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METHODOLOGIES AND APPLICATION

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Elliptic entropy of uncertain random variables with application to portfolio selection

Lin Chen[1] · **Rong Gao**[2] · **Yuxiang Bian**[3] · **Huafei Di**[4]

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Abstract

This paper investigates an elliptic entropy of uncertain random variables and its application in the area of portfolio selection. We first define the elliptic entropy to characterize the uncertainty of uncertain random variables and give some mathematical properties of the elliptic entropy. Then we derive a computational formula to calculate the elliptic entropy of function of uncertain random variables. Furthermore, we use the elliptic entropy to characterize the risk of investment and establish a mean-entropy portfolio selection model, in which the future security returns are described by uncertain random variables. Based on the chance theory, the equivalent form of the mean–entropy model is derived. To show the performance of the mean–entropy portfolio selection model, several numerical experiments are presented. We also numerically compare the mean–entropy model with the mean–variance model, the equi-weighted portfolio model, and the most diversified portfolio model by using three kinds of diversification indices. Numerical results show that the mean-entropy model outperforms the mean–variance model in selecting diversified portfolios no matter of using which diversification index.

Keywords Uncertainty theory · Elliptic entropy · Uncertain random variable · Chance theory · Mean-entropy model · Diversification index

1 Introduction

Shannon (1949) first initialized the entropy of random variables in logarithm form. After that, several scholars investigated the entropy in different angles. For example, Kullback and Leibler (1951) presented relative entropy to characterize the degree of difference between two random variables. Jaynes (1957) proposed the principle of maximum entropy and selected the probability distribution with maximum entropy from an infinite probability distribution that satisfied the given expected value and variance. Carbone and Stanley (2007) calculated the Shannon entropy of time series by using probability density function of long-range correlation cluster. Ponta and Carbone (2018) used the entropy measurement to implement the time series of prices and fluctuations in financial markets.

In the above literature of investigating entropy, a key theoretical assumption is that the indetermination is characterized by random variables (Gao et al. 2017; Rao et al. 2020; Rao and Yan 2020; Xiao et al. 2020). However, several evidences suggest that the probability distribution cannot always be used for characterizing the indeterminate phenomena (Liu 2009). To fill this research gap, Liu (2007) developed uncertainty theory to describe this type of indeterminate phenomena. Up to now, the uncertainty theory has gained considerable achievements in both theoretical and practical aspects (Zhang et al. 2016; Chen et al. 2017a, b; Cheng et al. 2017; Liu et al. 2017; Gao and Ralescu 2020). Interested readers may consult the book of Liu (2010) about the comprehensive development of uncertainty theory.

Within the framework of uncertainty theory, Liu (2009) first put forward the entropy of uncertain variables in log-

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✉ Rong Gao
gaor92@sina.com

✉ Huafei Di
dihuafei@yeah.net

1 College of Management and Economics, Tianjin University, Tianjin 300072, China

2 School of Economics and Management, Hebei University of Technology, Tianjin 300401, China

3 School of Business, East China University of Political Science and Law, Shanghai 201620, China

4 School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China

arithm form. After that, lots of scholars have done much work in this emerging field. Dai and Chen (2012) obtained a computational formula to calculate the entropy. Chen et al. (2012) investigated the cross-entropy to measure the divergence degree of uncertain variables and proposed the minimum cross-entropy principle. As a supplement of logarithm entropy, several types of entropies for uncertain variables have been investigated. For example, Yao et al. (2013) studied the sine entropy, Dai (2018) proposed the quadratic entropy, Gao et al. (2018) gave a generalized definition of cross-entropy for uncertain variables via uncertainty distributions.

With the complex process of the decision-making system, randomness and uncertainty need to be considered simultaneously (Mehralizade et al. 2020). Liu (2013a) put forward the chance theory to handle the complex decision-making system in which randomness and uncertainty coexisted. After that, several scholars have applied the chance theory into many areas, such as network optimization (Chen et al. 2018a; Jia et al. 2018), portfolio selection (Qin 2015; Ahmadzade and Gao 2020). Within the framework of chance theory, Sheng et al. (2017) first defined the entropy of uncertain random variables. Ahmadzade et al. (2017) proposed the partial entropy for uncertain random variables. Since then, many researchers have investigated the entropies for uncertain random variables from different perspectives. Ahmadzade et al. (2018) first put forward partial triangular entropy, and then applied the partial triangular entropy into the portfolio selection problem based on the chance theory. Based on absolute value function, Jia et al. (2018) investigated a new type of cross-entropy for uncertain random variables and discussed some mathematical properties of this new type of cross-entropy.

Entropy, as a quantitative estimate of diversity, has been widely applied in the area of portfolio selection (Deng and Pan 2018; Yao and Wang 2018; Li et al. 2020) and financial market (Zhou et al. 2013; Ponta and Carbone 2018). Huang (2008) used the entropy to characterize the risk and proposed two mean-entropy models with the framework of credibility measure. Huang (2012) introduced the proportion entropy to establish the mean–variance and mean–semivariance diversification models with credibilistic measure. Kar et al. (2017) established a multi-objective uncertain portfolio selection model by treating average return as expected value and divergence among security returns as cross-entropy. Based on the Minkowski measure, Yue and Wang (2017) investigated the third and fourth moments for fuzzy multi-objective portfolio selection model. Within the framework of goal programming, Aksarayli and Pala (2018) proposed a mean–variance–skewness–kurtosis-entropy for portfolio optimization. Deng et al. (2018a) established a fuzzy tri-objective mean–semivariance–entropy portfolio model with fuzzy return rates. Deng et al. (2018b) used the entropy to measure risk and proposed a fuzzy mean-entropy portfolio models with transaction costs. Within the framework of multiobjective optimization, Chen and Xu (2019) investigated a mean–semivariance–entropy model for the portfolio selection problem with fuzzy returns. Based on the optimistic and pessimistic criteria, Gupta et al. (2019) proposed two intuitionistic fuzzy portfolio selection models by considering the variance, skewness, and entropy.

In recent years, some researchers have investigated the multi-period portfolio selection based on entropy. For instance, taking return, transaction cost, risk and diversification degree of portfolio into consideration, Zhang et al. (2012) presented a mean–semivariance–entropy model for multi-period portfolio selection with fuzzy information. Considering that entropy can be seen as a measure of risk, Mehlawat (2016) investigated the multi-objective multi-period portfolio selection problems with fuzzy information. Liu et al. (2018) discussed a mean–semivariance–skewness model for multi-period fuzzy portfolio selection with considering the proportion entropy. Zhang and Li (2019) studied the impact of semi-entropy on the diversified multi-period portfolio selection with background risk. Except entropy, there are some other indicators to measure risk, such as high-order moment (Chen et al. 2017d), skewness (Chen et al. 2018b), semi-variance (Chen et al. 2017c, 2019), risk parity (Cesarone et al. 2020), conditional value-at-risk (Cesarone and Colucci 2018), absolute deviation (Zhang 2016, 2019), semi-absolute deviation (Zhang 2017; Yue et al. 2019), quadratic deviation (Wu et al. 2020), and semi-entropy (Zhou et al. 2016).

