $1 \leq p, q \leq n$ is a real number matrix of order $n \times n$ and each $Z_q$, $1 \leq q \leq n$ is a $LR$-BFCN, and each unknown parameter $h_p$, $1 \leq p \leq n$ is also $LR$-BFCN.

The CBFL system in (16) can be written in matrix form as:

$$L\widetilde{H} = \widetilde{Z}, \tag{17}$$

where $\widetilde{H}$ and $\widetilde{Z}$ are $LR$-BFCN.

We can rewrite the above $LR$-CBFL system(17) as follows:

$$L(\widetilde{U} + i\widetilde{V}) = \widetilde{E} + i\widetilde{F}, \tag{18}$$

where each $\widetilde{U}$, $\widetilde{V}$, $\widetilde{E}$ and $\widetilde{F}$ are $LR$-BFN.

If we compare the real and imaginary part of the given equation, then we obtain two $LR$-bipolar fuzzy linear systems given as:

$$L\widetilde{U}=\widetilde{E},\quad L\widetilde{V}=\widetilde{F}. \tag{19}$$

**Definition 20** A $LR$-BFCNs vector $H=(\widetilde{h}_1,\widetilde{h}_2,\ldots.\widetilde{h}_n)^T$ is the form of $H_q=\prec[u_q^*,u_{lq}^*,u_{rq}^*]_{LR},[u_q^\bullet,u_{lq}^\bullet,u_{rq}^\bullet]_{LR}\succ +i\prec[v_q^*,v_{lq}^*,v_{rq}^*]_{LR},[v_q^\bullet,v_{lq}^\bullet,v_{rq}^\bullet]_{LR}\succ=\widetilde{u}_q+i\widetilde{v}_q$, where $1\le q\le n$ is called the $LR$-bipolar fuzzy complex solution of $LR$-CBFL System of equations in (18) and (19) if,

$$\widetilde{U}'=(\widetilde{u}_1,\widetilde{u}_2,\ldots,\widetilde{u}_n),\qquad \widetilde{V}'=(\widetilde{v}_1,\widetilde{v}_2,\ldots,\widetilde{v}_n),$$

are the $LR$-bipolar fuzzy solution of $LR$-bipolar fuzzy linear system in (19), respectively, if

$$L\widetilde{U}'=\widetilde{E},\quad L\widetilde{V}'=\widetilde{F}, \tag{20}$$

and this solution vector also satisfied the system (16) given as:

$$\begin{cases}\sum_{q=1}^n l_{pq}u_q^* &= e_p^*,\\ \sum_{q=1}^n l_{pq}u_{lq}^* &= e_{lp}^*,\\ \sum_{q=1}^n l_{pq}u_{rq}^* &= e_{rp}^*,\\ \sum_{q=1}^n l_{pq}u_q^\bullet &= e_p^\bullet,\\ \sum_{q=1}^n l_{pq}u_{lq}^\bullet &= e_{lp}^\bullet,\\ \sum_{q=1}^n l_{pq}u_{rq}^\bullet &= e_{rp}^\bullet.\end{cases} \tag{21}$$

$$\begin{cases}\sum_{q=1}^n l_{pq}v_q^* &= f_p^*,\\ \sum_{q=1}^n l_{pq}v_{lq}^* &= f_{lp}^*,\\ \sum_{q=1}^n l_{pq}v_{rq}^* &= f_{rp}^*,\\ \sum_{q=1}^n l_{pq}v_q^\bullet &= f_p^\bullet,\\ \sum_{q=1}^n l_{pq}v_{lq}^\bullet &= f_{lp}^\bullet,\\ \sum_{q=1}^n l_{pq}v_{rq}^\bullet &= f_{rp}^\bullet.\end{cases} \tag{22}$$

### 4.1 Solution procedure to n × n *LR*-CBFL system with real coefficients

To solve the $LR$-CBFL system of equation with real coefficients in (16), first we should compare the real and imaginary part of system of equations to get two $n\times n$ $LR$-bipolar fuzzy linear (BFL) systems $L\widetilde{U}=\widetilde{E}$ and $L\widetilde{V}=\widetilde{F}$.

To solve real part of $LR$-CBFL system of equation, i.e., $L\widetilde{U}=\widetilde{E}$, first we should separate the positive($*$) and negative($\bullet$) part of system of equations to get two $n\times n$ $LR$-real linear systems given as

$$L(u^*,u_l^*,u_r^*)_{LR}=(e^*,e_l^*,e_r^*)_{LR}, \tag{23}$$
$$L(u^\bullet,u_l^\bullet,u_r^\bullet)_{LR}=(e^\bullet,e_l^\bullet,e_r^\bullet)_{LR}. \tag{24}$$

To solve $LR$-real linear system $L(u^*,u_l^*,u_r^*)_{LR}=(e^*,e_l^*,e_r^*)_{LR}$, we resolve this system into two systems, one is $n\times n$ mean value system and other is $2n\times 2n$ left-right spread system.

The $n\times n$ mean value system for the positive part of $L\widetilde{U}=\widetilde{E}$ is given as:

$$\begin{cases}l_{11}u_1^*+l_{12}u_2^*+\cdots+l_{1n}u_n^* &= e_1^*,\\ l_{21}u_1^*+l_{22}u_2^*+\cdots+l_{2n}u_n^* &= e_2^*,\\ . & \\ . & \\ . & \\ l_{n1}u_1^*+l_{n2}u_2^*+\cdots+l_{nn}u_n^* &= e_n^*.\end{cases} \tag{25}$$

The matrix notation of system (25) is given as

$$LU_q^*=E_p^*,$$

where $1\le p,q\le n$.

To solve left-right spread system of equation $L\widetilde{U}=\widetilde{E}$, we must solve the $2n\times 2n$ real linear system and where the right side column is function vector $(e_{l1}^*,e_{l2}^*,\ldots,e_{ln}^*,e_{r1}^*,e_{r2}^*,\ldots,e_{rn}^*)^T$. Now rearrange the left-right spread system, so the unknown parameters are $u_{lq}^*,u_{rq}^*$, $1\le q\le n$ and column of right side vector are

$$(e_{l1}^*,e_{l2}^*,\ldots,e_{ln}^*,e_{r1}^*,e_{r2}^*,\ldots,e_{rn}^*)^T.$$

We get first $2n\times 2n$ left-right spread system of positive($*$) part of $LR$-BFLS of equations $L\widetilde{U}=\widetilde{E}$ given as:

$$\begin{cases}t_{11}u_{l1}^*+t_{12}u_{l2}^*+\cdots+t_{1n}u_{ln}^*+t_{1,n+1}(-u_{r1}^*)+t_{1,n+2}\\ \quad(-u_{r2}^*)+\cdots+t_{1,2n}(-u_{rn}^*)=e_{l1}^*,\\ .\\ .\\ .\\ t_{n1}u_{l1}^*+t_{n2}u_{l2}^*+\cdots+t_{nn}u_{ln}^*+t_{n,n+1}(-u_{r1}^*)+t_{n,n+2}\\ \quad(-u_{r2}^*)+\cdots+t_{n,2n}(-u_{rn}^*)=e_{ln}^*,\\ t_{n+1,1}u_{l1}^*+t_{n+1,2}u_{l2}^*+\cdots+t_{n+1,n}u_{ln}^*+t_{n+1,n+1}(-u_{r1}^*)\\ \quad+t_{n+1,n+2}(-u_{r2}^*)+\cdots+t_{n+1,2n}(-u_{rn}^*)=e_{r1}^*,\\ .\\ .\\ .\\ t_{2n,1}u_{l1}^*+t_{2n,2}u_{l2}^*+\cdots+t_{2n,n}u_{ln}^*+t_{2n,n+1}(-u_{r1}^*)\\ \quad+t_{2n,n+2}(-u_{r2}^*)+\cdots+t_{2n,2n}(-u_{rn}^*)=e_{rn}^*,\end{cases}$$

where $l_{pq}$ are determined as follows:

$$\begin{aligned}l_{pq}\ge\ & 0\Longrightarrow t_{pq}=l_{pq},t_{p+n,q+n}=l_{pq},\\ l_{pq}<\ & 0\Longrightarrow t_{p,q+n}=-l_{pq},t_{p+n,q}=-l_{pq},\end{aligned} \tag{26}$$

where $1 \leq p, q \leq 2n$ and any $t_{pq}$ which is not determined from Eq. (26) is zero.
Using the matrix notation, we obtain

$$M(u_l^*, u_r^*) = (e_l^*, e_r^*), \tag{27}$$

where $M = l_{pq}$, $1 \leq p, q \leq 2n$

$$(u_{l1}^*, u_{l2}^*, \ldots, u_{ln}^*, u_{r1}^*, u_{r2}^*, \ldots, u_{rn}^*)^T \textit{and}$$
$$(e_{l1}^*, e_{l2}^*, \ldots, e_{ln}^*, e_{r1}^*, e_{r2}^*, \ldots, e_{rn}^*)^T.$$

We can also write the matrix form of Eq. (27) as:

$$\begin{pmatrix} A & -B \\ -B & A \end{pmatrix} \begin{pmatrix} u_l^* \\ u_r^* \end{pmatrix} = \begin{pmatrix} e_l^* \\ e_r^* \end{pmatrix}. \tag{28}$$

In the similar way we can take the negative($\bullet$) part of the $LR$-BFLS of equation given $L\widetilde{U} = \widetilde{E}$.

To solve $LR$-real linear system $L(u^\bullet, u_l^\bullet, u_r^\bullet)_{LR} = (e^\bullet, e_l^\bullet, e_r^\bullet)_{LR}$, we divide this system into two systems, one is $n \times n$ mean value system and other is $2n \times 2n$ left-right spread system for negative part of $L\widetilde{U} = \widetilde{E}$.

The $n \times n$ mean value system for the negative part of $L\widetilde{U} = \widetilde{E}$ is given as:

$$\begin{cases} l_{11}u_1^\bullet + l_{12}u_2^\bullet + \cdots + l_{1n}u_n^\bullet &= e_1^\bullet, \\ l_{21}u_1^\bullet + l_{22}u_2^\bullet + \cdots + l_{2n}u_n^\bullet &= e_2^\bullet, \\ . \\ . \\ . \\ l_{n1}u_1^\bullet + l_{n2}u_2^\bullet + \cdots + l_{nn}u_n^\bullet &= e_n^\bullet. \end{cases} \tag{29}$$

The matrix notation of system (29) is given as

$$LU_q^\bullet = E_p^\bullet,$$

where $1 \leq p, q \leq n$.

