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On the enumeration of Boolean functions with distinguished variables

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Abstract
Boolean functions have a fundamental role in neural networks and machine learning. Enumerating these functions and significant subclasses is a highly complex problem. Therefore, it is of interest to study subclasses that escape this limitation and can be enumerated by means of sequences depending on the number of variables. In this article, we obtain seven new formulas corresponding to enumerations of some subclasses of Boolean functions. The versatility of these functions does the problem interesting to several different fields as game theory, hypergraphs, reliability, cryptography or logic gates.

Keywords Boolean functions · 2-Monotonic Boolean functions · Simple games · Complete games · Enumeration of Boolean functions · Dedekind's problem

1 Introduction

Since its origins, mathematics has been interested in classifying and listing the set of solutions to a given problem. As quoted in Hardy (1999) 'to enumerate a set of objects satisfying some set of properties means to explicitly produce a listing of all such objects.' Enumerating special types of Boolean functions or simple games is useful for the design of circuits and real-world voting systems that fulfill some desirable properties. This paper concerns enumerations of these structures.

A Boolean function has as input n Boolean variables (that is, values that can be either false or true) and produces as output another Boolean variable. It is monotonic if, for every combination of inputs, switching one of the inputs from false to true can only cause the output to switch from false to true and not from true to false. More precisely, a monotonic Boolean function of n variables (or, for short, a function) is a mapping $f : \{0, 1\}^n \to \{0, 1\}$ such that: $x \le y$ implies $f(x) \le f(y)$. The Dedekind's problem is given by the sequence $M(n)$ and is the number of monotonic Boolean functions of n variables, or the number of antichains of subsets of an n-set, or the number of elements in a free distributive lattice on n generators, or the number of Sperner families. Recall that an antichain of sets (also known as a Sperner family) is a family of sets, none of which is contained in any other set. The values of the sequence $M(n)$ for the first eight integers are known: 2, 3, 6, 20, 168, 7581, 7,828,354, 2,414,682,040,998, 56,130,437,228,687,557,907,788 (see sequence A000372 in the On-Line Encyclopedia of Integer Sequences, Sloane 1964).

The (inequivalent) Dedekind's number $S(n)$ is the number of different monotonic Boolean functions on n variables that do not differ in the name of the variables. If two Boolean functions are only differentiated in the labels, they are said to be equivalent. Thus, $S(n)$ counts the number of inequivalent monotonic Boolean functions of n or fewer variables. The values of this sequence for the first eight integers are also known: 2, 3, 5, 10, 30, 210, 163,53, 490,013,148 (see sequence A003182 in the On-Line Encyclopedia of Integer Sequences, Sloane 1964). This paper concerns enumerations of subclasses of inequivalent monotonic Boolean functions.

Two subclasses that excel within inequivalent monotonic Boolean functions are: the threshold functions and the 2-monotonic (or regular) functions, a superclass of threshold functions. A function f is called 2-monotonic or regular if it satisfies the following condition at every x: if $i < j$, $x_i = 1$, $x_j = 0$ and $f(x) = 0$, then $f(x + e_j - e_i) = 0$, where e_k denotes the k-th unit vector of appropriate dimension. 2-monotonicity and related concepts have been studied under various names in such areas of applied mathematics as threshold logic (Hu 1965; Muroga 1971), game theory (Carreras

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and Freixas 1996; Einy 1985; Taylor and Zwicker 1999) or graph and hypergraph theory (Chvátal and Hammer 1977; Reiterman et al. 1985). The interest in 2-monotonic functions usually stems from their close relationship with threshold functions, a Boolean function $f(x)$ is called threshold if there exist $n+1$ integers $w_1 \geq w_2 \geq \cdots \geq w_n \geq 0$ (weights) and $q > 0$ (quota or threshold) such that: $f(x) = 1$ *if and only if* $\sum_{i=1}^{n} w_i \cdot x_i \geq q$.

The number of threshold functions and 2-monotonic functions are known up to $n = 9$ variables, see, respectively, Freixas et al. (2012) and Kurz (2012), Kurz and Tautenhahn (2013). Identification of 2-monotonic Boolean functions can be done in polynomial time (Boros et al. 1991, 1997), and computational studies of separation algorithms for clique inequalities can serve for determining threshold functions (Marzi et al. 2019).

Recent research in capacitive threshold logic, strongly studied in the sixties, see, e.g., Hu (1965), has revived interest in this area, and it has reintroduced some of the problems that have yet to be solved. One of the main issues of threshold logic is the application of neural networks to the problem of realizing Boolean functions (that is, to decide if a given Boolean function is threshold). The linear separability problem has been dealt with, among others, in Freixas et al. (2017), Freixas and Molinero (2009), Picton (1991), Roychowdhury et al. (1994), Siu et al. (1995), Taylor and Zwicker (1992), Taylor and Zwicker (1999). Neural networks are usually designed to have the ability to learn and generalize. Neural networks have been used with computers during as early as the 1950s. Through the years, many different neural network architectures have been presented. One of the pioneering contributions was the perceptron, which is the simplest form of a neural network used for the classification of linearly separable patterns (patterns that are located on the opposite sides of a hyperplane). Basically, it consists of a single neuron with adjustable weights and a threshold, that is, a threshold function. The algorithm used to adjust the free parameters of this network emerged as a Rosenblatt learning procedure for his model of the brain, the perceptron. The convergence proof of such algorithm is known as the perceptron convergence theorem.

Another criterion for choosing significant subclasses of inequivalent monotonic Boolean functions consists in considering the number of types of equivalent variables. Two variables that play the same role in the function are considered equivalent, and then the set of variables decomposes into equivalence classes. Functions with a moderate number of equivalence classes are very useful in applications. Inequivalent functions with two equivalence classes are called bipartite. Similarly, tripartite and quadripartite inequivalent Boolean functions are considered. Quite curiously, bipartite 2-monotonic Boolean functions are enumerated for all n and follow a variant of a Fibonacci sequence (see Freixas et al. 2012 and Eq. (1) in this paper) and therefore grow exponentially on the number of variables n. Bipartite inequivalent Boolean functions were enumerated recently in Freixas and Samaniego (2020) after obtaining the enumeration of bipartite functions non-being 2-monotonic. The purpose of this paper is to enumerate some tripartite and quadripartite inequivalent Boolean functions with dominant and/or dominated variables.

Boolean functions are very versatile structures that arise in many contexts and lend themselves to more diverse interpretations. Monotonic Boolean functions are equivalent to monotonic hypergraphs that can also be thought as simple games, coherent structures (see, e.g., Ramamurthy 1990), logic gates or reliability systems (see, e.g., Kuo and Zhu 2012, and Freixas and Puente 2002 where basic analogies between reliability and game theory were listed), access structure in a secret sharing (see, e.g., Gvozdeva et al. 2013; Simmons 1990; Stinson 1992; Tassa 2007), etc. Each of these fields has developed different theories motivated by their discipline challenges, so that a fruitful cooperation has taken place allowing for a great development in this area of research.