Although the existing literature has investigated the entropy application in the portfolio optimization, there still exists some research gap. For example, the existing literature didn't consider the role of elliptic entropy in the portfolio selection problem in which the future returns can be characterized as uncertain random variables. Thus, proposing a model that can exactly provide practical guidance for the stock market is significant. This paper presents elliptic entropy of uncertain random variables and provides some mathematical properties of the elliptic entropy. Moreover, we put forward a computational formula to calculate the elliptic entropy of function of uncertain random variables. Based on the definition of elliptic entropy, we establish a mean-entropy portfolio selection model in which the future returns are described by uncertain random variables. Finally, we give some numerical examples to show the performance of the mean-entropy portfolio selection model.

The main contributions of this paper can be summarized in three aspects. First, we give the definition of elliptic entropy for uncertain random variables and enrich the risk characterization index of uncertain random variables. Second, regarding the security returns as uncertain random variables, we establish a mean-entropy model for portfolio selection problem and derive the equivalent form of the proposed model. Finally, we numerically compare the mean-entropy model with the mean–variance model, the equi-weighted portfolio model, and the most diversified portfolio model by using three kinds of diversification indices, which are the complements of the Herfindahl index, the Rosenbluth index, and the comprehensive concentration index. Numerical results show that the mean-entropy model outperforms the traditional mean–variance model in selecting diversified portfolios regardless of which diversification index we use.

This paper is organized as follows. Section 2 presents some preliminaries about the uncertainty theory and chance theory. Section 3 puts forward the concept of elliptic entropy of uncertain random variables. Section 4 gives the computational formula for the elliptic entropy of function of uncertain random variables. In Sect. 5, we apply the elliptic entropy into portfolio optimization and conduct some numerical examples to show the application of elliptic entropy in the area of portfolio selection. We present concluding remarks together with suggestions about further research in Sect. 6.

2 Preliminary

In this section, we introduce some basic concepts and results about the uncertainty theory and chance theory, respectively. The former is a branch of axiomatic mathematics for dealing with belief degrees (Liu 2007), and the latter is a mathematical methodology for handling complex systems in which uncertainty and randomness coexist (Liu 2013a).

2.1 Uncertainty theory

Assume that Γ is a nonempty set and $\mathcal{L}$ represents a σ-algebra over Γ. Elements of $\mathcal{L}$ are called events. Liu (2007) presented an axiomatic uncertain measure $\mathcal{M}\{\Lambda\}$ to indicate the belief degree that uncertain event Λ occurs, where the uncertain measure $\mathcal{M}$: $\mathcal{L} \to [0, 1]$ satisfies the following three axioms (Liu 2007):

Axiom 1 $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.

Axiom 2 $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ, where Λ^c is the complementary set of Λ.

Axiom 3 For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots,$ we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is regarded as uncertainty space. The product uncertain measure $\mathcal{M}$ on the product σ-algebra $\mathcal{L}$ was defined by Liu (2009) as the following product axiom:

Axiom 4 Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. Then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\},$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.

Definition 2.1 (Liu 2007) An uncertain variable is a function $\xi(\gamma)$ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{\xi(\gamma) \in B\}$ is a measurable function of $\gamma \in \Gamma$ for any Borel set B of $\Re$.

Definition 2.2 (Liu 2007) The uncertainty distribution Φ of an uncertain variable ξ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \le x\}$$

for any $x \in \Re$.

Example 2.3 (Liu 2010) An uncertain variable ξ is called normal if the ξ has normal uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e - x)}{\sqrt{3}\sigma}\right)\right)^{-1}, x \in \Re$$

denoted by $\mathcal{N}(e, \sigma)$, where e and σ are real numbers with $\sigma > 0$, which is shown in Fig. 1 (Liu 2010). The inverse uncertainty distribution of $\mathcal{N}(e, \sigma)$ is shown as

$$\Phi^{-1}(\alpha) = e + \frac{\sigma\sqrt{3}}{\pi}\ln\frac{\alpha}{1 - \alpha}.$$

Example 2.4 (Liu 2010) An uncertain variable ξ is called linear if the ξ has linear uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le a \\ (x - a)/(b - a), & \text{if } a < x \le b \\ 1, & \text{if } x > b \end{cases}$$

denoted by $\mathcal{L}(a, b)(a < b)$, which is shown in Fig. 2 (Liu 2010). The inverse uncertainty distribution of $\mathcal{L}(a, b)$ is

$$\Phi^{-1}(\alpha) = (1 - \alpha)a + \alpha b.$$

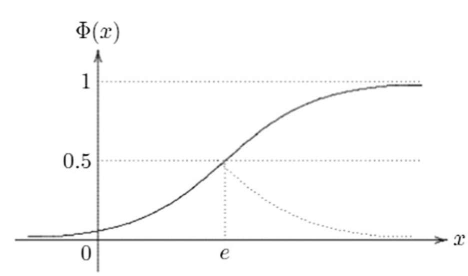

Fig. 1 Normal uncertainty distribution

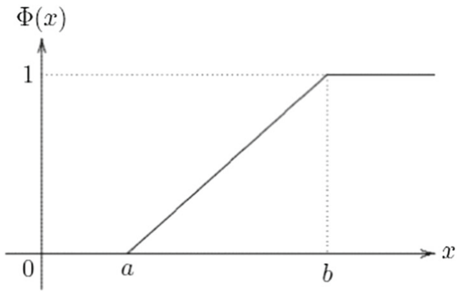

Fig. 2 Linear uncertainty distribution

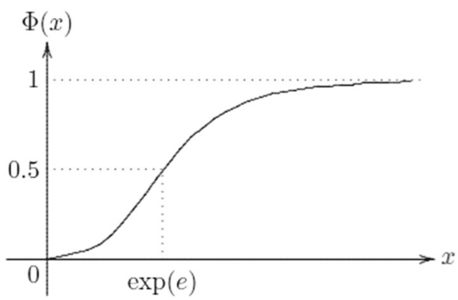

Fig. 3 Lognormal uncertainty distribution

Example 2.5 (Liu 2010) An uncertain variable ξ is called lognormal if lnx is a normal uncertain variable $\mathcal{N}(e, \sigma)$. In other words, a lognormal uncertain variable has an uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e - \ln x)}{\sqrt{3}\sigma}\right)\right)^{-1}, x \geq 0$$

denoted by $\mathcal{LOGN}(e, \sigma)$, where e and σ are real numbers with $\sigma > 0$, which is shown in Fig. 3 (Liu 2010). The inverse uncertainty distribution of $\mathcal{LOGN}(e, \sigma)$ is shown as

$$\Phi^{-1}(\alpha) = \exp\left(e + \frac{\sigma\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right).$$

2.2 Chance theory

In many cases, uncertainty and randomness usually appear simultaneously in a complex system. To describe this phenomenon, Liu (2013a) proposed the chance theory, which is a mathematical methodology for modeling complex systems in which uncertainty and randomness coexist.

Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and $(\Omega, \mathcal{A}, \Pr)$ be a probability space. The product $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ is said to be a chance space. Any element Θ in $\mathcal{L} \times \mathcal{A}$ is said to be an event in the chance space.

Definition 2.6 (Liu 2013a) The chance measure of event Θ is defined as

$$\text{Ch}\{\Theta\} = \int_0^1 \Pr\{\omega \in \Omega \mid \mathcal{M}\{\gamma \in \Gamma \mid (\gamma, \omega) \in \Theta\} \geq x\}\text{d}x.$$

Integrating uncertainty and randomness, an uncertain random variable was introduced by Liu (2013a) as follows.

Definition 2.7 (Liu 2013a) An uncertain random variable is a measurable function ξ from a chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ to the set of real numbers, i.e., $\{\xi \in B\}$ is an event in $\mathcal{L} \times \mathcal{A}$ for any Borel set B of real numbers.

Definition 2.8 (Liu 2013a) Let ξ be an uncertain random variable. Then its chance distribution is defined by

$$\Phi(x) = \text{Ch}\{\xi \leq x\}, \forall x \in \Re.$$

Liu (2013b) provided the following operation law to calculate the chance distribution of uncertain random variable.

Theorem 2.9 (Liu 2013b) *Assume that $\eta_1, \eta_2, \ldots, \eta_m$ are independent random variables with probability distributions $\Psi_1, \Psi_2, \ldots, \Psi_m$, respectively, and assume that $\tau_1, \tau_2, \ldots, \tau_n$ are independent uncertain variables. Then the uncertain random variable $\xi = f(\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n)$ has a chance distribution*

$$\Phi(x) = \int_{\Re^m} F(x; y_1, y_2, \ldots, y_m)\text{d}\Psi_1(y_1)\text{d}\Psi_2(y_2) \ldots \text{d}\Psi_m(y_m),$$

where $F(x; y_1, y_2, \ldots, y_m)$ is the uncertainty distribution of $f(y_1, y_2, \ldots, y_m, \tau_1, \tau_2, \ldots, \tau_n)$ for any real numbers $y_1, y_2, \ldots, y_m$.

Definition 2.10 (Liu 2013b) Let ξ be an uncertain random variable. Then its expected value is defined as

$$E[\xi] = \int_0^{+\infty} \text{Ch}\{\xi \geq x\}\text{d}x - \int_{-\infty}^0 \text{Ch}\{\xi \leq x\}\text{d}x$$

provided that at least one of the two integrals is finite.

Theorem 2.11 (Liu 2013b) *Let $\eta_1, \eta_2, \ldots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \ldots, \Psi_m$, and $\tau_1, \tau_2, \ldots, \tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n$, respectively. Then the uncertain random variable $\xi = f(\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n)$ has an expected value*

$$E[\xi] = \int_{\Re^m} \int_0^1 f\left(y_1, y_2, \ldots, y_m, \Upsilon_1^{-1}(\alpha), \Upsilon_2^{-1}(\alpha),\right.$$

$$\ldots, \Upsilon_n^{-1}(\alpha)\Big) \mathrm{d}\alpha \mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)\ldots \mathrm{d}\Psi_m(y_m)$$

provided that the function f is strictly increasing or decreasing with respect to $\tau_1, \tau_2, \ldots, \tau_n$.

3 Elliptic entropy of uncertain random variable

The purpose of this section is to consider a new type of entropy called elliptic entropy for uncertain random variable, as well as provide some mathematical properties of the elliptic entropy. In the chance theory, Ahmadzade et al. (2017) gave the definition of entropy for uncertain random variables as follows.

Definition 3.1 (Ahmadzade et al. 2017) Suppose that η_1, ..., η_m are independent random variables with probability distributions $\Psi_1, \ldots, \Psi_m$, respectively, and $\tau_1, \ldots, \tau_n$ are uncertain variables. Entropy of uncertain random variable $\xi = f(\eta_1, \ldots, \eta_m, \tau_1, \ldots, \tau_n)$ is defined as

$$H[\xi] = \int_{\Re^m} \int_{-\infty}^{+\infty} S(F(x, y_1, \ldots, y_m))\mathrm{d}x\mathrm{d}\Psi_1(y_1)\ldots \mathrm{d}\Psi_m(y_m),$$

where $S(t) = -t\ln t-(1-t)\ln(1-t)$ and $F(x, y_1, \ldots, y_m)$ is the uncertainty distribution of uncertain variable $f(y_1, \ldots, y_m, \tau_1, \ldots, \tau_n)$ for any real numbers $y_1, \ldots, y_m$.

Following the results of Ahmadzade et al. (2017), we define the elliptic entropy of uncertain random variable as follows.

Definition 3.2 Suppose that $\eta_1, \eta_2, \ldots, \eta_m$ are independent random variables with probability distributions $\Psi_1, \Psi_2, \ldots, \Psi_m$, respectively, and $\tau_1, \tau_2, \ldots, \tau_n$ are uncertain variables. Elliptic entropy of uncertain random variable $\xi = f(\eta_1, \ldots, \eta_m, \tau_1, \ldots, \tau_n)$ is defined as

$$H[\xi] = \int_{\Re^m} \int_{-\infty}^{+\infty} g(F(x, y_1, \ldots, y_m))\mathrm{d}x\mathrm{d}\Psi_1(y_1)\ldots \mathrm{d}\Psi_m(y_m),$$

where $g(t) = 2k\sqrt{t(1-t)}$ and $F(x, y_1, \ldots, y_m)$ is the uncertainty distribution of uncertain variable $f(\eta_1, \ldots, \eta_m, \tau_1, \ldots, \tau_n)$ for any real numbers $y_1, \ldots, y_m$.

Note that $g(t) = 2k\sqrt{t(1-t)}$ (shown in Fig. 4) is strictly increasing in [0, 0.5] and decreasing in [0.5, 1], and k is a given number determined by the decision-maker. Moreover, k is the half axis of ellipse and takes values on the interval $(0, +\infty)$. From Fig. 4, we can see that $g(t)$ is a function of k. Some special entropies can be induced for a given k. For instance, when $k = \frac{1}{2}$, the elliptic entropy becomes the circle entropy shown in Ahmadzade et al. (2017).