To solve negative part of left-right spread system of equation $L\widetilde{U} = \widetilde{E}$, we must solve the $2n \times 2n$ crisp linear system and where the right side column is function vector $(e_{l1}^\bullet, e_{l2}^\bullet, \ldots, e_{ln}^\bullet, e_{r1}^\bullet, e_{r2}^\bullet, \ldots, e_{rn}^\bullet)^T$. Now rearrange the left-right spread system, so the unknown parameters are $u_{lq}^\bullet, u_{rq}^\bullet$, $1 \leq q \leq n$ and column of right side vector are

$$(e_{l1}^\bullet, e_{l2}^\bullet, \ldots, e_{ln}^\bullet, e_{r1}^\bullet, e_{r2}^\bullet, \ldots, e_{rn}^\bullet)^T.$$

We get first $2n \times 2n$ left-right spread system of negative($\bullet$) part of $L\widetilde{U} = \widetilde{E}$ of equation,

$$\begin{cases} t_{11}u_{l1}^\bullet + t_{12}u_{l2}^\bullet + \cdots + t_{1n}u_{ln}^\bullet + t_{1,n+1}(-u_{r1}^\bullet) + t_{1,n+2} \\ \quad (-u_{r2}^\bullet) + \cdots + t_{1,2n}(-u_{rn}^\bullet) = e_{l1}^\bullet, \\ . \\ . \\ . \\ t_{n1}u_{l1}^\bullet + t_{n2}u_{l2}^\bullet + \cdots + t_{nn}u_{ln}^\bullet + t_{n,n+1}(-u_{r1}^\bullet) + t_{n,n+2} \\ \quad (-u_{r2}^\bullet) + \cdots + t_{n,2n}(-u_{rn}^\bullet) = e_{ln}^\bullet, \\ t_{n+1,1}u_{l1}^\bullet + t_{n+1,2}u_{l2}^\bullet + \cdots + t_{n+1,n}u_{ln}^\bullet + t_{n+1,n+1}(-u_{r1}^\bullet) \\ \quad + t_{n+1,n+2}(-u_{r2}^\bullet) + \cdots + t_{n+1,2n}(-u_{rn}^\bullet) = e_{r1}^\bullet, \\ . \\ . \\ . \\ t_{2n,1}u_{l1}^\bullet + t_{2n,2}u_{l2}^\bullet + \cdots + t_{2n,n}u_{ln}^\bullet + t_{2n,n+1}(-u_{r1}^\bullet) \\ \quad + t_{2n,n+2}(-u_{r2}^\bullet) + \cdots + t_{2n,2n}(-u_{rn}^\bullet) = e_{rn}^\bullet, \end{cases}$$

where $l_{pq}$ are determined as follows:

$$\begin{aligned} l_{pq} \geq 0 &\Longrightarrow t_{pq} = l_{pq}, t_{p+n,q+n} = l_{pq}, \\ l_{pq} < 0 &\Longrightarrow t_{p,q+n} = -l_{pq}, t_{p+n,q} = -l_{pq}, \end{aligned} \tag{30}$$

where $1 \leq p, q \leq 2n$ and any $t_{pq}$ which is not determined from Eq. (30) is zero.

Using the matrix notation, we obtain

$$M(u_l^\bullet, u_r^\bullet) = (e_l^\bullet, e_r^\bullet), \tag{31}$$

where $M = l_{pq}$, $1 \leq p, q \leq 2n$

$$(u_{l1}^\bullet, u_{l2}^\bullet, \ldots, u_{ln}^\bullet, u_{r1}^\bullet, u_{r2}^\bullet, \ldots, u_{rn}^\bullet)^T \textit{and}$$
$$(e_{l1}^\bullet, e_{l2}^\bullet, \ldots, e_{ln}^\bullet, e_{r1}^\bullet, e_{r2}^\bullet, \ldots, e_{rn}^\bullet)^T.$$

We can also write the matrix form of Eq. (31) as:

$$\begin{pmatrix} A & -B \\ -B & A \end{pmatrix} \begin{pmatrix} u_l^\bullet \\ u_r^\bullet \end{pmatrix} = \begin{pmatrix} e_l^\bullet \\ e_r^\bullet \end{pmatrix}. \tag{32}$$

To find the solution of real part of $LR$-CBFL system with real coefficients in (16), we can find the solution of Eqs. (25), (28), (29) and (32) by using matrix inverse method.

Similarly, we can solve imaginary part of $LR$-CBFL system of equation, i.e., $L\widetilde{V} = \widetilde{F}$, first we separate the positive($*$) and negative($\bullet$) part of system of equations to get two $n \times n$ $LR$-real linear systems given as

$$L(v^*, v_l^*, v_r^*)_{LR} = (f^*, f_l^*, f_r^*)_{LR}, \tag{33}$$
$$L(v^\bullet, v_l^\bullet, v_r^\bullet)_{LR} = (f^\bullet, f_l^\bullet, f_r^\bullet)_{LR}. \tag{34}$$

To solve $LR$-real linear system $L(v^*, v_l^*, v_r^*)_{LR} = (f^*, f_l^*, f_r^*)_{LR}$, we restructure this system into two systems, one is $n \times n$ mean value system and other is $2n \times 2n$ left-right spread system.

The $n \times n$ mean value system for the positive part of $L\widetilde{V} = \widetilde{F}$ is given as:

$$\begin{cases} l_{11}v_1^* + l_{12}v_2^* + \cdots + l_{1n}v_n^* &= f_1^*, \\ l_{21}v_1^* + l_{22}v_2^* + \cdots + l_{2n}v_n^* &= f_2^*, \\ . \\ . \\ . \\ l_{n1}v_1^* + l_{n2}v_2^* + \cdots + l_{nn}v_n^* &= f_n^*. \end{cases} \tag{35}$$

The matrix notation of system (35) is given as

$$LV_q^* = F_p^*,$$

where $1 \leq p, q \leq n$.

To solve left-right spread system of equation $L\widetilde{V} = \widetilde{F}$, we must solve the $2n \times 2n$ real linear system and where the right side column is function vector $(f_{l1}^*, f_{l2}^*, \ldots, f_{ln}^*, f_{r1}^*, f_{r2}^*, \ldots, f_{rn}^*)^T$. Now rearrange the left-right spread system, so the unknown parameters are $v_{lq}^*, v_{rq}^*, 1 \leq q \leq n$ and column of right side vector are

$$(f_{l1}^*, f_{l2}^*, \ldots, f_{ln}^*, f_{r1}^*, f_{r2}^*, \ldots, f_{rn}^*)^T.$$

We get first $2n \times 2n$ left-right spread system of positive($*$) part of $LR$-BFLS of equation $L\widetilde{V} = \widetilde{F}$ given as:

$$\begin{cases} t_{11}v_{l1}^* + t_{12}v_{l2}^* + \cdots + t_{1n}v_{ln}^* + t_{1,n+1}(-v_{r1}^*) \\ \quad +t_{1,n+2}(-v_{r2}^*) + \cdots + t_{1,2n}(-v_{rn}^*) = f_{l1}^*, \\ . \\ . \\ . \\ t_{n1}v_{l1}^* + t_{n2}v_{l2}^* + \cdots + t_{nn}v_{ln}^* + t_{n,n+1}(-v_{r1}^*) + t_{n,n+2} \\ \quad (-v_{r2}^*) + \cdots + t_{n,2n}(-v_{rn}^*) = f_{ln}^*, \\ t_{n+1,1}v_{l1}^* + t_{n+1,2}v_{l2}^* + \cdots + t_{n+1,n}v_{ln}^* + t_{n+1,n+1}(-v_{r1}^*) \\ \quad +t_{n+1,n+2}(-v_{r2}^*) + \cdots + t_{n+1,2n}(-v_{rn}^*) = f_{r1}^*, \\ . \\ . \\ . \\ t_{2n,1}v_{l1}^* + t_{2n,2}v_{l2}^* + \cdots + t_{2n,n}v_{ln}^* + t_{2n,n+1}(-v_{r1}^*) \\ \quad +t_{2n,n+2}(-v_{r2}^*) + \cdots + t_{2n,2n}(-v_{rn}^*) = f_{rn}^*, \end{cases}$$

where $l_{pq}$ are determined as follows:

$$\begin{aligned} &l_{pq} \geq 0 \Longrightarrow t_{pq} = l_{pq}, t_{p+n,q+n} = l_{pq}, \\ &l_{pq} < 0 \Longrightarrow t_{p,q+n} = -l_{pq}, t_{p+n,q} = -l_{pq}, \end{aligned} \tag{36}$$

where $1 \leq p, q \leq 2n$ and any $t_{pq}$ which is not determined from Eqs. (36) is zero.

Using the matrix notation, we obtain

$$M(v_l^*, v_r^*) = (f_l^*, f_r^*), \tag{37}$$

where $M = l_{pq}, 1 \leq p, q \leq 2n$

$$(v_{l1}^*, v_{l2}^*, \ldots, v_{ln}^*, v_{r1}^*, v_{r2}^*, \ldots, v_{rn}^*)^T \, and$$
$$(f_{l1}^*, f_{l2}^*, \ldots, f_{ln}^*, f_{r1}^*, f_{r2}^*, \ldots, f_{rn}^*)^T.$$

We can also write the matrix form of Eq. (37) as:

$$\begin{pmatrix} A & -B \\ -B & A \end{pmatrix} \begin{pmatrix} v_l^* \\ v_r^* \end{pmatrix} = \begin{pmatrix} f_l^* \\ f_r^* \end{pmatrix}. \tag{38}$$

In the similar way we can take the negative($\bullet$) part of the $LR$-BFLS of equation given $L\widetilde{V} = \widetilde{F}$.

To solve $LR$-real linear system $L(v^\bullet, v_l^\bullet, v_r^\bullet)_{LR} = (f^\bullet, f_l^\bullet, f_r^\bullet)_{LR}$, we divide this system into two systems, one is $n \times n$ mean value system and other is $2n \times 2n$ left-right spread system for negative part of $L\widetilde{V} = \widetilde{F}$.

The $n \times n$ mean value system for the negative part of $L\widetilde{V} = \widetilde{F}$ is given as:

$$\begin{cases} l_{11}v_1^\bullet + l_{12}v_2^\bullet + \cdots + l_{1n}v_n^\bullet &= f_1^\bullet, \\ l_{21}v_1^\bullet + l_{22}v_2^\bullet + \cdots + l_{2n}v_n^\bullet &= f_2^\bullet, \\ . \\ . \\ . \\ l_{n1}v_1^\bullet + l_{n2}v_2^\bullet + \cdots + l_{nn}v_n^\bullet &= f_n^\bullet. \end{cases} \tag{39}$$

The matrix notation of system (39) is given as

$$LV_q^\bullet = F_p^\bullet,$$

where $1 \leq p, q \leq n$.

To solve negative part of left-right spread system of equation $L\widetilde{V} = \widetilde{F}$, we must solve the $2n \times 2n$ real linear system and where the right side column is function vector $(f_{l1}^\bullet, f_{l2}^\bullet, \ldots, f_{ln}^\bullet, f_{r1}^\bullet, f_{r2}^\bullet, \ldots, f_{rn}^\bullet)^T$. Now rearrange the left-right spread system, so the unknown parameters are $v_{lq}^\bullet, v_{rq}^\bullet, 1 \leq q \leq n$ and column of right side vector are

$$(f_{l1}^\bullet, f_{l2}^\bullet, \ldots, f_{ln}^\bullet, f_{r1}^\bullet, f_{r2}^\bullet, \ldots, f_{rn}^\bullet)^T.$$

We get first $2n \times 2n$ left-right spread system of negative($\bullet$) part of $L\widetilde{V} = \widetilde{F}$ of equation,

$$\begin{cases} t_{11}v^{\bullet}_{l1} + t_{12}v^{\bullet}_{l2} + \cdots + t_{1n}v^{\bullet}_{ln} + t_{1,n+1} \\ \quad (-v^{\bullet}_{r1}) + t_{1,n+2}(-v^{\bullet}_{r2}) + \cdots + t_{1,2n}(-v^{\bullet}_{rn}) = f^{\bullet}_{l1}, \\ . \\ . \\ . \\ t_{n1}v^{\bullet}_{l1} + t_{n2}v^{\bullet}_{l2} + \cdots + t_{nn}v^{\bullet}_{ln} + t_{n,n+1}(-v^{\bullet}_{r1}) + t_{n,n+2} \\ \quad (-v^{\bullet}_{r2}) + \cdots + t_{n,2n}(-v^{\bullet}_{rn}) = f^{\bullet}_{ln}, \\ t_{n+1,1}v^{\bullet}_{l1} + t_{n+1,2}v^{\bullet}_{l2} + \cdots + t_{n+1,n}v^{\bullet}_{ln} + t_{n+1,n+1}(-v^{\bullet}_{r1}) \\ \quad + t_{n+1,n+2}(-v^{\bullet}_{r2}) + \cdots + t_{n+1,2n}(-v^{\bullet}_{rn}) = f^{\bullet}_{r1}, \\ . \\ . \\ . \\ t_{2n,1}v^{\bullet}_{l1} + t_{2n,2}v^{\bullet}_{l2} + \cdots + t_{2n,n}v^{\bullet}_{ln} + t_{2n,n+1}(-v^{\bullet}_{r1}) \\ \quad + t_{2n,n+2}(-v^{\bullet}_{r2}) + \cdots + t_{2n,2n}(-v^{\bullet}_{rn}) = f^{\bullet}_{rn}, \end{cases}$$

where $l_{pq}$ are determined as follows:

$$\begin{aligned} l_{pq} \geq 0 &\Longrightarrow t_{pq} = l_{pq}, t_{p+n,q+n} = l_{pq}, \\ l_{pq} < 0 &\Longrightarrow t_{p,q+n} = -l_{pq}, t_{p+n,q} = -l_{pq}, \end{aligned} \tag{40}$$

where $1 \leq p, q \leq 2n$ and any $t_{pq}$ which is not determined from Eq. (40) is zero.