Simple games are at the core of voting systems, in them a single alternative, such as a bill or an amendment, is pitted against the status quo, and voters can vote for or against the bill. Due to its importance, naturalness and significance of its applications (see, e.g., Cheung and Ng 2014; Felsenthal and Machover 1998; Freixas 2004; Kilgour 1983; Kurz and Napel 2016; Le Breton et al. 2012; Leech 2002; Taylor and Pacelli 2008; Taylor and Zwicker 1993; von Neumann and Morgenstern 1944, we have adopted in this paper the language of simple games. We also point out that a simple game can be seen as a particular case of a cooperative game, see, e.g., Peters (2015), whenever the image set of the characteristic function $v : 2N \to \mathbb{R}$ is binary, usually described by $\{0, 1\}$, in which the value 0 is interpreted as a losing result and the value 1 as a winning result, so that any coalition S with $v(S) = 0$ is losing and any coalition with $v(S) = 1$ is winning. Nevertheless, the theory of simple games has been mostly developed independently of cooperative games since the outputs 'losing' and 'winning' are qualitative rather than quantitative as occurs for cooperative games.

We consider that this paper is relevant for the readers of Soft Computing because the most relevant problem in Boolean functions is the separability problem that consists in determining whether a given function is threshold. As 2-monotonicity is a necessary, but not sufficient condition for the function to be threshold, it is of great interest to enumerate all 2-monotonic functions and then check one-by-one which of them are threshold since this checking can be done in polynomial time, cf. Crama (1987). Nevertheless, it is a very complex problem listing 2-monotonic functions. Developments of techniques in soft computing may be helpful in future developments.

The paper is organized as follows. Section 2 sets out the main terminology and notions that will be used in the article. Section 3 collects the known results on enumerations for bipartite, tripartite and quadripartite simple games. Sections 4, 5 and 6 are devoted to find new enumerations of tripartite simple games with either vetoes or nulls. Sections 7, 8 and 9 are devoted to find new enumerations of quadripartite simple games with vetoes and nulls. Section 10 concludes the paper by summarizing the main findings and indicating some open problems to encourage future research.

2 Background on simple games

Definition 1 A *simple game* is a pair (N, W) in which $N = \{1, 2, \ldots, n\}$ and W is a collection of subsets of N that satisfies: $N \in W$, $\emptyset \notin W$ and, the monotonicity property, if $S \in W$ and $S \subseteq T \subseteq N$ then $T \in W$.

A (monotonic) simple game corresponds to a (monotonic) Boolean function in the field of Boolean algebra.

The set N is called the *grand coalition*. Members of N are called *players* or *voters*, and subsets of N are called *coalitions*, the coalitions that belong to W are called *winning coalitions*, the subfamily of *minimal* winning coalitions is $W^m = \{S \in W : \forall T \subset S \Rightarrow T \notin W\}$. The minimal winning coalitions form an antichain of subsets that allows to generate the simple game. By $|S|$, we mean the cardinality of a coalition S and use $|S| = s$ whenever there is no confusion.

Definition 2 Two simple games (N, W) and (N', W') are *isomorphic* if there exists a one-to-one correspondence $f : N \to N'$ such that $S \in W$ *if and only if* $f(S) \in W'$; f is called and *isomorphism* of simple games.

Two isomorphic simple games only differ in the labels, so from now on we will only consider simple games up to isomorphisms. (Monotonic) simple games up to isomorphisms correspond to inequivalent (monotonic) Boolean functions. Two distinguished types of voters frequently appear in simple games.

Definition 3 A voter i has *veto* whenever $i \in S$ for all $S \in W$. A voter i is *null* whenever $i \notin S$ for all $S \in W^m$.

In any democratic voting system represented by a simple game, the veto players (if any) are the most powerful ones in the system. Oppositely, null voters (if any) do not exert any influence in the game. In Boolean algebra, the null voters correspond to irrelevant variables and the veto players to essential variables.

One common idea of the most voting systems used in practice is the concept of influence, i.e., that a particular voting system may give one voter more influence than another. The so-called desirability relation, precisely stated in Definition 4, on the set of voters is a way to make influence precise. Isbell already used it in Isbell (1958).

Definition 4 Let (N, W) be a simple game. Player i is *at least as desirable* as j ($i \succsim j$, in short) in (N, W) if: $S \cup \{j\} \in W \Rightarrow S \cup \{i\} \in W$, for all $S \subseteq N \setminus \{i, j\}$. Players i and j are *equally desirable* ($i \approx j$, in short) in (N, W) if: $i \succsim j$ and $j \succsim i$. Player i is *strictly more desirable* than player j ($i \succ j$, in short) in (N, W) if: $i \succsim j$ and $i \not\approx j$.

Note that the desirability relation is a preordering, a reflexive and transitive relation, on the set of players N and that each subset $N_i \subseteq N$ formed by equally desirable players is an equivalence class. We refer to N_i as an equally desirable class. We also speak of the number of classes or types, t, of voters meaning the number of equivalence classes. This is a fundamental parameter in our study. The veto players (if any) form the strongest equivalence class, N_1, whereas null players (if any) form the weakest equivalence class, N_t.

Definition 5 A simple game (N, W) is *complete* or *linear* if the desirability relation is a complete preordering.

Complete simple games correspond to 2-monotonic Boolean functions in Boolean algebra. Let t be the number of types of a complete game, we can always assume $N_1 > N_2 > \cdots > N_t$ understanding that $N_k > N_{k+1}$ if and only if $i \succ j$ for all $i \in N_k$ and $j \in N_{k+1}$.

Definition 6 Let (N, W) be a simple game with t equally desirable classes. If $t = 2$ the game is called *bipartite*, if $t = 3$ it is called *tripartite* and if $t = 4$ it is called *quadripartite*.

In this paper, we are concerned with tripartite simple games having either voters with right to veto (vetoes, for short) or null voters (nulls, for short) and with quadripartite simple games having vetoes and nulls. As we shall see in Proposition 1, all the enumerations of this paper can be done assuming the presence of veto players in the game. Thus, for the tripartite case we will have: $N_1 > N_2 > N_3$ if the game is complete and $N_1 > N_i$ for $i = 2, 3$ if the game is not complete because N_1 is formed by vetoes. For quadripartite games with vetoes (voters in N_1) and nulls (voters in N_4) we will have $N_1 > N_2 > N_3 > N_4$ if the game is complete and $N_1 > N_i$ for $i = 2, 3, 4$ and $N_i > N_4$ for $i = 2, 3$ if the game is not complete, but neither $N_2 > N_3$ nor $N_3 > N_2$ are met.

Example 1 Let N be a Parliament formed by the president and two chambers, for example, the House and the Senate. To pass a proposal, it is required the approval of the president and the approval of an absolute majority of the members in each chamber. Assume the sizes of the chambers are 9 and 15, thus, $|N| = 25$. Then, the approval is achieved if the president votes in favor, at least 5 members of the House vote in favor and at least 8 members of the Senate vote in favor of the proposal submitted at hand. Let $N_1 = \{1\}$, $N_2 = \{2, \ldots, 10\}$ and $N_3 = \{11, \ldots, 25\}$, where 1 represents the president, the elements in N_2 are those in the House, and the elements in N_3 are those in the Senate. Formally,

$$W^m = \{S \subseteq N : 1 \in S, \ |S \cap N_2| = 5, \ \text{and} \ |S \cap N_3| = 8\}.$$

Note that the president has veto because $1 \in S$ for all $S \in W$ and $1 \succ i$ for all $i \in N \setminus \{1\}$, but neither $i \succsim j$ nor $j \succsim i$ for all $i \in N_2$ and $j \in N_3$. Thus, the game is a tripartite non-complete game with a veto player.