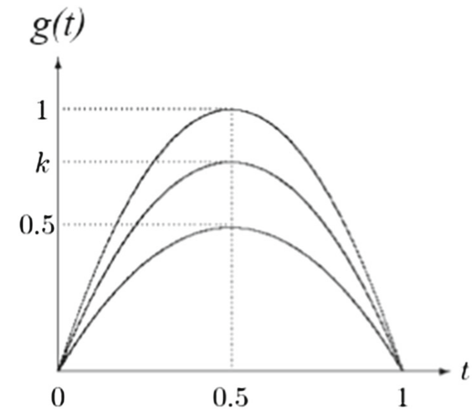

Fig. 4 The function $g(t)$

Theorem 3.3 *Let $\eta_1, \eta_2, \ldots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \ldots, \Psi_m$, respectively, and $\tau_1, \tau_2, \ldots, \tau_n$ be independent uncertain variables. For a measurable function f, the uncertain random variable $\xi = f(\eta_1, \ldots, \eta_m, \tau_1, \ldots, \tau_n)$ has elliptic entropy*

$$H[\xi] = k\int_{\Re^m} \int_{-\infty}^{+\infty} F^{-1}(\alpha, y_1, \ldots, y_m)\frac{2\alpha-1}{\sqrt{\alpha(1-\alpha)}}\mathrm{d}\alpha\mathrm{d}\Psi_1(y_1)\ldots \mathrm{d}\Psi_m(y_m).$$

Proof Let $g(\alpha) = 2k\sqrt{\alpha(1-\alpha)}$. Then $g(\alpha)$ is a derivable function with $g'(\alpha) = k\frac{1-2\alpha}{\sqrt{\alpha(1-\alpha)}}$. Since

$$g(F(x, y_1, \ldots, y_m)) = \int_0^{F(x,y_1,\ldots,y_m)} g'(\alpha)\mathrm{d}\alpha = -\int_{F(x,y_1,\ldots,y_m)}^1 g'(\alpha)\mathrm{d}\alpha,$$

we have

$$\begin{aligned} H[\xi] &= \int_{\Re^m} \int_{-\infty}^{+\infty} g(F(x, y_1, \ldots, y_m))\mathrm{d}x\mathrm{d}\Psi_1(y_1)\ldots \mathrm{d}\Psi_m(y_m) \\ &= \int_{\Re^m} \int_{-\infty}^{0} \int_0^{F(x,y_1,\ldots,y_m)} g'(\alpha)\mathrm{d}\alpha\mathrm{d}x\mathrm{d}\Psi_1(y_1)\ldots \mathrm{d}\Psi_m(y_m) \\ &\quad - \int_{\Re^m} \int_{0}^{+\infty} \int_{F(x,y_1,\ldots,y_m)}^1 g'(\alpha)\mathrm{d}\alpha\mathrm{d}x\mathrm{d}\Psi_1(y_1)\ldots \mathrm{d}\Psi_m(y_m). \end{aligned}$$

It follows from Fubini theorem (Chen et al. 2012) that

$$\begin{aligned} H[\xi] = &\int_{\Re^m}\int_0^{F(0,y_1,\ldots,y_m)}\int_{F^{-1}(\alpha,y_1,\ldots,y_m)}^{0} g'(\alpha) \\ &\mathrm{d}x\mathrm{d}\alpha\mathrm{d}\Psi_1(y_1)\ldots\mathrm{d}\Psi_m(y_m) \\ &-\int_{\Re^m}\int_{F(0,y_1,\ldots,y_m)}^{1}\int_0^{F^{-1}(\alpha,y_1,\ldots,y_m)} g'(\alpha) \\ &\mathrm{d}x\mathrm{d}\alpha\mathrm{d}\Psi_1(y_1)\ldots\mathrm{d}\Psi_m(y_m) \\ = &-\int_{\Re^m}\int_0^{F(0,y_1,\ldots,y_m)} F^{-1}(\alpha,y_1,\ldots,y_m)g'(\alpha) \\ &\mathrm{d}\alpha\mathrm{d}\Psi_1(y_1)\ldots\mathrm{d}\Psi_m(y_m) \\ &-\int_{\Re^m}\int_{F(0,y_1,\ldots,y_m)}^{1} F^{-1}(\alpha,y_1,\ldots,y_m)g'(\alpha) \\ &\mathrm{d}\alpha\mathrm{d}\Psi_1(y_1)\ldots\mathrm{d}\Psi_m(y_m) \\ = &-\int_{\Re^m}\int_0^1 F^{-1}(\alpha,y_1,\ldots,y_m)g'(\alpha) \\ &\mathrm{d}\alpha\mathrm{d}\Psi_1(y_1)\ldots\mathrm{d}\Psi_m(y_m) \\ = &\,k\int_{\Re^m}\int_0^1 F^{-1}(\alpha,y_1,\ldots,y_m)\frac{2\alpha-1}{\sqrt{\alpha(1-\alpha)}} \\ &\mathrm{d}\alpha\mathrm{d}\Psi_1(y_1)\ldots\mathrm{d}\Psi_m(y_m). \end{aligned}$$

Thus the proof is verified. □

We next summarize some mathematical properties of elliptic entropy of uncertain random variables.

Theorem 3.4 *Let τ be an uncertain variable with uncertainty distribution function Φ and η be a random variable with probability distribution function Ψ. If $\xi = \tau\eta$, then $H[\xi] = H[\tau]E[\eta]$.*

Proof If $\xi = \tau\eta$, then $F^{-1}(\alpha, y) = \Phi^{-1}(\alpha)y$. Therefore, by using Theorem 3.3, we obtain

$$\begin{aligned} H[\xi] &= k\int_{\Re}\int_0^1 \Phi^{-1}(\alpha)y\frac{2\alpha-1}{\sqrt{\alpha(1-\alpha)}}\mathrm{d}\alpha\mathrm{d}\Psi(y) \\ &= k\int_0^1 \Phi^{-1}(\alpha)\frac{2\alpha-1}{\sqrt{\alpha(1-\alpha)}}\mathrm{d}\alpha\int_{\Re} y\mathrm{d}\Psi(y) \\ &= H[\tau]E[\eta]. \end{aligned}$$

Thus the proof is completed. □

Theorem 3.5 *Let τ be an uncertain variable with uncertainty distribution function Φ and η be a random variable with probability distribution function Ψ. If $\xi = \eta + \tau$, then $H[\xi] = H[\tau]$.*