Using the matrix notation, we obtain

$$M(v^{\bullet}_l, v^{\bullet}_r) = (f^{\bullet}_l, f^{\bullet}_r), \tag{41}$$

where $M = l_{pq}, 1 \leq p, q \leq 2n$

$$\begin{aligned} &(v^{\bullet}_{l1}, v^{\bullet}_{l2}, \ldots, v^{\bullet}_{ln}, v^{\bullet}_{r1}, v^{\bullet}_{r2}, \ldots, v^{\bullet}_{rn})^T \textit{ and} \\ &(f^{\bullet}_{l1}, f^{\bullet}_{l2}, \ldots, f^{\bullet}_{ln}, f^{\bullet}_{r1}, f^{\bullet}_{r2}, \ldots, f^{\bullet}_{rn})^T. \end{aligned}$$

We can also write the matrix form of Eq. (41) as:

$$\begin{pmatrix} A & -B \\ -B & A \end{pmatrix} \begin{pmatrix} v^{\bullet}_l \\ v^{\bullet}_r \end{pmatrix} = \begin{pmatrix} f^{\bullet}_l \\ f^{\bullet}_r \end{pmatrix}. \tag{42}$$

To find the solution of imaginary part of $LR$-CBFL system with real coefficients in (16), we can find the solution of equations (35), (38), (39) and (42) by using matrix inverse method.

**Theorem 3** *For any arbitrary $LR$-bipolar fuzzy complex vector $\widetilde{Z}$ the unique solution $\widetilde{H}$ is a $LR$-bipolar fuzzy complex vector of $LR$-CBFL system of equations with real coefficients in (16) if and only if the matrix* $M = \begin{pmatrix} A & -B \\ -B & A \end{pmatrix}$ *is non-singular, the necessary and sufficient condition that the matrices $(A+B)^{-1}$ and $(A-B)^{-1}$ exist.*

***Example 4*** Consider the $2 \times 2$ $LR$-CBFL system with real coefficients

$$\begin{aligned} \widetilde{h}_1 - \widetilde{h}_2 &= \prec [-3, 3, 2]_{LR}, [-3, 3, 3]_{LR} \succ \\ &\quad + i \prec [-4, 4, 4]_{LR}, [-5, 2, 3]_{LR} \succ, \\ \widetilde{h}_1 + 2\widetilde{h}_2 &= \prec [21, 4, 3]_{LR}, [15, 6, 3]_{LR} \succ \\ &\quad + i \prec [32, 7, 4]_{LR}, [25, 3, 4]_{LR} \succ. \end{aligned}$$

To solve this $LR$-CBFL system first we put $\widetilde{h}_1 = \widetilde{u}_1 + i\widetilde{v}_1$ and $\widetilde{h}_2 = \widetilde{u}_2 + i\widetilde{v}_2$ in original system and get,

$$\begin{aligned} &(\widetilde{u}_1 + i\widetilde{v}_1) - (\widetilde{u}_2 + i\widetilde{v}_2) \\ &\quad = \prec [-3, 3, 2]_{LR}, [-3, 3, 3]_{LR} \succ \\ &\quad + i \prec [-4, 4, 4]_{LR}, [-5, 2, 3]_{LR} \succ, \\ &(\widetilde{u}_1 + i\widetilde{v}_1) + 2(\widetilde{u}_2 + i\widetilde{v}_2) \\ &\quad = \prec [21, 4, 3]_{LR}, [15, 6, 3]_{LR} \succ \\ &\quad + i \prec [32, 7, 4]_{LR}, [25, 3, 4]_{LR} \succ, \end{aligned}$$

compare the real and imaginary parts to both side of the equations and we obtain two $LR$-BFLS of equations given as:

$$\begin{cases} \widetilde{u}_1 - \widetilde{u}_2 = \prec [-3, 3, 2]_{LR}, [-3, 3, 3]_{LR} \succ, \\ \widetilde{u}_1 + 2\widetilde{u}_2 = \prec [21, 4, 3]_{LR}, [15, 6, 3]_{LR} \succ, \end{cases} \tag{43}$$

and,

$$\begin{cases} v_1 - v_2 = \prec [-4, 4, 4]_{LR}, [-5, 2, 3]_{LR} \succ, \\ v_1 + 2v_2 = \prec [32, 7, 4]_{LR}, [25, 3, 4]_{LR} \succ, \end{cases} \tag{44}$$

To solve real part of $LR$-CBFL system with real coefficients given in (43) first we put $\widetilde{u}_1 = \prec [u^*_1, u^*_{l1}, u^*_{r1}]_{LR}, [u^{\bullet}_1, u^{\bullet}_{l1}, u^{\bullet}_{r1}]_{LR} \succ$ and $\widetilde{u}_2 = \prec [u^*_2, u^*_{l2}, u^*_{r2}]_{LR}, [u^{\bullet}_2, u^{\bullet}_{l2}, u^{\bullet}_{r2}]_{LR} \succ$ in original system and get,

$$\begin{aligned} &\prec [u^*_1, u^*_{l1}, u^*_{r1}]_{LR}, [u^{\bullet}_1, u^{\bullet}_{l1}, u^{\bullet}_{r1}]_{LR} \succ \\ &\quad - \prec [u^*_2, u^*_{l2}, u^*_{r2}]_{LR}, [u^{\bullet}_2, u^{\bullet}_{l2}, u^{\bullet}_{r2}]_{LR} \succ \\ &\quad = \prec [-3, 3, 2]_{LR}, [-3, 3, 3]_{LR} \succ, \\ &\prec [u^*_1, u^*_{l1}, u^*_{r1}]_{LR}, [u^{\bullet}_1, u^{\bullet}_{l1}, u^{\bullet}_{r1}]_{LR} \succ \\ &\quad + 2 \prec [u^*_2, u^*_{l2}, u^*_{r2}]_{LR}, [u^{\bullet}_2, u^{\bullet}_{l2}, u^{\bullet}_{r2}]_{LR} \succ \\ &\quad = \prec [21, 4, 3]_{LR}, [15, 6, 3]_{LR} \succ. \end{aligned}$$

To solve the $LR$-BFLS of equation in (43), first we take the positive($*$) part of the system (43) given as:

$$\begin{aligned} &(u^*_1, u^*_{l1}, u^*_{r1})_{LR} - (u^*_2, u^*_{l2}, u^*_{r2})_{LR} = (-3, 2, 2)_{LR}, \\ &(u^*_1, u^*_{l1}, u^*_{r1})_{LR} + 2(u^*_2, u^*_{l2}, u^*_{r2})_{LR} = (21, 4, 3)_{LR}. \end{aligned}$$

To solve this positive part we can divide this system into two systems, one is mean value system and other is left-right spread system.

The mean value system for positive part of (43) is given as:

$$u^*_1 - u^*_2 = -3,$$

$u_1^* + 2u_2^* = 21,$

by solving above equations, we have $u_1^* = 5$ and $u_2^* = 8$. Now the extended $4 \times 4$ form of positive part of left-right spread is given as:

$$\begin{aligned}
&1(u_{l1}^*) + 0(u_{l2}^*) + 0(u_{r1}^*) + 1(u_{r2}^*) = 3,\\
&1(u_{l1}^*) + 2(u_{l2}^*) + 0(u_{r1}^*) + 0(u_{r2}^*) = 4,\\
&0(u_{l1}^*) + 1(u_{l2}^*) + 1(u_{r1}^*) + 0(u_{r2}^*) = 2,\\
&0(u_{l1}^*) + 0(u_{l2}^*) + 1(u_{r1}^*) + 2(u_{r2}^*) = 3.
\end{aligned}$$

The matrix form of the above-mentioned system is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} u_{l1}^* \\ u_{l2}^* \\ u_{r1}^* \\ u_{r2}^* \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 2 \\ 3 \end{pmatrix},$$

this can also be written as:

$$M(u_l^*, u_r^*) = (e_l^*, e_r^*),$$

and the solution of above system is,

$$\begin{pmatrix} u_{l1}^* \\ u_{l2}^* \\ u_{r1}^* \\ u_{r2}^* \end{pmatrix} = \begin{pmatrix} 1.3333 & -0.3333 & 0.6667 & -0.6667 \\ -0.6667 & 0.6667 & -0.3333 & 0.3333 \\ 0.6667 & -0.6667 & 1.3333 & -0.3333 \\ -0.3333 & 0.3333 & -0.6667 & 0.6667 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ 2 \\ 3 \end{pmatrix}$$

$$\begin{aligned}
&u_{l1}^* = 2, \quad u_{l2}^* = 1,\\
&u_{r1}^* = 1, \quad u_{r2}^* = 1.
\end{aligned}$$

Similarly, we can take the negative($\bullet$) part of the system (43) given as:

$$\begin{aligned}
&(u_1^\bullet, u_{l1}^\bullet, u_{r1}^\bullet)_{LR} - (u_2^\bullet, u_{l2}^\bullet, u_{r2}^\bullet)_{LR} = (-3, 3, 3)_{LR},\\
&(u_1^\bullet, u_{l1}^\bullet, u_{r1}^\bullet)_{LR} + 2(u_2^*, u_{l2}^\bullet, u_{r2}^\bullet)_{LR} = (15, 6, 3)_{LR}.
\end{aligned}$$

To solve this negative part of $LR$-BFLS of equation we can restructure this system into two systems, one is mean value system and other is left-right spread system.

