METHODOLOGIES AND APPLICATION

A novel interval-valued fuzzy soft decision-making method based on CoCoSo and CRITIC for intelligent healthcare management evaluation

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Abstract

The intelligent healthcare management is of great concern to mobilize the enthusiasm of individuals and groups, and effectively use limited resources to achieve maximum health improvement by AI technology. When considering the intelligent healthcare management evaluation, the primary issues involve many uncertainties. Interval-valued fuzzy soft set, depicted by membership degree with interval form, is a more resultful means for capturing uncertainty. In this paper, the comparison issue in interval-valued fuzzy soft environment is disposed of by proposing novel score function. Later, some new properties for interval-valued fuzzy soft matrix are investigated in detail. Moreover, the objective weight is calculated by CRITIC (Criteria Importance Through Inter-criteria Correlation) method. Meanwhile, the combined weight is determined by reflecting both subjective weight and the objective weight. Then, interval-valued fuzzy soft decision-making algorithm-based CoCoSo (Combined Compromise Solution) is developed. Lastly, the validity of algorithm is expounded by the healthcare management industry evaluation issue, along with their sensitivity analysis. The main characteristics of the presented algorithm are: (1) without counterintuitive phenomena; (2) no division by zero problem; (3) have strong ability to distinguish alternatives.

Keywords Intelligent healthcare management · Interval-valued fuzzy soft sets · Score function · CoCoSo · CRITIC

1 Introduction

Intelligent healthcare management (IHM) is based on the results of health checkups, the establishment of a dedicated health record, and targeted personalized health management programs (Coyle 2010). With the development of artificial intelligence technology, the health monitoring, health assessment and health intervention of intelligent detection instruments for users are realized.

Recently, IHM has already been successfully employed in diverse fields, such as social commerce open innovation (Davies et al. 2019), blood and marrow transplant (Repaczki-Jones et al. 2019), professionalization in medicine (Gerard 2019), education (Bean et al. 2019), logistics (Pohjosenpera et al. 2019), airway management (Schnittker et al.

2019), supply chain management (Scavarda et al. 2019). Nanshan Zhong, as an academician of the Chinese academy of engineering and a famous respiratory expert, once said, "Intelligent healthcare management is the future of life technology." The healthcare management companies are looking for business opportunities in the blue ocean of intelligent healthcare market. However, it is not enough to rely themselves on taking the intelligent healthcare market quickly. As a consequence, they should choose an amazing artificial intelligence (AI) company to join in cooperation for upgrading their technical level. As AI leaders, Toutiao, Sensetime, Megvii, Cloudwalk and Ubtech are certainly great cooperative partners to be considered. Therefore, I propose to choose the right cooperative company to solve the problem of MCDM (multi-criteria decision making). Nevertheless, the increasingly complex decision-making environment and indecisive decision makers (DMs) make it difficult to express uncertain decision information when solving the above multi-criteria decision making (MCDM) problems (Si et al. 2019).

Interval-valued fuzzy soft set (IVFSS), firstly introduced by Son (2007), has been deemed to be a more resultful means to depict uncertain information in a greater range point when compared with soft set (SS) (Molodtsov 1999). In view of

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such advantage (Feng et al. 2019) of the IVFSS, it is fast becoming a hot study topic, including algebraic structure (Liu et al. 2014, ?), matrix theory (Rajarajeswari and Dhanalakshmi 2014), parameter reduction (Ma et al. 2013; Qin and Ma 2018), information measure (Jiang et al. 2013; Wang et al. 2015; Peng and Yang 2015), data filling (Qin and Ma 2019) and decision-making technologies (Qin and Ma 2018; Yang and Peng 2017; Yang et al. 2009; Feng et al. 2010; Yuan and Hu 2012; Xiao et al. 2013; Peng and Yang 2017; Tripathy et al. 2017; Peng et al. 2017; Chen and Zou 2017; Peng and Garg 2018; Sooraj and Tripathy 2018).

When disposing of MCDM issues in interval-valued fuzzy soft environment, there are four limiting factors that are most advanced, which constitute our motivations.

(1) The existing decision-making methods (Qin and Ma 2018; Yang and Peng 2017; Yang et al. 2009; Feng et al. 2010; Yuan and Hu 2012; Xiao et al. 2013; Peng and Yang 2017; Tripathy et al. 2017; Peng et al. 2017; Chen and Zou 2017; Peng and Garg 2018), disposing of dealing MCDM problems, have counterintuitive phenomena (Yang et al. 2009; Yuan and Hu 2012; Xiao et al. 2013), division by zero problem (Yang and Peng 2017) and low resolution in differentiating the optimal alternative (Peng et al. 2017; Son 2007; Chen and Zou 2017; Peng and Garg 2018). It may be unaccommodated or unfitted for experts to choose best alternative(s). The CoCoSo (Combined Compromise Solution) method, initially explored by Yazdani et al. (2019), is an effective algorithm to deal the uncertain information in a logical and commonsensible way (Wen et al. 2019). Therefore, the first motivation is to deal MCDM issues by presenting novel decision making algorithm out of three deficiencies.

(2) The existing score functions (Peng and Yang 2017) can accurately rank the common alternatives. However, it cannot distinguish two IVFNs in some special cases. Hence, the second motivation is to introduce a new score function, which can have strong sense of differentiation.

(3) The existing interval-valued fuzzy soft weighting determine models only focus on objective weight (Yang and Peng 2017; Xiao et al. 2013; Chen and Zou 2017) or subjective weight (Tripathy et al. 2017; Sooraj and Tripathy 2018). The subjective weight is offered by DMs based on their experience, while they neglect the underlying weight information conveyed by the evaluation information. The objective weight can be obtained from the evaluation information by some innovative methods, while they fail to take the preference information of experts into account. Moreover, it may lead drastically to increasing criteria weight of that criteria when one value in decision matrix is above/below the other values for 20–30% (within the same criteria). How to combine them is an interesting topic (Peng and Yang 2017; Peng et al. 2017; Peng and Garg 2018). Hence, the third motivation is to develop the combined weight model, which can integrate both subjective weight and objective weight.

(4) Some matrix operations (Liu et al. 2014; Rajarajeswari and Dhanalakshmi 2014) on IVFSSs are not abundant enough, and their properties are not enough wonderful. Hence, the fourth motivation of this article is to present some matrix operations for IVFSSs and also discuss their interesting properties in detail.

According to the above argumentation and the characteristics of IHM industry evaluation, we present a revised MCDM method in the interval-valued fuzzy soft environment. The contributions of paper are shown as follows:

(1) A novel interval-valued fuzzy soft IHM industry decision making issue based on CoCoSo method is explored, which can obtain the best alternative without counterintuitive phenomena and division by zero problem, and have a strong ability to differentiate alternatives.
(2) The novel score function of IVFN is presented and their interesting properties are explored in detail.
(3) The combined weight model is based on CRITIC (Criteria Importance Through Inter-criteria Correlation) and the linear weighted comprehensive method, which can reflect both subjective weight and objective weight information.
(4) Some new theorems and relations based on interval-valued fuzzy soft matrix are discussed.

To process our discussion, the rest part is listed in the following. In Sect. 2, some basic notions of interval-valued fuzzy set and IVFSS are primitively retrospected. In Sect. 3, the novel score function of IVFN is explored and its properties are proved. In Sect. 4, the novel matrix-based IVFSSs are defined and discussed in detail. In Sect. 5, we develop a novel interval-valued fuzzy soft-based CoCoSo with CRITIC. In Sect. 6, a case study in intelligent health management evaluation method is presented, and the sensitive analysis is shown. In Sect. 7, some examples to illustrate the effectiveness of developed method under interval-valued fuzzy soft environment are given. Section 8 makes some conclusions.

2 Preliminaries

This section primarily reviews some notions of interval-valued fuzzy set and interval-valued fuzzy soft set.

Definition 1 (Zadeh 1975) An interval-valued fuzzy set (IVFS) I in U is given by

$$I = \{< \chi, [I^-(\chi), I^+(\chi)] >| \chi \in U\}, \tag{1}$$

where $0 \leq I^-(\chi) \leq I^+(\chi) \leq 1$. For simplicity, we call $i = [I^-(\chi), I^+(\chi)]$ an interval-valued fuzzy number (IVFN) denoted by $i = [i^-, i^+]$.

Definition 2 (Xu and Da 2002) Let $\chi = [\chi^-, \chi^+]$ and $y = [y^-, y^+]$ be two IVFNs, $\lambda \in [0, 1]$, then their operational laws are defined as follows:

(1) $\chi + y = [\chi^-, \chi^+] + [y^-, y^+] = [\chi^- + y^-, \chi^+ + y^+]$;
(2) $\lambda \chi = [\lambda \chi^-, \lambda \chi^+]$;
(3) $\chi^\lambda = [(\chi^-)^\lambda, (\chi^+)^\lambda]$;
(4) $\chi \cdot y = [\chi^-, \chi^+] \cdot [y^-, y^+] = [\chi^- \cdot y^-, \chi^+ \cdot y^+]$;
(5) $\chi = y$ iff $\chi^- = y^-$ and $\chi^+ = y^+$;
(6) $\chi^c = [1 - \chi^+, 1 - \chi^-]$;
(7) $-\chi = [-\chi^+, -\chi^-]$.

Definition 3 (Son 2007) Let U be an initial universe and A be a set of parameters, a pair (F, A) is called an interval-valued fuzzy soft set over $\widetilde{P}(U)$, where F is a mapping given by $F : A \rightarrow \widetilde{P}(U)$. $\widetilde{P}(U)$ denotes the set of all interval-valued fuzzy subsets of U.

For $\forall \varepsilon \in A, F(\varepsilon)$ is an interval-valued fuzzy subset of U, and it is called an interval-valued fuzzy value set of parameter ε. Let $F(\varepsilon)(x)$ denote the membership value that object x holds parameter ε, then $F(\varepsilon)$ can be written as an interval-valued fuzzy set that $F(\varepsilon) = \{x/F(\varepsilon)(x) \mid x \in U\} = \{x/[F^-(\varepsilon)(x), F^+(\varepsilon)(x)] \mid x \in U\}$.

Definition 4 (Son 2007) The complement of an interval-valued fuzzy soft set (F, A), is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, A)$. $\forall \varepsilon \in A, x \in U, F^c(\varepsilon)(x) = [1 - F^+(\varepsilon)(x), 1 - F^-(\varepsilon)(x)]$.

For comparing the two IVFNs ($m = n = 1$), Peng and Yang (2017) derived the following comparative law-based score function.

Definition 5 (Peng and Yang 2017) For two interval-value fuzzy numbers χ and y,

(1) if $s(\chi) > s(y)$, then $\chi \succ y$;
(2) if $s(\chi) < s(y)$, then $\chi \prec y$;
(3) if $s(\chi) = s(y)$, then $\chi \sim y$.

3 A novel interval-valued fuzzy score function

This section describes some existing score functions (Peng and Yang 2017; Peng and Garg 2018) and develops a new score function by taking the discrimination power into consideration.

3.1 Some existing score functions

Definition 6 (Peng and Yang 2017) Let $F(\varepsilon)(x)$ be an IVFN, then the score function of $F(\varepsilon)(x)$ is defined as follows:

$$s_{p1}(F(\varepsilon)(x)) = \frac{F^-(\varepsilon)(x) + F^+(\varepsilon)(x) - 1}{2}. \tag{2}$$

Definition 7 (Peng and Garg 2018) Let $F(\varepsilon)(x)$ be an IVFN, then the score function of $F(\varepsilon)(x)$ is defined as follows:

$$s_{p2}(F(\varepsilon)(x)) = \frac{e^{F^-(\varepsilon)(x) + F^+(\varepsilon)(x)} - 1}{1 + F^+(\varepsilon)(x) - F^-(\varepsilon)(x)}. \tag{3}$$

3.2 The novel score function

Definition 8 For an IVFN $F(\varepsilon)(x) = [F^-(\varepsilon)(x), F^+(\varepsilon)(x)]$, the novel score function can be denoted as follows:

$$s_{pxd}(F(\varepsilon)(x))$$
$$= \frac{F^-(\varepsilon)(x) + F^+(\varepsilon)(x) - \ln\left(1 + F^+(\varepsilon)(x) - F^-(\varepsilon)(x)\right)}{2} \tag{4}$$

, where $s_{pxd}(F(\varepsilon)(x)) \in [0, 1]$.

Theorem 1 *For an IVFN $F(\varepsilon)(x) = [F^-(\varepsilon)(x), F^+(\varepsilon)(x)]$, $s_{pxd}(F(\varepsilon)(x))$ monotonically increases along with the increase of $F^-(\varepsilon)(x)$ and $F^+(\varepsilon)(x)$.*

Proof Based on Eq.(4), we obtain the first partial derivative of $s_{pxd}(F(\varepsilon)(x))$ with $F^-(\varepsilon)(x)$,
$\frac{\partial s_{pxd}(F(\varepsilon)(x))}{\partial F^-(\varepsilon)(x)} = \frac{F^+(\varepsilon)(x) - F^-(\varepsilon)(x)}{2(1 + F^+(\varepsilon)(x) - F^-(\varepsilon)(x))} \geq 0$.
Analogously, we can have the first partial derivative of $s_{pxd}(F(\varepsilon)(x))$ with $F^+(\varepsilon)(x)$,
$\frac{\partial s_{pxd}(F(\varepsilon)(x))}{\partial F^+(\varepsilon)(x)} = \frac{2 + F^+(\varepsilon)(x) - F^-(\varepsilon)(x)}{2(1 + F^+(\varepsilon)(x) - F^-(\varepsilon)(x))} \geq 0$.
Consequently, we can obtain that $s_{pxd}(F(\varepsilon)(x))$ monotonically increases along with the increase of $F^-(\varepsilon)(x)$ and $F^+(\varepsilon)(x)$. $\square$

Theorem 2 *For an IVFN $F(\varepsilon)(x) = [F^-(\varepsilon)(x), F^+(\varepsilon)(x)]$, the novel score function $s_{pxd}(F(\varepsilon)(x))$ complies with the following properties.*

(1) $0 \leq s_{pxd}(F(\varepsilon)(x)) \leq 1$;
(2) $s_{pxd}(F(\varepsilon)(x)) = 1$ *iff* $F(\varepsilon)(x) = [1, 1]$;
(3) $s_{pxd}(F(\varepsilon)(x)) = -1$ *iff* $F(\varepsilon)(x) = [0, 0]$.

Proof Based on Theorem 1, we can see that if we only just take $F^-(\varepsilon)(x)$ or $F^+(\varepsilon)(x)$ into consideration, $s_{pxd}(F(\varepsilon)(x))$ can have the min-value or max-value when $F(\varepsilon)(x) = [0, 0]$ or $F(\varepsilon)(x) = [1, 1]$. In other words, $s_{pxd}(F(\varepsilon)(x))_{\min} = 0$ and $s_{pxd}(F(\varepsilon)(x))_{\max} = 1$. Hence, $0 \leq s_{pxd}(F(\varepsilon)(x)) \leq 1$. □

Theorem 3 *Let* $F(\varepsilon)(x) = (F^-(\varepsilon)(x), F^+(\varepsilon)(x))$ *and* $G(\varepsilon)(x) = (G^-(\varepsilon)(x), G^+(\varepsilon)(x))$ *be two IVFNs. If* $F^-(\varepsilon)(x) > G^-(\varepsilon)(x)$ *and* $F^+(\varepsilon)(x) > G^+(\varepsilon)(x)$, *then* $s_{pxd}(F(\varepsilon)(x)) > s_{pxd}(G(\varepsilon)(x))$.

Proof According to Theorem 1, we can find that $s_{pxd}(F(\varepsilon)(x))$ monotonically increases along with the increase of $F^-(\varepsilon)(x)$ and $F^+(\varepsilon)(x)$.

Hence, if $F^-(\varepsilon)(x) > G^-(\varepsilon)(x)$ and $F^+(\varepsilon)(x) > G^+(\varepsilon)(x)$, then $s_{pxd}(F(\varepsilon)(x)) > s_{pxd}(G(\varepsilon)(x))$.

To test the feasibility of the proposed score function s_{pxd} for ranking IVFNs, Table 1 shows the comparison between the results achieved by the proposed score function s_{pxd} and the cases of the score functions s_{p1} and s_{p2} obtained by Peng and Yang (2017) and Peng and Garg (2018), respectively. □

From Table 1, we can conclude that the presented score function s_{pxd} can effectively solve the deficiencies of s_{p1} proposed by Peng and Yang (2017). In other words, the presented score function can distinguish the difference of alternatives when the existing score function s_{p1} cannot deal with. Moreover, the presented score function can keep the same result as s_{p1} proposed by Peng and Garg (2018).

4 Novel operations on interval-valued fuzzy soft matrix

Matrix plays an all-important role in the broad area of science, medical diagnosis and engineering. However, the classical matrix theory cannot deal the cases involving diverse types of uncertainties. Inspired by Rajarajeswari and Dhanalakshmi (2014), we continue to mine some novel interesting operators and new properties for interval-valued fuzzy soft matrix.