Proof If $\xi = \eta + \tau$, then $F^{-1}(\alpha, y) = \Phi^{-1}(\alpha) + y$. Therefore, by using Theorem 3.3, we obtain

$$\begin{aligned} H[\xi] &= k\int_{\Re}\int_0^1 (\Phi^{-1}(\alpha)+y)\frac{2\alpha-1}{\sqrt{\alpha(1-\alpha)}}\mathrm{d}\alpha\mathrm{d}\Psi(y) \\ &= k\int_{\Re}\int_0^1 \Phi^{-1}(\alpha)\frac{2\alpha-1}{\sqrt{\alpha(1-\alpha)}}\mathrm{d}\alpha\mathrm{d}\Psi(y) \\ &\quad + k\int_{\Re}\int_0^1 y\frac{2\alpha-1}{\sqrt{\alpha(1-\alpha)}}\mathrm{d}\alpha\mathrm{d}\Psi(y) \\ &= k\int_0^1 \Phi^{-1}(\alpha)\frac{2\alpha-1}{\sqrt{\alpha(1-\alpha)}}\mathrm{d}\alpha\int_{\Re}\mathrm{d}\Psi(y) \\ &\quad + k\int_0^1 \frac{2\alpha-1}{\sqrt{\alpha(1-\alpha)}}\mathrm{d}\alpha\int_{\Re} y\mathrm{d}\Psi(y) \\ &= H[\tau]. \end{aligned}$$

Thus the proof is completed. □

4 Elliptic entropy of function of uncertain random variable

Our attention of this section is to discuss the elliptic entropy of function of uncertain random variables, and then verify the positive linearity property of elliptic entropy. Following Jia et al. (2018) and Sheng et al. (2018) in the area of chance theory and Barberis (2000), Brandt et al. (2005), Brandt and Santa-Clara (2006), and Martellini and Urosevic (2006) in financial market, we consider that the random variables and uncertain variables are both independent in the elliptic entropy of function of uncertain random variables.

Theorem 4.1 *Let $\eta_1, \eta_2, \ldots, \eta_n$ be independent random variables, and $\tau_1, \tau_2, \ldots, \tau_n$ be independent uncertain variables. Suppose that*

$$\xi_1 = f_1(\eta_1, \tau_1), \xi_2 = f_2(\eta_2, \tau_2), \ldots, \xi_n = f_n(\eta_n, \tau_n).$$

If $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect $x_{m+1}, x_{m+2}, \ldots, x_n$, then $\xi = f(\eta_1, \ldots, \eta_m, \tau_1, \ldots, \tau_n)$ has the elliptic entropy

$$\begin{aligned} H[\xi] = k\int_{\Re^m}\int_0^1 & f\Big(F_1^{-1}(\alpha, y_1), \ldots, F_m^{-1}(\alpha, y_m), \\ & F_{m+1}^{-1}(1-\alpha, y_{m+1}), \ldots, F_n^{-1}(1-\alpha, y_n)\Big) \\ & \frac{2\alpha-1}{\sqrt{\alpha(1-\alpha)}}\mathrm{d}\alpha\mathrm{d}\Psi_1(y_1)\ldots\mathrm{d}\Psi_n(y_n), \end{aligned}$$

where $F_i^{-1}(\alpha, y_i)$ is the inverse uncertainty distribution of uncertain variable $f_i(\tau_i, y_i)$ for any real number y_i, $i = 1, 2, \ldots, n$.

Proof Based on the mathematical properties of inverse uncertainty distribution of uncertain variable shown in Liu (2010), we can obtain that

$$F^{-1}(\alpha, y_1, \ldots, y_m) = f\Big(F_1^{-1}(\alpha, y_1), \ldots, F_m^{-1}(\alpha, y_m),$$

$$F_{m+1}^{-1}(1-\alpha, y_{m+1}), \ldots, F_n^{-1}(1-\alpha, y_n)\Big).$$

Applying Theorem 3.3, this theorem is verified. □

Based on Theorem 4.1, we next present two corollaries for the elliptic entropy with strictly increasing or decreasing functions.

Corollary 4.2 *Let $\eta_1, \eta_2, \ldots, \eta_n$ be independent random variables, and $\tau_1, \tau_2, \ldots, \tau_n$ be independent uncertain variables. Suppose that*

$$\xi_1 = f_1(\eta_1, \tau_1), \xi_2 = f_2(\eta_2, \tau_2), \ldots, \xi_n = f_n(\eta_n, \tau_n).$$

If $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_n$, then $\xi = f(\eta_1, \eta_2, \ldots, \tau_n)$ has an elliptic entropy

$$H[\xi] = k \int_{\Re^m} \int_0^1 f(F_1^{-1}(\alpha, y_1), F_2^{-1}(\alpha, y_2), \ldots, F_n^{-1}(\alpha, y_n)) \frac{2\alpha - 1}{\sqrt{\alpha(1-\alpha)}} \mathrm{d}\alpha \mathrm{d}\Psi_1(y_1) \ldots \mathrm{d}\Psi_n(y_n).$$

Corollary 4.3 *Let $\eta_1, \eta_2, \ldots, \eta_n$ be independent random variables, and $\tau_1, \tau_2, \ldots, \tau_n$ be independent uncertain variables. Suppose that*

$$\xi_1 = f_1(\eta_1, \tau_1), \xi_2 = f_2(\eta_2, \tau_2), \ldots, \xi_n = f_n(\eta_n, \tau_n).$$

If $f(x_1, x_2, \ldots, x_n)$ is strictly decreasing with respect to $x_1, x_2, \ldots, x_n$, then $\xi = f(\eta_1, \eta_2, \ldots, \tau_n)$ has an elliptic entropy

$$H[\xi] = k \int_{\Re^m} \int_0^1 f(F_1^{-1}(1-\alpha, y_1), F_2^{-1}(1-\alpha, y_2), \ldots, F_n^{-1}(1-\alpha, y_n)) \frac{2\alpha - 1}{\sqrt{\alpha(1-\alpha)}} \mathrm{d}\alpha \mathrm{d}\Psi_1(y_1) \ldots \mathrm{d}\Psi_n(y_n).$$

According to the results shown in Theorem 4.1, Corollary 4.2, and Corollary 4.3, we present the next theorem about the computational formula to calculate the elliptic entropy of function and provide the theoretical basis for the mean-entropy portfolio selection model.