The mean value system for negative part is given as:

$$\begin{aligned}
&u_1^\bullet - u_2^\bullet = -3,\\
&u_1^\bullet + 2u_2^\bullet = 15,
\end{aligned}$$

by solving above equations, we have $u_1^\bullet = 3$ and $u_2^\bullet = 6$. Now the extended $4 \times 4$ form of negative part of left-right spread is given as:

$$\begin{aligned}
&1(u_{l1}^\bullet) + 0(u_{l2}^\bullet) + 0(u_{r1}^\bullet) + 1(u_{r2}^\bullet) = 3,\\
&1(u_{l1}^\bullet) + 2(u_{l2}^\bullet) + 0(u_{r1}^\bullet) + 0(u_{r2}^\bullet) = 6,\\
&0(u_{l1}^\bullet) + 1(u_{l2}^\bullet) + 1(u_{r1}^\bullet) + 0(u_{r2}^\bullet) = 3,\\
&0(u_{l1}^\bullet) + 0(u_{l2}^\bullet) + 1(u_{r1}^\bullet) + 2(u_{r2}^\bullet) = 3.
\end{aligned}$$

The matrix form of the above-mentioned system is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} u_{l1}^\bullet \\ u_{l2}^\bullet \\ u_{r1}^\bullet \\ u_{r2}^\bullet \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 3 \\ 3 \end{pmatrix},$$

this can also be written as:

$$M(u_l^\bullet, u_r^\bullet) = (e_l^\bullet, e_r^\bullet),$$

and the solution of above system is,

$$\begin{pmatrix} u_{l1}^\bullet \\ u_{l2}^\bullet \\ u_{r1}^\bullet \\ u_{r2}^\bullet \end{pmatrix} = \begin{pmatrix} 1.3333 & -0.3333 & 0.6667 & -0.6667 \\ -0.6667 & 0.6667 & -0.3333 & 0.3333 \\ 0.6667 & -0.6667 & 1.3333 & -0.3333 \\ -0.3333 & 0.3333 & -0.6667 & 0.6667 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \\ 3 \\ 3 \end{pmatrix}$$

$$\begin{aligned}
&u_{l1}^\bullet = 2, \quad u_{l2}^\bullet = 2,\\
&u_{r1}^\bullet = 1, \quad u_{r2}^\bullet = 1.
\end{aligned}$$

Hence, the solution for real part of $LR$-CBFL system of equation with real coefficients in (4) is given as:

$$\begin{aligned}
\widetilde{U}_1 &= \prec [u_1^*, u_{l1}^*, u_{r1}^*]_{LR}, [u_1^\bullet, u_{l1}^\bullet, u_{r1}^\bullet]_{LR} \succ\\
&= \prec [5, 2, 1]_{LR}, [3, 2, 1]_{LR} \succ,\\
\widetilde{U}_2 &= \prec [u_2^*, u_{l2}^*, u_{r2}^*]_{LR}, [u_2^\bullet, u_{l2}^\bullet, u_{r2}^\bullet]_{LR} \succ\\
&= \prec [8, 1, 1]_{LR}, [6, 2, 1]_{LR} \succ .
\end{aligned}$$

Similarly, we can solve imaginary part of $LR$-CBFL system with real coefficients given in (44) first we put $\widetilde{v}_1 = \prec [v_1^*, v_{l1}^*, v_{r1}^*]_{LR}, [v_1^\bullet, v_{l1}^\bullet, v_{r1}^\bullet]_{LR} \succ$ and $\widetilde{v}_2 = \prec [v_2^*, v_{l2}^*, v_{r2}^*]_{LR}, [v_2^\bullet, v_{l2}^\bullet, v_{r2}^\bullet]_{LR} \succ$ in original system and get,

$$\begin{aligned}
&\prec [v_1^*, v_{l1}^*, v_{r1}^*]_{LR}, [v_1^\bullet, v_{l1}^\bullet, v_{r1}^\bullet]_{LR} \succ\\
&\quad - \prec [v_2^*, v_{l2}^*, v_{r2}^*]_{LR}, [v_2^\bullet, v_{l2}^\bullet, v_{r2}^\bullet]_{LR} \succ\\
&\quad = \prec [-4, 4, 4]_{LR}, [-5, 2, 3]_{LR} \succ,\\
&\prec [v_1^*, v_{l1}^*, v_{r1}^*]_{LR}, [v_1^\bullet, v_{l1}^\bullet, v_{r1}^\bullet]_{LR} \succ\\
&\quad +2 \prec [v_2^*, v_{l2}^*, v_{r2}^*]_{LR}, [v_2^\bullet, v_{l2}^\bullet, v_{r2}^\bullet]_{LR} \succ\\
&\quad = \prec [32, 7, 4]_{LR}, [25, 3, 4]_{LR} \succ .
\end{aligned}$$

To solve the $LR$-BFLS equation in (44), first we take the positive($*$) part of the system (44) given as:

$$(v_1^*, v_{l1}^*, v_{r1}^*)_{LR} - (v_2^*, v_{l2}^*, v_{r2}^*)_{LR} = (-4, 4, 4)_{LR},$$
$$(v_1^*, v_{l1}^*, v_{r1}^*)_{LR} + 2(v_2^*, v_{l2}^*, v_{r2}^*)_{LR} = (32, 7, 4)_{LR}.$$

To solve this positive part we can divide this system into two systems, one is mean value system and other is left-right spread system.

The mean value system for positive part of (44) is given as:

$$v_1^* - v_2^* = -4,$$
$$v_1^* + 2v_2^* = 32,$$

by solving above equations, we have $v_1^* = 8$ and $v_2^* = 12$. Now the extended $4 \times 4$ form of positive part of left-right spread is given as:

$$1(v_{l1}^*) + 0(v_{l2}^*) + 0(v_{r1}^*) + 1(v_{r2}^*) = 4,$$
$$1(v_{l1}^*) + 2(v_{l2}^*) + 0(v_{r1}^*) + 0(v_{r2}^*) = 7,$$
$$0(v_{l1}^*) + 1(v_{l2}^*) + 1(v_{r1}^*) + 0(v_{r2}^*) = 4,$$
$$0(v_{l1}^*) + 0(v_{l2}^*) + 1(v_{r1}^*) + 2(v_{r2}^*) = 4.$$

The matrix form of the above-mentioned system is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} v_{l1}^* \\ v_{l2}^* \\ v_{r1}^* \\ v_{r2}^* \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ 4 \\ 4 \end{pmatrix},$$

this can also be written as:

$$M(v_l^*, v_r^*) = (f_l^*, f_r^*),$$

and the solution of above system is,

$$\begin{pmatrix} v_{l1}^* \\ v_{l2}^* \\ v_{r1}^* \\ v_{r2}^* \end{pmatrix} = \begin{pmatrix} 1.3333 & -0.3333 & 0.6667 & -0.6667 \\ -0.6667 & 0.6667 & -0.3333 & 0.3333 \\ 0.6667 & -0.6667 & 1.3333 & -0.3333 \\ -0.3333 & 0.3333 & -0.6667 & 0.6667 \end{pmatrix} \begin{pmatrix} 4 \\ 7 \\ 4 \\ 4 \end{pmatrix}$$

$$v_{l1}^* = 2, \quad v_{l2}^* = 1,$$
$$v_{r1}^* = 1, \quad v_{r2}^* = 1.$$

Similarly, we can take the negative($\bullet$) part of the system (44) given as:

$$(v_1^\bullet, v_{l1}^\bullet, v_{r1}^\bullet)_{LR} - (v_2^\bullet, v_{l2}^\bullet, v_{r2}^\bullet)_{LR} = (-5, 2, 3)_{LR},$$
$$(v_1^\bullet, v_{l1}^\bullet, v_{r1}^\bullet)_{LR} + 2(v_2^*, v_{l2}^\bullet, v_{r2}^\bullet)_{LR} = (25, 3, 4)_{LR}.$$

To solve this negative part of $LR$-BFLS of equation we can divide this system into two systems, one is mean value system and other is left-right spread system.

The mean value system for negative part is given as:

$$v_1^\bullet - v_2^\bullet = -5,$$
$$v_1^\bullet + 2v_2^\bullet = 25,$$

by solving above equations, we have $v_1^\bullet = 5$ and $v_2^\bullet = 10$. Now the extended $4 \times 4$ form of negative part of left-right spread is given as:

$$1(v_{l1}^\bullet) + 0(v_{l2}^\bullet) + 0(v_{r1}^\bullet) + 1(v_{r2}^\bullet) = 2,$$
$$1(v_{l1}^\bullet) + 2(v_{l2}^\bullet) + 0(v_{r1}^\bullet) + 0(v_{r2}^\bullet) = 3,$$
$$0(v_{l1}^\bullet) + 1(v_{l2}^\bullet) + 1(v_{r1}^\bullet) + 0(v_{r2}^\bullet) = 3,$$
$$0(v_{l1}^\bullet) + 0(v_{l2}^\bullet) + 1(v_{r1}^\bullet) + 2(v_{r2}^\bullet) = 4.$$

The matrix form of the above-mentioned system is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} v_{l1}^\bullet \\ v_{l2}^\bullet \\ v_{r1}^\bullet \\ v_{r2}^\bullet \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 3 \\ 4 \end{pmatrix},$$

this can also be written as:

$$M(v_l^\bullet, v_r^\bullet) = (f_l^\bullet, f_r^\bullet),$$

and the solution of above system is,

$$\begin{pmatrix} v_{l1}^\bullet \\ v_{l2}^\bullet \\ v_{r1}^\bullet \\ v_{r2}^\bullet \end{pmatrix} = \begin{pmatrix} 1.3333 & -0.3333 & 0.6667 & -0.6667 \\ -0.6667 & 0.6667 & -0.3333 & 0.3333 \\ 0.6667 & -0.6667 & 1.3333 & -0.3333 \\ -0.3333 & 0.3333 & -0.6667 & 0.6667 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 3 \\ 4 \end{pmatrix}$$

$$v_{l1}^\bullet = 1, \quad v_{l2}^\bullet = 1,$$
$$v_{r1}^\bullet = 2, \quad v_{r2}^\bullet = 1.$$

Hence, the solution for imaginary part of $LR$-CBFL system of equations with real coefficients in (4) is given as:

$$\widetilde{V}_1 = \prec [v_1^*, v_{l1}^*, v_{r1}^*]_{LR}, [v_1^\bullet, v_{l1}^\bullet, v_{r1}^\bullet]_{LR} \succ$$
$$= \prec [8, 3, 2]_{LR}, [5, 1, 2]_{LR} \succ,$$
$$\widetilde{V}_2 = \prec [v_2^*, v_{l2}^*, v_{r2}^*]_{LR}, [v_2^\bullet, v_{l2}^\bullet, v_{r2}^\bullet]_{LR} \succ$$

$$= \prec [12, 2, 1]_{LR}, [10, 1, 1]_{LR} \succ .$$

Hence, the solution for $LR$-CBFL system of equation with real coefficients is given as:

$$\begin{aligned}\widetilde{H}_1 &= \widetilde{u}_1 + i\widetilde{v}_1 = \prec [5, 2, 1]_{LR}, [3, 2, 1]_{LR} \succ \\ &\quad + i \prec [8, 3, 2]_{LR}, [5, 1, 2]_{LR} \succ, \\ \widetilde{H}_2 &= \widetilde{u}_2 + i\widetilde{v}_2 = \prec [8, 1, 1]_{LR}, [6, 2, 1]_{LR} \succ \\ &\quad + i \prec [12, 2, 1]_{LR}, [10, 1, 1]_{LR} \succ .\end{aligned}$$

### 4.2 Solution procedure to n × n *LR*-CBFL system with complex coefficients

**Definition 21** Consider the $n \times n$ complex linear system of equations given as:

$$\begin{cases} l_{11}h_1 + l_{12}h_2 + \cdots + l_{1n}h_n &= z_1, \\ l_{21}h_1 + l_{22}h_2 + \cdots + l_{2n}h_n &= z_2, \\ . \\ . \\ . \\ l_{n1}h_1 + l_{n2}h_2 + \cdots + l_{nn}h_n &= z_n, \end{cases} \tag{45}$$

is the $LR$-complex bipolar fuzzy linear (CBFL) system of equations with complex coefficients, if all the coefficient elements $l_{pq}, 1 \le p, q \le n$ is a complex number matrix of order $n \times n$ and each $Z_q$, $1 \le q \le n$ is a $LR$-BFCN, and each unknown parameter $h_p$, $1 \le p \le n$ is also $LR$-BFCN.