A simple game can be presented in a more convenient way. Given the equivalence classes of the simple game $N_1, \ldots, N_t$ such that $N_{i+1} > N_i$ does not hold for every $i = 1, \ldots, t-1$, the vector $\overline{n} = (n_1, n_2, \ldots, n_t)$ represents the grand coalition and any vector $\overline{s} = (s_1, s_2, \ldots, s_t)$ with $0 \leq s_i \leq n_i$ represents the set of coalitions $\{S : |S \cap N_i| = s_i, \text{ for all } i = 1, \ldots, t\}$ identified with $\overline{s}$. Coalitions represented by the *model* $\overline{s}$ are either all winning or all are not. Thus, we write either $\overline{s} \in \overline{W}$ or $\overline{s} \notin \overline{W}$, where $\overline{W} = \{\overline{s} \in \Lambda(\overline{n}) : S \in W\}$ and $\Lambda(\overline{n})$ is the set of models of coalitions, i.e., $\{\overline{m} \in (\mathbb{N} \cup \{0\})^t : \overline{0} \leq \overline{m} \leq \overline{n}\}$. The elements in $\overline{s} \in \overline{W}$, which are minimal componentwise, represent minimal winning coalitions and write $\overline{s} \in \overline{W^m}$ for them, where $\overline{W^m} = \{\overline{s} \in \Lambda(\overline{n}) : S \in W^m\}$.

Let $\overline{m} \in \Lambda(\overline{n})$ and $\overline{p} \in \Lambda(\overline{n})$. By $\overline{m} \leq \overline{p}$ we mean $m_i \leq p_i$ for all $i = 1, \ldots, t$, and by $\overline{m} < \overline{p}$ we mean $\overline{m} \leq \overline{p}$ and $\overline{m} \neq \overline{p}$. By $\overline{0}$ we mean the t-dimensional vector formed by zeros. The next conditions meet the requirements of monotonic simple games:

(i) $\overline{n} \in \overline{W}$,
(ii) $\overline{0} \notin \overline{W}$, and
(iii) $\overline{m} \leq \overline{p}$ with $\overline{m} \in \overline{W}$ implies $\overline{p} \in \overline{W}$.

See Carreras and Freixas (1996) for further details on this presentation of simple games. If we know that $N_i > N_{i+1}$, then the next condition is also fulfilled:

(iv) Let $\overline{m} = (m_1, \ldots, m_{i-1}, m_i, m_{i+1}, \ldots, m_t) \in \overline{W}$ with $m_i < n_i$ and $m_{i+1} > 0$, then $(m_1, \ldots, m_{i-1}, m_i + 1, m_{i+1} - 1, \ldots, m_t) \in \overline{W}$.

Note that condition (iv) captures the idea that if a player is replaced in a winning coalition by another player being more desirable than her, then the new coalition becomes also winning. Observe that in this argument the condition $N_i > N_{i+1}$ is essential.

Another fundamental parameter in our study for each game is r, the number of representatives of minimal winning coalitions of the simple game, that is the cardinality of $\overline{W^m}$.

The following nomenclature is used in the rest of the paper. Each description depends on either (n, r) or simply n. By (n, r) we refer to n voters and r representatives of minimal winning coalitions.

Thus, by BCG(n, r) we mean the class of all bipartite complete games with n voters and r representatives of minimal winning coalitions, up to isomorphisms, and by $BCG(n, r)$ its cardinality. By BCG(n), we mean the class of all bipartite complete games with n voters, up to isomorphisms, and by $BCG(n)$ its cardinality. The rest of notations in Table 1 are analogous.

Example 2 (*Example 1 revisited*) As noticed, the game belongs to TNCGV for $n = 25$ and $r = 1$ since the game has 25 players and there is only one representative of minimal winning coalitions, which is $(1, 5, 8)$. The winning representatives are the vectors in the set:

$$\{(1, h, s) \in \Lambda(1, 9, 15) : 5 \leq h \leq 9 \text{ and } 8 \leq s \leq 15\}.$$

Note that there are $\binom{1}{1} \cdot \binom{9}{5} \cdot \binom{15}{8} = 810810$ minimal winning coalitions with representative $(1, 5, 8)$. This simple example illustrates the convenience of using the succinct representation of the game by the vector $\overline{n}$ and the set $\overline{W^m}$ instead of the standard presentation given by the set of voters N and the set of minimal winning coalitions W^m.

It obviously holds the following property:

Proposition 1 *For all n and r we have:*

a. $TCGV(n, r) = TCGN(n, r)$
b. $TNCGV(n, r) = TNCGN(n, r)$

Proof Let $(N, W) \in$ TCGV(n, r) and without null voters, let (N, W') be obtained from (N, W) defined as

$$S \in W' \text{ if and only if } S \cup N_1 \in W$$

so that the vetoes in N_1 for (N, W) have been converted into nulls in (N, W'). The 'if and only if' clause guarantees the one-to-one correspondence. The reasoning done does not depend on n or r. The second part is *mutatis mutandis* the same.

Corollary 1 *For all n and r, we have:*

a. $TSGV(n, r) = TSGN(n, r)$,
b. $TCGV(n) = TCGN(n)$,
c. $TNCGV(n) = TNCGN(n)$, *and*
d. $TSGV(n) = TSGN(n)$.

Table 1 The classes of games we deal with in this paper

Nomenclature	Description
BCG	Bipartite complete games
BNCG	Bipartite non-complete games with n voters and r minimal winning representatives
BSG	Bipartite complete games with n voters and r minimal winning representatives
TCGV	Tripartite complete games with vetoes
TNCGV	Tripartite non-complete games with vetoes
TSGV	Tripartite simple games with vetoes
TCGN	Tripartite complete games with nulls
TNCGN	Tripartite non-complete games with nulls
TSGN	Tripartite simple games with nulls
QCGVN	Quatripartite complete games with vetoes and nulls
QNCGVN	Quatripartite non-complete games with vetoes and nulls
QSGVN	Quatripartite simple games with vetoes and nulls

Hence, by Proposition 1 and Corollary 1, the enumerations for tripartite games with vetoes for either complete or non-complete games coincide with the respective enumerations for tripartite games with nulls for complete and non-complete games.

3 Known enumerations for simple games with less than 5 equivalence classes

If only anonymous or symmetric (i.e., any pair of voters are equally desirable) voters are considered for simple games with n voters, we have $t = 1$ and $USG(n) = UCG(n) = n$; here, $USG(n)$ denotes the number of unipartite simple games and $UCG(n)$ the number of unipartite complete games.

The number of bipartite complete games with n voters, $BCG(n)$, was enumerated in Freixas et al. (2012) and later on in Kurz and Tautenhahn (2013), giving a proof based on generating functions:

$$BCG(n) = F(n+6) - (n^2 + 4n + 8), \qquad (1)$$

where $F(n)$ are the Fibonacci numbers which constitute a well-known sequence of integer numbers defined by the following recurrence relation: $F(0) = 0$, $F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$ for all $n > 1$ from which the formula

$$F(n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right], \quad n \geq 0$$

is deduced.

The number of tripartite complete games with vetoes and n voters, $TCGV(n)$ (and the number of tripartite complete games with nulls and n voters, $TCGN(n)$), was enumerated in Freixas and Kurz (2013):

$$TCGV(n) = TCGN(n) = F(n+7) - \frac{1}{2}(n^3 + 2n^2 + 13n + 26), \qquad (2)$$

whenever $n \geq 4$.