Theorem 4.4 *Let η_1 and η_2 be independent random variables with probability distribution functions Ψ_1 and Ψ_2, respectively, and τ_1 and τ_2 be independent uncertain variables with uncertainty distribution functions Φ_1 and Φ_2, respectively. If $\xi_1 = \eta_1 + \tau_1$ and $\xi_2 = \eta_2 + \tau_2$, then*

$$H[\xi_1 \xi_2] = H[\tau_1 \tau_2] + H[\tau_2]E[\eta_1] + H[\tau_1]E[\eta_2].$$

Proof It is clear that $F_1^{-1}(\alpha, y_1) = y_1 + \Phi_1^{-1}(\alpha)$ and $F_2^{-1}(\alpha, y_2) = y_2 + \Phi_2^{-1}(\alpha)$. Based on the results shown in Theorem 4.1, we have

$$\begin{aligned}
H[\xi] &= k \int_{\Re^2} \int_0^1 F_1^{-1}(\alpha, y_1) F_2^{-1}(\alpha, y_2) \frac{2\alpha - 1}{\sqrt{\alpha(1-\alpha)}} \mathrm{d}\alpha \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2) \\
&= k \int_{\Re^2} \int_0^1 (y_1 + \Phi_1^{-1}(\alpha))(y_2 + \Phi_2^{-1}(\alpha)) \frac{2\alpha - 1}{\sqrt{\alpha(1-\alpha)}} \mathrm{d}\alpha \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2) \\
&= k \int_{\Re^2} \int_0^1 \Phi_1^{-1}(\alpha) \Phi_2^{-1}(\alpha) \frac{2\alpha - 1}{\sqrt{\alpha(1-\alpha)}} \mathrm{d}\alpha \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2) \\
&\quad + k \int_{\Re^2} \int_0^1 y_1 \Phi_2^{-1}(\alpha) \frac{2\alpha - 1}{\sqrt{\alpha(1-\alpha)}} \mathrm{d}\alpha \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2) \\
&\quad + k \int_{\Re^2} \int_0^1 y_2 \Phi_1^{-1}(\alpha) \frac{2\alpha - 1}{\sqrt{\alpha(1-\alpha)}} \mathrm{d}\alpha \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2) \\
&= k \int_0^1 \Phi_1^{-1}(\alpha) \Phi_2^{-1}(\alpha) \frac{2\alpha - 1}{\sqrt{\alpha(1-\alpha)}} \mathrm{d}\alpha \int_{\Re^2} \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2) \\
&\quad + k \int_0^1 \Phi_2^{-1}(\alpha) \frac{2\alpha - 1}{\sqrt{\alpha(1-\alpha)}} \mathrm{d}\alpha \int_{\Re^2} y_1 \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2) \\
&\quad + k \int_0^1 \Phi_1^{-1}(\alpha) \frac{2\alpha - 1}{\sqrt{\alpha(1-\alpha)}} \mathrm{d}\alpha \int_{\Re^2} y_2 \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2) \\
&= H[\tau_1 \tau_2] + H[\tau_2]E[\eta_1] + H[\tau_1]E[\eta_2].
\end{aligned}$$

The next theorem summarizes the positive linearity property for the elliptic entropy of uncertain random variables. □

Theorem 4.5 (Positive linearity) *Let η_1 and η_2 be independent random variables with probability distribution functions Ψ_1 and Ψ_2 respectively, and τ_1 and τ_2 be independent uncertain variables with uncertainty distribution functions Φ_1 and Φ_2, respectively. Suppose that $\xi_1 = f(\eta_1, \tau_1)$ and $\xi_2 = f(\eta_2, \tau_2)$. Then for any real numbers a and b, we* *have*

$$H[a\xi_1 + b\xi_2] = |a|H[\xi_1] + |b|H[\xi_2].$$

Proof We prove this theorem by three steps.

Step 1 We prove $H[a\xi_1] = |a|H[\xi_1]$. If $a > 0$, then $af(\tau_1, y_1)$ has an inverse uncertainty distribution

$$F^{-1}(\alpha, y_1) = aF_1^{-1}(\alpha, y_1),$$

where $F^{-1}(\alpha, y_1)$ is the inverse uncertainty distribution of $f_1(\tau_1, y_1)$. It follows from Theorem 4.1 that

$$H[a\xi] = ak \int_{\Re} \int_0^1 F_1^{-1}(\alpha, y_1) \frac{2\alpha - 1}{\sqrt{\alpha(1-\alpha)}} \mathrm{d}\alpha \mathrm{d}\Psi_1(y_1) = |a|H[\xi_1].$$

If $a < 0$, then $af(\tau_1, y_1)$ has an inverse uncertainty distribution

$$F^{-1}(\alpha, y_1) = aF_1^{-1}(1-\alpha, y_1).$$

It follows from Theorem 4.1 that

$$\begin{aligned} H[a\xi] &= ak \int_{\Re} \int_0^1 F_1^{-1}(1-\alpha, y_1) \frac{2\alpha - 1}{\sqrt{\alpha(1-\alpha)}} \mathrm{d}\alpha \mathrm{d}\Psi_1(y_1) \\ &= ak \int_{\Re} \int_1^0 F_1^{-1}(\alpha, y_1) \frac{1 - 2\alpha}{\sqrt{\alpha(1-\alpha)}} \mathrm{d}(-\alpha) \mathrm{d}\Psi_1(y_1) \\ &= -ak \int_{\Re} \int_0^1 F_1^{-1}(\alpha, y_1) \frac{2\alpha - 1}{\sqrt{\alpha(1-\alpha)}} \mathrm{d}\alpha \mathrm{d}\Psi_1(y_1) = |a|H[\xi_1]. \end{aligned}$$

If $a = 0$, we then immediately have $H[a\xi_1] = 0 = |a|H[\xi_1]$. Thus we obtain $H[a\xi_1] = |a|H[\xi_1]$.

Step 2 We prove $H[\xi_1+\xi_2] = H[\xi_1]+H[\xi_2]$. The inverse uncertainty distribution of $f_1(\tau_1, y_1) + f_2(\tau_2, y_2)$ is

$$F^{-1}(\alpha, y_1, y_2) = F^{-1}(\alpha, y_1) + F^{-1}(\alpha, y_2).$$

It follows from Theorem 4.1 that

$$\begin{aligned} H[\xi_1 + \xi_2] &= k \int_{\Re^2} \int_0^1 (F_1^{-1}(\alpha, y_1) + F_2^{-1}(\alpha, y_2)) \frac{2\alpha - 1}{\sqrt{\alpha(1-\alpha)}} \mathrm{d}\alpha \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2) \\ &= k \int_{\Re^2} \int_0^1 F_1^{-1}(\alpha, y_1) \frac{2\alpha - 1}{\sqrt{\alpha(1-\alpha)}} \mathrm{d}\alpha \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2) \\ &\quad + k \int_{\Re^2} \int_0^1 F_2^{-1}(\alpha, y_2) \frac{2\alpha - 1}{\sqrt{\alpha(1-\alpha)}} \mathrm{d}\alpha \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2) \\ &= H[\xi_1] + H[\xi_2]. \end{aligned}$$

Step 3 For any real numbers a and b, combining Step 1 and Step 2, we derive

$$H[a\xi_1 + b\xi_2] = |a|H[\xi_1] + |b|H[\xi_2].$$

The theorem is proved. □

5 Application to uncertain random portfolio selection problem

In this section, we apply the elliptic entropy of uncertain random variable into the portfolio selection problem. Old stocks and new stocks have always coexisted in the real stock market (Qin 2015; Qin et al. 2017). For old stocks, we can rely on historical data to obtain the probability distribution. For new stocks, however, we have to rely on experts' estimations to predict the security returns. Following Qin (2015), Qin et al. (2017), Ahmadzade et al. (2018), and Ahmadzade and Gao (2020), we employ the chance theory to investigate the optimal portfolio selection problem in such a complex security market.