The matrix form of the $LR$-CBFL system of equation with complex coefficients in (45) given as:

$$L\widetilde{H}_p = \widetilde{Z}_q. \tag{46}$$

A $LR$-bipolar fuzzy complex number vector

$$\widetilde{H} = (\widetilde{h}_1, \widetilde{h}_2, \ldots, \widetilde{h}_n)^T, \tag{47}$$

where $\widetilde{h}_q = \widetilde{u}_q + i\widetilde{v}_q$, $1 \le q \le n$ is called a $LR$-bipolar fuzzy complex solution of the $LR$-CBFL system given in (45), if $\widetilde{H}$ satisfies (46).

**Definition 22** An arbitrary $LR$-bipolar fuzzy complex vector should be represented as $\widetilde{h}_q = \widetilde{u}_q + i\widetilde{v}_q$, where $\widetilde{u}_q = \prec [u^*_q, u^*_{lq}, u^*_{rq}]_{LR}, [u^\bullet_q, u^\bullet_{lq}, u^\bullet_{rq}]_{LR} \succ$ and $\widetilde{v}_q = \prec [v^*_q, v^*_{lq}, v^*_{rq}]_{LR}, [v^\bullet_q, v^\bullet_{lq}, v^\bullet_{rq}]_{LR} \succ$ are two $LR$-bipolar fuzzy number vectors. The $LR$-bipolar fuzzy complex vector can be written as:

$$\begin{aligned}\widetilde{H}_q &= \prec [u^*_q, u^*_{lq}, u^*_{rq}]_{LR}, [u^\bullet_q, u^\bullet_{lq}, u^\bullet_{rq}]_{LR} \succ \\ &\quad + i \prec [v^*_q, v^*_{lq}, v^*_{rq}]_{LR}, [v^\bullet_q, v^\bullet_{lq}, v^\bullet_{rq}]_{LR} \succ .\end{aligned}$$

**Theorem 4** *The $n \times n$ $LR$-CBFL system of equations with complex coefficients in (45) is equivalent to pair of $2n \times 2n$ order $LR$-bipolar fuzzy linear system*

$$M\widetilde{h}^*_q = \widetilde{z}^*_q, \quad M\widetilde{h}^\bullet_q = \widetilde{z}^\bullet_q, \tag{48}$$

*where,*

$$M = \begin{pmatrix} L_1 & -L_2 \\ L_2 & L_1 \end{pmatrix}, \quad \widetilde{h}^*_q = \begin{pmatrix} \widetilde{u}^*_q \\ \widetilde{v}^*_q \end{pmatrix}, \quad \widetilde{z}^*_q = \begin{pmatrix} \widetilde{e}^*_g \\ \widetilde{f}^*_q \end{pmatrix},$$
$$h^*_q = \begin{pmatrix} \widetilde{u}^*_q \\ \widetilde{v}^*_q \end{pmatrix}, \quad \widetilde{z}^*_q = \begin{pmatrix} \widetilde{e}^*_g \\ \widetilde{f}^*_q \end{pmatrix}. \tag{49}$$

***Proof*** We defined $L = L_1 + iL_2$ where $L_1, L_2 \epsilon \Re$ and $\widetilde{z}_q = \widetilde{e}_q + i\widetilde{f}_q$ where $\widetilde{e}_q$ and $\widetilde{f}_q$ are $LR$-bipolar fuzzy number vector. We also suppose the unknown vector $\widetilde{H}_q = \widetilde{u}_q + i\widetilde{v}_q$, where $\widetilde{u}_q$ and $\widetilde{v}_q$ are two unknown $LR$-bipolar fuzzy number vectors.

First we take the positive($*$) part of the system given in (46), i.e.,

$$L\widetilde{H}^*_q = \widetilde{z}^*_q,$$

the above-mentioned equation can also be written as

$$(L_1 + iL_2)(\widetilde{u}^*_q + i\widetilde{v}^*_q) = \widetilde{e}^*_q + i\widetilde{f}^*_q.$$

That is,

$$(L_1\widetilde{u}^*_q - L_2\widetilde{v}^*_q) + i(L_1\widetilde{v}^*_q + L_2\widetilde{u}^*_q) = \widetilde{e}^*_q + i\widetilde{f}^*_q.$$

Compare real and complex coefficients of above equation we get,

$$\begin{aligned}L_1\widetilde{u}^*_q - L_2\widetilde{v}^*_q &= \widetilde{e}^*_q, \\ L_1\widetilde{v}^*_q + L_2\widetilde{u}^*_q &= \widetilde{f}^*_q.\end{aligned} \tag{50}$$

The matrix form of above-mentioned system is give as

$$\begin{pmatrix} L_1 & -L_2 \\ L_2 & L_1 \end{pmatrix} \begin{pmatrix} \widetilde{u}^*_q \\ \widetilde{v}^*_q \end{pmatrix} = \begin{pmatrix} \widetilde{e}^*_g \\ \widetilde{f}^*_q \end{pmatrix}, \tag{51}$$

which is $2n \times 2n$ order $LR$-BFLS equations.

we can express it in matrix form as:

$$M\widetilde{h}^*_q = \widetilde{z}^*_q.$$

Similarly, we can take the negative ($\bullet$) part of the system given in (46), i.e.,

$$L\widetilde{H}^\bullet_q = \widetilde{z}^\bullet_q,$$

the above-mentioned equation can also be written as

$$(L_1 + iL_2)(\widetilde{u}_q^{\bullet} + i\widetilde{v}_q^{\bullet}) = \widetilde{e}_q^{\bullet} + i\widetilde{f}_q^{\bullet}.$$

That is,

$$(L_1\widetilde{u}_q^{\bullet} - L_2\widetilde{v}_q^{\bullet}) + i(L_1\widetilde{v}_q^{\bullet} + L_2\widetilde{u}_q^{\bullet}) = \widetilde{e}_q^{\bullet} + i\widetilde{f}_q^{\bullet}.$$

Compare real and complex coefficients of above equation we get,

$$\begin{aligned} L_1\widetilde{u}_q^{\bullet} - L_2\widetilde{v}_q^{\bullet} &= \widetilde{e}_q^{\bullet}, \\ L_1\widetilde{v}_q^{\bullet} + L_2\widetilde{u}_q^{\bullet} &= \widetilde{f}_q^{\bullet}. \end{aligned} \tag{52}$$

The matrix form of above-mentioned system is give as

$$\begin{pmatrix} L_1 & -L_2 \\ L_2 & L_1 \end{pmatrix} \begin{pmatrix} \widetilde{u}_q^{\bullet} \\ \widetilde{v}_q^{\bullet} \end{pmatrix} = \begin{pmatrix} \widetilde{e}_g^{\bullet} \\ \widetilde{f}_q^{\bullet} \end{pmatrix}, \tag{53}$$

which is $2n \times 2n$ order $LR$-BFLS of equation.
we can express it in matrix form as:

$$M\widetilde{h}_q^{\bullet} = \widetilde{z}_q^{\bullet}.$$

□

**Theorem 5** *The $LR$-bipolar fuzzy linear systems in (48) can be extended into the following linear system of matrix equations:*

$$(M^+ + M^-)h_q^* = z_q^*, \quad (M^+ + M^-)h_q^{\bullet} = z_q^{\bullet},$$

$$\begin{pmatrix} M^+ & -M^- \\ -M^- & M^+ \end{pmatrix} \begin{pmatrix} h_{lq}^* \\ h_{rq}^* \end{pmatrix} = \begin{pmatrix} z_{lq}^* \\ z_{rq}^* \end{pmatrix},$$

$$\begin{pmatrix} M^+ & -M^- \\ -M^- & M^+ \end{pmatrix} \begin{pmatrix} h_{lq}^{\bullet} \\ h_{rq}^{\bullet} \end{pmatrix} = \begin{pmatrix} z_{lq}^{\bullet} \\ z_{rq}^{\bullet} \end{pmatrix},$$

*where*

$$M = \begin{pmatrix} M^+ & -M^- \\ -M^- & M^+ \end{pmatrix} = M^+ + M^-,$$

$$\begin{aligned} \widetilde{h}_q &= \prec [h_q^*, h_{lq}^*, h_{rq}^*]_{LR}, [h_q^{\bullet}, h_{lq}^{\bullet}, h_{rq}^{\bullet}]_{LR} \succ \\ &= \prec \left( \begin{pmatrix} u_q^* \\ v_q^* \end{pmatrix} \begin{pmatrix} u_{lq}^* \\ v_{lq}^* \end{pmatrix} \begin{pmatrix} u_{rq}^* \\ v_{rq}^* \end{pmatrix} \right), \\ &\quad \left( \begin{pmatrix} u_q^{\bullet} \\ v_q^{\bullet} \end{pmatrix} \begin{pmatrix} u_{lq}^{\bullet} \\ v_{lq}^{\bullet} \end{pmatrix} \begin{pmatrix} u_{rq}^{\bullet} \\ v_{rq}^{\bullet} \end{pmatrix} \right) \succ, \\ \widetilde{z}_q &= \prec [z_q^*, z_{lq}^*, z_{rq}^*]_{LR}, [z_q^{\bullet}, z_{lq}^{\bullet}, z_{rq}^{\bullet}]_{LR} \succ \\ &= \prec \left( \begin{pmatrix} e_q^* \\ f_q^* \end{pmatrix} \begin{pmatrix} e_{lq}^* \\ f_{lq}^* \end{pmatrix} \begin{pmatrix} e_{rq}^* \\ f_{rq}^* \end{pmatrix} \right), \\ &\quad \left( \begin{pmatrix} e_q^{\bullet} \\ f_q^{\bullet} \end{pmatrix} \begin{pmatrix} e_{lq}^{\bullet} \\ f_{lq}^{\bullet} \end{pmatrix} \begin{pmatrix} e_{rq}^{\bullet} \\ f_{rq}^{\bullet} \end{pmatrix} \right) \succ, \end{aligned}$$

*where the elements $m_{pq}^+$ and $m_{pq}^-$ are determined as follow: if $m_{pq} \geq 0$, $m_{pq}^+ = m_{pq}$ otherwise $m_{pq}^+ = 0$, $1 \leq p, q \leq 2n$; if $m_{pq} < 0$, $m_{pq}^- = m_{pq}$ otherwise $m_{pq}^- = 0$, $1 \leq p, q \leq 2n$.*

***Proof*** We denote

$$\begin{aligned} \widetilde{H}_q &= \widetilde{u}_q + i\widetilde{v}_q = \prec [u_q^*, u_{lq}^*, u_{rq}^*]_{LR}, [u_q^{\bullet}, u_{lq}^{\bullet}, u_{rq}^{\bullet}]_{LR} \succ \\ &\quad + i \prec [v_q^*, v_{lq}^*, v_{rq}^*]_{LR}, [v_q^{\bullet}, v_{lq}^{\bullet}, v_{rq}^{\bullet}]_{LR} \succ, \\ \widetilde{z}_q &= \widetilde{u}_q + i\widetilde{v}_q = \prec [e_q^*, e_{lq}^*, e_{rq}^*]_{LR}, [e_q^{\bullet}, e_{lq}^{\bullet}, e_{rq}^{\bullet}]_{LR} \succ \\ &\quad + i \prec [f_q^*, f_{lq}^*, f_{rq}^*]_{LR}, [f_q^{\bullet}, f_{lq}^{\bullet}, f_{rq}^{\bullet}]_{LR} \succ, \end{aligned}$$

where $u_q^*$, $v_q^*$ are the mean values and $u_{lq}^*$, $v_{lq}^*$, $u_{rq}^*$ and $v_{rq}^*$ are left-right spread values of $LR$-bipolar fuzzy number vector $\widetilde{u}_q, \widetilde{v}_q$, respectively.