The number of quadripartite complete games with vetoes and nulls with n voters, $QCGVN(n)$, was also enumerated in Freixas and Kurz (2013):

$$QCGVN(n) = CGVN(n, 4) = F(n+8) - \frac{1}{6}(n^4 - 2n^3 + 26n^2 + 47n + 132) \qquad (3)$$

whenever $n \geq 5$.

Note that $BCG(n)$, $TCGV(n)$, and $QCGVN(n)$ belong to $\Theta\left(\left(\frac{1+\sqrt{5}}{2}\right)^n\right)$, which means that the limits:

$$\lim_{n\to\infty} \frac{BCG(n)}{((1+\sqrt{5})/2)^n}, \quad \lim_{n\to\infty} \frac{TCGV(n)}{((1+\sqrt{5})/2)^n}, \quad \lim_{n\to\infty} \frac{QCGVN(n)}{((1+\sqrt{5})/2)^n} \qquad (4)$$

are real numbers. This is because the sequences in the numerators are functions of n which are the sums of two exponentials with different basis and a polynomial. As n tends to infinity, one may disregard the exponential term with lower basis in absolute value and the polynomial. In particular, the three respective limits in (4) are the positive numbers: $9 + 4\sqrt{5}$, $(1/2)(29 + 13\sqrt{5})$, and $(1/2)(47 + 21\sqrt{5})$.

The sequence $BCG(n)$ appears in Sloane (1964) with code A163250, but also in A053808, which suggests another equivalent way to interpret the numbers in the first column of Table 3.

Table 2 The positive numbers of bipartite non-complete games $BNCG(n, r)$, up to isomorphisms, for $n < 14$

n ↓ / r →	1	2	3	4	5	6
4	1	1				
5	2	4				
6	6	18	3			
7	10	45	16			
8	19	107	72	6		
9	28	206	210	39		
10	44	381	543	190	10	
11	60	634	1190	633	76	
12	85	1025	2425	1817	406	15
13	110	1556	4528	4480	1522	130

The combinations of n and r that give rise to non-complete games are determined by the following result in Freixas and Samaniego (2020), which will be used from Sects. 4 to 9. The following trivial lemma provides an upper bound for r as a function of n.

Lemma 1 $BNCG(n, r) \neq 0$ if and only if $n > 3$ *and* $r \leq \left\lfloor \frac{n}{2} \right\rfloor$.

From Lemma 1, the following result was immediately deduced.

Corollary 2

$$BNCG(n) = \begin{cases} 0 & \text{if } n \leq 3 \\ \sum_{r=1}^{\lfloor n/2 \rfloor} BNCG(n, r) & \text{if } n > 3 \end{cases}$$

The number of bipartite non-complete games with n voters, $BNCG(n)$, was recently studied in Freixas and Samaniego (2020). A parameterization for these games allows to generate all games of this type for small values of n and r. Table 2 shows the positive numbers $BNCG(n, r)$ for all pairs (n, r) with $n < 14$. The first column indicates the number of players, n, and the first row the number of representatives of minimal winning coalitions, r. The numbers in Table 2 together with those for complete games in (1) allow to determine the number of bipartite simple games $BSG(n)$ for small values of n, see Table 3.

3.1 Related work on bipartite simple games for $r = 1$ and r being maximal

We summarize here the closed-form expressions of bipartite games that will be useful to generate enumerations of tripartite and quadripartite games from Sects. 4 to 9. We consider the extreme cases for r, either minimal ($r = 1$) or maximal.

Table 3 The numbers of bipartite non-complete $BNCG(n)$, bipartite complete $BCG(n)$, and bipartite simple games $BSG(n)$, up to isomorphisms, for $n < 14$

n	$BCG(n)$	$BNCG(n)$	$BSG(n)$
1	0	0	0
2	1	0	1
3	5	0	5
4	15	2	17
5	36	6	42
6	76	27	103
7	148	71	219
8	273	204	477
9	485	483	968
10	839	1168	2007
11	1424	2593	4017
12	2384	5773	8157
13	3952	12,326	16,278

Formulas for $r = 1$:

Proposition 2 *The number of bipartite complete games with one representative of minimal winning coalitions is:*

$$BCG(n, r = 1) = (n - 1)^2$$

Proof As the game is bipartite and complete (we assume w.l.o.g. that $N_1 > N_2$), $r = 1$ implies that the game has vetoes or nulls. Otherwise, if (a, b) was the unique representative of minimal winning coalitions it would be: $a < n_1$ ($a = n_1$ means that the n_1 voters in N_1 are vetoes) and $b > 0$ ($b = 0$ means that the n_2 voters in N_2 are nulls). Then, $(a + 1, b - 1)$ would be another representative of minimal winning coalitions. As neither $(a, b) \geq (a + 1, b - 1)$ nor $(a + 1, b - 1) \geq (a, b)$ we have a contradiction with $r = 1$.

For each (n_1, n_2) with $n_1 + n_2 = n$ ($n_i \geq 1$, for $i = 1, 2$) there are $n - 1$ games with a unique representative of minimal winning coalitions:

- 1 game with vetoes and nulls, which is $(n_1, 0)$,
- $n_2 - 1$ games with vetoes but not nulls, which are (n_1, b) with $1 \leq b \leq n_2 - 1$,
- $n_1 - 1$ games with nulls but not vetoes, which are $(a, 0)$ with $1 \leq a \leq n_1 - 1$.

Hence, each decomposition (n_1, n_2) allows for $n - 1$ games with $r = 1$. As the number of decompositions (n_1, n_2) is $n - 1$, we conclude that there exist $(n - 1)^2$ bipartite games with $r = 1$.

Proposition 3 *The number of bipartite non-complete games with one representative of minimal winning coalitions is:*

$$BNCG(n, r = 1) = \begin{cases} \dfrac{(n-2)(n^2-4n+6)}{12}, & \textit{if } n \textit{ is even} \\[2ex] \dfrac{(n-1)(n-2)(n-3)}{12}, & \textit{if } n \textit{ is odd} \end{cases}$$

Proof Let $r = 1$ and $n = n_1 + n_2$. We claim that:

a. there are $(n_1 - 1)(n_2 - 1)$ bipartite non-complete games with $r = 1$ for each vector decomposition (n_1, n_2) such that $n_1 > n_2$,
b. there are $n(n-2)/8$ bipartite non-complete games with $r = 1$ for each vector decomposition (n_1, n_2) such that $n_1 = n_2$.

This is because the game is not complete and whenever (a, b) is the unique representative of minimal winning coalitions it holds that the game has neither vetoes nor nulls (see proof of Proposition 2) so that $0 < a < n_1$ and $0 < b < n_2$. The non-completeness of the game implies that neither $(a+1, b-1)$ nor $(a-1, b+1)$ are winning representatives. From all this we deduce that there $(n_1 - 1)(n_2 - 1)$ bipartite non-complete games with $r = 1$ for each vector decomposition (n_1, n_2) with $n_1 > n_2$, condition required to avoid duplicities of the game. For the particular case, $n_1 = n_2$ we need to subtract from $(n_1 - 1)(n_2 - 1) = (n-2)^2/4$ those models (a, b) with $a < b$ (to avoid duplicities). Thus, we obtain $n(n-2)/8$ when n is even and $n_1 = n_2 = n/2$.

By summing up the number of games for each feasible decomposition (n_1, n_2) for n, we obtain the expression in $BNCG(n, r = 1)$.