Definition 9 (Rajarajeswari and Dhanalakshmi 2014) Let $U = \{x_1, x_2, \cdots, x_m\}$ be a universal set and E be the set of parameters given by $E = \{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\}$. Suppose that $A \subseteq E$ and (F, A) be an IVFSS over U. Then the IVFSS can be expressed in interval-valued fuzzy soft matrix (IVFSM) form as $I_{n \times m} = (F(\varepsilon_j)(x_i))_{n \times m}$, where

$$F(\varepsilon_j)(x_i) = \begin{cases} [F^-(\varepsilon_j)(x_i), F^+(\varepsilon_j)(x_i)], & \text{if } \varepsilon_j \in E, \\ [0, 0], & \text{if } \varepsilon_j \notin E. \end{cases}$$

Definition 10 (Rajarajeswari and Dhanalakshmi 2014) If $I_{n \times m}^A$ and $I_{n \times m}^B$ be two IVFSMs, then

(1) $I_{n \times m}^A \& I_{n \times m}^B$
$= \left(\left[\frac{F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i)}{2}, \frac{F_A^+(\varepsilon_j)(x_i) + F_B^+(\varepsilon_j)(x_i)}{2} \right] \right)_{n \times m}$;

(2) $I_{n \times m}^A \$ I_{n \times m}^B$
$= \left(\left[\sqrt{F_A^-(\varepsilon_j)(x_i) F_B^-(\varepsilon_j)(x_i)}, \right. \right.$
$\left. \left. \sqrt{F_A^+(\varepsilon_j)(x_i) F_B^+(\varepsilon_j)(x_i)} \right] \right)_{n \times m}$;

(3) $I_{n \times m}^A @ I_{n \times m}^B$
$= \left(\left[2 \frac{F_A^-(\varepsilon_j)(x_i) F_B^-(\varepsilon_j)(x_i)}{F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i)}, \right. \right.$
$\left. \left. 2 \frac{F_A^+(\varepsilon_j)(x_i) F_B^+(\varepsilon_j)(x_i)}{F_A^+(\varepsilon_j)(x_i) + F_B^+(\varepsilon_j)(x_i)} \right] \right)_{n \times m}$;

(4) $I_{n \times m}^A \cup I_{n \times m}^B$
$= \left(\left[\max\{F_A^-(\varepsilon_j)(x_i), F_B^-(\varepsilon_j)(x_i)\}, \right. \right.$
$\left. \max\{F_A^+(\varepsilon_j)(x_i), F_B^+(\varepsilon_j)(x_i)\} \right]_{n \times m}$;

(5) $I_{n \times m}^A \cap I_{n \times m}^B$
$= \left(\left[\min\{F_A^-(\varepsilon_j)(x_i), F_B^-(\varepsilon_j)(x_i)\}, \right. \right.$
$\left. \min\{F_A^+(\varepsilon_j)(x_i), F_B^+(\varepsilon_j)(x_i)\} \right]_{n \times m}$.

Definition 11 If $I_{n \times m}^A$ and $I_{n \times m}^B$ be two IVFSMs, then

(1) $I_{n \times m}^A \oplus I_{n \times m}^B = ([F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i) F_B^-(\varepsilon_j)(x_i), F_A^+(\varepsilon_j)(x_i) + F_B^+(\varepsilon_j)(x_i) - F_A^+(\varepsilon_j)(x_i) F_B^+(\varepsilon_j)(x_i)])_{n \times m}$;
(2) $I_{n \times m}^A \otimes I_{n \times m}^B = ([F_A^-(\varepsilon_j)(x_i) F_B^-(\varepsilon_j)(x_i), F_A^+(\varepsilon_j)(x_i) F_B^+(\varepsilon_j)(x_i)])_{n \times m}$;
(3) $I_{n \times m}^A \boxplus I_{n \times m}^B$
$= \left(\left[\frac{F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) + 1)}, \right. \right.$
$\left. \left. \frac{F_A^+(\varepsilon_j)(x_i) + F_B^+(\varepsilon_j)(x_i)}{2(F_A^+(\varepsilon_j)(x_i) + F_B^+(\varepsilon_j)(x_i) + 1)} \right] \right)_{n \times m}$.

Example 1 Let $I_{3 \times 3}^A = \begin{pmatrix} [0.3,0.4] \ [0.1,0.2] \ [0.2,0.6] \\ [0.4,0.5] \ [0.3,0.4] \ [0.3,0.5] \\ [0.2,0.4] \ [0.6,0.7] \ [0.2,0.5] \end{pmatrix}$ and
$I_{3 \times 3}^B = \begin{pmatrix} [0.4,0.6] \ [0.2,0.2] \ [0.2,0.3] \\ [0.2,0.3] \ [0.3,0.5] \ [0.4,0.6] \\ [0.3,0.5] \ [0.3,0.4] \ [0.4,0.6] \end{pmatrix}$ be two IVFSMs,
then the operations on IVFSMs are listed in Table 2.

Lemma 1 *(Peng 2019) For any number $a, b, c \in [0, 1]$, then*

$$c^2(2 - a^2 - b^2 - c^2) + a^2 b^2 \geq 0. \tag{5}$$

Theorem 4 *For every interval-valued fuzzy soft matrix $I_{n \times m}^A$, $I_{n \times m}^B$ and $I_{n \times m}^C$, then*

(1) $(I_{n \times m}^A \oplus I_{n \times m}^B) \& I_{n \times m}^C \subseteq (I_{n \times m}^A \& I_{n \times m}^C) \oplus (I_{n \times m}^B \& I_{n \times m}^C)$;

Table 1 The comparison of score functions

Ranking Examples	Score functions s_{p1} (Peng and Yang 2017)	s_{p2} (Peng and Garg 2018)	s_{pxd}
$F(\varepsilon)(x) = [0.4, 0.6]$			
	$F(\varepsilon)(x) = G(\varepsilon)(x)$	$F(\varepsilon)(x) < G(\varepsilon)(x)$	$F(\varepsilon)(x) < G(\varepsilon)(x)$
$G(\varepsilon)(x) = [0.5, 0.5]$			
$F(\varepsilon)(x) = [0.4, 0.8]$			
	$F(\varepsilon)(x) = G(\varepsilon)(x)$	$F(\varepsilon)(x) < G(\varepsilon)(x)$	$F(\varepsilon)(x) < G(\varepsilon)(x)$
$G(\varepsilon)(x) = [0.5, 0.7]$			

Table 2 The operations over IVFSMs

Operations	Results
$I_{3\times3}^A \ \& \ I_{3\times3}^B$	$\begin{pmatrix} [0.3500,0.5000] & [0.1500,0.2000] & [0.2000,0.4500] \\ [0.3000,0.4000] & [0.3000,0.4500] & [0.3500,0.5500] \\ [0.2500,0.4500] & [0.4500,0.5500] & [0.3000,0.5500] \end{pmatrix}$
$I_{3\times3}^A \ \$ \ I_{3\times3}^B$	$\begin{pmatrix} [0.3464,0.4899] & [0.1414,0.2000] & [0.2000,0.4243] \\ [0.2828,0.3873] & [0.3000,0.4472] & [0.3464,0.5477] \\ [0.2449,0.4472] & [0.4243,0.5292] & [0.2828,0.5477] \end{pmatrix}$
$I_{3\times3}^A \ @ \ I_{3\times3}^B$	$\begin{pmatrix} [0.3429,0.4800] & [0.1333,0.2000] & [0.2000,0.4000] \\ [0.2667,0.3750] & [0.3000,0.4444] & [0.3429,0.5455] \\ [0.2400,0.4444] & [0.4000,0.5091] & [0.2667,0.5455] \end{pmatrix}$
$I_{3\times3}^A \ \cup \ I_{3\times3}^B$	$\begin{pmatrix} [0.4000,0.6000] & [0.2000,0.2000] & [0.2000,0.6000] \\ [0.4000,0.5000] & [0.3000,0.5000] & [0.4000,0.6000] \\ [0.3000,0.5000] & [0.6000,0.7000] & [0.4000,0.6000] \end{pmatrix}$
$I_{3\times3}^A \ \cap \ I_{3\times3}^B$	$\begin{pmatrix} [0.3000,0.4000] & [0.1000,0.2000] & [0.2000,0.3000] \\ [0.2000,0.3000] & [0.3000,0.4000] & [0.3000,0.5000] \\ [0.2000,0.4000] & [0.3000,0.4000] & [0.2000,0.5000] \end{pmatrix}$
$I_{3\times3}^A \ \oplus \ I_{3\times3}^B$	$\begin{pmatrix} [0.5800,0.7600] & [0.2800,0.3600] & [0.3600,0.7200] \\ [0.5200,0.6500] & [0.5100,0.7000] & [0.5800,0.8000] \\ [0.4400,0.7000] & [0.7200,0.8200] & [0.5200,0.8000] \end{pmatrix}$
$I_{3\times3}^A \ \otimes \ I_{3\times3}^B$	$\begin{pmatrix} [0.1200,0.2400] & [0.0200,0.0400] & [0.0400,0.1800] \\ [0.0800,0.1500] & [0.0900,0.2000] & [0.1200,0.3000] \\ [0.0600,0.2000] & [0.1800,0.2800] & [0.0800,0.3000] \end{pmatrix}$
$I_{3\times3}^A \maltese I_{3\times3}^B$	$\begin{pmatrix} [0.2059,0.2500] & [0.1154,0.1429] & [0.1429,0.2368] \\ [0.1875,0.2222] & [0.1875,0.2368] & [0.2059,0.2619] \\ [0.1667,0.2368] & [0.2368,0.2619] & [0.1875,0.2619] \end{pmatrix}$

(2) $(I_{n\times m}^A \otimes I_{n\times m}^B) \ \& \ I_{n\times m}^C \supseteq (I_{n\times m}^A \ \& \ I_{n\times m}^C) \otimes (I_{n\times m}^B \ \& \ I_{n\times m}^C)$;

(3) $(I_{n\times m}^A \oplus I_{n\times m}^B) \ \$ \ I_{n\times m}^C \subseteq (I_{n\times m}^A \ \& \ I_{n\times m}^C) \oplus (I_{n\times m}^B \$ I_{n\times m}^C)$;

(4) $(I_{n\times m}^A \otimes I_{n\times m}^B) \ \$ \ I_{n\times m}^C \supseteq (I_{n\times m}^A \ \& \ I_{n\times m}^C) \otimes (I_{n\times m}^B \$ I_{n\times m}^C)$;

(5) $(I_{n\times m}^A \oplus I_{n\times m}^B) \maltese I_{n\times m}^C \subseteq (I_{n\times m}^A \maltese I_{n\times m}^C) \oplus (I_{n\times m}^B \maltese I_{n\times m}^C)$;

(6) $(I_{n\times m}^A \otimes I_{n\times m}^B) \maltese I_{n\times m}^C \supseteq (I_{n\times m}^A \maltese I_{n\times m}^C) \otimes (I_{n\times m}^B \maltese I_{n\times m}^C)$.

Proof We only prove the (1), (3) and (5) in detail; the (2), (4) and (6) can be proved in similar way.

(1) Let $I_{n\times m}^A$, $I_{n\times m}^B$ and $I_{n\times m}^C$ be three given IVFSM, then

$$(I_{n \times m}^A \oplus I_{n \times m}^B) \, \& \, I_{n \times m}^C = \left(\left[F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i), \, F_A^+(\varepsilon_j)(x_i) \right. \right.$$

$$\left. \left. + F_B^+(\varepsilon_j)(x_i) - F_A^+(\varepsilon_j)(x_i)F_B^+(\varepsilon_j)(x_i) \right] \right)_{n \times m} \, \& \, I_{n \times m}^C$$

$$= \left(\left[\frac{F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)}{2}, \right. \right.$$

$$\left. \left. \frac{F_A^+(\varepsilon_j)(x_i) + F_B^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i) - F_A^+(\varepsilon_j)(x_i)F_B^+(\varepsilon_j)(x_i)}{2} \right] \right)_{n \times m},$$

$$(I_{n \times m}^A \, \& \, I_{n \times m}^C) \oplus (I_{n \times m}^B \, \& \, I_{n \times m}^C)$$

$$= \left(\left[\frac{F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i)}{2}, \frac{F_A^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i)}{2} \right] \right)_{n \times m}$$

$$\oplus \left(\left[\frac{F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i)}{2}, \frac{F_B^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i)}{2} \right] \right)_{n \times m}$$

$$= \left(\left[\frac{F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) + 2F_C^-(\varepsilon_j)(x_i) - (F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i))(F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i))}{4}, \right. \right.$$

$$\left. \left. \frac{F_A^+(\varepsilon_j)(x_i) + F_B^+(\varepsilon_j)(x_i) + 2F_C^+(\varepsilon_j)(x_i) - (F_A^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i))(F_B^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i))}{4} \right] \right)_{n \times m}.$$

Let $f = \frac{F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) + 2F_C^-(\varepsilon_j)(x_i) - (F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i))(F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i))}{4}$
$- \frac{F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)}{2}$, then

$$f = \frac{F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i)\left(2 - F_A^-(\varepsilon_j)(x_i) - F_B^-(\varepsilon_j)(x_i) - F_C^-(\varepsilon_j)(x_i)\right)}{4} \geq$$

0 (Lemma 1).

Hence, we can have

$$\frac{F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) + 2F_C^-(\varepsilon_j)(x_i) - (F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i))(F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i))}{4}$$

$$\geq \frac{F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)}{2}$$

Similarly,

$$\frac{F_A^+(\varepsilon_j)(x_i) + F_B^+(\varepsilon_j)(x_i) + 2F_C^+(\varepsilon_j)(x_i) - (F_A^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i))(F_B^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i))}{4}$$

$$\geq \frac{F_A^+(\varepsilon_j)(x_i) + F_B^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i) - F_A^+(\varepsilon_j)(x_i)F_B^+(\varepsilon_j)(x_i)}{2}$$

According to the proposed score function s_{pxd}, we can have

$$(I_{n \times m}^A \oplus I_{n \times m}^B) \, \& \, I_{n \times m}^C \subseteq (I_{n \times m}^A \, \& \, I_{n \times m}^C) \oplus (I_{n \times m}^B \, \& \, I_{n \times m}^C).$$

(3) Let $I_{n\times m}^A$, $I_{n\times m}^B$ and $I_{n\times m}^C$ be three given IVFSM, then

$$(I_{n\times m}^A \oplus I_{n\times m}^B) \$ I_{n\times m}^C = \left(\left[F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i),\right.\right.$$

$$\left.\left.F_A^+(\varepsilon_j)(x_i) + F_B^+(\varepsilon_j)(x_i) - F_A^+(\varepsilon_j)(x_i)F_B^+(\varepsilon_j)(x_i)\right]\right)_{n\times m} \$ I_{n\times m}^C$$

$$= \left(\left[\sqrt{(F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i))F_C^-(\varepsilon_j)(x_i)},\right.\right.$$

$$\left.\left.\sqrt{(F_A^+(\varepsilon_j)(x_i) + F_B^+(\varepsilon_j)(x_i) - F_A^+(\varepsilon_j)(x_i)F_B^+(\varepsilon_j)(x_i))F_C^+(\varepsilon_j)(x_i)}\right]\right)_{n\times m},$$

$$(I_{n\times m}^A \, \& \, I_{n\times m}^C) \oplus (I_{n\times m}^B \$ I_{n\times m}^C) = \left(\left[\sqrt{F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)},\right.\right.$$

$$\left.\left.\sqrt{F_A^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i)}\right]\right)_{n\times m} \oplus \left(\left[\sqrt{F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)}, \sqrt{F_B^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i)}\right]\right)_{n\times m}$$

$$= \left(\left[\sqrt{F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)} + \sqrt{F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)} - \sqrt{F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)},\right.\right.$$

$$\left.\left.\sqrt{F_A^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i)} + \sqrt{F_B^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i)} - \sqrt{F_A^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i)F_B^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i)}\right]\right)_{n\times m}.$$

Let

$$f = \left(\sqrt{F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)} + \sqrt{F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)} - \sqrt{F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)}\right)^2$$

$$- \left(\sqrt{(F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i))F_C^-(\varepsilon_j)(x_i)}\right)^2,$$

then

$$f = F_C^-(\varepsilon_j)(x_i)\sqrt{F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)}\left(2 - 2\sqrt{F_C^-(\varepsilon_j)(x_i)}\left(\sqrt{F_A^-(\varepsilon_j)(x_i)} + \sqrt{F_B^-(\varepsilon_j)(x_i)}\right)\right.$$

$$\left. + \sqrt{F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)}\left(1 + F_C^-(\varepsilon_j)(x_i)\right)\right) \geq F_C^-(\varepsilon_j)(x_i)\sqrt{F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)}$$

$$\left(2 - 2\sqrt{F_C^-(\varepsilon_j)(x_i)}\left(\sqrt{F_A^-(\varepsilon_j)(x_i)} + \sqrt{F_B^-(\varepsilon_j)(x_i)}\right) + \sqrt{F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)}2\sqrt{F_C^-(\varepsilon_j)(x_i)}\right)$$

$$= 2F_C^-(\varepsilon_j)(x_i)\sqrt{F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)}\left(1 - \left(\sqrt{F_A^-(\varepsilon_j)(x_i)} + \sqrt{F_B^-(\varepsilon_j)(x_i)}\right.\right.$$

$$\left.\left. - \sqrt{F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)}\right)\sqrt{F_C^-(\varepsilon_j)(x_i)}\right) \geq 0.$$

Hence, we can have

$$\sqrt{F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)} + \sqrt{F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)} - \sqrt{F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)}$$

$$\geq \sqrt{(F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i))F_C^-(\varepsilon_j)(x_i)}.$$