First we take the positive($*$) part of $LR$-bipolar fuzzy linear system $M\widetilde{h}_q^* = \widetilde{z}_q^*$ is

$$M(h_q^*, h_{lq}^*, h_{rq}^*)_{LR} = (z_q^*, z_{lq}^*, z_{rq}^*)_{LR}. \tag{54}$$

Let

$$M = \begin{pmatrix} L_1 & -L_2 \\ L_2 & L_1 \end{pmatrix} = M^+ + M^-, \tag{55}$$

where $m_{pq}^+$ and $m_{pq}^-$ are determined as follows: if $m_{pq} \geq 0$, $m_{pq}^+ = m_{pq}$ otherwise $m_{pq}^+ = 0, 1 \leq p, q \leq 2n$; if $m_{pq} < 0$, $m_{pq}^- = m_{pq}$ other $m_{pq}^- = 0$, $1 \leq p, q \leq 2n$.
Since

$$c\widetilde{h}_q = \begin{cases} (ch, ch_{lq}, ch_{rq})_{LR}, & c \geq 0, \\ (ch, -ch_{rq}, -ch_{lq})_{LR}, & c \leq 0, \end{cases} \tag{56}$$

we have

$$G\widetilde{h}_q^* = \begin{cases} (Gh_q^*, Gh_{lq}^*, Gh_{rq}^*)_{LR}, & G \geq 0, \\ (Gh_q^*, -Gh_{rq}^*, -Gh_{lq}^*)_{LR}, & G \leq 0. \end{cases} \tag{57}$$

Equation (54) becomes

$$(M^+ + M^-)(h_q^*, h_{lq}^*, h_{rq}^*)_{LR} = (z_q^*, z_{lq}^*, z_{rq}^*)_{LR},$$

which is equivalent to,

$$\begin{aligned} &M^+(h_q^*, h_{lq}^*, h_{rq}^*)_{LR} + M^-(h_q^*, h_{lq}^*, h_{rq}^*)_{LR} \\ &\quad = (z_q^*, z_{lq}^*, z_{rq}^*)_{LR}, \\ &(M^+h_q^*, M^+h_{lq}^*, M^+h_{rq}^*)_{LR} + (M^-h_q^*, -M^-h_{rq}^*, -M^-h_{lq}^*)_{LR} \\ &\quad = (z_q^*, z_{lq}^*, z_{rq}^*)_{LR}, \\ &(M^+h_q^* + M^-h_q^*, M^+h_{lq}^* - M^-h_{rq}^*, M^+h_{rq}^* - M^-h_{lq}^*)_{LR} \\ &\quad = (z_q^*, z_{lq}^*, z_{rq}^*)_{LR}, \end{aligned}$$

by comparing we have,

$$(M^+ + M^-)h_q^* = z_q^*,$$

$$M^+h^*_{lq} - M^-h^*_{rq} = z^*_{lq},$$
$$M^+h^*_{rq} - M^-h^*_{lq} = z^*_{rq}. \tag{58}$$

Representing the system of Eq. (58) in matrix form, we have

$$(M^+ + M^-)\begin{pmatrix} u^*_q \\ v^*_q \end{pmatrix} = \begin{pmatrix} e^*_q \\ f^*_q \end{pmatrix},$$
$$\begin{pmatrix} M^+ & -M^- \\ -M^- & M^+ \end{pmatrix}\begin{pmatrix} u^*_{lq} \\ v^*_{lq} \\ u^*_{rq} \\ v^*_{rq} \end{pmatrix} = \begin{pmatrix} e^*_{lq} \\ f^*_{lq} \\ e^*_{rq} \\ f^*_{rq} \end{pmatrix}.$$

Similarly, we can take the negative($\bullet$) part of $LR$-bipolar fuzzy linear system $M\widetilde{h}^\bullet_q = \widetilde{z}^\bullet_q$ is

$$M(h^\bullet_q, h^\bullet_{lq}, h^\bullet_{rq})_{LR} = (z^\bullet_q, z^\bullet_{lq}, z^\bullet_{rq})_{LR}. \tag{59}$$

Let

$$M = \begin{pmatrix} L_1 & -L_2 \\ L_2 & L_1 \end{pmatrix} = M^+ + M^-, \tag{60}$$

where $m^+_{pq}$ and $m^-_{pq}$ are determined as follows: if $m_{pq} \geq 0$, $m^+_{pq} = m_{pq}$ otherwise $m^+_{pq} = 0, 1 \leq p, q \leq 2n$; if $m_{pq} < 0$, $m^-_{pq} = m_{pq}$ otherwise $m^-_{pq} = 0, 1 \leq p, q \leq 2n$.
Since

$$c\widetilde{h}_q = \begin{cases} (ch, ch_{lq}, ch_{rq})_{LR}, & c \geq 0, \\ (ch, -ch_{rq}, -ch_{lq})_{LR}, & c \leq 0, \end{cases} \tag{61}$$

we have

$$G\widetilde{h}^\bullet_q = \begin{cases} (Gh^\bullet_q, Gh^\bullet_{lq}, Gh^\bullet_{rq})_{LR}, & G \geq 0, \\ (Gh^\bullet_q, -Gh^\bullet_{rq}, -Gh^\bullet_{lq})_{LR}, & G \leq 0. \end{cases} \tag{62}$$

Equation (59) becomes

$$(M^+ + M^-)(h^\bullet_q, h^\bullet_{lq}, h^\bullet_{rq})_{LR} = (z^\bullet_q, z^\bullet_{lq}, z^\bullet_{rq})_{LR},$$

which is equivalent to,

$$M^+(h^*_q, h^\bullet_{lq}, h^\bullet_{rq})_{LR} + M^-(h^\bullet_q, h^\bullet_{lq}, h^\bullet_{rq})_{LR}$$
$$= (z^\bullet_q, z^\bullet_{lq}, z^\bullet_{rq})_{LR},$$
$$(M^+h^\bullet_q, M^+h^\bullet_{lq}, M^+h^\bullet_{rq})_{LR}$$
$$+(M^-h^\bullet_q, -M^-h^\bullet_{rq}, -M^-h^\bullet_{lq})_{LR}$$
$$= (z^\bullet_q, z^\bullet_{lq}, z^\bullet_{rq})_{LR},$$
$$(M^+h^\bullet_q + M^-h^\bullet_q, M^+h^\bullet_{lq} - M^-h^\bullet_{rq}, M^+h^\bullet_{rq} - M^-h^\bullet_{lq})_{LR}$$
$$= (z^\bullet_q, z^\bullet_{lq}, z^\bullet_{rq})_{LR},$$

by comparing we have,

$$(M^+ + M^-)h^\bullet_q = z^\bullet_q,$$
$$M^+h^\bullet_{lq} - M^-h^\bullet_{rq} = z^\bullet_{lq},$$
$$M^+h^\bullet_{rq} - M^-h^\bullet_{lq} = z^\bullet_{rq}. \tag{63}$$

Representing the system of Eq. (63) in matrix form, we have

$$(M^+ + M^-)\begin{pmatrix} u^\bullet_q \\ v^\bullet_q \end{pmatrix} = \begin{pmatrix} e^\bullet_q \\ f^\bullet_q \end{pmatrix},$$
$$\begin{pmatrix} M^+ & -M^- \\ -M^- & M^+ \end{pmatrix}\begin{pmatrix} u^\bullet_{lq} \\ v^\bullet_{lq} \\ u^\bullet_{rq} \\ v^\bullet_{rq} \end{pmatrix} = \begin{pmatrix} e^\bullet_{lq} \\ f^\bullet_{lq} \\ e^\bullet_{rq} \\ f^\bullet_{rq} \end{pmatrix}.$$

□

***Example 5*** We consider a simple RLC circuit (Rahgooy et al. 2009) in which current and source all are in terms of $LR$-BFCN. A circuit with $LR$-BFCN source and current is shown in Figure 1. The source and current are in terms of bipolar fuzzy number, the positive part shows likelihood of source and current and negative part shows unlikelihood of source and current.

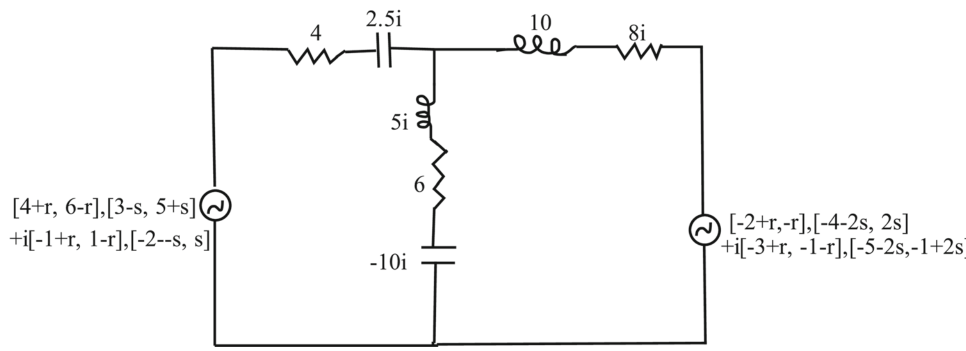
We use Kirchhoff's second law for the above circuit and write the linear equations for first and second loop is given as:

$$\begin{cases} (10 - 7.5i)I_1 - (6 - 5i)I_2 = \prec [5, 1, 1]_{LR}, [4, 1, 1]_{LR} \succ \\ \quad +i \prec [0, 1, 1]_{LR}, [-1, 1, 1]_{LR} \succ, \\ -(6 - 5i)I_1 + (16 + 3i)I_2 = \prec [-1, , 1, 1]_{LR}, \\ \quad [-2, 1, 1]_{LR} \succ +i \prec [-2, 1, 1]_{LR}, [-3, 1, 1]_{LR} \succ . \end{cases}$$

We can put $\widetilde{I}_1 = \widetilde{u}_1 + i\widetilde{v}_1 = \prec [u^*_1, u^*_{l1}, u^*_{r1}]_{LR}, [u^\bullet_1, u^\bullet_{l1}, u^\bullet_{r1}]_{LR} \succ +i \prec [v^*_1, v^*_{l1}, v^*_{r1}]_{LR}, [v^\bullet_1, v^\bullet_{l1}, v^\bullet_{r1}]_{LR} \succ$, and $\widetilde{I}_2 = \widetilde{u}_2 + i\widetilde{v}_2 = \prec [u^*_2, u^*_{l2}, u^*_{r2}]_{LR}, [u^\bullet_2, u^\bullet_{l2}, u^\bullet_{r2}]_{LR} \succ +i \prec [v^*_2, v^*_{l2}, v^*_{r2}]_{LR}, [v^\bullet_2, v^\bullet_{l2}, v^\bullet_{r2}]_{LR} \succ$, in above-mentioned system then system become,
$(10 - 7.5i)\{\prec [u^*_1, u^*_{l1}, u^*_{r1}]_{LR}, [u^\bullet_1, u^\bullet_{l1}, u^\bullet_{r1}]_{LR} \succ +i \prec [v^*_1, v^*_{l1}, v^*_{r1}]_{LR}, [v^\bullet_1, v^\bullet_{l1}, v^\bullet_{r1}]_{LR} \succ\} - (6 - 5i)\ \{\prec [u^*_2, u^*_{l2}, u^*_{r2}]_{LR}, [u^\bullet_2, u^\bullet_{l2}, u^\bullet_{r2}]_{LR} \succ +i \prec [v^*_2, v^*_{l2}, v^*_{r2}]_{LR}, [v^\bullet_2, v^\bullet_{l2}, v^\bullet_{r2}]_{LR} \succ\} = \prec [5, 1, 1]_{LR}, [4, 1, 1]_{LR} \succ +i \prec [0, 1, 1]_{LR}, [-1, 1, 1]_{LR} \succ, -(6-5i)\{\prec [u^*_1, u^*_{l1}, u^*_{r1}]_{LR}, [u^\bullet_1, u^\bullet_{l1}, u^\bullet_{r1}]_{LR} \succ +i \prec [v^*_1, v^*_{l1}, v^*_{r1}]_{LR}, [v^\bullet_1, v^\bullet_{l1}, v^\bullet_{r1}]_{LR} \succ\} + (16 + 3i)\ \{\prec [u^*_2, u^*_{l2}, u^*_{r2}]_{LR}, [u^\bullet_2, u^\bullet_{l2}, u^\bullet_{r2}]_{LR} \succ +i \prec [v^*_2, v^*_{l2}, v^*_{r2}]_{LR}, [v^\bullet_2, v^\bullet_{l2}, v^\bullet_{r2}]_{LR} \succ\} = \prec [-1, , 1, 1]_{LR}, [-2, 1, 1]_{LR} \succ +i \prec [-2, 1, 1]_{LR}, [-3, 1, 1]_{LR} \succ$.