It is very interesting to note that the sequence $BNCG(n, r = 1)$ is described in www.oeis.org as A005993, which was also was obtained when enumerating some other problems of different motivation than ours. Observe, therefore, that some enumerating apparently different problems are equivalent to the enumeration $BNCG(n, r = 1)$.

Corollary 3 *The number of bipartite simple games with one representative of minimal winning coalitions is:*

$$BSG(n, r = 1) = \begin{cases} \dfrac{(n-2)(n^2-4n+6)}{12} + (n-1)^2, & \textit{if } n \textit{ is even} \\[2ex] \dfrac{(n-1)(n-2)(n-3)}{12} + (n-1)^2, & \textit{if } n \textit{ is odd} \end{cases}$$

The next three results for r being maximal for bipartite complete games are proved in Freixas and Samaniego (2020), so we only state them here.

Lemma 2 *For $n > 1$, $BCG(n, r) = 0$ if $r > \lceil \frac{n}{2} \rceil$.*

Proposition 4 *Let $n > 1$. The number of bipartite complete games with a maximal number of minimal winning pairs $\left\lceil \frac{n}{2} \right\rceil$ is:*

$$\text{BCG}\left(n, r = \left\lceil \frac{n}{2} \right\rceil\right) = \begin{cases} \dfrac{1}{8}(n^2 + 14n - 24), & \textit{if } n \textit{ is even} \\[2ex] \dfrac{1}{2}(n-1), & \textit{if } n \textit{ is odd} \end{cases}$$

Theorem 1 *The number of bipartite non-complete simple games, for $n \geq 4$, with a maximal number of minimal winning representatives $\lfloor \frac{n}{2} \rfloor$, is:*

$$BNCG\left(n, r = \left\lfloor \frac{n}{2} \right\rfloor\right) = \begin{cases} \dfrac{1}{8} n(n-2), & \textit{if } n \textit{ is even} \\[2ex] \dfrac{1}{16}(n^3 + n^2 - 25n + 39), & \textit{if } n \textit{ is odd} \end{cases}$$

Note that the polynomials for n even have degree 2, while the polynomials for n odd have degree 3. This different behavior for n even and n odd also occurs in this paper when we deal with the case of a maximal number of minimal winning representatives, r. The intuition follows from the fact that r is closer to n for n even, than for n odd. This more demanding condition when n is even for r maximal limits the number of games with respect to the odd case for n. Observe, for example, that $BNCG(4, 2) = 1$ and this game is $\overline{n} = (2, 2)$ and $(2, 0)$, $(0, 2)$ as representatives of minimal winning coalitions, i.e., the bipartite non-complete games defined as $W^m = \{\{1, 2\}, \{3, 4\}\}$. $BNCG(5, 2) = 4$ and these games are represented by $\overline{n} = (3, 2)$ and the respective representatives of minimal winning coalitions:

a. $(3, 0)$, $(1, 2)$,
b. $(3, 0)$, $(1, 1)$,
c. $(3, 0)$, $(0, 2)$,
d. $(2, 0)$, $(0, 2)$.

which, respectively, correspond to the bipartite non-complete games:

a. $W^m = \{\{1, 2, 3\}, \{1, 4, 5\}, \{2, 4, 5\}, \{3, 4, 5\}\}$,
b. $W^m = \{\{1, 2, 3\}, \{1, 4\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}\}$,
c. $W^m = \{\{1, 2, 3\}, \{4, 5\}\}$,

d. $W^m = \{\{1,2\},\{1,3\},\{2,3\},\{4,5\}\}$.

Corollary 4, deduced from Proposition 4 and Theorem 1, provides the number of bipartite simple games as a function of n for the maximal number of minimal winning representatives.

Corollary 4

$$BSG\left(n, r = \left\lceil \frac{n}{2} \right\rceil\right) = \begin{cases} \frac{1}{4}(n^2+6n-12), & \text{if } n \text{ even} \\ \frac{1}{2}(n-1), & \text{if } n \text{ odd} \end{cases}$$

4 Enumerations of tripartite simple games with vetoes or nulls

As pointed out, in Corollary 1, $TNCGV(n,r) = TNCGN(n,r)$. Thus, in the rest of this section we only refer to one of them, $TNCGV(n,r)$.

Lemma 3 *Let n be the number of players and r be the number of representatives of minimal winning coalitions, then we have*

$$TNCGV(n,r) = \sum_{i=4}^{n-1} BNCG(i,r)$$

Proof If a bipartite game has either vetoes or nulls then it is a complete game. Thus, all bipartite non-complete games have neither vetoes nor nulls.

From each non-complete bipartite game with m players ($m < n$) we can add $n-m$ vetoes (or nulls) to get a tripartite non-complete game with n players and vetoes (or nulls) and this is how all non-complete tripartite games with vetoes are generated. Hence, $TNCGV(n,r) = \sum_{i=4}^{n-1} BNCG(i,r)$.

From Lemmas 1 and 3, we can determine the pairs (n,r), which lead to $TNCGV(n,r)$ being different of zero.

Corollary 5 $TNCGV(n,r) \neq 0$ if and only if $n > 4$ *and* $r \leq \left\lfloor \frac{n-1}{2} \right\rfloor$.

The number of tripartite non-complete games with vetoes as a function of n can be written more precisely by only using those $BNCG(n,r)$ which are not equal to zero.

Corollary 6

$$TNCGV(n) = \begin{cases} 0 & \text{if } n \leq 4 \\ \sum_{i=4}^{n-1} \sum_{r=1}^{\lfloor i/2 \rfloor} BNCG(i,r) & \text{if } n > 4 \end{cases}$$

Table 4 The positive numbers of tripartite non-complete games with vetoes (or nulls) $TNCGV(n,r)$, up to isomorphisms, for $n < 15$

n ↓ / r →	1	2	3	4	5	6
5	1	1				
6	3	5				
7	9	23	3			
8	19	68	19			
9	38	175	91	6		
10	66	381	301	45		
11	110	762	844	235	10	
12	170	1396	2034	868	86	
13	255	2421	4459	2685	492	15
14	365	3977	8987	7165	2014	145

Table 5 The numbers of tripartite games with veto (or null) players: complete $TCGV(n)$, tripartite non-complete $TNCGV(n)$, and tripartite simple games $TSGV(n)$, up to isomorphisms, for $n < 15$

n	$TCGV(n)$	$TNCGV(n)$	$TSGV(n)$
1	0	0	0
2	0	0	0
3	0	0	0
4	1	0	1
5	11	2	13
6	37	8	45
7	98	35	133
8	225	106	331
9	470	310	780
10	919	793	1712
11	1713	1961	3674
12	3082	4554	7636
13	5400	10,327	15,727
14	9274	22,653	31,927

Thus, from Corollary 6 and Table 2 we deduce, in Table 4, the number of tripartite non-complete games for all feasible combinations (n,r) whenever $n < 15$.

Table 4 together with Corollary 6 allows to determine the number of tripartite simple games with vetoes (or with nulls) $TSGV$ for small values of n. See Table 5.

5 Enumeration of tripartite simple games with either nulls or vetoes with a unique representative of minimal winning coalitions

The goal of this section is to enumerate the number of tripartite simple games with either nulls or vetoes with a unique representative of minimal winning coalitions. The procedure to follow in this and the subsequent sections is, firstly, determine the number of complete games of this type and, secondly, determine the number of non-complete games of this type. By the addition of the two results, we will obtain the enumeration of tripartite simple games with either vetoes (or nulls) with a unique representative of minimal winning coalitions. This is the first goal of the paper, and as far as we know, a new enumeration.