In the traditional financial market, the returns on investment were quantified as expected value and risk as variance (Qin 2015; Qin et al. 2017). However, several evidences indicated that entropy is more general as an efficient measure to characterize risk than variance (Huang 2012; Zhang et al. 2012; Kar et al. 2017; Chen and Xu 2019). Motivated by the above observations, we establish an entropy optimization model for the uncertain random portfolio selection problem, in which the elliptic entropy is employed to reflect risk associated with investment. For better understanding, Table 1 summarizes the notations used in the mean-entropy portfolio selection model.

Let $\xi_1, \xi_2, \ldots, \xi_n$ be the independent uncertain random return rates and $x_1, x_2, \ldots, x_n$ be the investment proportions in the securities. The aim of mean-entropy model is to find out the most desirable portfolio by regarding the expected value of the total return as the investment return and using the entropy to measure the investment risk. Following Huang (2008) and Huang (2012), we choose a portfolio with maximum investment return under the condition of a given tolerable risk level. Under this framework, the mean-entropy model for uncertain random portfolio selection problem can

Table 1 Summary of notations for the portfolio selection model

Notation	Description
n	The number of securities
i	The index of securities, $i = 1, 2, \ldots, n$
ξ_i	The return rate of the security i
x_i	The investment proportion of the security i
F_i^{-1}	Inverse uncertainty distribution of the uncertain variable f_i
γ	The maximum entropy level
E	Expected value operator
V	Variance operator
H	Entropy operator

be established as follows:

$$\begin{cases} \max_{x_1,\ldots,x_n} & E[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \\ \text{subject to:} & \\ & H[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \le \gamma \\ & x_1 + x_2 + \cdots + x_n = 1 \\ & x_i \ge 0, \quad i = 1, 2, \ldots, n, \end{cases} \tag{1}$$

where E is the expected value, H represents the elliptic entropy, and γ denotes the maximum entropy level. The constraint $H[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \le \gamma$ means that the entropy value of the portfolio must be lower than or equal to a predetermined safety level γ.

Let $\eta_1, \eta_2, \ldots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \ldots, \Psi_m$, and $\tau_1, \tau_2, \ldots, \tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_n$, respectively. We consider $\xi_i = f_i(\eta_i, \tau_i)$ $(i = 1, 2, \ldots, n)$ as uncertain random variables, $F_i^{-1}(\alpha, y_i)$ as the inverse uncertainty distribution of the uncertain variable $f_i(y_i, \tau_i)$. According to Theorem 2.11, we can transform the objective function into

$$E[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] = \sum_{i=1}^{n} x_i \int_{\Re}\int_0^1 F_i^{-1}(\alpha, y_i)\mathrm{d}\alpha\mathrm{d}\Psi_i(y_i).$$

By using Theorem 4.1 and Theorem 4.5, we can transform the entropy constraint into the following one:

$$\sum_{i=1}^{n} x_i \int_{\Re}\int_0^1 \frac{2\alpha - 1}{\sqrt{\alpha(1-\alpha)}} F_i^{-1}(\alpha, y_i)\mathrm{d}\alpha\mathrm{d}\Psi_i(y_i) \le \frac{\gamma}{k}.$$

Therefore, Model (1) can be converted into the equivalent form

$$\begin{cases} \max_{x_1,\ldots,x_n} & \sum_{i=1}^{n} x_i \int_{\Re}\int_0^1 F_i^{-1}(\alpha, y_i)\mathrm{d}\alpha\mathrm{d}\Psi_i(y_i) \\ \text{subject to :} & \\ & \sum_{i=1}^{n} x_i \int_{\Re}\int_0^1 \frac{2\alpha-1}{\sqrt{\alpha(1-\alpha)}} F_i^{-1}(\alpha, y_i)\mathrm{d}\alpha\mathrm{d}\Psi_i(y_i) \le \frac{\gamma}{k} \\ & x_1 + x_2 + \cdots + x_n = 1 \\ & x_i \ge 0, \quad i = 1, 2, \ldots, n. \end{cases} \tag{2}$$

Note that we can directly solve Model (2), because it is linear programming, which can be solved precisely by MATLAB and other software. In the following, we present three numerical examples to show the performance of the mean-entropy portfolio selection Model (2). Example 5.1 presents the situation that the investor has 4 securities for portfolio investment, where the risky returns are described by uncertain random variables. Similar to Woerheide and Persson (1993), we employ the complements of Herfindahl index (Woerheide and Persson 1993), the Rosenbluth index (Rosenbluth 1961), and the comprehensive concentration index (Horvath 1970) to characterize the diversification degree of our mean-entropy model and other models such as the mean-entropy model (Ahmadzade and Gao 2020), the equi-weighted portfolio model (DeMiguel et al. 2009), and the most diversified portfolio model (Choueifaty and Coignard 2008; Choueifaty et al. 2013; Froidure et al. 2019). Example 5.2 investigates the situation that the investor has 10 securities for portfolio investment. Example 5.3 presents the situation in which the random returns are characterized as normal random distribution and uncertain returns are characterized as various uncertainty distributions such as linear, normal, and lognormal, to show the robustness of results. In other words, Example 5.3 shows that the performance of our model does not depend on the distribution assumption. Although any finite number of stocks can be considered, we respectively choose 4 and 10 stocks in Example 5.1 and Example 5.2 to reduce the complexity of the presentation. The experiments are performed on a personal computer with Windows 10 and Intel (R) Core (TM) i7-4790 CPU 3.60 GHz and 2.0 GB memory. The numerical examples are implemented in MATLAB 2017b.

Before processing the numerical examples, we first summarize the three concentration indices, and then present three diversification indices. Three types of concentration indices are summarized as follows.

(1) Herfindahl index, which is the most widely used measure of economic concentration, takes the shares of the all individual firms into account (Woerheide and Persson 1993). The Herfindahl index is

$$\mathrm{HI} = \sum_{i=1}^{n} x_i^2, \tag{3}$$

where $x_i (i = 1, 2, \ldots, n)$ are the investment proportions in the securities.

(2) Rosenbluth index ranks of firms as weights with security holdings ranked in descending order by size with the i-th firm receiving rank i (Rosenbluth 1961). The Rosenbluth index is

$$\mathrm{RI} = \frac{1}{2\sum_{i=1}^{n} i x_i}, \tag{4}$$

where $x_i (i = 1, 2, \ldots, n)$ are the investment proportions in the securities.