Similarly,

$$\sqrt{F_A^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i)} + \sqrt{F_B^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i)} - \sqrt{F_A^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i)F_B^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i)}$$

$$\geq \sqrt{(F_A^+(\varepsilon_j)(x_i) + F_B^+(\varepsilon_j)(x_i) - F_A^+(\varepsilon_j)(x_i)F_B^+(\varepsilon_j)(x_i))F_C^+(\varepsilon_j)(x_i)}.$$

According to the proposed score function s_{pxd}, we can have

$$(I_{n \times m}^A \oplus I_{n \times m}^B) \,\$\, I_{n \times m}^C \subseteq (I_{n \times m}^A \,\&\, I_{n \times m}^C) \oplus (I_{n \times m}^B \,\$\, I_{n \times m}^C).$$

(5) Let $I_{n \times m}^A$, $I_{n \times m}^B$ and $I_{n \times m}^C$ be three given IVFSM, then

$$(I_{n \times m}^A \oplus I_{n \times m}^B) \maltese I_{n \times m}^C = \left(\left[\frac{F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i) + 1)}, \right. \right.$$

$$\left. \left. \frac{F_A^+(\varepsilon_j)(x_i) + F_B^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i) - F_A^+(\varepsilon_j)(x_i)F_B^+(\varepsilon_j)(x_i)}{2(F_A^+(\varepsilon_j)(x_i) + F_B^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i) - F_A^+(\varepsilon_j)(x_i)F_B^+(\varepsilon_j)(x_i) + 1)} \right] \right)_{n \times m},$$

$$(I_{n \times m}^A \maltese I_{n \times m}^C) \oplus (I_{n \times m}^B \maltese I_{n \times m}^C)$$

$$= \left(\left[\frac{F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) + 1)} + \frac{F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i)}{2(F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) + 1)} \right. \right.$$

$$\left. - \frac{F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) + 1)} \frac{F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i)}{2(F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) + 1)}, \right.$$

$$\frac{F_A^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i)}{2(F_A^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i) + 1)} + \frac{F_B^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i)}{2(F_B^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i) + 1)}$$

$$\left. \left. - \frac{F_A^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i)}{2(F_A^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i) + 1)} \frac{F_B^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i)}{2(F_B^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i) + 1)} \right] \right)_{n \times m}.$$

Let

$$f = \frac{F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) + 1)} + \frac{F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i)}{2(F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) + 1)}$$

$$- \frac{F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) + 1)} \frac{F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i)}{2(F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) + 1)}$$

$$- \frac{F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)}{2((F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i) + 1))},$$

then

$$
\begin{aligned}
f &= \frac{F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) + 1)} - \frac{F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i) + 1)} \\
&\quad + \frac{F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i)}{2(F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) + 1)}\left(1 - \frac{F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) + 1)}\right) \\
&\geq \frac{F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i) + 1)} \\
&\quad - \frac{F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i) + 1)} + \frac{F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i)}{2(F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) + 1)} \\
&\quad \left(1 - \frac{F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) + 1)}\right) = \frac{(F_A^-(\varepsilon_j)(x_i) - 1)F_B^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i) + 1)} \\
&\quad + \frac{F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i)}{2(F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) + 1)}\left(1 - \frac{F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) + 1)}\right) \\
&= \frac{(F_A^-(\varepsilon_j)(x_i) - 1)F_B^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i)\left(1 - F_B^-(\varepsilon_j)(x_i)\right) + F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) + 1)} + \frac{F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i)}{2(F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) + 1)} \\
&\quad \left(1 - \frac{F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) + 1)}\right) \geq \frac{(F_A^-(\varepsilon_j)(x_i) - 1)F_B^-(\varepsilon_j)(x_i)}{2(F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) + 1)} \\
&\quad + \frac{F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i)}{2(F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) + 1)}\left(1 - \frac{F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) + 1)}\right) \\
&= \frac{1}{4(F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) + 1)(F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) + 1)}\left(2(F_A^-(\varepsilon_j)(x_i))^2 F_B^-(\varepsilon_j)(x_i)\right. \\
&\quad + 2F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i) + F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i) \\
&\quad \left. + F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i) + (F_C^-(\varepsilon_j)(x_i))^2 + 2F_C^-(\varepsilon_j)(x_i)(1 - F_B^-(\varepsilon_j)(x_i))\right) \geq 0.
\end{aligned}
$$

Hence, we can have

$$
\begin{aligned}
&\frac{F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) + 1)} + \frac{F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i)}{2(F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) + 1)} \\
&\quad - \frac{F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) + 1)} \frac{F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i)}{2(F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) + 1)} \\
&\geq \frac{F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i) + 1)}.
\end{aligned}
$$

Similarly, we can have

$$
\begin{aligned}
&\frac{F_A^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i)}{2(F_A^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i) + 1)} + \frac{F_B^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i)}{2(F_B^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i) + 1)} \\
&\quad - \frac{F_A^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i)}{2(F_A^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i) + 1)} \frac{F_B^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i)}{2(F_B^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i) + 1)} \\
&\geq \frac{F_A^+(\varepsilon_j)(x_i) + F_B^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i) - F_A^+(\varepsilon_j)(x_i)F_B^+(\varepsilon_j)(x_i)}{2(F_A^+(\varepsilon_j)(x_i) + F_B^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i) - F_A^+(\varepsilon_j)(x_i)F_B^+(\varepsilon_j)(x_i) + 1)}.
\end{aligned}
$$

According to the proposed score function s_{pxd}, we can have

$$(I_{n\times m}^A \oplus I_{n\times m}^B) \maltese I_{n\times m}^C \subseteq (I_{n\times m}^A \maltese I_{n\times m}^C)$$
$$\oplus (I_{n\times m}^B \maltese I_{n\times m}^C).$$

□

Theorem 5 *For every interval-valued fuzzy soft matrix $I_{n\times m}^A$, $I_{n\times m}^B$ and $I_{n\times m}^C$, then*

(1) $(I_{n\times m}^A \& I_{n\times m}^B) \oplus I_{n\times m}^C = (I_{n\times m}^A \oplus I_{n\times m}^C) \& (I_{n\times m}^B \oplus I_{n\times m}^C)$;

(2) $(I_{n\times m}^A \& I_{n\times m}^B) \otimes I_{n\times m}^C = (I_{n\times m}^A \otimes I_{n\times m}^C) \& (I_{n\times m}^B \otimes I_{n\times m}^C)$;

(3) $(I_{n\times m}^A \$ I_{n\times m}^B) \oplus I_{n\times m}^C \subseteq (I_{n\times m}^A \oplus I_{n\times m}^C) \$ (I_{n\times m}^B \oplus I_{n\times m}^C)$;

(4) $(I_{n\times m}^A \$ I_{n\times m}^B) \otimes I_{n\times m}^C \supseteq (I_{n\times m}^A \otimes I_{n\times m}^C) \$ (I_{n\times m}^B \otimes I_{n\times m}^C)$;

(5) $(I_{n\times m}^A @ I_{n\times m}^B) \oplus I_{n\times m}^C \subseteq (I_{n\times m}^A \oplus I_{n\times m}^C) @ (I_{n\times m}^B \oplus I_{n\times m}^C)$;

(6) $(I_{n\times m}^A @ I_{n\times m}^B) \otimes I_{n\times m}^C \supseteq (I_{n\times m}^A \otimes I_{n\times m}^C) @ (I_{n\times m}^B \otimes I_{n\times m}^C)$;

(7) $(I_{n\times m}^A \maltese I_{n\times m}^B) \oplus I_{n\times m}^C \supseteq (I_{n\times m}^A \oplus I_{n\times m}^C) \maltese (I_{n\times m}^B \oplus I_{n\times m}^C)$;

(8) $(I_{n\times m}^A \maltese I_{n\times m}^B) \otimes I_{n\times m}^C \subseteq (I_{n\times m}^A \otimes I_{n\times m}^C) \maltese (I_{n\times m}^B \otimes I_{n\times m}^C)$.

Proof We only prove the (1), (3), (5) and (7) in detail; the (2), (4), (6) and (8) can be proved in similar way.

(1) Let $I_{n\times m}^A$, $I_{n\times m}^B$ and $I_{n\times m}^C$ be three given IVFSM, then

$$+F_C^-(\varepsilon_j)(x_i) - \sqrt{F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)}F_C^-(\varepsilon_j)(x_i),$$
$$\sqrt{F_A^+(\varepsilon_j)(x_i)F_B^+(\varepsilon_j)(x_i)} + F_C^+(\varepsilon_j)(x_i)$$
$$\left. -\sqrt{F_A^+(\varepsilon_j)(x_i)F_B^+(\varepsilon_j)(x_i)}F_C^+(\varepsilon_j)(x_i)\right] \right)_{n\times m}$$

$$(I_{n\times m}^A \oplus I_{n\times m}^C) \$ (I_{n\times m}^B \oplus I_{n\times m}^C)$$
$$= \left(\left[\sqrt{F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)}\right.\right.$$
$$\sqrt{F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)},$$
$$\sqrt{F_A^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i) - F_A^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i)}$$
$$\left.\left.\sqrt{F_B^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i) - F_B^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i)}\right]\right)_{n\times m}.$$

Let

$$f = \left(\sqrt{F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)}\right.$$
$$\left.\sqrt{F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)}\right)^2$$
$$-\left(\sqrt{F_A^+(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)} + F_C^-(\varepsilon_j)(x_i)\right.$$
$$\left.-\sqrt{F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)}F_C^-(\varepsilon_j)(x_i)\right)^2,$$

then

$$f = F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)$$
$$- F_B^-(\varepsilon_j)(x_i)(F_C^-(\varepsilon_j)(x_i))^2 - F_A^-(\varepsilon_j)(x_i)(F_C^-(\varepsilon_j)(x_i))^2$$
$$+ 2\sqrt{F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)(F_C^-(\varepsilon_j)(x_i))^2}$$

$$(I_{n\times m}^A \& I_{n\times m}^B) \oplus I_{n\times m}^C = \left(\left[\frac{F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i)}{2} + F_C^-(\varepsilon_j)(x_i) - \frac{F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i)}{2}F_C^-(\varepsilon_j)(x_i),\right.\right.$$
$$\left.\left.\frac{F_A^+(\varepsilon_j)(x_i) + F_B^+(\varepsilon_j)(x_i)}{2} + F_C^+(\varepsilon_j)(x_i) - \frac{F_A^+(\varepsilon_j)(x_i) + F_B^+(\varepsilon_j)(x_i)}{2}F_C^+(\varepsilon_j)(x_i)\right]\right)_{n\times m},$$

$$(I_{n\times m}^A \oplus I_{n\times m}^C) \& (I_{n\times m}^B \oplus I_{n\times m}^C)$$
$$= \left(\left[\frac{F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)}{2},\right.\right.$$
$$\left.\left.\frac{F_A^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i) - F_A^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i) + F_B^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i) - F_B^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i)}{2}\right]\right)_{n\times m}$$
$$= \left(\left[\frac{F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i)}{2} + F_C^-(\varepsilon_j)(x_i) - \frac{F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i)}{2}F_C^-(\varepsilon_j)(x_i), \frac{F_A^+(\varepsilon_j)(x_i) + F_B^+(\varepsilon_j)(x_i)}{2}\right.\right.$$
$$\left.\left.+F_C^+(\varepsilon_j)(x_i) - \frac{F_A^+(\varepsilon_j)(x_i) + F_B^+(\varepsilon_j)(x_i)}{2}F_C^+(\varepsilon_j)(x_i)\right]\right)_{n\times m} = (I_{n\times m}^A \& I_{n\times m}^B) \oplus I_{n\times m}^C.$$

(3) Let $I_{n\times m}^A$, $I_{n\times m}^B$ and $I_{n\times m}^C$ be three given IVFSM, then

$$(I_{n\times m}^A \$ I_{n\times m}^B) \oplus I_{n\times m}^C = \left(\left[\sqrt{F_A^+(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)}\right.\right.$$

$$-2\sqrt{F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)}F_C^-(\varepsilon_j)(x_i) = F_C^-(\varepsilon_j)(x_i)(1 - F_C^-(\varepsilon_j)(x_i))\left(\sqrt{F_A^-(\varepsilon_j)(x_i)} - \sqrt{F_B^-(\varepsilon_j)(x_i)}\right)^2 \geq 0.$$

Hence, we can have

$$\sqrt{F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)}\sqrt{F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)}$$
$$\geq \sqrt{F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)} + F_C^-(\varepsilon_j)(x_i) - \sqrt{F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)}F_C^-(\varepsilon_j)(x_i).$$

Similarly, we can obtain

$$\sqrt{F_A^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i) - F_A^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i)}\sqrt{F_B^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i) - F_B^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i)}$$
$$\geq \sqrt{F_A^+(\varepsilon_j)(x_i)F_B^+(\varepsilon_j)(x_i)} + F_C^+(\varepsilon_j)(x_i) - \sqrt{F_A^+(\varepsilon_j)(x_i)F_B^+(\varepsilon_j)(x_i)}F_C^+(\varepsilon_j)(x_i).$$

According to the proposed score function s_{pxd}, we can have

$$(I_{n\times m}^A \$ I_{n\times m}^B) \oplus I_{n\times m}^C \subseteq (I_{n\times m}^A \oplus I_{n\times m}^C)\$(I_{n\times m}^B \oplus I_{n\times m}^C).$$

(5) Let $I_{n\times m}^A$, $I_{n\times m}^B$ and $I_{n\times m}^C$ be three given IVFSM, then

$$(I_{n\times m}^A @ I_{n\times m}^B) \oplus I_{n\times m}^C = \left(\left[\frac{2F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)}{F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i)} + F_C^-(\varepsilon_j)(x_i) - \frac{2F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)}{F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i)}F_C^-(\varepsilon_j)(x_i),\right.\right.$$
$$\left.\left.\frac{2F_A^+(\varepsilon_j)(x_i)F_B^+(\varepsilon_j)(x_i)}{F_A^+(\varepsilon_j)(x_i) + F_B^+(\varepsilon_j)(x_i)} + F_C^+(\varepsilon_j)(x_i) - \frac{2F_A^+(\varepsilon_j)(x_i)F_B^+(\varepsilon_j)(x_i)}{F_A^+(\varepsilon_j)(x_i) + F_B^+(\varepsilon_j)(x_i)}F_C^+(\varepsilon_j)(x_i)\right]\right)_{n\times m}$$
$$(I_{n\times m}^A \oplus I_{n\times m}^C) @ (I_{n\times m}^B \oplus I_{n\times m}^C)$$
$$= \left(\left[\frac{2(F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i))(F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i))}{F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)},\right.\right.$$
$$\left.\left.\frac{2(F_A^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i) - F_A^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i))(F_B^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i) - F_B^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i))}{F_A^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i) - F_A^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i) + F_B^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i) - F_B^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i)}\right]\right)_{n\times m}$$

Let

$$f = \frac{2F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)}{F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i)} + F_C^-(\varepsilon_j)(x_i) - \frac{2F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)}{F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i)}F_C^-(\varepsilon_j)(x_i)$$
$$-\frac{2(F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i))(F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i))}{F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)},$$

then

$$f = \frac{(1 - F_C^-(\varepsilon_j)(x_i))(F_A^-(\varepsilon_j)(x_i) - F_B^-(\varepsilon_j)(x_i))^2 F_C^-(\varepsilon_j)(x_i)}{(F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i))(F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i)) + 2F_C^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i) - F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)}$$
$$\geq 0.$$

Hence, we can have

$$\frac{2F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)}{F_A^-(\varepsilon_j)(x_i)+F_B^-(\varepsilon_j)(x_i)}+F_C^-(\varepsilon_j)(x_i)-\frac{2F_A^-(\varepsilon_j)(x_i)F_B^-(\varepsilon_j)(x_i)}{F_A^-(\varepsilon_j)(x_i)+F_B^-(\varepsilon_j)(x_i)}F_C^-(\varepsilon_j)(x_i)$$

$$\geq\frac{2(F_A^-(\varepsilon_j)(x_i)+F_C^-(\varepsilon_j)(x_i)-F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i))(F_B^-(\varepsilon_j)(x_i)+F_C^-(\varepsilon_j)(x_i)-F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i))}{F_A^-(\varepsilon_j)(x_i)+F_C^-(\varepsilon_j)(x_i)-F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)+F_B^-(\varepsilon_j)(x_i)+F_C^-(\varepsilon_j)(x_i)-F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)}.$$

Similarly, we can have

$$\frac{2F_A^+(\varepsilon_j)(x_i)F_B^+(\varepsilon_j)(x_i)}{F_A^+(\varepsilon_j)(x_i)+F_B^+(\varepsilon_j)(x_i)}+F_C^+(\varepsilon_j)(x_i)-\frac{2F_A^+(\varepsilon_j)(x_i)F_B^+(\varepsilon_j)(x_i)}{F_A^+(\varepsilon_j)(x_i)+F_B^+(\varepsilon_j)(x_i)}F_C^+(\varepsilon_j)(x_i)$$

$$\geq\frac{2(F_A^+(\varepsilon_j)(x_i)+F_C^+(\varepsilon_j)(x_i)-F_A^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i))(F_B^+(\varepsilon_j)(x_i)+F_C^+(\varepsilon_j)(x_i)-F_B^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i))}{F_A^+(\varepsilon_j)(x_i)+F_C^+(\varepsilon_j)(x_i)-F_A^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i)+F_B^+(\varepsilon_j)(x_i)+F_C^+(\varepsilon_j)(x_i)-F_B^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i)}.$$

According to the proposed score function s_{pxd}, we can have

$$(I_{n\times m}^A\,@\,I_{n\times m}^B)\oplus I_{n\times m}^C\supseteq(I_{n\times m}^A\oplus I_{n\times m}^C)@(I_{n\times m}^B\oplus I_{n\times m}^C).$$