Proposition 5 *The number of tripartite complete games with vetoes (or nulls) with one representative of minimal winning coalitions is:*

$$TCGV(n, r = 1) = \frac{1}{6}(n^3 - 6n^2 + 11n - 6) \quad \textit{if } n \geq 3$$

Proof As the game is complete, we have $N_1 > N_2 > N_3$ and $n_1 + n_2 + n_3 = n$ with $n_i > 0$ for each $i = 1, 2, 3$. Let (m_1, m_2, m_3) be the unique minimal winning representative. As the game has vetoes, it is $m_1 = n_1$. As the voters in the second class are dominated by the desirability relation by the veto players, it must be $m_2 < n_2$, otherwise $m_2 = n_2$ and voters in N_2 would also be veto players, a contradiction with $N_1 > N_2$. As the voters in the second class dominate by the desirability relation the voters in the third class, it must be $m_2 > 0$ and $m_3 < n_3$; otherwise, if $m_2 = 0$ or $m_3 = n_3$, the voters in N_3 would be equally desirable to those in N_2, a contradiction with $N_2 > N_3$. Then, $n_1 \geq 1$, $n_2 \geq 2$, and $n_3 \geq 1$.

Hence, given n we need to count all these complete games. For (n_1, n_2, n_3), with $n_1 \geq 1$, $n_2 \geq 2$, and $n_3 \geq 1$, we have $n_2 - 1$ choices for m_2 and $n - n_2 - 1$ choices for the pairs (n_1, n_3). By using the subindex i instead of $n_2 - 1$, we can write:

$$TCGV(n, r = 1) = \sum_{i=1}^{n-3} i \cdot (n - i - 2),$$

which, by induction, is equivalent to

$$TCGV(n, r = 1) = \frac{1}{6}(n^3 - 6n^2 + 11n - 6)$$

Proposition 6 *The number of tripartite non-complete games with one representative of minimal winning coalitions is:*

$$TNCGV(n, r = 1) = \begin{cases} \dfrac{1}{48}(n^4 - 10n^3 + 38n^2 - 68n + 48), & \textit{if } n \textit{ is even} \\[2ex] \dfrac{1}{48}(n^4 - 10n^3 + 38n^2 - 62n + 33), & \textit{if } n \textit{ is odd} \end{cases}$$

for $n \geq 5$ and TNCGV(n,r=1)=0 otherwise.

Proof By Lemma 6, we know the numbers $TNCGV(n, r = 1)$. We have:

$$TNCGV(n, r = 1) = \sum_{i=4}^{n-1} BNCG(i, r = 1)$$

for $n \geq 5$ and $TNCGV(n, r = 1) = 0$ otherwise. To use the result on bipartite non-complete games in Proposition 3, we need to distinguish between the even case and the odd case. Let n be even. We have:

$$\begin{aligned} TNCGV(n, r = 1) &= \sum_{i=1}^{\frac{n}{2}-2} BNCG(n - 2i, r = 1) \\ &\quad + BNCG(n - (2i - 1), r = 1) \\ TNCGV(n, r = 1) &= \frac{1}{12} \sum_{i=1}^{\frac{n}{2}-2} [(n - 2i)(n - 2i - 1)(n - 2i - 2) \\ &\quad + (n - 2i - 2)((n - 2i)^2 - 4(n - 2i) + 6)] \\ &= \frac{1}{48}(n^4 - 10n^3 + 38n^2 - 68n + 48) \end{aligned}$$

Let n be odd. We have

$$\begin{aligned} TNCGV(n, r = 1) &= \sum_{i=1}^{(n-3)/2} BNCG(n - (2i - 1), r = 1) \\ &\quad + \sum_{i=1}^{(n-5)/2} BNCG(n - 2i, r = 1) \\ &= 1 + \sum_{i=1}^{(n-5)/2} [\, BNCG(n - (2i - 1), r = 1) \\ &\quad + BNCG(n - 2i, r = 1)\,] \end{aligned}$$

and by applying the result in Proposition 3 it results:

$$\begin{aligned} TNCGV(n, r = 1) &= 1 + \frac{1}{12} \sum_{i=1}^{(n-5)/2} [(n - 2i - 1)((n - 2i + 1)^2 \\ &\quad - 4(n - 2i + 1) + 6)((n - 2i - 1) \\ &\quad (n - 2i - 2)(n - 2i - 3)] \\ &= \frac{1}{48}(n^4 - 10n^3 + 38n^2 - 62n + 33). \end{aligned}$$

The addition of the two results obtained in Propositions 5 and 6 allows to enumerate the class of tripartite simple games with vetoes (or nulls) with a unique minimal winning representative.

Corollary 7 *The number of tripartite simple games with vetoes with one representative of minimal winning coalitions is:*

$$TSGV(n, r = 1)$$

$$= \begin{cases} \frac{1}{48} n \cdot (n^3 - 2n^2 - 10n + 20), & \textit{if } n \textit{ is even} \\ \frac{1}{48}\,(n^4 - 2n^3 - 10n^2 + 26n - 15), & \textit{if } n \textit{ is odd} \end{cases}$$

Observe that the two polynomials of degree 4 only differ in the linear part. Note that the game described in Example 1 is one of the 5346 tripartite non-complete games with vetoes that are counted in $TNCGV(n = 25, r = 1)$ and one of the 7370 tripartite simple games with vetoes that there are counted in $TSGV(n = 25, r = 1)$.

The enumerations obtained in Propositions 5 and 6, and Corollary 7 are, as far as we know, new and candidates to be entered in www.oeis.org.

6 Enumeration of tripartite simple games with either vetoes or nulls with a maximal number of minimal winning representatives

The goal of this section is to enumerate the number of tripartite simple games with either vetoes or nulls with a maximal number of representatives of minimal winning coalitions. We, firstly, determine the number of complete games of this type and, secondly, determine the number of non-complete games of this type. By the addition of the two results, we will obtain the enumeration of tripartite simple games with either vetoes (or nulls) with a maximal number of representatives of minimal winning coalitions. This is the second goal of the paper, and as far as we know, a new enumeration.

We start with the complete simple game case. As for $r > 1$ there are not bipartite complete games with vetoes, we have:

$$TCGV(n, r) = \sum_{i<n} BCG(i, r)$$

whenever $r > 1$. The maximal value for r as a function of n is achieved for $n = \lfloor \frac{n}{2} \rfloor$ because of Lemma 2 and that

$$TCGV\left(n, r = \left\lfloor \frac{n}{2} \right\rfloor\right) = \begin{cases} BCG\left(n-1, \lfloor \frac{n}{2} \rfloor\right) & \text{if } n \text{ is even} \\ BCG\left(n-1, \lfloor \frac{n}{2} \rfloor\right) + BCG\left(n-2, \lfloor \frac{n}{2} \rfloor\right) & \text{if } n \text{ is odd} \end{cases}$$

From this equality and Proposition 4, we deduce the next result.

Proposition 7 *For $n \geq 4$:*

$$TCGV\left(n, r = \left\lfloor \frac{n}{2} \right\rfloor\right) = \begin{cases} \frac{1}{2}(n-2), & \textit{if } n \textit{ is even} \\ \frac{1}{8}(n^2 + 16n - 49), & \textit{if } n \textit{ is odd} \end{cases}$$

We consider now non-complete games.