First, we take the positive($*$) part of the above-mentioned system of equations,

**Fig. 1** A circuit with $LR$-BFCN source and current




$(10-7.5i)\{(u_1^*, u_{l1}^*, u_{r1}^*)_{LR} + i(v_1^*, v_{l1}^*, v_{r1}^*)_{LR}\} - (6-5i)\{(u_2^*, u_{l2}^*, u_{r2}^*)_{LR} + i(v_2^*, v_{l2}^*, v_{r2}^*)_{LR}\} = (5,1,1)_{LR} + i(0,1,1)_{LR}$,
$-(6-5i)\{(u_1^*, u_{l1}^*, u_{r1}^*)_{LR} + i(v_1^*, v_{l1}^*, v_{r1}^*)_{LR}\} + (16+3i)\{(u_2^*, u_{l2}^*, u_{r2}^*)_{LR} + i(v_2^*, v_{l2}^*, v_{r2}^*)_{LR}\} = (-1,,1,1)_{LR} + i(-2,1,1)_{LR}$.

According to Theorem (4), the $LR$-CBFL system of equation with complex coefficients is converted into positive part of $LR$-bipolar linear system $M\widetilde{I}_q^* = \widetilde{z}_q^*$, given as

$$\begin{pmatrix} 10 & -6 & 7.5 & -5 \\ -6 & 16 & -5 & -3 \\ -7.5 & 5 & 10 & -6 \\ 5 & 3 & -6 & 16 \end{pmatrix} \begin{pmatrix} \widetilde{u}_1^* \\ \widetilde{u}_2^* \\ \widetilde{v}_1^* \\ \widetilde{v}_2^* \end{pmatrix} = \begin{pmatrix} (5,1,1)_{LR} \\ (-1,1,1)_{LR} \\ (0,1,1)_{LR} \\ (-2,1,1)_{LR} \end{pmatrix}.$$

Applying Theorem (5), we get two linear system one is mean value system and other is left-right spread system given as:

$$\begin{pmatrix} 10 & -6 & 7.5 & -5 \\ -6 & 16 & -5 & -3 \\ -7.5 & 5 & 10 & -6 \\ 5 & 3 & -6 & 16 \end{pmatrix} \begin{pmatrix} u_1^* \\ u_2^* \\ v_1^* \\ v_2^* \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 0 \\ -2 \end{pmatrix}.$$

$$\begin{pmatrix} 10 & 0 & 7.5 & 0 & 0 & 6 & 0 & 5 \\ 0 & 16 & 0 & 0 & 6 & 0 & 5 & 3 \\ 0 & 5 & 10 & 0 & 7.5 & 0 & 0 & 6 \\ 5 & 3 & 0 & 16 & 0 & 0 & 6 & 0 \\ 0 & 6 & 0 & 5 & 10 & 0 & 7.5 & 0 \\ 6 & 0 & 5 & 3 & 0 & 16 & 0 & 0 \\ 7.5 & 0 & 0 & 6 & 0 & 5 & 10 & 0 \\ 0 & 0 & 6 & 0 & 5 & 3 & 0 & 16 \end{pmatrix} \begin{pmatrix} u_{l1}^* \\ u_{l2}^* \\ v_{l1}^* \\ v_{l2}^* \\ u_{r1}^* \\ u_{r2}^* \\ v_{r1}^* \\ v_{r2}^* \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Solution for mean value system is given as

$$\begin{pmatrix} u_1^* \\ u_2^* \\ v_1^* \\ v_2^* \end{pmatrix} = \begin{pmatrix} 0.0876 & 0.0389 & -0.0328 & 0.0224 \\ 0.0389 & 0.0640 & 0.0224 & -0.0325 \\ 0.0328 & -0.0224 & 0.0876 & 0.0389 \\ -0.0224 & -0.0325 & 0.0389 & 0.0640 \end{pmatrix} \begin{pmatrix} 5 \\ -1 \\ 0 \\ -2 \end{pmatrix}$$
$$= \begin{pmatrix} 0.3542 \\ 0.0654 \\ 0.1086 \\ -0.2072 \end{pmatrix}.$$

Solution for left-right spread system given as

$$\begin{pmatrix} u_{l1}^* \\ u_{l2}^* \\ v_{l1}^* \\ v_{l2}^* \\ u_{r1}^* \\ u_{r2}^* \\ v_{r1}^* \\ v_{r2}^* \end{pmatrix} = \begin{pmatrix} 0.1694 & -0.0015 & -0.0971 & 0.0061 & 0.0819 & -0.0404 & -0.0643 & -0.0612 \\ -0.0015 & 0.0725 & 0.0061 & 0.0171 & -0.0404 & 0.0085 & -0.0162 & -0.0154 \\ -0.0643 & -0.0162 & 0.1694 & -0.0015 & -0.971 & 0.0061 & 0.0819 & -0.0404 \\ -0.0162 & -0.0154 & -0.0015 & 0.0725 & 0.0061 & 0.0171 & -0.0404 & 0.0085 \\ 0.0819 & -0.0404 & -0.0643 & -0.0162 & 0.1694 & -0.0015 & -0.0971 & 0.0061 \\ -0.0404 & 0.0085 & -0.0162 & -0.0154 & -0.0015 & 0.0725 & 0.0061 & 0.0171 \\ -0.0971 & 0.0061 & 0.0819 & -0.0404 & -0.0643 & -0.0162 & 0.1694 & -0.0015 \\ -0.0061 & 0.0171 & -0.0404 & 0.0085 & -0.0162 & -0.0154 & -0.0015 & 0.0725 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} u_{l1}^* \\ u_{l2}^* \\ v_{l1}^* \\ v_{l2}^* \\ u_{r1}^* \\ u_{r2}^* \\ v_{r1}^* \\ v_{r2}^* \end{pmatrix} = \begin{pmatrix} 0.0378 \\ 0.0307 \\ 0.0378 \\ 0.0307 \\ 0.0378 \\ 0.0307 \\ 0.0378 \\ 0.0307 \end{pmatrix}.$$

Similarly, we can take the negative($\bullet$) part of the $LR$-CBFL system of equations with complex coefficients,
$(10-7.5i)\{(u_1^\bullet, u_{l1}^\bullet, u_{r1}^\bullet)_{LR} + i(v_1^\bullet, v_{l1}^\bullet, v_{r1}^\bullet)_{LR}\} - (6-5i)\{(u_2^\bullet, u_{l2}^\bullet, u_{r2}^\bullet)_{LR} + i(v_2^\bullet, v_{l2}^\bullet, v_{r2}^\bullet)_{LR}\} = (4,1,1)_{LR} + i(-1,1,1)_{LR}$,

$-(6-5i)\{(u_1^\bullet, u_{l1}^\bullet, u_{r1}^\bullet)_{LR} + i(v_1^\bullet, v_{l1}^\bullet, v_{r1}^\bullet)_{LR}\} + (16+3i)\{(u_2^\bullet, u_{l2}^\bullet, u_{r2}^\bullet)_{LR} + i(v_2^\bullet, v_{l2}^\bullet, v_{r2}^\bullet)_{LR}\} = (-2,1,1)_{LR} + i(-3,1,1)_{LR}$.

According to Theorem (4), the $LR$-CBFL system of equation with complex coefficients is converted into negative part of $LR$-bipolar linear system $M\widetilde{I}_q^\bullet = \widetilde{z}_q^\bullet$, given as

$$\begin{pmatrix} 10 & -6 & 7.5 & -5 \\ -6 & 16 & -5 & -3 \\ -7.5 & 5 & 10 & -6 \\ 5 & 3 & -6 & 16 \end{pmatrix} \begin{pmatrix} \widetilde{u}_1^\bullet \\ \widetilde{u}_2^\bullet \\ \widetilde{v}_1^\bullet \\ \widetilde{v}_2^\bullet \end{pmatrix} = \begin{pmatrix} (4,1,1)_{LR} \\ (-2,1,1)_{LR} \\ (-1,1,1)_{LR} \\ (-3,1,1)_{LR} \end{pmatrix}.$$

Applying Theorem (5), we get two linear system one is mean value system and other is left-right spread system given as:

$$\begin{pmatrix} 10 & -6 & 7.5 & -5 \\ -6 & 16 & -5 & -3 \\ -7.5 & 5 & 10 & -6 \\ 5 & 3 & -6 & 16 \end{pmatrix} \begin{pmatrix} u_1^\bullet \\ u_2^\bullet \\ v_1^\bullet \\ v_2^\bullet \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ -1 \\ -3 \end{pmatrix}.$$

$$\begin{pmatrix} 10 & 0 & 7.5 & 0 & 0 & 6 & 0 & 5 \\ 0 & 16 & 0 & 0 & 6 & 0 & 5 & 3 \\ 0 & 5 & 10 & 0 & 7.5 & 0 & 0 & 6 \\ 5 & 3 & 0 & 16 & 0 & 0 & 6 & 0 \\ 0 & 6 & 0 & 5 & 10 & 0 & 7.5 & 0 \\ 6 & 0 & 5 & 3 & 0 & 16 & 0 & 0 \\ 7.5 & 0 & 0 & 6 & 0 & 5 & 10 & 0 \\ 0 & 0 & 6 & 0 & 51 & 3 & 0 & 16 \end{pmatrix} \begin{pmatrix} u_{l1}^\bullet \\ u_{l2}^\bullet \\ v_{l1}^\bullet \\ v_{l2}^\bullet \\ u_{r1}^\bullet \\ u_{r2}^\bullet \\ v_{r1}^\bullet \\ v_{r2}^\bullet \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Solution for mean value system is given as

$$\begin{pmatrix} u_1^\bullet \\ u_2^\bullet \\ v_1^\bullet \\ v_2^\bullet \end{pmatrix} = \begin{pmatrix} 0.0876 & 0.0389 & -0.0328 & 0.0224 \\ 0.0389 & 0.0640 & 0.0224 & -0.0325 \\ 0.0328 & -0.0224 & 0.0876 & 0.0389 \\ -0.0224 & -0.0325 & 0.0389 & 0.0640 \end{pmatrix} \begin{pmatrix} 4 \\ -2 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 0.2382 \\ -0.0924 \\ -0.0283 \\ -0.2552 \end{pmatrix}.$$