(7)

$$(I_{n\times m}^A\maltese I_{n\times m}^B)\oplus I_{n\times m}^C=\left(\left[\frac{F_A^-(\varepsilon_j)(x_i)+F_B^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i)+F_B^-(\varepsilon_j)(x_i)+1)}+F_C^-(\varepsilon_j)(x_i)\right.\right.$$

$$\left.-\frac{F_A^-(\varepsilon_j)(x_i)+F_B^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i)+F_B^-(\varepsilon_j)(x_i)+1)}F_C^-(\varepsilon_j)(x_i),\frac{F_A^+(\varepsilon_j)(x_i)+F_B^+(\varepsilon_j)(x_i)}{2(F_A^+(\varepsilon_j)(x_i)+F_B^+(\varepsilon_j)(x_i)+1)}+F_C^+(\varepsilon_j)(x_i)\right.$$

$$\left.\left.-\frac{F_A^+(\varepsilon_j)(x_i)+F_B^+(\varepsilon_j)(x_i)}{2(F_A^+(\varepsilon_j)(x_i)+F_B^+(\varepsilon_j)(x_i)+1)}F_C^+(\varepsilon_j)(x_i)\right]\right)_{n\times m}$$

$$(I_{n\times m}^A\oplus I_{n\times m}^C)\maltese(I_{n\times m}^B\oplus I_{n\times m}^C)$$

$$=\left(\left[\frac{F_A^-(\varepsilon_j)(x_i)+F_C^-(\varepsilon_j)(x_i)-F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)+F_B^-(\varepsilon_j)(x_i)+F_C^-(\varepsilon_j)(x_i)-F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i)+F_C^-(\varepsilon_j)(x_i)-F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)+F_B^-(\varepsilon_j)(x_i)+F_C^-(\varepsilon_j)(x_i)-F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)+1)},\right.\right.$$

$$\left.\left.\frac{F_A^+(\varepsilon_j)(x_i)+F_C^+(\varepsilon_j)(x_i)-F_A^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i)+F_B^+(\varepsilon_j)(x_i)+F_C^+(\varepsilon_j)(x_i)-F_A^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i)}{2(F_A^+(\varepsilon_j)(x_i)+F_C^+(\varepsilon_j)(x_i)-F_A^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i)+F_B^+(\varepsilon_j)(x_i)+F_C^+(\varepsilon_j)(x_i)-F_B^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i)+1)}\right]\right)_{n\times m}.$$

Let

$$f=\frac{F_A^-(\varepsilon_j)(x_i)+F_B^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i)+F_B^-(\varepsilon_j)(x_i)+1)}+F_C^-(\varepsilon_j)(x_i)-\frac{F_A^-(\varepsilon_j)(x_i)+F_B^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i)+F_B^-(\varepsilon_j)(x_i)+1)}F_C^-(\varepsilon_j)(x_i)$$

$$-\frac{F_A^-(\varepsilon_j)(x_i)+F_C^-(\varepsilon_j)(x_i)-F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)+F_B^-(\varepsilon_j)(x_i)+F_C^-(\varepsilon_j)(x_i)-F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i)+F_C^-(\varepsilon_j)(x_i)-F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)+F_B^-(\varepsilon_j)(x_i)+F_C^-(\varepsilon_j)(x_i)-F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)+1)},$$

then

$$f=\frac{F_A^-(\varepsilon_j)(x_i)+F_B^-(\varepsilon_j)(x_i)+F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)+F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)+2F_C^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i)+F_B^-(\varepsilon_j)(x_i)+1)}$$

$$-\frac{F_A^-(\varepsilon_j)(x_i)+F_C^-(\varepsilon_j)(x_i)-F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)+F_B^-(\varepsilon_j)(x_i)+F_C^-(\varepsilon_j)(x_i)-F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i)+F_C^-(\varepsilon_j)(x_i)-F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)+F_B^-(\varepsilon_j)(x_i)+F_C^-(\varepsilon_j)(x_i)-F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)+1)}$$

$$\geq\frac{F_A^-(\varepsilon_j)(x_i)+F_B^-(\varepsilon_j)(x_i)+F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)+F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)+2F_C^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i)+F_C^-(\varepsilon_j)(x_i)-F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)+F_B^-(\varepsilon_j)(x_i)+F_C^-(\varepsilon_j)(x_i)-F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)+1)}$$

$$-\frac{F_A^-(\varepsilon_j)(x_i)+F_C^-(\varepsilon_j)(x_i)-F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)+F_B^-(\varepsilon_j)(x_i)+F_C^-(\varepsilon_j)(x_i)-F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i)+F_C^-(\varepsilon_j)(x_i)-F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)+F_B^-(\varepsilon_j)(x_i)+F_C^-(\varepsilon_j)(x_i)-F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)+1)}$$

$$= \frac{F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)}{(F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i) + 1)}$$
$$\geq 0.$$

Hence, we can have

$$\frac{F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) + 1)} + F_C^-(\varepsilon_j)(x_i) - \frac{F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) + 1)}F_C^-(\varepsilon_j)(x_i)$$
$$\geq \frac{F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i)}{2(F_A^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_A^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i) + F_B^-(\varepsilon_j)(x_i) + F_C^-(\varepsilon_j)(x_i) - F_B^-(\varepsilon_j)(x_i)F_C^-(\varepsilon_j)(x_i) + 1)}.$$

Similarly, we can have

$$\frac{F_A^+(\varepsilon_j)(x_i) + F_B^+(\varepsilon_j)(x_i)}{2(F_A^+(\varepsilon_j)(x_i) + F_B^+(\varepsilon_j)(x_i) + 1)} + F_C^+(\varepsilon_j)(x_i) - \frac{F_A^+(\varepsilon_j)(x_i) + F_B^+(\varepsilon_j)(x_i)}{2(F_A^+(\varepsilon_j)(x_i) + F_B^+(\varepsilon_j)(x_i) + 1)}F_C^+(\varepsilon_j)(x_i)$$
$$\geq \frac{F_A^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i) - F_A^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i) + F_B^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i) - F_B^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i)}{2(F_A^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i) - F_A^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i) + F_B^+(\varepsilon_j)(x_i) + F_C^+(\varepsilon_j)(x_i) - F_B^+(\varepsilon_j)(x_i)F_C^+(\varepsilon_j)(x_i) + 1)}.$$

According to the proposed score function s_{pxd}, we can have

$$(I_{n\times m}^A \boxplus I_{n\times m}^B) \oplus I_{n\times m}^C \supseteq (I_{n\times m}^A \oplus I_{n\times m}^C)$$
$$\boxplus (I_{n\times m}^B \oplus I_{n\times m}^C).$$

$\square$

Definition 12 Let $I_{n\times m}$ be an IVFSM and for every $a, b \in N$, we define

$$I_{n\times m}^{(a,b)} = \left(\left[\frac{(F^-(\varepsilon_j)(x_i))^a}{b}, \frac{(F^+(\varepsilon_j)(x_i))^a}{b} \right] \right)_{n\times m}.$$

From above definition, we can easily have the following relations:

(1) If $a \leq a_1$, then $I_{n\times m}^{(a,b)} \supseteq I_{n\times m}^{(a_1,b)}$;
(2) If $b \leq b_1$, then $I_{n\times m}^{(a,b)} \supseteq I_{n\times m}^{(a,b_1)}$;
(3) If $a \leq a_1$ and $b \leq b_1$, then $I_{n\times m}^{(a,b)} \supseteq I_{n\times m}^{(a_1,b_1)}$;
(4) If $I_{n\times m}^A \subseteq I_{n\times m}^B$, then $(I_{n\times m}^A)^{(a,b)} \subseteq (I_{n\times m}^B)^{(a,b)}$.

Theorem 6 Let $I_{n\times m}$ be an IVFSM and for every $a, b \in N$, then

(1) $(I_{n\times m}^{(a,1)})^{(b,1)} = I_{n\times m}^{(ab,1)} = (I_{n\times m}^{(b,1)})^{(a,1)}$;
(2) $(I_{n\times m}^{(1,a)})^{(1,b)} = I_{n\times m}^{(1,ab)} = (I_{n\times m}^{(1,b)})^{(1,a)}$;
(3) $(I_{n\times m}^{(a,1)})^{(1,b)} = I_{n\times m}^{(a,b)}$;
(4) $(I_{n\times m}^{(1,a)})^{(b,1)} = I_{n\times m}^{(b,a^b)}$.

Proof It is trial. $\square$

Theorem 7 Let $I_{n\times m}$ be an IVFSM and for every $a, b, a_1, b_1 \in N$, then

(1) $(I_{n\times m}^{(a,b)})^{(a_1,b_1)} = I_{n\times m}^{(aa_1, b_1 b^{a_1})}$;
(2) $(I_{n\times m}^{(a,b)})^{(a_1,b_1)} \otimes (I_{n\times m}^{(b,a)})^{b_1,a_1} = I_{n\times m}^{(aa_1 + bb_1, a_1 b_1 a^{b_1} b^{a_1})}$.

Proof We only prove the (1), and the (2) can be proved in similar way.

(1) According to the Definition 12, we can have

$$(I_{n\times m}^{(a,b)})^{(a_1,b_1)}$$
$$= (([\frac{(F^-(\varepsilon_j)(x_i))^a}{b}, \frac{(F^+(\varepsilon_j)(x_i))^a}{b}])_{n\times m})^{(a_1,b_1)}$$
$$= ([\frac{(F^-(\varepsilon_j)(x_i))^{aa_1}}{b_1 b^{a_1}}, \frac{(F^+(\varepsilon_j)(x_i))^{aa_1}}{b_1 b^{a_1}}])_{n\times m}$$
$$= I_{n\times m}^{(aa_1, b_1 b^{a_1})}.$$

$\square$

Theorem 8 Let $I_{n\times m}^A$ and $I_{n\times m}^B$ be two IVFSMs and for every $a, b \in N$, then

(1) $(I_{n\times m}^A \cap I_{n\times m}^B)^{(a,b)} \oplus (I_{n\times m}^A \cup I_{n\times m}^B)^{(a,b)} = (I_{n\times m}^A)^{(a,b)} \oplus (I_{n\times m}^B)^{(a,b)}$;
(2) $(I_{n\times m}^A \cap I_{n\times m}^B)^{(a,b)} \otimes (I_{n\times m}^A \cup I_{n\times m}^B)^{(a,b)} = (I_{n\times m}^A)^{(a,b)} \otimes (I_{n\times m}^B)^{(a,b)}$;
(3) $(I_{n\times m}^A \cap I_{n\times m}^B)^{(a,b)} \& (I_{n\times m}^A \cup I_{n\times m}^B)^{(a,b)} = (I_{n\times m}^A)^{(a,b)} \& (I_{n\times m}^B)^{(a,b)}$;
(4) $(I_{n\times m}^A \cap I_{n\times m}^B)^{(a,b)} @ (I_{n\times m}^A \cup I_{n\times m}^B)^{(a,b)} = (I_{n\times m}^A)^{(a,b)} @ (I_{n\times m}^B)^{(a,b)}$;
(5) $(I_{n\times m}^A \cap I_{n\times m}^B)^{(a,b)} \$ (I_{n\times m}^A \cup I_{n\times m}^B)^{(a,b)} = (I_{n\times m}^A)^{(a,b)} \$ (I_{n\times m}^B)^{(a,b)}$;
(6) $(I_{n\times m}^A \cap I_{n\times m}^B)^{(a,b)} \boxplus (I_{n\times m}^A \cup I_{n\times m}^B)^{(a,b)} = (I_{n\times m}^A)^{(a,b)} \boxplus (I_{n\times m}^B)^{(a,b)}$.

Proof We only prove the (1) in detail, the (2)–(6) can be proved in similar way.

(1) Let $I_{n\times m}^A$ and $I_{n\times m}^B$ be two IVFSMs, and $\max\{x, y\} + \min\{x, y\} = x + y$, $\max\{x, y\}\min\{x, y\} = xy$, then

$$
(I_{n\times m}^A \cap I_{n\times m}^B)^{(a,b)} \oplus (I_{n\times m}^A \cup I_{n\times m}^B)^{(a,b)}
$$

$$
= \left(\left[\frac{\min\{F_A^-(\varepsilon_j)(x_i), F_B^-(\varepsilon_j)(x_i)\}^a}{b}, \frac{\min\{F_A^+(\varepsilon_j)(x_i), F_B^+(\varepsilon_j)(x_i)\}^a}{b}\right]\right)_{n\times m} \oplus
$$

$$
\left(\left[\frac{\max\{F_A^-(\varepsilon_j)(x_i), F_B^-(\varepsilon_j)(x_i)\}^a}{b}, \frac{\max\{F_A^+(\varepsilon_j)(x_i), F_B^+(\varepsilon_j)(x_i)\}^a}{b}\right]\right)_{n\times m}
$$

$$
= \left(\left[\frac{\min\{F_A^-(\varepsilon_j)(x_i), F_B^-(\varepsilon_j)(x_i)\}^a + \max\{F_A^-(\varepsilon_j)(x_i), F_B^-(\varepsilon_j)(x_i)\}^a - \min\{F_A^-(\varepsilon_j)(x_i), F_B^-(\varepsilon_j)(x_i)\}^a\max\{F_A^-(\varepsilon_j)(x_i), F_B^-(\varepsilon_j)(x_i)\}^a}{b},\right.\right.
$$

$$
\left.\left.\frac{\min\{F_A^+(\varepsilon_j)(x_i), F_B^+(\varepsilon_j)(x_i) + \max\{F_A^+(\varepsilon_j)(x_i), F_B^+(\varepsilon_j)(x_i) - \min\{F_A^+(\varepsilon_j)(x_i), F_B^+(\varepsilon_j)(x_i)\max\{F_A^+(\varepsilon_j)(x_i), F_B^+(\varepsilon_j)(x_i)\}^a}{b}\right]\right)_{n\times m}
$$

$$
= \left(\left[\frac{(F_A^-(\varepsilon_j)(x_i))^a + (F_B^-(\varepsilon_j)(x_i))^a - (F_A^-(\varepsilon_j)(x_i))^a(F_B^-(\varepsilon_j)(x_i))^a}{b},\right.\right.
$$

$$
\left.\left.\frac{(F_A^+(\varepsilon_j)(x_i))^a + (F_B^+(\varepsilon_j)(x_i))^a - (F_A^+(\varepsilon_j)(x_i))^a(F_B^+(\varepsilon_j)(x_i))^a}{b}\right]\right)_{n\times m} = (I_{n\times m}^A)^{(a,b)} \oplus (I_{n\times m}^B)^{(a,b)}. \qquad \square
$$

Theorem 9 *Let $I_{n\times m}^A$ and $I_{n\times m}^B$ be two IVFSMs and for every $a, b \in N$, then*

(1) $\left((I_{n\times m}^A)^{(a,b)} \oplus (I_{n\times m}^B)^{(a,b)}\right) \,\&\, \left((I_{n\times m}^A)^{(a,b)} \otimes (I_{n\times m}^B)^{(a,b)}\right)$
$= (I_{n\times m}^A)^{(a,b)} \,\&\, (I_{n\times m}^B)^{(a,b)}$;

(2) $\left((I_{n\times m}^A)^{(a,b)} \oplus (I_{n\times m}^B)^{(a,b)}\right) \maltese \left((I_{n\times m}^A)^{(a,b)} \otimes (I_{n\times m}^B)^{(a,b)}\right)$
$= (I_{n\times m}^A)^{(a,b)} \maltese (I_{n\times m}^B)^{(a,b)}$;

(3) $[\left((I_{n\times m}^A)^{(a,b)} \oplus (I_{n\times m}^B)^{(a,b)}\right) \cup \left((I_{n\times m}^A)^{(a,b)} \otimes (I_{n\times m}^B)^{(a,b)}\right)]$
$\,\&\, [\left((I_{n\times m}^A)^{(a,b)} \oplus (I_{n\times m}^B)^{(a,b)}\right) \cap \left((I_{n\times m}^A)^{(a,b)} \otimes (I_{n\times m}^B)^{(a,b)}\right)] = (I_{n\times m}^A)^{(a,b)} \,\&\, (I_{n\times m}^B)^{(a,b)}$;

(4) $[\left((I_{n\times m}^A)^{(a,b)} \oplus (I_{n\times m}^B)^{(a,b)}\right) \cup \left((I_{n\times m}^A)^{(a,b)} \otimes (I_{n\times m}^B)^{(a,b)}\right)]$
$\maltese [\left((I_{n\times m}^A)^{(a,b)} \oplus (I_{n\times m}^B)^{(a,b)}\right) \cap \left((I_{n\times m}^A)^{(a,b)} \otimes (I_{n\times m}^B)^{(a,b)}\right)] = (I_{n\times m}^A)^{(a,b)} \maltese (I_{n\times m}^B)^{(a,b)}$;

(5) $[((I_{n\times m}^A)^{(a,b)} \otimes (I_{n\times m}^B)^{(a,b)}) \,\&\, ((I_{n\times m}^A)^{(a,b)} \otimes (I_{n\times m}^B)^{(a,b)})^c] \,\&\, [((I_{n\times m}^A)^{(a,b)} \oplus (I_{n\times m}^B)^{(a,b)}) \,\&\, ((I_{n\times m}^A)^{(a,b)} \oplus (I_{n\times m}^B)^{(a,b)})^c] = ([\frac{1}{2}, \frac{1}{2}])_{n\times m}$;

(6) $[((I_{n\times m}^A)^{(a,b)} \otimes (I_{n\times m}^B)^{(a,b)}) \maltese ((I_{n\times m}^A)^{(a,b)} \otimes (I_{n\times m}^B)^{(a,b)})^c] \maltese [((I_{n\times m}^A)^{(a,b)} \oplus (I_{n\times m}^B)^{(a,b)}) \maltese ((I_{n\times m}^A)^{(a,b)} \oplus (I_{n\times m}^B)^{(a,b)})^c] = ([\frac{1}{4}, \frac{1}{4}])_{n\times m}$.