Proposition 8 *For $n \geq 5$:*

$$TNCGV\left(n, r = \left\lfloor \frac{n-1}{2} \right\rfloor\right) = \begin{cases} \frac{1}{16}(n^3 - 36n + 80), & \textit{if } n \textit{ is even} \\ \frac{1}{8}(n-1)(n-3), & \textit{if } n \textit{ is odd} \end{cases}$$

Proof We apply Theorem 1 and Corollary 6 on bipartite non-complete, and we distinguish between the odd and even case. Let n be an odd number, then

$$\begin{aligned} TNCGV\left(n, r = \left\lfloor \frac{n-1}{2} \right\rfloor\right) &= BNCG(n-1, (n-2)/2) + BNCG(n-2, (n-2)/2) \\ &= \frac{1}{16}((n-1)^3 + (n-1)^2 - 25(n-1) + 39) \\ &\quad + \frac{1}{8}(n-2)(n-4) \\ &= \frac{1}{16}(n^3 - 36n + 80). \end{aligned}$$

Let n be an odd number, then

$$\begin{aligned} TNCGV\left(n, r = \left\lfloor \frac{n-1}{2} \right\rfloor\right) &= BNCG(n-1, (n-1)/2) \\ &= \frac{1}{8}(n-1)(n-3). \end{aligned}$$

Now the polynomials for n even have degree 3 and the polynomials for n odd have degree 2 (oppositely to what occurs for $BNCG(n, r = \lfloor n/2 \rfloor)$). But observe that for consecutive numbers of n, r is closer to n for n odd than for n even. Thus, the number of non-complete tripartite games with vetoes increases faster for n even than for n odd.

By using the results obtained in Propositions 7 and 8, we deduce the sequence for the number of simple games with a maximal number of minimal winning representatives.

Corollary 8 *For $n \geq 4$:*

$$TSGV\left(n, r = \left\lfloor \frac{n}{2} \right\rfloor\right) = \begin{cases} \frac{1}{2}(n-2), & \textit{if } n \textit{ is even} \\ \frac{1}{4}(n^2 + 6n - 23), & \textit{if } n \textit{ is odd} \end{cases}$$

Table 6 The positive numbers of quadripartite non-complete games with veto players and nulls $QNCGVN(n, r)$, up to isomorphisms, for $n < 16$

n ↓ / r →	1	2	3	4	5	6
6	1	1				
7	4	6				
8	13	29	3			
9	32	97	22			
10	70	272	113	6		
11	136	653	414	51		
12	246	1415	1258	286	10	
13	416	2811	3292	1154	96	
14	671	5242	5232	3839	588	15
15	1036	9249	16,738	11,004	2602	160

The enumerations obtained in Propositions 7 and 8 and in Corollary 8 are, as far as we know, new candidates to be entered in www.oeis.org.

Table 7 The numbers of quadripartite games with vetoes and nulls: complete games $QCGVN(n)$, non-complete games $QNCGVN(n)$, and simple games $QSGVN(n)$, up to isomorphisms, for $n < 16$

n	$QCGVN$(n)	$QNCGVN$(n)	$QSGVN$(n)
1	0	0	0
2	0	0	0
3	0	0	0
4	0	0	0
5	0	0	0
6	8	2	10
7	35	10	45
8	113	45	158
9	303	151	454
10	717	461	1178
11	1552	1254	2806
12	3145	3215	6360
13	6062	7769	13,831
14	11,242	15,587	26,829
15	20,230	40,789	61,019

7 Enumeration of quadripartite simple games with vetoes and nulls

The next Lemma 4 links non-complete quadripartite simple games with non-complete bipartite simple games so that we can easily enumerate these games for small combinations of n and r.

Lemma 4 *Given $n > 5$ and $r \leq \lfloor \frac{n}{2} \rfloor - 1$,*

$$QNCGVN(n, r) = \sum_{i=1}^{n-(2r+1)} i \cdot BNCG(n - i - 1, r)$$

Proof A bipartite non-complete game does not have vetoes or nulls. On the other hand, any quadripartite non-complete game with vetoes and nulls of n players and r representatives of minimal winning coalitions can be obtained from a bipartite non-complete game of r representatives and with j voters, with $j < n - 1$. It is only needed to add k vetoes and $j - k$ nulls to the bipartite game, where $0 < k < j$. Thus, $n - j - 1$ games of this type are obtained from bipartite games with j voters. Let $j = n - i - 1$, then it holds

$$QNCGVN(n, r) = \sum_{i=1}^{n-(2r+1)} i \cdot BNCG(n - i - 1, r)$$

From Lemma 4 and the enumeration of non-complete bipartite games in Table 2, we can determine the number of quadripartite non-complete games with vetoes and nulls in Table 6.

The numbers in Table 6 together with the enumeration for quadripartite complete games with vetoes and nulls, in Eq. (3), lead to the enumeration of the number of quadripartite simple games with vetoes and nulls $QSGVN$ for small values of n in Table 7.

8 Enumeration of quadripartite simple games with vetoes and nulls with a unique representative of minimal winning coalitions

The goal of this section is to enumerate the number of quadripartite simple games with vetoes and nulls with a unique representative of minimal winning coalitions.

Proposition 9 *There are no quadripartite complete games with vetoes and nulls with a unique minimal winning representative.*

Proof Assume there exists a complete game of this type. Then, by completeness it is $N_1 > N_2 > N_3 > N_4$. Let (m_1, m_2, m_3, m_4) be the unique minimal winning representative. It must verify:

a. $m_1 = n_1$, because the strongest players by the desirability relation are vetoes.
b. $m_2 < n_2$, otherwise the players in N_2 would also be vetoes, a contradiction with $N_1 > N_2$.
c. $m_2 > 0$, otherwise the players in N_2 and in N_3 would belong to the same equivalence class, a contradiction with $N_2 > N_3$.

d. $m_3 < n_3$, otherwise the players in N_3 and in N_2 would belong to the same equivalence class, a contradiction with $N_2 > N_3$.
e. $m_3 > 0$ otherwise the players in N_3 would be nulls too, a contradiction.
f. $m_4 = 0$, because these players are nulls.

But, then $(n_1, m_2 + 1, m_3 - 1, 0)$ would also be a minimal winning representative. Thus, a contradiction with the assumed condition $r = 1$.

As a consequence of Proposition 9, we have $QNCGVN(n, r = 1) = QSGVN(n, r = 1)$. We now compute these numbers.