Solution for left-right spread system given as

$$\begin{pmatrix} u_{l1}^\bullet \\ u_{l2}^\bullet \\ v_{l1}^\bullet \\ v_{l2}^\bullet \\ u_{r1}^\bullet \\ u_{r2}^\bullet \\ v_{r1}^\bullet \\ v_{r2}^\bullet \end{pmatrix} = \begin{pmatrix} 0.1694 & -0.0015 & -0.0971 & 0.0061 & 0.0819 & -0.0404 & -0.0643 & -0.0612 \\ -0.0015 & 0.0725 & 0.0061 & 0.0171 & -0.0404 & 0.0085 & -0.0162 & -0.0154 \\ -0.0643 & -0.0162 & 0.1694 & -0.0015 & -0.971 & 0.0061 & 0.0819 & -0.0404 \\ -0.0162 & -0.0154 & -0.0015 & 0.0725 & 0.0061 & 0.0171 & -0.0404 & 0.0085 \\ 0.0819 & -0.0404 & -0.0643 & -0.0162 & 0.1694 & -0.0015 & -0.0971 & 0.0061 \\ -0.0404 & 0.0085 & -0.0162 & -0.0154 & -0.0015 & 0.0725 & 0.0061 & 0.0171 \\ -0.0971 & 0.0061 & 0.0819 & -0.0404 & -0.0643 & -0.0162 & 0.1694 & -0.0015 \\ -0.0061 & 0.0171 & -0.0404 & 0.0085 & -0.0162 & -0.0154 & -0.0015 & 0.0725 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} u_{l1}^* \\ u_{l2}^* \\ v_{l1}^* \\ v_{l2}^* \\ u_{r1}^* \\ u_{r2}^* \\ v_{r1}^* \\ v_{r2}^* \end{pmatrix} = \begin{pmatrix} 0.0378 \\ 0.0307 \\ 0.0378 \\ 0.0307 \\ 0.0378 \\ 0.0307 \\ 0.0378 \\ 0.0307 \end{pmatrix}.$$

Hence, the solution is $LR$-BFCN for the $LR$-CBFL system of equation with complex coefficients given as
$\widetilde{I}_1 = \prec [0.3542, 0.0378, 0.0378]_{LR}, [0.2382, 0.0378, 0.0378]_{LR} \succ + i \prec [0.1086, 0.0378, 0.0378]_{LR}, [-0.0283, 0.0378, 0.0378]_{LR} \succ$,
$\widetilde{I}_2 = \prec [0.0654, 0.0307, 0.0307]_{LR}, [-0.0924, 0.0307, 0.0307]_{LR} \succ + i \prec [-0.2072, 0.0307, 0.0307]_{LR}, [-0.2552, 0.0307, 0.0307]_{LR} \succ$,

The positive part of solution shows likelihood (in percentage) and negative part of solution shows unlikelihood (in percentage) of current in circuit.

## 5 Conclusions

In real-world problems, most of the applications in different areas are dealt with system of linear equations to find the unknown parameters. In different situations the unknown parameters of the systems are uncertain or vague. We represent the parameters in terms of BFN and BFCN which contain parameters r and s. We usually face the problems of calculation to solve these systems due to interruption of parameters $r$ and $s$. We have discussed different methods to overcome the problems of parameters and solved the system in terms of $LR$-fuzzy numbers, $LR$-bipolar fuzzy numbers and $LR$-bipolar fuzzy complex numbers which are free from parameters $r$ and $s$. We have discussed a technique to solve $LR$-BFLS of equation, $LR$-CBFL system with real coefficients and $LR$-CBFL system with complex coefficients. In the first section, we have discussed a technique to solve the $LR$-BFLS of equations in which the $n \times n$ coefficient matrix of the system is represented by real numbers and right side vector is represented by $LR$-BFN. To solve the $LR$-BFLS of equations we replace the original system into the pair of positive($*$) and negative($\bullet$) of two $n \times n$ $LR$-fuzzy linear system and solve these system we use mean values and left-right spread systems. In the next section, first we have discussed a technique to solve the $LR$-CBFL system with real coefficients in which we can replace $n \times n$ $LR$-CBFL system with real coefficients by pair of real and imaginary parts of $n \times n$ $LR$-BFLS of equation and then solve these systems by using mean value and left-right spread systems. Further, we have extended the technique (Guo and Zhang 2016) to solve the $LR$-CBFL system with complex coefficients in which the $n \times n$ coefficient matrix of the system is represented by regular complex numbers and right side vector is represented by $LR$-BFCN. To solve the system we can replace $n \times n$ $LR$-CBFL system with complex coefficients by pair of positive($*$) and negative($\bullet$) of two $2n \times 2n$ mean value system and $4n \times 4n$ left-right spread system. Some numerical examples have also been solved to show the effectiveness of the technique. We plan to extend our study to solve: (1) circuit analysis and models of artificial intelligence, (2) numerical solution of bipolar fuzzy initial value problems, (3) bipolar fuzzy linear programming problems, and (4) numerical solution of $m$-polar fuzzy initial value problems.

## Compliance of ethical standards

**Conflict of interest** The authors declare that they have no conflict of interest.

**Ethical approval** This article does not contain any studies with human participants or animals performed by any of the authors.

## References


Abbasbandy S, Alavi M (2005) A method for solving fuzzy linear systems. Fuzzy Sets Syst 2(2):37–43

Abbasbandy S, Jafarian A (2006) Steepest descent method for system of fuzzy linear equations. Appl Math Comput 175(1):823–833

Akram M (2011) Bipolar fuzzy graphs. Inf Sci 24(181):5548–5564

Akram M, Arshad M (2019) A novel trapezoidal bipolar fuzzy TOPSIS method for group decision-making. Group Decis Negot 28(3):565–584

Akram M, Muhammad G, Allahviranloo T (2019a) Bipolar fuzzy linear system of equationss. Comput Appl Math 38(2):69

Akram M, Muhammad G, Koam AN, Hussain N (2019b) Iterative methods for solving a system of linear equations in a bipolar fuzzy environment. Mathematics 7(8):728

Akram M, Muhammad G, Hussian N (2019c) Bipolar fuzzy system of linear equations with polynomial parametric form. J Intell Fuzzy Syst 37(6):8275–8287

Akram M, Saleem D, Allahviranloo T (2019d) Linear system of equations in m-polar fuzzy environment. J Intell Fuzzy Syst 37(6):8251–8266

Akram M, Sarwar M, Dudek WA (2021) Graphs for the analysis of bipolar fuzzy information. Stud Fuzz Soft Comput 401:1–452

Akram M, Ali M, Allahviranloo T (2020a) Certain methods to solve bipolar fuzzy linear system of equations. Comput Appl Math 39:213

Akram M, Muhammad G, Allahviranloo T, Hussain N (2020b) LU decomposition method to solve bipolar fuzzy linear systems. J Intell Fuzzy Syst 39(3):3329–3349

Allahviranloo T, Otadi M, Mosleh M (2008) Iterative method for fuzzy equations. Soft Comput 12(10):935–939

Allahviranloo T, Hosseinzadeh AA, Ghanbari M, Haghi E, Nuraei R (2014) On the new solutions for a fully fuzzy linear system. Soft Comput 18(1):95–107

Allahviranloo T (2004) Numerical methods for fuzzy system of linear equations. Appl Math Comput 155(2):493–502

Allahviranloo T (2005a) Successive over relaxation iterative method for fuzzy system of linear equations. Appl Math Comput 162(1):189–196

Allahviranloo T (2005b) The adomian decomposition method for fuzzy system of linear equations. Appl Math Comput 163(2):553–563

Allahviranloo T (2019) Uncertain information and linear systems, studies in systems. Studies in systems, decision and control, vol 254. Springer, Berlin

Allahviranloo T, Hosseinzadeh F, Khorasani M, Khezerloo M (2013) On the fuzzy solution of LR fuzzy linear systems. Appl Math Model 37(3):1170–1176

Alghamdi MA, Alshehri NO, Akram M (2018) Multi-criteria decision-making methods in bipolar fuzzy environment. Int J Fuzzy System 20(6):2057–2064

Amirfakhrian M (2012) Analyzing the solution of a system of fuzzy linear equations by a fuzzy distance. Soft Comput 16(6):1035–1041

Amirfakhrian M, Fallah M, Rodriguez-Lopez R (2018) A method for solving fuzzy matrix equations. Soft Comput 22:2095–2103

Behera D, Chakraverty S (2012) A new method for solving real and complex fuzzy systems of linear equations. Comput Math Model 23(4):507–518

Buckley JJ (1989) Fuzzy complex number. Fuzzy Sets Syst 33:333–345

Dehghan M, Hashemi B, Ghatee M (2007) Solution of fully fuzzy linear systems using iterative techniques. Chaos Solitons Fractals 34(2):316–336

Dubois D, Prade H (1978) Operations on fuzzy numbers. Int J Syst Sci 9:613–626

Ezzati R (2011) Solving fuzzy linear systems. Soft Comput 15(1):193–197

Fariborzi Araghi MA, Fallahzadeh A (2013) Inherited LU factorization for solving fuzzy system of linear equations. Soft Comput 17(1):159–163

Friedman M, Ma Ming M, Kandel A (1998) Fuzzy linear systems. Fuzzy Sets Syst 96:201–209

Ghanbari M, Allahviranloo T, Pedrycz W (2020) On the rectangular fuzzy complex linear systems. Appl Soft Comput 91:106196

Guo X, Zhang K (2016) Minimal solution of complex fuzzy linear systems. Adv Fuzzy Syst 2016:9. https://doi.org/10.1155/2016/5293917

Jahantigh MA, Khezerloo S, Khezerloo M (2010) Complex fuzzy linear systems. Int J Ind Math 2(1):21–28

Koam NA, Akram M, Muhammad G, Hussain N (2020) LU decomposition scheme for solving m-polar fuzzy system of linear equations. Math Problems Eng Article ID 8384593

Jha A (2018) Meromorphic function of fuzzy complex variables. J New Theory 20:1–12

Moloudzadeh S, Allahviranloo T, Darabi P (2013) A new method for solving an arbitrary fully fuzzy linear system. Soft Comput 17(9):1725–1731

Minc H (1988) Nonnegative matrices. Wiley, New York

Rahgooy T, Yazdi HS, Monsefi R (2009) Fuzzy complex system of linear equations applied to circuit analysis. Int J Comput Electr Eng 1(5):535

Rao C, Lin H, Liu M (2020a) Design of comprehensive evaluation index system for P2P credit risk of three rural borrowers. Soft Comput 24:11493–11509

Rao C, Liu M, Goh M, Wen J (2020b) 2-stage modified random forest model for credit risk assessment of P2P network lending to Three Rurals borrowers. Appl Soft Comput 95:106570

Saqib M, Akram M, Shahida B, Allahviranloo T (2020a) Numerical solution of bipolar fuzzy initial value problem. J Intell Fuzzy Syst. https://doi.org/10.3233/JIFS201619

Saqib M, Akram M, Shahida B (2020b) Certain efficient iterative methods for bipolar fuzzy system of linear equations. J Intell Fuzzy Syst 39(3):3971–3985

Zadeh LA (1965) Fuzzy sets. Inf Control 8(3):338–353

Zadeh LA (1971) Similarity relations and fuzzy orderings. Inf Sci 3(2):177–200

Zadeh LA (1975) The concept of a linguistic and application to approximate reasoning-I. Inf Sci 8:199–249

Zhang WR (1994) Bipolar fuzzy sets and relations: a computational framework forcognitive modeling and multiagent decision analysis. Proceedings of IEEE Conference 305–309

Zhang, W. R.: YinYang Bipolar fuzzy sets, Fuzzy Systems Proceedings, IEEE World Congress on Computational Intelligence, (1998), 835-840

Zheng B, Wang K (2006) Inconsistent fuzzy linear systems. Appl Math Comput 181:973–981

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.