Proof It is trial. $\qquad \square$

Theorem 10 *Let $I_{n\times m}^A$ and $I_{n\times m}^B$ be two IVFSMs and for every $a, b \in N$, then*

(1) $[(I_{n\times m}^A \cup I_{n\times m}^B)^{(a,b)} @ (I_{n\times m}^A \cap I_{n\times m}^B)^{(a,b)}]\$[(I_{n\times m}^A \cup I_{n\times m}^B)^{(a,b)} \,\&\, (I_{n\times m}^A \cap I_{n\times m}^B)^{(a,b)}] = (I_{n\times m}^A)^{(a,b)} \$ (I_{n\times m}^B)^{(a,b)}$;

(2) $[(I_{n\times m}^A)^{(a,b)} @ (I_{n\times m}^B)^{(a,b)}]\$[(I_{n\times m}^A)^{(a,b)} \,\&\, (I_{n\times m}^B)^{(a,b)}] = (I_{n\times m}^A)^{(a,b)} \$ (I_{n\times m}^B)^{(a,b)}$.

Proof We only prove the (1) in detail; the (2) can be proved in similar way.

(1) Let $I_{n\times m}^A$ and $I_{n\times m}^B$ be two IVFSMs, and $\max\{x, y\} + \min\{x, y\} = x + y$, $\max\{x, y\}\min\{x, y\} = xy$, then

Table 3 The interval-valued fuzzy soft set (F, E)

	ε_1	ε_2	$\cdots$	ε_n
x_1	$F(\varepsilon_1)(x_1)$	$F(\varepsilon_2)(x_1)$	$\cdots$	$F(\varepsilon_n)(x_1)$
x_2	$F(\varepsilon_1)(x_2)$	$F(\varepsilon_2)(x_2)$	$\cdots$	$F(\varepsilon_n)(x_2)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
x_m	$F(\varepsilon_1)(x_m)$	$F(\varepsilon_2)(x_m)$	$\cdots$	$F(\varepsilon_n)(x_m)$

5.2 Approach to determine combined weights

5.2.1 Determine objective weights: CRITIC method

Criteria can be served as an important source of information in decision-making issues. Important parameter weights can reveal the wealth of information contained in each, called "objective weights." The CRITIC (Criteria Importance Through Inter-criteria Correlation) is a method for calcu-

$$
[(I_{n\times m}^A \cup I_{n\times m}^B)^{(a,b)}@(I_{n\times m}^A \cap I_{n\times m}^B)^{(a,b)}] = \left(\left[\frac{2\frac{\min\{F_A^-(\varepsilon_j)(x_i), F_B^-(\varepsilon_j)(x_i)\}^a}{b} \frac{\max\{F_A^-(\varepsilon_j)(x_i), F_B^-(\varepsilon_j)(x_i)\}^a}{b}}{\frac{\min\{F_A^-(\varepsilon_j)(x_i), F_B^-(\varepsilon_j)(x_i)\}^a}{b} + \frac{\max\{F_A^-(\varepsilon_j)(x_i), F_B^-(\varepsilon_j)(x_i)\}^a}{b}}, \right. \right.
$$

$$
\left. \frac{2\frac{\min\{F_A^+(\varepsilon_j)(x_i), F_B^+(\varepsilon_j)(x_i)\}^a}{b} \frac{\max\{F_A^+(\varepsilon_j)(x_i), F_B^+(\varepsilon_j)(x_i)\}^a}{b}}{\frac{\min\{F_A^+(\varepsilon_j)(x_i), F_B^+(\varepsilon_j)(x_i)\}^a}{b} + \frac{\max\{F_A^+(\varepsilon_j)(x_i), F_B^+(\varepsilon_j)(x_i)\}^a}{b}} \right]\right)_{n\times m}
$$

$$
= \left(\left[\frac{2\frac{(F_A^-(\varepsilon_j)(x_i) F_B^-(\varepsilon_j)(x_i))^a}{b}}{(F_A^-(\varepsilon_j)(x_i))^a + (F_B^-(\varepsilon_j)(x_i))^a}, \frac{2\frac{(F_A^+(\varepsilon_j)(x_i) F_B^+(\varepsilon_j)(x_i))^a}{b}}{(F_A^+(\varepsilon_j)(x_i))^a + (F_B^+(\varepsilon_j)(x_i))^a} \right]\right)_{n\times m},
$$

$$
[(I_{n\times m}^A \cup I_{n\times m}^B)^{(a,b)}\&(I_{n\times m}^A \cap I_{n\times m}^B)^{(a,b)}] = \left(\left[\frac{\min\{F_A^-(\varepsilon_j)(x_i), F_B^-(\varepsilon_j)(x_i)\}^a + \max\{F_A^-(\varepsilon_j)(x_i), F_B^-(\varepsilon_j)(x_i)\}^a}{2}, \right. \right.
$$

$$
\left. \left. \frac{\min\{F_A^+(\varepsilon_j)(x_i), F_B^+(\varepsilon_j)(x_i)\}^a + \max\{F_A^+(\varepsilon_j)(x_i), F_B^+(\varepsilon_j)(x_i)\}^a)}{2} \right] \right)_{n\times m}
$$

$$
= \left(\left[\frac{(F_A^-(\varepsilon_j)(x_i))^a + (F_B^-(\varepsilon_j)(x_i))^a}{2b}, \frac{(F_A^+(\varepsilon_j)(x_i))^a + (F_B^+(\varepsilon_j)(x_i))^a}{2b} \right] \right)_{n\times m}, [(I_{n\times m}^A \cup I_{n\times m}^B)^{(a,b)}@(I_{n\times m}^A \cap I_{n\times m}^B)^{(a,b)}]
$$

$$
\$[(I_{n\times m}^A \cup I_{n\times m}^B)^{(a,b)}\&(I_{n\times m}^A \cap I_{n\times m}^B)^{(a,b)}] = \left(\left[\sqrt{\frac{(F_A^-(\varepsilon_j)(x_i))^a}{b} \frac{(F_B^-(\varepsilon_j)(x_i))^a}{b}}, \right. \right.
$$

$$
\left. \left. \sqrt{\frac{(F_A^+(\varepsilon_j)(x_i))^a}{b} \frac{(F_B^+(\varepsilon_j)(x_i))^a}{b}} \right] \right)_{n\times m} = (I_{n\times m}^A)^{(a,b)} \$ (I_{n\times m}^B)^{(a,b)}. \qquad \square
$$

5 Novel interval-valued fuzzy soft decision-making method based on CoCoSo with CRITIC

5.1 The introduction of decision-making issue

Suppose that $U = \{x_1, x_2, \cdots, x_m\}$ be a set of alternatives, $E = \{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\}$ be a discrete set of parameters, and $W = \{w_1, w_2, \cdots, w_n\}$ be weight vector with $w_j \in [0, 1], \sum_{j=1}^{n} w_j = 1$. Suppose that the assessment of alternative x_i with respect to attribute ε_j be denoted by interval-valued fuzzy number $F(\varepsilon_j)(x_i) = [F^-(\varepsilon_j)(x_i), F^+(\varepsilon_j)(x_i)]$, which can be shown in Table 3.

lating the objective weight of a given criterion in MCDM problems (Diakoulaki et al. 1995). The objective weights obtained through the above methods combine the strength comparison of each criterion with the conflict between the criteria. The standard intensity contrast is considered to be the standard deviation, and the conflict between them is calculated by the correlation coefficient. In this section, we extend this method to the interval value fuzzy soft environment.

Suppose that $F(\varepsilon_j)(x_i)$ denotes the interval-valued fuzzy soft performance value of ith alternative according to jth parameter ($i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$), w_j^o represents the fuzzy objective weight of jth parameter, C is a series of cost parameters, and B is a series of benefit parameters. In the following, the steps of computing interval-valued fuzzy soft objective weights-based CRITIC are listed.

Step 1: Calculate score function $F = (\tau_{ij})_{m\times n}(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$ of each IVFN $F(\varepsilon_j)(x_i) = [F^-(\varepsilon_j)(x_i), F^+(\varepsilon_j)(x_i)]$ by Eq. (6).

$$\tau_{ij} = \frac{F^-(\varepsilon_j)(x_i) + F^+(\varepsilon_j)(x_i) - \ln\left(1 + F^+(\varepsilon_j)(x_i) - F^-(\varepsilon_j)(x_i)\right)}{2}.$$

$$(6)$$

Step 2: Switch the matrix F into a standard interval-valued fuzzy soft matrix $F' = (\tau'_{ij})_{m \times n}$ by Eq. (7)

$$\tau'_{ij} = \begin{cases} \frac{\tau_{ij} - \tau_j^-}{\tau_j^+ - \tau_j^-}, \text{ if } j \in B, \\ \frac{\tau_j^+ - \tau_{ij}}{\tau_j^+ - \tau_j^-}, \text{ if } j \in C, \end{cases} \quad (7)$$

where $\tau_j^- = \underset{i}{min}\ \tau_{ij}$ and $\tau_j^+ = \underset{i}{max}\ \tau_{ij}$

Step 3: Compute the criteria standard deviations by Eq. (8).

$$\sigma_j = \sqrt{\frac{\sum\limits_{i=1}^{m}(\tau'_{ij} - \overline{\tau}_j)^2}{m}}, \quad (8)$$

where $\overline{\tau}_j = \frac{\sum\limits_{i=1}^{m} \tau'_{ij}}{m}$.

Step 4: Determine the correlation between criteria pairs using Eq. (9).

$$\rho_{jk} = \frac{\sum\limits_{i=1}^{m}(\tau'_{ij} - \overline{\tau}_j)(\tau'_{ik} - \overline{\tau}_k)}{\sqrt{\sum\limits_{i=1}^{m}(\tau'_{ij} - \overline{\tau}_j)^2 \sum\limits_{i=1}^{m}(\tau'_{ik} - \overline{\tau}_k)^2}}. \quad (9)$$

Step 5: Compute the quantity of information of each criterion as follows:

$$c_j = \sigma_j \sum_{k=1}^{n}(1 - \rho_{jk}). \quad (10)$$

The larger the c_j is, the more information a certain criterion contains, so the weight of this evaluation criterion is greater than that of other criteria.

Step 6: Calculate the objective weight of each criterion as follows:

$$\omega_j = \frac{c_j}{\sum\limits_{j=1}^{n} c_j}. \quad (11)$$

5.2.2 Determine combined weights: linear weighted comprehensive method

Suppose that the subjective weight, given by experts, is $w = \{w_1, w_2, \cdots, w_n\}$, where $\sum\limits_{j=1}^{n} w_j = 1, 0 \leq w_j \leq$

1. The objective weight, calculated by Eq.(11), is $\omega = \{\omega_1, \omega_2, \cdots, \omega_n\}$, where $\sum\limits_{j=1}^{n} \omega_j = 1, 0 \leq \omega_j \leq 1$.

Therefore, combined weight $\varpi = \{\varpi_1, \varpi_2, \cdots, \varpi_n\}$ can be denoted as follows:

$$\varpi_j = \frac{w_j * \omega_j}{\sum\limits_{j=1}^{n} w_j * \omega_j}, \quad (12)$$

where $\sum\limits_{j=1}^{n} \varpi_j = 1, 0 \leq \varpi_j \leq 1$.

The subjective weights are unified with the objective weights by using a nonlinear weighted synthesis method. According to multiplier effect, the larger the value of subjective weight and objective weight, the larger the combined weight, and vice versa. In additional, it is easily seen that Eq. (12) promotes restrictions that only consider objective or subjective effects. The advantage of Eq. (12) is that the parameter weights of alternatives can simultaneously show objective information and subjective information (Wang and Li 2020; Lin et al. 2020).

5.3 The interval-valued fuzzy soft CoCoSo method

CoCoSo (Combined Compromise Solution) is a new and resultful MCDM method, which is explored by Yazdani et al. (2019) recently. It has successfully applied in diverse domains such as hospital service quality (Roy et al. 2018), location selection (Yazdani et al. 2019, 2020), stock management (Erceg et al. 2019), sustainability assessment (Ecer et al. 2020) and financial risk evaluation (Peng and Huang 2020). The presented approach is based on an integrated exponentially weighted product (EWP) and simple additive weighting (SAW) model, which can be a compendium of compromise solutions. In order to deal the MCDM issue, we develop an IVFS-CoCoSo approach.

Generally speaking, the IVFS-CoCoSo method has the following steps.

Remark 1 It should be noted that the interval-value fuzzy soft CoCoSo method whose information in the decision matrix is denoted by IVFNs to express the preference of DMs. IVFN is very effective in capturing the uncertainties and inaccuracies of experts or DM in MCDM problems. In addition, the IVFS-CoCoSo method is a valuable tool for dealing with DM problems, which has a strong ability to distinguish optimal alternative from nominated alternatives, without counterintuitive phenomena, and no division by zero problem (Peng and Garg 2018). Nevertheless, others MCDM methods based on interval-valued fuzzy soft environments have no such valuable features. The framework of proposed method is presented in Fig. 1.
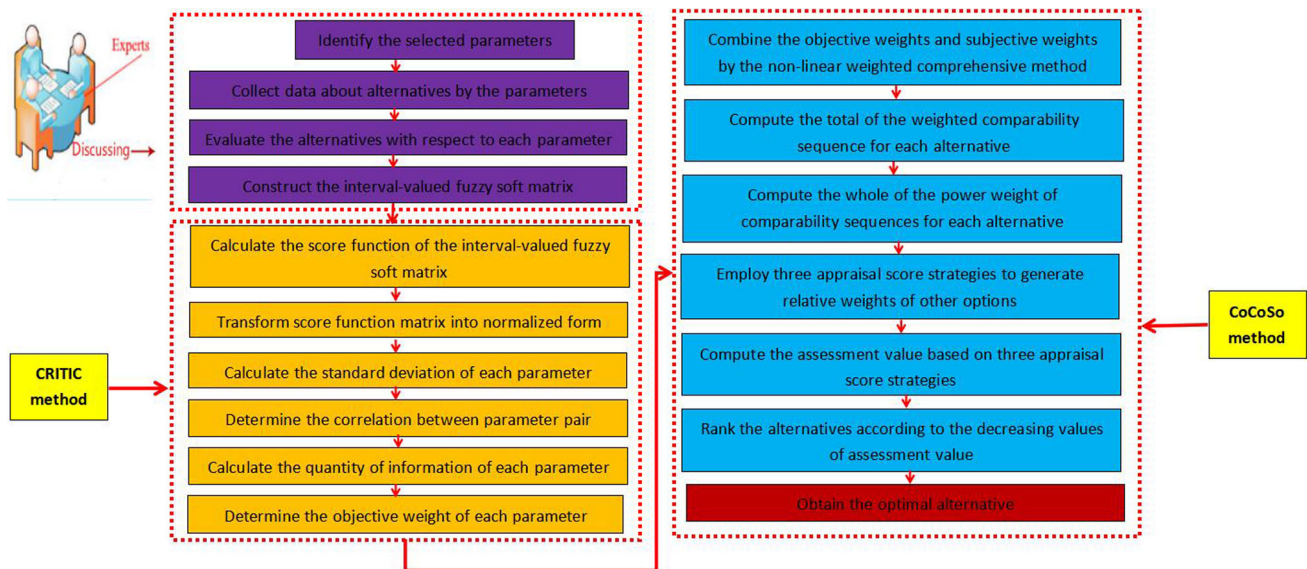
Fig. 1 The framework for using the proposed method

6 A case study in intelligent health management evaluation

Intelligent health management (IHM) is a new industry and new format formed by the integration of health management, artificial intelligence, big data and other new generation of information technology (Fig. 2).

6.1 The development status of health management

Most people's understanding of health management (Mardani et al. 2019) has some misunderstanding as follows:

(1) Health management is considered to be a higher-level healthcare program for the economically well-off, which is an additional burden for the economically less well-off.

However, the fact is: the experience of more than 20 years' study on health management in the USA shows that there is a relationship between 90% and 10% of health management for any enterprise and individual. China is no exception, with the least healthy 1% of the population and the 19% with chronic diseases sharing 70% of medical expenses. Therefore, accurate understanding of their health status and potential risks through health management can not only effectively reduce the risk of disease, but also save medical expenses.

(2) It is believed that health management is only for the provision of services to the sick groups in medical institutions who need special care.

However, in fact, compared with interventional therapy,

health management pays more attention to "preventing diseases before they happen," and determines the different emphasis of health management according to the characteristics of different populations and different health risks. Therefore, the main service objects of health management are the healthy and sub-healthy people in the society, as well as the groups with chronic diseases and diseases before recovery.

(3) Health management is equal to physical examination, equal to the healthcare products seen everywhere in the market, that there is no necessary connection with disease prevention.

But in fact, the WHO study data show that one-third human disease through preventive health care can be avoided, one-third can be effectively controlled through early discovery, one-third by information communicate effectively improve therapeutic effect. Therefore, the occurrence of all kinds of diseases has its certain reasons and rules, which are determined by congenital genetic factors and influenced by acquired behaviors and lifestyles.