Proposition 10 *The number of quadripartite non-complete games with vetoes and nulls with one representative of minimal winning coalitions is:*

$$QNCGVN(n, r = 1) = \begin{cases} \frac{1}{240}(n^5 - 15n^4 + 90n^3 - 270n^2 + 404n - 240), & \text{if } n \text{ is even} \\ \frac{1}{240}(n^5 - 15n^4 + 90n^3 - 270n^2 + 389n - 195), & \text{if } n \text{ is odd} \end{cases}$$

Proof By using Lemma 4 for $r = 1$, we have

$$QNCGVN(n, r = 1) = \sum_{i=1}^{n-5} i \cdot BNCG(n - i - 1, r = 1)$$

We now replace $BNCG(n - i - 1, r = 1)$ for its value by distinguishing the even from the odd case.
Let n be even. We have:

$$QNCGVN(n, r = 1) = \frac{1}{12}\sum_{i=1}^{n/2-2} (2i - 1)(n - 2i - 2)((n - 2i)^2 - 4(n - 2i) + 6) + \frac{1}{12}\sum_{i=1}^{n/2-3} (2i)(n - 2i - 2)((n - 2i - 3)(n - 2i - 4).$$

Let n be odd. We have

$$QNCGVN(n, r = 1) = \frac{1}{12}\sum_{i=1}^{(n-1)/2-2} (2i - 1)(n - 2i - 1)(n - 2i - 2)(n - 2i - 3) + \frac{1}{12}\sum_{i=1}^{(n-1)/2-2} (2i)(n - 2i - 3)((n - 2i - 1)^2 - 4(n - 2i - 1) + 6).$$

Expressions that after simplification become

$$QNCGVN(n, r = 1) = \begin{cases} \frac{1}{240}(n^5 - 15n^4 + 90n^3 - 270n^2 + 404n - 240), & \text{if } n \text{ is even} \\ \frac{1}{240}(n^5 - 15n^4 + 90n^3 - 270n^2 + 389n - 195), & \text{if } n \text{ is odd} \end{cases}$$

Note that these two polynomials of degree 5 just differ in the linear part.

As there are no quadripartite complete games with vetoes and nulls for $r = 1$, according to Proposition 9, the next consequence trivially follows.

Corollary 9 *The number of quadripartite simple games with vetoes and nulls with one representative of minimal winning coalitions is:*

$$QSGVN(n, r = 1) = \begin{cases} \frac{1}{240}(n^5 - 15n^4 + 90n^3 - 270n^2 + 404n - 240), & \text{if } n \text{ is even} \\ \frac{1}{240}(n^5 - 15n^4 + 90n^3 - 270n^2 + 389n - 195), & \text{if } n \text{ is odd} \end{cases}$$

The enumeration obtained in Corollary 9 is, as far as we know, a new and a candidate to be entered in www.oeis.org.

9 Enumeration of quadripartite simple games with vetoes and nulls with a maximal number of minimal winning representatives

We start by computing the number of complete games of this type. The maximal value for r as a function of n when computing $QCGVN(n, r)$ is achieved for $r = \lfloor \frac{n-1}{2} \rfloor$ because of Lemma 2 and that it is necessary to add at least a veto player and a null player in bipartite complete games with a maximal r, which implies that

$$QCGVN\left(n, r = \left\lfloor \frac{n-1}{2} \right\rfloor\right) = \begin{cases} BCG\left(n - 2, \lfloor \frac{n-1}{2} \rfloor\right) + 2 \cdot BCG(n - 3, \lfloor \frac{n-1}{2} \rfloor) & \text{if } n \text{ is even} \\ BCG\left(n - 2, \lfloor \frac{n-1}{2} \rfloor\right) & \text{if } n \text{ is odd} \end{cases}$$

From this equality and Proposition 4, we deduce the next result.

Proposition 11 *For $n \geq 5$:*

$$QCGVN\left(n, r = \left\lfloor \frac{n-1}{2} \right\rfloor\right)$$

$$= \begin{cases} \frac{1}{8}(n^2 + 18n - 80), & \textit{if } n \textit{ is even} \\ \frac{1}{2}(n - 3), & \textit{if } n \textit{ is odd} \end{cases}$$

Proposition 12 *The number of quadripartite non-complete games with vetoes and nulls with a maximal number of minimal winning representatives is:*

$$QNCGVN\,(n, r = \lfloor n/2 \rfloor - 1) = \begin{cases} \frac{1}{8}(n - 2)(n - 4), & \textit{if } n \textit{ is even} \\ \frac{1}{16}(n^3 - n^2 - 49n + 145), & \textit{if } n \geq 5 \textit{ is odd} \end{cases}$$

Proof Let n be an even number. We have:

$$QNCGVN(n, r = n/2 - 1) = BNCG(n - 2, r = (n - 2)/2) = \frac{1}{8}(n - 2)(n - 4).$$

Let n be an odd number. We have:

$$\begin{aligned} QNCGVN\left(n, r = \frac{n-3}{2}\right) &= 2 \cdot BNCG\left(n - 3, \frac{n-3}{2}\right) \\ &\quad + 1 \cdot BNCG\left(n - 2, \frac{n-3}{2}\right) \\ &= \frac{2}{8}(n - 3)(n - 5) \\ &\quad + \frac{1}{16}((n - 2)^3 + (n - 2)^2 - 25(n - 2) + 39) \\ &= \frac{1}{16}(n^3 - n^2 - 49n + 145). \end{aligned}$$

Note again that for consecutive values of n, r is closer to n for n even than for n odd, which allows a faster increasing of the number of games for n odd.

By adding the results in Proposition 11 and Proposition 12, we obtain, in Corollary 10, the number of quadripartite simple games with vetoes and nulls.

Corollary 10 *For* $n \geq 5$*:*

$$QSGVN\left(n, r = \left\lfloor \frac{n-1}{2} \right\rfloor\right) = \begin{cases} \frac{1}{4}(n^2 + 6n - 36), & \textit{if } n \textit{ is even} \\ \frac{1}{2}(n - 3), & \textit{if } n \textit{ is odd} \end{cases}$$

The enumerations obtained in Proposition 12 and Corollary 10 are, as far as we know, new and candidates to be entered in www.oeis.org.

10 Conclusion and open problems

This article finds concise closed-form expressions for certain classes of monotonic Boolean functions or, equivalently, certain classes of simple games. Specifically, games are enumerated, in which the players are grouped into three or four equivalence classes and contain either very powerful voters (with the right to veto) or weak voters without the ability to influence (nulls). The results that are obtained are polynomial.

From a theoretical point of view, the results obtained contribute to enlarging and expanding the known enumerations of some subclasses of Boolean functions, which constitute a variant of Dedekind's problem on the enumeration of monotonic inequivalent Boolean functions or simple games.

From a practical point of view, the results obtained contribute to the design of suitable voting systems that meet certain conditions desired a priori. For instance, if one wants to design a tricameral voting system with veto players and with a given number of representatives, then he/she can generate all of them through the results of this article and select the most convenient one. Similarly, the results can be used to design circuits or reliability systems with certain restrictions.

Listing monotonic Boolean functions is a tremendously complex problem, Dedekind already noticed it. Enumerating subclasses of them is not easy either; for example, the number of bipartite simple games is already exponential. We conclude this paper by indicating two open problems that derive from this paper and that are likely to be studied in the near future.

We have determined the number of tripartite and quadripartite simple games with special types of voters. It is an open and challenging problem to determine the number of tripartite simple games and quadripartite simple games. Given its complexity, soft computing techniques might be very useful to achieve this ambitious goal.

The most relevant problem in Boolean functions is to determine whether a given function is threshold. As 2-monotonicity is a necessary, but not sufficient condition for the function to be threshold, it is of great interest to enumerate classes of 2-monotonic functions and then check one-by-one which of them are threshold, since this checking can be done in polynomial time. A direct continuation of this paper would be to enumerate the number of weighted tripartite simple games with vetoes (or nulls) and the number of quadripartite simple games with vetoes and nulls.

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Compliance with ethical standards

Conflict of interest I declare that I has no conflict of interest.

Ethical approval (in case of humans or animals were involved) This article does not contain any studies with human participants or animals performed by any of the authors.

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