Therefore, this paper believes that the meaning of health management contains three characteristics: (1) it is a process of overall management of individual or group health risk factors; (2) it is a non-medical means to help people approach the level of complete physical and mental health; (3) it can mobilize individual and collective initiative, effectively use limited resources to achieve the maximum health improvement effect.
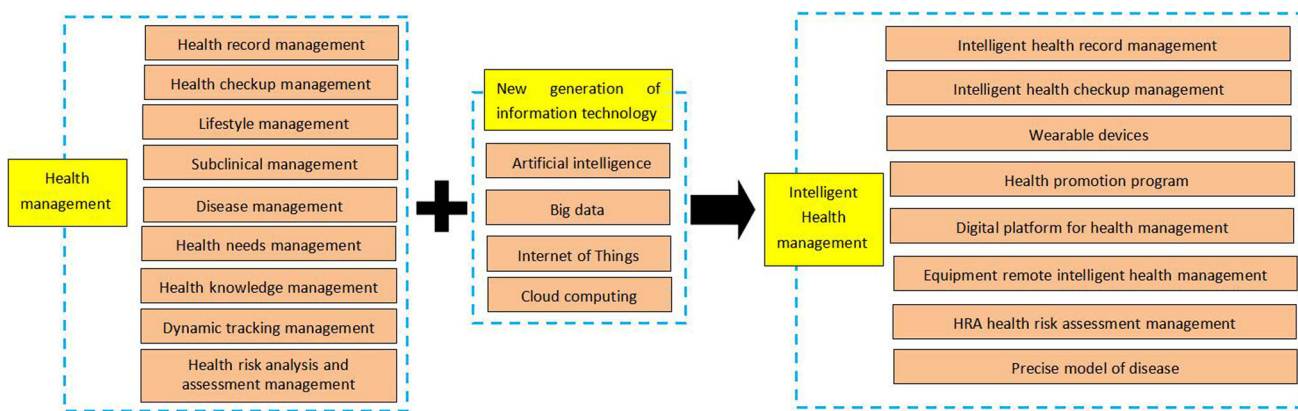
Fig. 2 The framework of intelligent health management
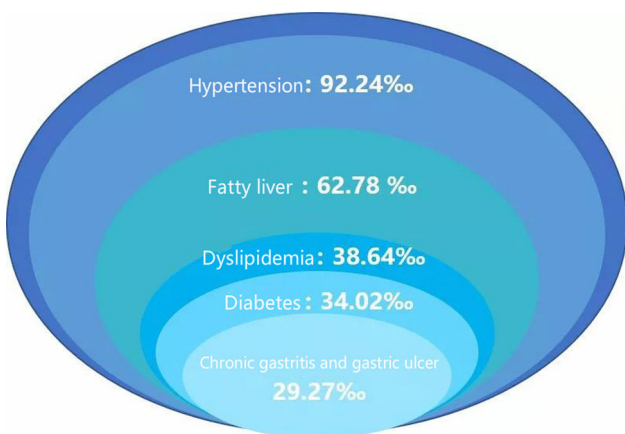
Fig. 3 Types of chronic diseases in Chinese physical examination population

Countries around the world have introduced laws and policies related to health management industry and held relevant competitions to promote the rapid development of the industry. Health management services in China are still dominated by physical examination, and the incidence of chronic diseases remains high (Fig. 3), showing the characteristics of large number of patients and high medical costs. According to the blue book of health management 2018, the number of outpatients in China's medical services was estimated to exceed 8 billion in 2017, while the number of health management (physical examination) patients was only 500 million. Moreover, the health management service is single. More than 95% of the health management service is still physical examination, lacking of post-examination service. According to the statistical bulletin of health development in China in 2017, the total health expenditure reached 5159.88 billion Yuan in 2017, accounting for 6.2% of the GDP. Compared with 2010, the compound annual growth rate was as high as 15%, and the total health expenditure showed a rapid growth trend.

6.2 The development status of intelligent health management

1. Combination of artificial intelligence and health management

(1) Big data and flu prediction (health management + big data)
    Methods: establish related database, intelligent analysis model, etc.
(2) Machine learning and blood glucose management (health management + machine learning algorithm)
    Methods: accurate diabetes model was established by machine learning algorithm and other techniques.
(3) Database technology and health factor detection (artificial intelligence + genotype + health management)
    Methods: establish health big data platform, database, etc.
(4) Improvement of health management and quality of life (artificial intelligence algorithm + big data + health management)
    Methods: health management optimization platform and health promotion scheme were established.

2. Policy, market and other factors drive the development of intelligent health management industry

The accelerated aging of the population, the introduction of national policies and the improvement of artificial intelligence technology will promote the intelligent and personalized development of health management.

(1) The aging of the population
    According to data released by the national bureau of statistics, 249.49 million people aged 60 or above were born in 2018, 8.59 million more than the previous year, accounting for 17.9% of the total population. Among them, the population aged 65 or above accounted for

Algorithm 1 :CoCoSo

1: Achieve the interval-valued fuzzy soft set (F, A), which is shown in Table 3.
2: Calculate the score function $F = (\tau_{ij})_{m \times n}$ of each IVFN $F(\varepsilon_j)(x_i) = [F^-(\varepsilon_j)(x_i), F^+(\varepsilon_j)(x_i)]$ by Eq. (6).
3: Switch the matrix $F = (\tau_{ij})_{m \times n}$ into a standard interval-valued fuzzy soft matrix $F' = (\tau'_{ij})_{m \times n}$ by Eq. (7).
4: Determine combined weight ϖ by Eq. (12).
5: Calculate the total of the weighted comparability sequence for every alternative as S_i:

$$S_i = \sum_{j=1}^{n} \varpi_j * \tau'_{ij} \qquad (13)$$

6: Calculate the whole of the power weight of comparability sequences for each alternative as P_i:

$$P_i = \sum_{j=1}^{n} (\tau'_{ij})^{\varpi_j} \qquad (14)$$

7: Relative weights of alternatives employing the below aggregation strategies are calculated. In such step, three appraisal score strategies are employed for producing relative weights of other options by Eqs. (15)-(17):

$$k_{ia} = \frac{P_i + S_i}{\sum_{i=1}^{m}(P_i + S_i)}, \qquad (15)$$

$$k_{ib} = \frac{S_i}{\min_i S_i} + \frac{P_i}{\min_i P_i}, \qquad (16)$$

$$k_{ic} = \frac{\lambda S_i + (1 - \lambda) P_i}{\lambda \max_i S_i + (1 - \lambda) \max_i P_i}, 0 \le \lambda \le 1, \qquad (17)$$

where k_{ia} denotes the arithmetic mean of sums of WSM and WPM scores, k_{ib} reveals a sum of relative scores of WPM and WSM compared to the optimal and k_{ic} releases the balanced compromise of WPM and WSM models scores.
8: Calculate the assessment value k_i by Eq. (18).

$$k_i = \sqrt[3]{k_{ia}k_{ib}k_{ic}} + \frac{k_{ia} + k_{ib} + k_{ic}}{3} \qquad (18)$$

9: Order alternatives by the decreasing values of assessed value $k_i (i = 1, 2, \cdots, m)$.

11.9% of the total population, an increase of 8.27 million over the previous year. According to the 2017 world population outlook released by the United Nations, the number of people aged 60 and above will double by 2050 and more than triple by 2100, with 65% of them in Asia.

(2) Health management services have great potential in the future

While 70% of people in the USA have access to health management services, less than 0.1% of people in China have access to health management services. As China's economy continues to improve, the demand for health management services will expand.

(3) State policy support and industry planning

In 2017, the state council issued the notice on the development plan of a new generation of artificial intelligence, which specifically pointed out to strengthen group intelligent health management, break through key technologies such as health big data analysis and Internet of things, and promote the transformation of health management from point-like monitoring to continuous monitoring, and from short process management to long process management.

(4) The promotion of artificial intelligence technology

The deep learning algorithm based on deep convolution neural network accelerates the development of artificial intelligence technology and promotes the cross-border integration between health management and it.

Health management services include health record management, lifestyle management, dynamic tracking management and many other services (Fig. 2). Currently, the main services in the market are only health checkup management and disease management.

In the process of evaluating the health checkup of people, it is very necessary to design a reasonable assessment system to guarantee the effective and scientific evaluation results (Shi et al. 2018). This section constructs and depicts an evaluation criteria of health checkup as $\varepsilon_j (j = 1, 2, \cdots, 16)$. The description of each parameter is briefly stated in Table 4.

Parameter comprehensive evaluation model is realized by establishing parameter scoring system. Whether the index score is accurate and reasonable, the key is to find sensitive and specific grade cut points. Here, the epidemiological data of the above 16 main parameters are sorted and summarized, and the latest general normal value standards of clinical internal medicines are reviewed. Various types of cardiovascular disease, hypertension, dyslipidemia, diabetes, obesity prevention and treatment guidelines, and the cutoff points provided in various clinical diagnostic criteria, the cut-point suitability is measured by the percentage of attributable risk of different cut-point populations. Find the point where the sensitivity and specificity are relatively good and the false positive rate is relatively low. The principle of the health management process is shown in Fig. 4.

According to the health management procedure principle of Fig. 4, the selected 16 important physiological parameters are classified into five levels according to health (excellent), sub-health (general), alert (raise), symptoms (early disease) and onset (disease). The parameter score value is interval-valued fuzzy set to [0,1], and the 5 levels are assigned [0,0.2], [0.2,0.4], [0.4,0.6], [0.6,0.8] and [0.8,1] by the equalization assignment method.

First, the latest general-purpose parameter normal value is used as the benchmark value of each parameter, and it

Table 4 The evaluation criteria of health checkup

Parameters	Brief description
Body mass index (ε_1)	Obesity is not only an independent disease, but also a risk factor of type 2 diabetes, cardiovascular disease, hypertension, stroke and many kinds of cancer. Studies show that the body mass index (BMI) of most people has a significant correlation with the percentage of body fat, which can better reflect the degree of obesity
Waist circumference (ε_2)	It is the simplest and most practical measure of the accumulation of fat in the abdomen, known as central obesity. The distribution of fat in the body, especially the degree of abdominal fat accumulation, has a stronger correlation with obesity-related diseases and is an independent predictor of risk, which has a stronger prediction accuracy than BMI. The use of both waist circumference and BMI can better assess a variety of related chronic diseases
Heart rate (ε_3)	It is the basic vital sign. When you are at rest, heart rate is an important indicator to measure the cardiovascular function of the human body. It is an important basis for formulating exercise prescriptions. Studies have shown that in patients with a heart rate of more than 90 beats/min, the mortality rate after acute myocardial infarction and myocardial infarction is significantly increased, which is more than twice that of normal heart rate
Blood pressure (ε_4)	Hypertension is the most common chronic disease, and it is also the most important risk factor for cardiovascular and cerebrovascular diseases. The complications caused by complications such as stroke, myocardial infarction, heart failure and chronic kidney disease are high. Practice has shown that monitoring blood pressure indicators can reduce blood pressure levels in patients with hypertension, significantly reducing stroke and heart disease
Electrocardiogram (ε_5)	It is the most important first choice for the diagnosis of heart disease such as coronary heart disease, myocardial infarction, severe arrhythmia, ventricular hypertrophy and heart failure. Correct analysis of ECG is the basis for early intervention or a series of follow-up diagnostic tests
Pulmonary X-ray (ε_6)	In the diagnosis and treatment of disease prevention, pulmonary X-ray examination has obvious effects and simple methods. It is widely used in health examination, especially in professional examination. Regular pulmonary X-ray examination can be used to detect pulmonary tuberculosis, pneumoconiosis, lung cancer with no obvious symptoms and chronic respiratory diseases such as bronchitis and emphysema. Pulmonary X-ray examination plays an important role in the prevention and diagnosis of respiratory diseases
White blood cell count (ε_7)	White blood cells are an important part of the body's immune defense system. The number of white blood cells in the normal blood is very stable. Inflammation, bacterial and viral infection, tissue injury, regeneration disorder, anemia, leukemia, etc., can often cause changes in the quantity and quality of white blood cells
Hemoglobin (ε_8)	It is the main content of red blood cells, which plays an important role in transporting oxygen and carbon dioxide. Hemoglobin reduction is the landmark index of diagnosis of various types of anemia, and according to the hemoglobin reduction degree of anemia severity classification classification. Abnormal increase of hemoglobin can be seen in plateau disease, red blood cell number, abnormal increase of hemoglobin content can also cause increased blood viscosity, increased blood flow resistance, aggravating heart load, thrombosis, resulting in cardiovascular and cerebral vascular disease
Transaminase (ε_9)	Liver is an important organ for the transformation and metabolism of human nutrients. Transaminase is widely present in liver tissues and cells and plays an important role in the synthesis and catabolism of liver amino acids. Abnormal increase of blood transaminase indicates that liver tissues and cells are damaged. Index of transaminase elevates, besides concern with hepatitis, also cause fatty liver irrationally with dietary structure, the toxic hepatitis that harm of long-term alcoholism, chemical poison brings about is concerned. The content of aspartate transaminase in cardiac muscle cell is higher, and aspartate transaminase can also be increased when cardiomyopathy changes
Total bilirubin (ε_{10})	The formation of bilirubin after the decomposition of aging red blood cell hemoglobin in the body, liver cells plays an important role in bilirubin metabolism through uptake, binding, transport and excretion, and any one of the obstacles in the process can cause the accumulation of bilirubin in the blood, resulting in hyperbilirubinemia and jaundice. Hyperbilirubinemia is common in hepatitis, obstructive jaundice, cirrhosis, cholelithiasis, pancreatic head cancer, hemolytic jaundice and other diseases. Total bilirubin is an important index of hepatobiliary metabolism
Blood creatinine (ε_{11})	It is a muscle metabolite of human body, which can be filtered through the glomerular and almost completely discharged with urine. The glomerular filtration rate (GFR) can be estimated according to the blood creatinine concentration, which is an important indicator for detecting impaired renal function. Elevated serum creatinine is associated with impaired kidney function and an inability to expel harmful toxins from the body, such as glomerulonephritis, uremia, hypertensive nephropathy and diabetic nephropathy

Table 4 continued

Parameters	Brief description
Urinary protein (ε_{12})	Due to the reabsorption of renal tubules, there is little protein left in normal urine, which cannot be detected. Proteinuria occurs when glomerulonephritis, nephrotic syndrome, pyelonephritis and other causes glomerular or tubular damage. Meanwhile, microalbuminuria has been proven to be an independent predictor of cardiovascular events. In the Chinese guidelines for the prevention and treatment of hypertension, urinary protein, creatinine and GFR are all important and simple sensitive indicators for the damage of the target organ of hypertensive kidney
Lipids (ε_{13})	They are a general term for total cholesterol (TC), triglyceride (TG) and lipids in plasma. The cholesterol that is closely related to disease is TC, TG, low-density lipoprotein cholesterol (LDL-C) and high-density lipoprotein cholesterol (HDL-C), which are important risk factors of ischemic cardiovascular disease and are the recommended basic serum lipid indexes. A number of prospective cohort studies and a large number of clinical studies in China have confirmed that elevated serum TC or LDL-C or decreased serum HDL-C can increase the risk of cardiovascular disease, and serum TC or LDL-C levels in the population are positively correlated with ischemic cardiovascular disease, while HDL-C levels are negatively correlated with ischemic cardiovascular disease
Blood glucose (ε_{14})	Cardiovascular and microvascular complications caused by hyperglycemia are the main causes of disability and death in diabetic patients. The incidence and death of cardiovascular disease in diabetic patients were 2~4 times higher than those with normal glucose tolerance. Hypertension with diabetes mellitus has twice the risk of cardiovascular disease compared with non-diabetes mellitus. Prospective population studies have shown that the risk of cardiovascular disease in patients with diabetes but no history of coronary heart disease is equal to that in patients with myocardial infarction but no history of diabetes. The mortality of diabetic patients after myocardial infarction is significantly higher than that of non-diabetic patients
Homocysteine (ε_{15})	High homocysteine may promote the proliferation of vascular smooth muscle cells by damaging vascular endothelial cells, promote the accumulation of apolipoprotein in the vascular wall and affect the fibrinolytic activity, and promote the development of atherosclerosis. High homocysteine can significantly increase the incidence of stroke in hypertensive patients, which may be one of the reasons why the incidence of stroke in China is higher than that in Europe and America. Chinese guidelines for the prevention and treatment of hypertension listed high homocysteine as a potential risk factor for cardiovascular disease
Uric acid (ε_{16})	High uric acid crystals deposit in joints and kidneys, causing gout arthritis and gouty nephropathy. Microcrystals deposit in vascular walls, damage vascular intima, promote the formation and development of atherosclerosis and cause hypertension

Fig. 4 The principle of health management process cut-point on Health-Sub-health-Onset
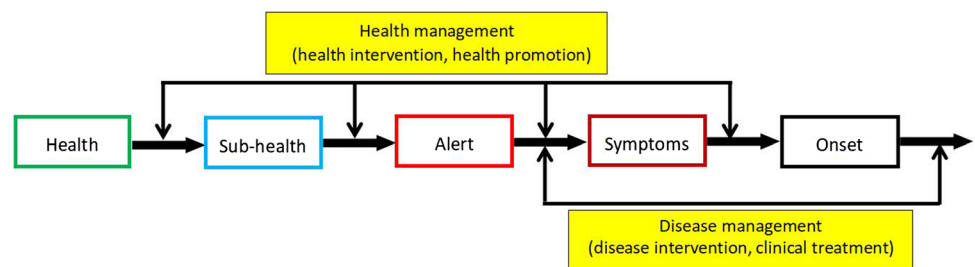
is established as an excellent health [0.8,1] cut point. Then, the key research parameters raised the alert and the disease occurred at two cut points. The warning of the increase in the index is sub-health downstream. The significance of the cutoff point is to provide a boundary that should be vigilant and begin to intervene, prevent the risk factors of most chronic diseases from rising early and do not cause excessive psychological pressure on health check individuals. The margins of the parameters proposed in various guidelines are elevated, and the risk of related diseases is slightly increased, but the clinical diagnostic criteria are not met. This category belongs to this level, which is in the reversible stage of the body's physiological compensation, such as the normal high value of hypertension, blood sugar, damage and elevated blood lipids. The division of the disease occurrence

point is based on the principle that the measured value of the parameter can clarify the diagnosis of the relevant disease, and the disease process has passed the pre-disease stage, and the body may have a qualitative pathological damage, such as hypertension level 2, moderate anemia, obesity, symptoms, hyperlipidemia, severe liver and kidney function damage. The parameter is between the rising point of warning and the point at which the disease occurs. It is generally in the pre-clinical stage of the disease. Clinical symptoms may occur, and the risk of related diseases is moderately increased. It is classified as the early cut point of the disease. According to the above principles, the scores of important physiological parameters for health check-up are presented. The detailed data are shown in Table 5.

Table 5 The important physiological parameters of health checkup score sheet

Indicators	Unit	Excellent (health)	General (alert)	Raise (symptoms)	Early disease	Disease (onset)
Parameters	Unit	[0.8,1]	[0.6,0.8]	[0.4,0.6]	[0.2,0.4]	[0,0.2]
Body mass index	kg/m²	18.50–22.99	23.0–23.99	24.0–27.99 <18.50	28.0–29.99	≥30
Waist circumference	cm	<85/80	85–89/80–84	90–94/85–89	95–99/90–95	≥100/95
Heart rate	times/min	50–89	90–99	100–119 <50	120–159	≥160
Blood pressure (systolic/diastolic)	mmHG	90–119/60–79	120–129/80–84	130–139/85–89 <90/60	140–159/90–99	≥160/100
Electrocardiogram		Normal and roughly normal	Abnormal score ≤2	Abnormal score 3–4	Abnormal score 5–7	Abnormal score ≥8
Pulmonary X-ray		Normal	Abnormal score 1	Abnormal score 2	Abnormal score 3	Abnormal score ≥4
White blood cell count	×10⁹/L	5.0–7.99	4.0–4.99 8.0–9.99	2.0–3.99 ≥10	1.01–1.99	≤1.0
Hemoglobin	g/L	120–159	160–179 101–119	180–199 91–100	200–239 ≤90	≥240
Transaminase	u/L	<40	40–59	60–79	80–119	≥120
Total bilirubin	mol/L	<15	15–17	18–20	21–30	≥30
Blood creatinine	mol/L	50–99	100–114	115–132 <50	133–185	≥186
Urinary protein	mg/24h	negative – <20	Weakly positive± 20–29	+ 30–299	++ ~ + + + 300–499	++++ ≥500
Lipids (TC)	mmol/L	<3.63	3.63–5.17	5.18–6.19	6.20–6.98	≥6.99
Blood glucose	mmol/L	2.80–5.59	5.60–6.09	6.10–6.99 <2.80	7.0–8.99	≥9.0
Homocysteine	mol/L	<10	10–15	16–19	20–29	≥30
Uric acid	mol/L	210–399	400–429	430–479 <210	480–539	≥540

Example 2 Suppose that there are six individuals $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ to be considered for health check-up. The parameter set $E = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8, \varepsilon_9, \varepsilon_{10}, \varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{14}, \varepsilon_{15}, \varepsilon_{16}\}$ is employed in assessing the individuals by advanced AI equipments, which ε_1 (body mass index), ε_2 (waist circumference), ε_3 (heart rate), ε_4 (blood pressure), ε_5 (electrocardiogram), ε_6 (pulmonary X-ray), ε_7 (white blood cell count), ε_8 (hemoglobin), ε_9 (transaminase), ε_{10} (total bilirubin), ε_{11} (blood creatinine), ε_{12} (urinary protein), ε_{13} (lipids), ε_{14} (blood glucose), ε_{15} (homocysteine) and ε_{16} (uric acid). There is no need to distinguish the so-called benefit parameters and cost parameters because all parameters are converted to corresponding IVFNs with uniform scale. Based on years of health check-up experience and mature technology, the weight information is assigned as $W = (0.04, 0.05, 0.03, 0.06, 0.1, 0.05, 0.07, 0.1, 0.04, 0.08, 0.12, 0.06, 0.04, 0.06, 0.05, 0.05)$. The assessments for health check-up individuals arising from the medical machine detection employing advanced technology and the medical examination data with its tabular form presented in Table 6.

Next, we employ the developed algorithm ($\lambda = 0.5$) to choose the healthiest check-up individual under interval-valued fuzzy soft information.

Step 1. Obtain the interval-valued fuzzy soft set (F, A) employing the 5 levels by Table 5, as shown in Table 7.

Step 2. Compute the score function $F = (\tau_{ij})_{6\times16}$ of each IVFN $F(\varepsilon_j)(x_i) = [F^-(\varepsilon_j)(x_i), F^+(\varepsilon_j)(x_i)]$ by Eq. (6).

Table 6 The medical examination data in Example 2

	ε_1	ε_2	ε_3	ε_4	ε_5	ε_6	ε_7	ε_8
x_1	21.5	82/78	92	100/63	1	1	6.3	123
x_2	21.4	80/76	65	121/83	1	1	6.4	130
x_3	21.5	83/75	64	124/82	2	1	7.33	129
x_4	22	82/74	70	123/83	2	1	8.2	135
x_5	19	86/83	110	122/81	2	1	9.1	139
x_6	23.2	86/82	45	123/82	3	2	9.32	163

	ε_9	ε_{10}	ε_{11}	ε_{12}	ε_{13}	ε_{14}	ε_{15}	ε_{16}
x_1	35	13	113	18	3.52	2.93	9	413
x_2	39	14	114	23	3.42	4.27	13	423
x_3	38	16	130	24	3.28	5.12	14	415
x_4	36	17	133	28	3.79	5.83	15	426
x_5	45	17	120	21	3.89	6.02	12	431
x_6	47	17	48	27	5.13	5.71	13	409

$S_1 = 0.9823, S_2 = 0.8226, S_3 = 0.6096, S_4 = 0.4472, S_5 = 0.2661, S_6 = 0.0671.$

Step 6. Compute the whole of the power weight of comparability sequences for each check-up individual as P_i:

$P_1 = 15.9758, P_2 = 13, P_3 = 11, P_4 = 8, P_5 = 4, P_6 = 1.$

Step 7. Three appraisal score strategies are used to generate relative weights of other options, which are derived using Eqs. (15)-(17) and shown as follows:

$F = (\tau_{ij})_{6\times16}$

$$= \begin{pmatrix} 0.8088 & 0.8088 & 0.6088 & 0.8088 & 0.6088 & 0.6088 & 0.8088 & 0.8088 & 0.8088 & 0.8088 & 0.6088 & 0.8088 & 0.8088 & 0.8088 & 0.8088 & 0.6088 \\ 0.8088 & 0.8088 & 0.8088 & 0.6088 & 0.6088 & 0.6088 & 0.8088 & 0.8088 & 0.8088 & 0.8088 & 0.6088 & 0.6088 & 0.8088 & 0.8088 & 0.6088 & 0.6088 \\ 0.8088 & 0.8088 & 0.8088 & 0.6088 & 0.6088 & 0.6088 & 0.8088 & 0.8088 & 0.8088 & 0.6088 & 0.4088 & 0.6088 & 0.8088 & 0.8088 & 0.6088 & 0.6088 \\ 0.8088 & 0.8088 & 0.8088 & 0.6088 & 0.6088 & 0.6088 & 0.6088 & 0.8088 & 0.8088 & 0.6088 & 0.4088 & 0.6088 & 0.6088 & 0.6088 & 0.6088 & 0.6088 \\ 0.8088 & 0.6088 & 0.4088 & 0.6088 & 0.6088 & 0.6088 & 0.6088 & 0.8088 & 0.6088 & 0.6088 & 0.4088 & 0.6088 & 0.6088 & 0.6088 & 0.6088 & 0.4088 \\ 0.6088 & 0.6088 & 0.4088 & 0.6088 & 0.4088 & 0.4088 & 0.6088 & 0.6088 & 0.6088 & 0.6088 & 0.4088 & 0.6088 & 0.6088 & 0.6088 & 0.6088 & 0.6088 \end{pmatrix}.$$

Step 3. There is no need to switch the matrix due to all parameters are benefit parameters.

Step 4. Calculate combined weight ϖ by Eq. (12) as follows:

$\varpi_1 = 0.0367, \varpi_2 = 0.0436, \varpi_3 = 0.0354, \varpi_4 = 0.0626,$

$\varpi_5 = 0.0918, \varpi_6 = 0.0459, \varpi_7 = 0.0669, \varpi_8 = 0.0918,$

$\varpi_9 = 0.0349, \varpi_{10} = 0.0852, \varpi_{11} = 0.1278, \varpi_{12} = 0.0626,$

$\varpi_{13} = 0.0382, \varpi_{14} = 0.0573, \varpi_{15} = 0.0522, \varpi_{16} = 0.0671.$

Step 5. Compute the total of the weighted comparability sequence for every check-up individual as S_i:

$k_{1a} = 0.3019, k_{2a} = 0.2461, k_{3a} = 0.2067, k_{4a} = 0.1504, k_{5a} = 0.0759, k_{6a} = 0.019.$

$k_{1b} = 30.6106, k_{2b} = 25.2562, k_{3b} = 20.0827, k_{4b} = 14.6628, k_{5b} = 7.9651, k_{6b} = 2.$

$k_{1c} = 1.0000, k_{2c} = 0.8151, k_{3c} = 0.6846, k_{4c} = 0.4981, k_{5c} = 0.2516, k_{6c} = 0.0629.$

Step 8. Compute the assessment value k_i by Eq. (18) as follows:

$k_1 = 12.736, k_2 = 10.4899, k_3 = 8.4077, k_4 = 6.1355, k_5 = 3.2981, k_6 = 0.8277.$

Step 9. Rank the check-up individuals according to the decreasing values of assessment value k_i as follows:

$x_1 \succ x_2 \succ x_3 \succ x_4 \succ x_5 \succ x_6.$

Table 7 The transformed interval-valued fuzzy soft set in Example 2

	ε_1	ε_2	ε_3	ε_4	ε_5	ε_6	ε_7	ε_8
x_1	[0.8,1]	[0.8,1]	[0.6,0.8]	[0.8,1]	[0.6,0.8]	[0.6,0.8]	[0.8,1]	[0.8,1]
x_2	[0.8,1]	[0.8,1]	[0.8,1]	[0.6,0.8]	[0.6,0.8]	[0.6,0.8]	[0.8,1]	[0.8,1]
x_3	[0.8,1]	[0.8,1]	[0.8,1]	[0.6,0.8]	[0.6,0.8]	[0.6,0.8]	[0.8,1]	[0.8,1]
x_4	[0.8,1]	[0.8,1]	[0.8,1]	[0.6,0.8]	[0.6,0.8]	[0.6,0.8]	[0.6,0.8]	[0.8,1]
x_5	[0.8,1]	[0.6,0.8]	[0.4,0.6]	[0.6,0.8]	[0.6,0.8]	[0.6,0.8]	[0.6,0.8]	[0.8,1]
x_6	[0.6,0.8]	[0.6,0.8]	[0.4,0.6]	[0.6,0.8]	[0.4,0.6]	[0.4,0.6]	[0.6,0.8]	[0.6,0.8]

	ε_9	ε_{10}	ε_{11}	ε_{12}	ε_{13}	ε_{14}	ε_{15}	ε_{16}
x_1	[0.8,1]	[0.8,1]	[0.6,0.8]	[0.8,1]	[0.8,1]	[0.8,1]	[0.8,1]	[0.6,0.8]
x_2	[0.8,1]	[0.8,1]	[0.6,0.8]	[0.6,0.8]	[0.8,1]	[0.8,1]	[0.6,0.8]	[0.6,0.8]
x_3	[0.8,1]	[0.6,0.8]	[0.4,0.6]	[0.6,0.8]	[0.8,1]	[0.8,1]	[0.6,0.8]	[0.6,0.8]
x_4	[0.8,1]	[0.6,0.8]	[0.4,0.6]	[0.6,0.8]	[0.6,0.8]	[0.6,0.8]	[0.6,0.8]	[0.6,0.8]
x_5	[0.6,0.8]	[0.6,0.8]	[0.4,0.6]	[0.6,0.8]	[0.6,0.8]	[0.6,0.8]	[0.6,0.8]	[0.4,0.6]
x_6	[0.6,0.8]	[0.6,0.8]	[0.4,0.6]	[0.6,0.8]	[0.6,0.8]	[0.6,0.8]	[0.6,0.8]	[0.6,0.8]

6.3 The analysis of weight information

In the following, we give some comparisons of the original weight, the objective weights (Yang and Peng 2017; Xiao et al. 2013; Chen and Zou 2017), the existing combined weights (Peng et al. 2017; Peng and Garg 2018) and the proposed combined weights.

From Fig. 5, the developed combined weights determining method can both availably reveal the objective preference (CRITIC) and subjective preference to some extent. For objective weights proposed by Yang and Peng (2017), the objective weights w_3, w_5, w_8, w_9, w_{11} and w_{13} are almost half or double of the original weight values, which may lead to diverse ranking results or optimal alternative in the process of decision making. The objective weights, proposed by Xiao et al. (2013), encounter the same situation for w_3, w_4 (double or more) and w_5, w_7, w_8, w_{11} (half or less). Moreover, the objective weights in Xiao et al. (2013) are not actually true due to the number m of alternatives is less than the number n of parameters. In other words, the equation $c_{ij} = \sum_{k=1}^{n}(f_{ik} - f_{jk})(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$ in Definition 8 (Xiao et al. 2013) is no way to calculate. For the objective weights, proposed by Chen and Zou (2017), encounter the same situation for w_3 (double or more) and w_5, w_8, w_{11} (half or less). As a consequence, the proposed combined weights not only have a certain difference compared with original weights, but also have kept it within reasonable limits.

From Fig. 6, the combined weights, proposed by Peng and Garg (2018), are gradually keep a relatively steady state (tend to some fixed values) when p increases. For combined weight w_{11}, the weight values are almost 4 times different at first. Although it gradually tends to 2 times different, it also may lead to diverse ranking results or optimal alternative in

the process of decision making. Hence, the existing combined weights (Peng and Garg 2018) fail to obey the rule of differentiation within a certain range. However, the proposed combined weights have no such issue. Moreover, it is also hard to choose a suitable parameter value p for obtaining optimal alternative and ranking.

From Fig. 7, the combined weights, proposed by Peng et al. (2017), are almost the same as the original weights with very subtle changes. Especially, the combined weights are almost approaching to the original weights when q is increasing. In other words, it cannot reflect the idea of objective weights. But for our proposed combined weights, they can have appropriate difference, which take the objective information into consideration compared with the original weights.

7 Comparison with some existing methods

7.1 The discrimination degrees of some existing methods

For a better comparison with some existing methods (Son 2007; Yang and Peng 2017; Yang et al. 2009; Yuan and Hu 2012; Xiao et al. 2013; Peng et al. 2017; Chen and Zou 2017; Peng and Garg 2018), the comparison results are shown in Fig. 8 by employing the Example 2.

For some existing decision-making methods [TOPSIS (Yang and Peng 2017), Choice value-1 (Yang et al. 2009), CODAS (Peng et al. 2017)], they have stronger power of discrimination degrees during the process of decision making. However, the other parts of decision-making methods [WDBA (Peng and Garg 2018), similarity measure-1 (Peng and Garg 2018), MABAC (Peng et al. 2017), EDAS (Peng

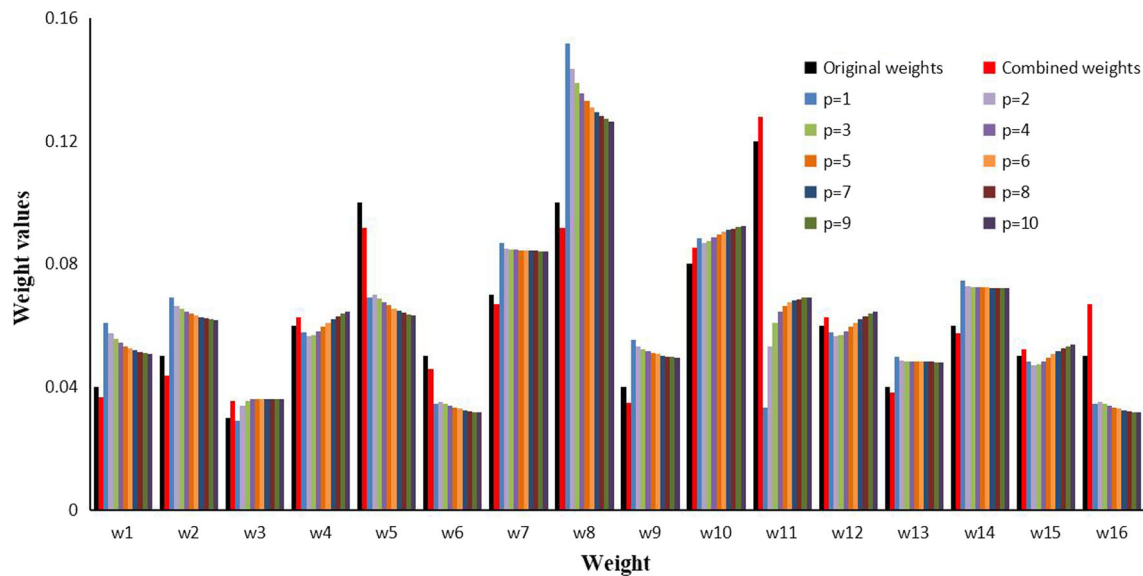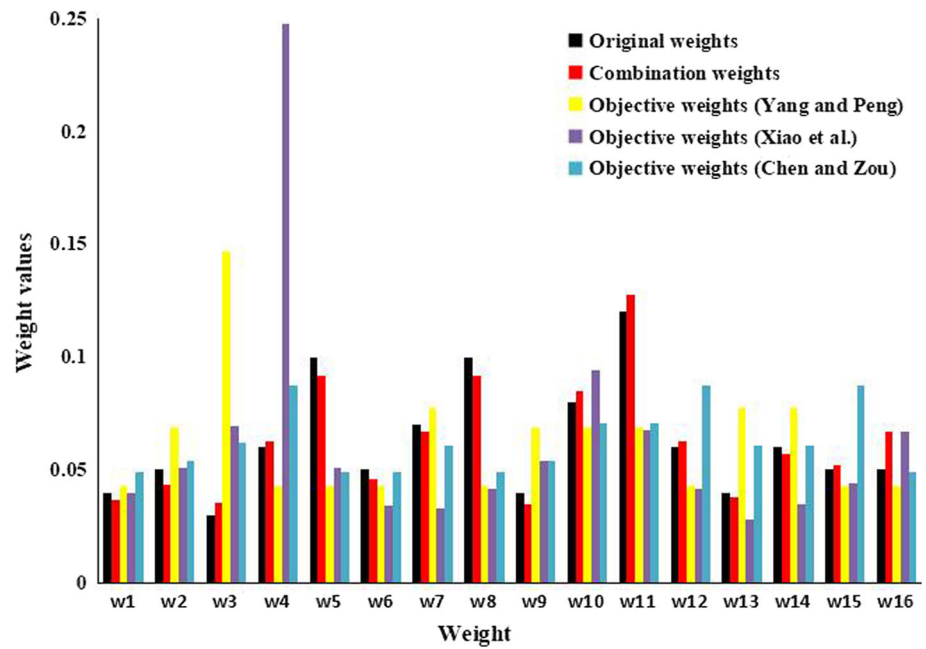**Fig. 5** The comparison of the weight information

Fig. 6 The comparison of the weight information (Peng and Garg 2018)

et al. 2017), similarity measure-2 (Peng et al. 2017), choice value-2 (Son 2007), GRA (Chen and Zou 2017)] fail to such advantages, which make the DMs or experts hardly choose the optimal health check-up individual in convincing and resultful way. Moreover, it is easily known that the proposed MCDM method-based CoCoSo possesses high degree of differentiation compared with some existing MCDM methods.

7.2 The division by zero problem of some existing methods

Example 3 Suppose that there have another medical machine detection employing advanced technology and the medical examination data with its tabular form presented in Table 8.

Remark 2 From Table 9, we can find the conclusive ranking and optimal health check-up individual by the developed method are in agreement with the decision results of WDBA (Peng and Garg 2018), CODAS (Peng and Garg 2018), similarity measure (Peng and Garg 2018), MABAC (Peng et al. 2017), EDAS (Peng et al. 2017), similarity measure (Peng
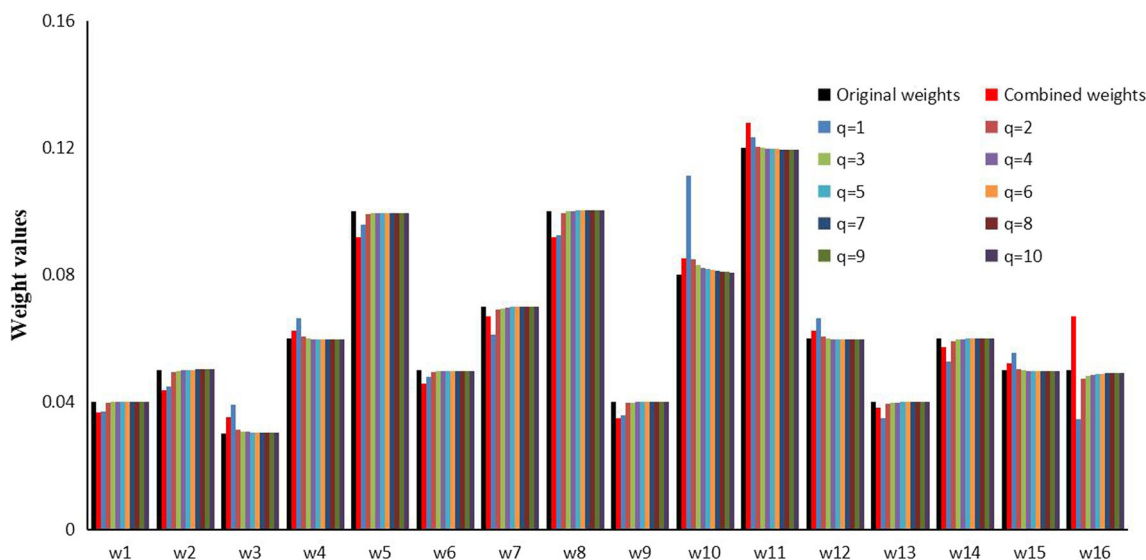
Fig. 7 The comparison of the weight information (Peng et al. 2017)
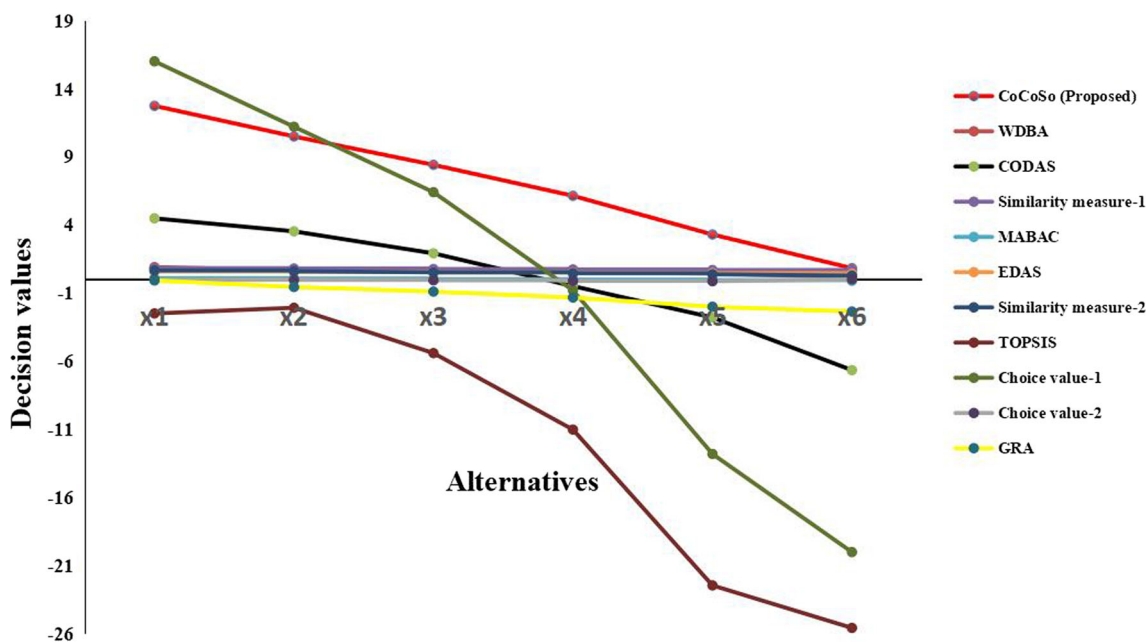
Fig. 8 The comparison of the existing decision-making methods

et al. 2017), choice value-1 (Yang et al. 2009), comparison table (Son 2007; Yuan and Hu 2012), and GRA (Chen and Zou 2017). In additional, for the TOPSIS (Yang and Peng 2017), they fail to achieve the optimal health check-up individual and ranking order because the "division by zero problem."

7.3 The counterintuitive phenomena of some existing methods

Example 4 Suppose that there have another medical machine detection employing advanced technology and the medical examination data (physical examination items (ε_3, ε_4 and ε_7 with weight $w = (0.3, 0.3, 0.4)$) specified by health check-up individuals) with its tabular form presented in Table 10.

Remark 3 From Table 11, it can be easily seen that the optimal health check-up individual computed by the developed method is in agreement with the decision results of WDBA

Table 8 The transformed interval-valued fuzzy soft set in Example 3

	ε_1	ε_2	ε_3	ε_4	ε_5	ε_6	ε_7	ε_8
x_1	[0.8,1]	[0.8,1]	[0.8,1]	[0.8,1]	[0.6,0.8]	[0.6,0.8]	[0.8,1]	[0.8,1]
x_2	[0.8,1]	[0.8,1]	[0.8,1]	[0.6,0.8]	[0.6,0.8]	[0.6,0.8]	[0.8,1]	[0.8,1]
x_3	[0.8,1]	[0.8,1]	[0.8,1]	[0.6,0.8]	[0.6,0.8]	[0.6,0.8]	[0.8,1]	[0.8,1]
x_4	[0.8,1]	[0.8,1]	[0.8,1]	[0.6,0.8]	[0.6,0.8]	[0.6,0.8]	[0.6,0.8]	[0.8,1]
x_5	[0.8,1]	[0.6,0.8]	[0.4,0.6]	[0.6,0.8]	[0.6,0.8]	[0.6,0.8]	[0.6,0.8]	[0.8,1]
x_6	[0.6,0.8]	[0.6,0.8]	[0.4,0.6]	[0.6,0.8]	[0.4,0.6]	[0.4,0.6]	[0.6,0.8]	[0.6,0.8]

	ε_9	ε_{10}	ε_{11}	ε_{12}	ε_{13}	ε_{14}	ε_{15}	ε_{16}
x_1	[0.8,1]	[0.8,1]	[0.6,0.8]	[0.8,1]	[0.8,1]	[0.8,1]	[0.8,1]	[0.6,0.8]
x_2	[0.8,1]	[0.8,1]	[0.6,0.8]	[0.6,0.8]	[0.8,1]	[0.8,1]	[0.6,0.8]	[0.6,0.8]
x_3	[0.8,1]	[0.6,0.8]	[0.4,0.6]	[0.6,0.8]	[0.8,1]	[0.8,1]	[0.6,0.8]	[0.6,0.8]
x_4	[0.8,1]	[0.6,0.8]	[0.4,0.6]	[0.6,0.8]	[0.6,0.8]	[0.6,0.8]	[0.6,0.8]	[0.6,0.8]
x_5	[0.6,0.8]	[0.6,0.8]	[0.4,0.6]	[0.6,0.8]	[0.6,0.8]	[0.6,0.8]	[0.6,0.8]	[0.4,0.6]
x_6	[0.6,0.8]	[0.6,0.8]	[0.4,0.6]	[0.6,0.8]	[0.6,0.8]	[0.6,0.8]	[0.6,0.8]	[0.6,0.8]

Table 9 Ranking results from different methods for Example 3

Methods	Ranking	Optimal alternative
WDBA (Peng and Garg 2018)	$x_1 \succ x_2 \succ x_3 \succ x_4 \succ x_5 \succ x_6$	x_1
CODAS (Peng and Garg 2018)	$x_1 \succ x_2 \succ x_3 \succ x_4 \succ x_5 \succ x_6$	x_1
Similarity measure-1 (Peng and Garg 2018)	$x_1 \succ x_2 \succ x_3 \succ x_4 \succ x_5 \succ x_6$	x_1
MABAC (Peng et al. 2017)	$x_1 \succ x_2 \succ x_3 \succ x_4 \succ x_5 \succ x_6$	x_1
EDAS (Peng et al. 2017)	$x_1 \succ x_2 \succ x_3 \succ x_4 \succ x_5 \succ x_6$	x_1
Similarity measure-2 (Peng et al. 2017)	$x_1 \succ x_2 \succ x_3 \succ x_4 \succ x_5 \succ x_6$	x_1
TOPSIS (Yang and Peng 2017)	N/A	$*$
Choice value-1 (Yang et al. 2009)	$x_1 \succ x_2 \succ x_3 \succ x_4 \succ x_5 \succ x_6$	x_1
Comparison table (Son 2007)	$x_1 \succ x_2 \succ x_3 \succ x_4 \succ x_5 \succ x_6$	x_1
Comparison table (Yuan and Hu 2012)	$x_1 \succ x_2 \succ x_3 \succ x_4 \succ x_5 \succ x_6$	x_1
GRA (Chen and Zou 2017)	$x_1 \succ x_2 \succ x_3 \succ x_4 \succ x_5 \succ x_6$	x_1
CoCoSo (proposed)	$x_1 \succ x_2 \succ x_3 \succ x_4 \succ x_5 \succ x_6$	x_1

$p = 2, t = 2$ in Peng and Garg (2018); $q = 2$ in Yang and Peng (2017); $\varepsilon = 0.1$ in Xiao et al. (2013); $\alpha = 1, \beta = 0$ in Chen and Zou (2017);
"N/A" signifies the "division by zero problem"
"*" denotes that it has no optimal alternative to select

Table 10 The transformed interval-valued fuzzy soft set in Example 4

	ε_3	ε_4	ε_7
x_1	[0.0,0.2]	[0.6,0.8]	[0.6,0.8]
x_2	[0.2,0.4]	[0.4,0.6]	[0.6,0.8]
x_3	[0.2,0.4]	[0.4,0.6]	[0.4,0.6]

(Peng and Garg 2018), CODAS (Peng and Garg 2018), similarity measure-1 (Peng and Garg 2018), MABAC (Peng et al. 2017), EDAS (Peng et al. 2017), similarity measure-2 (Peng et al. 2017). For TOPSIS (Yang and Peng 2017) and GRA (Chen and Zou 2017), the optimal health check-up individuals are all x_1, which is different from the proposed method. The main reason is that the weight determining method influence the final results. Moreover, we also find that the Choice value-1 (Yang et al. 2009) and Comparison table (Son 2007; Yuan and Hu 2012) cannot determine the optimal health check-up individual due to their drawbacks of counterintuitive phenomena, which has been discussed in Peng and Garg (2018).

8 Conclusion

The key contributions can be concluded below.

(1) The new score function for IVFN is proposed, which has strong power in distinguishing two IVFNs;
(2) The combined weight model is proposed based on CRITIC and linear weighted comprehensive method,

Table 11 Ranking results from different methods for Example 4

Methods	Ranking	Optimal alternative
WDBA (Peng and Garg 2018)	$x_2 \succ x_1 \succ x_3$	x_2
CODAS (Peng and Garg 2018)	$x_2 \succ x_1 \succ x_3$	x_2
Similarity measure-1 (Peng and Garg 2018)	$x_2 \succ x_3 \succ x_1$	x_2
MABAC (Peng et al. 2017)	$x_2 \succ x_1 \succ x_3$	x_2
EDAS (Peng et al. 2017)	$x_2 \succ x_3 \succ x_1$	x_2
Similarity measure-2 (Peng et al. 2017)	$x_2 \succ x_1 \succ x_3$	x_2
TOPSIS (Yang and Peng 2017)	$x_1 \succ x_2 \succ x_3$	x_1
Choice value-1 (Yang et al. 2009)	$x_2 \sim x_1 \succ x_3$	**
Comparison table (Son 2007)	$x_2 \sim x_1 \succ x_3$	**
Comparison table (Yuan and Hu 2012)	$x_2 \sim x_1 \succ x_3$	**
GRA (Chen and Zou 2017)	$x_1 \succ x_2 \succ x_3$	x_1
CoCoSo (proposed)	$x_2 \succ x_1 \succ x_3$	x_2

$p = 2, t = 2$ in Peng and Garg (2018); $q = 2$ in Yang and Peng (2017); $\varepsilon = 0.1$ in Xiao et al. (2013); $\alpha = 1, \beta = 0$ in Chen and Zou (2017);

"**" denotes that it has no certain optimal alternative to select

which can simultaneously consider subjective preference and objective preference;

(3) Some matrix operations on IVFSSs are presented and their interesting properties are proved in detail;

(4) The novel interval-valued fuzzy soft MCDM method based on CoCoSo is proposed, which can obtain the best alternative without counterintuitive phenomena, achieve the decision results without division by zero problem and possess a strong ability to differentiate the optimal alternative.

However, the proposed MCDM method consumes a certain amount of time complexity, and lacks the self-adjusting weight mechanism (Huang and Liang 2019). How to reduce the time complexity and fuse the idea of self-paced learning are key research directions in the future. Moreover, the outstanding CoCoSo method for handling the intelligent healthcare management decision-making issues under diverse fuzzy environment (Shen et al. 2020; Zhan and Alcantud 2019; Alcantud et al. 2019; Wang et al. 2020) is an interesting topic.

Compliances with ethical standards

Conflict of interest None.

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