(3) Comprehensive concentration index indicates the combination of both discrete measures and summary measures (Horvath 1970). The comprehensive concentration index is

$$\mathrm{CCI} = x_1 + \sum_{i=2}^{n} x_i^2[1 + (1 - x_i)], \tag{5}$$

where x_1 is the proportion of the largest firm, $x_i, i = 2, 3, \ldots, n$ are ranked in descending order.

Following Woerheide and Persson (1993), we employ the complements of the Herfindahl index, Rosenbluth index, and comprehensive concentration index to characterize the diversification degree of the portfolio. The complements of the Herfindahl index, Rosenbluth index, and comprehensive concentration index are shown as:

$$HI^C = 1 - \sum_{i=1}^{n} x_i^2,\ RI^C = 1 - \frac{1}{2\sum_{i=1}^{n} i x_i},\ CCI^C = 1 - x_1 - \sum_{i=2}^{n} x_i^2[1 + (1 - x_i)]. \tag{6}$$

In the above three types of diversification indices, the larger the value of the diversification index, the more diversified the portfolio of the investor.

We next summarize the mean-entropy model, the equi-weighted portfolio model, and the most diversified portfolio model. In the mean–variance model, the portfolio return and risk are characterized as expected value and variance, respectively (Ahmadzade and Gao 2020). Mathematically, the mean–variance model can be established by

$$\begin{cases} \max_{x_1,\ldots,x_n} & E[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \\ \text{subject to:} & \\ & V[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \le \gamma \\ & x_1 + x_2 + \cdots + x_n = 1 \\ & x_i \ge 0, \quad i = 1, 2, \ldots, n, \end{cases} \tag{7}$$

where E is the expected value, V represents the variance, and γ denotes the maximum risk level. The constraint $V[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \le \gamma$ means that the risk of the portfolio must be lower than or equal to a predetermined safety level γ.

In the equi-weighted portfolio model, the investor assigns each portfolio with equal weight (DeMiguel et al. 2009). For n stocks, the equi-weighted portfolio weights are

$$x_i = \frac{1}{n}, \quad i = 1, 2, \ldots, n. \tag{8}$$

In the most diversified portfolio model, the investor seeks to maximize the diversification ratio, which is defined as the ratio of the portfolio's weighted average volatility to its overall volatility (Choueifaty and Coignard 2008; Choueifaty et al. 2013). That is,

$$\begin{cases} \max_{x_1,\ldots,x_n} & \frac{\overline{\sigma}\overline{X}}{\sqrt{\overline{X}'U\overline{X}}} \\ \text{subject to:} & \\ & x_1 + x_2 + \cdots + x_n = 1 \\ & x_i \ge 0, \quad i = 1, 2, \ldots, n, \end{cases} \tag{9}$$

where $\overline{X} = (x_1, x_2, \ldots, x_n)'$ are the weights, $\overline{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ are the standard deviations of returns on the stocks, and U is the variance–covariance matrix of returns on the stocks (Pai 2017).

Example 5.1 According to the data in security markets and the experts' knowledge, we consider that the investor chooses 4 securities from different industries for investment, among which the distributions of uncertain and random returns are normal and uniform, respectively. We consider the data in the numerical example of Ahmadzade and Gao (2020), in which the 4 securities are assumed to be uncertain random variables with $\xi_i = \eta_i + \tau_i (i = 1, 2, 3, 4)$. The data of the uncertain random security returns are shown in Table 2. Note that $\mathcal{U}$ denotes the uniform random distribution and $\mathcal{N}$ represents the normal uncertainty distribution shown in Example 2.3. Similar to Gao and Ralescu (2018), we set $k = \frac{1}{2}$.

Based on the data shown in Table 2, the optimal portfolios under the mean-entropy model with different limits of the maximal entropy of the overall return γ can be obtained as shown in Table 3. We can see from Table 3 that when the maximal entropy achieves 148, all securities will be selected. In particular, the manager should invest Securities 1 and 3 with proportions around 25%, Security 2 with proportion around 43.53%, and Security 4 with proportion less than 7%. As γ goes up, more and more investment will be concentrated in Securities 2 and 3. We can also observe from Table 3 that the optimal revenue for the 4 security is increasing with γ. However, the risk is also increasing with γ because the investment will be concentrated.

Table 2 Uncertain random returns of four securities of Example 5.1

Security (i)	Random term	Uncertain term	Inverse uncertainty distribution
1	$\mathcal{U}(108, 132)$	$\mathcal{N}(100, 15)$	$F_1^{-1}(\alpha, y_1) = y_1 + 100 + \frac{15\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}$
2	$\mathcal{U}(165, 195)$	$\mathcal{N}(115, 13)$	$F_2^{-1}(\alpha, y_2) = y_2 + 115 + \frac{13\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}$
3	$\mathcal{U}(240, 260)$	$\mathcal{N}(125, 14)$	$F_3^{-1}(\alpha, y_3) = y_3 + 125 + \frac{14\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}$
4	$\mathcal{U}(162, 188)$	$\mathcal{N}(130, 20)$	$F_4^{-1}(\alpha, y_4) = y_4 + 130 + \frac{20\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}$

Table 3 Allocation of money to 4 securities of Example 5.1. (%)

γ	Security 1	Security 2	Security 3	Security 4	Objective value
148	24.36	43.53	25.17	6.94	297.4380
150	11.14	49.15	34.72	4.99	314.9200
152	4.39	51.50	37.18	6.93	322.1445
154	0.00	52.37	39.90	7.73	327.6625
156	0.00	49.50	44.73	5.77	331.3610
158	0.00	47.37	48.30	4.33	334.0730
160	0.00	45.62	51.27	3.11	336.3270

Considering all the future security returns are described by uncertain random variables, we next compare our mean-entropy model with the mean–variance model (Ahmadzade and Gao 2020), the equi-weighted portfolio model (DeMiguel et al. 2009), and the most diversified portfolio model (Choueifaty and Coignard 2008; Choueifaty et al. 2013) by using the Herfindahl index (Woerheide and Persson 1993), the Rosenbluth index (Rosenbluth 1961), and the comprehensive concentration index (Horvath 1970). Based on the data shown in Table 2, we obtain that when the predetermined safety level $\gamma = 150$, under the mean–variance model, the optimal portfolio plan is $x_1 = 0.203$, $x_2 = 0.103$, $x_3 = 0.694$, and $x_4 = 0$. Under the equi-weighted portfolio model, the optimal portfolio plan is $x_1 = 0.25$, $x_2 = 0.25$, $x_3 = 0.25$, and $x_4 = 0.25$. Under the most diversified portfolio model, the optimal portfolio plan is $x_1 = 0.354$, $x_2 = 0.143$, $x_3 = 0.451$, and $x_4 = 0.052$.

According to the diversification index shown in Equation (6), the diversification degree under various different measures for the mean-entropy model (ME model in short), the mean–variance model (MV model), the equi-weighted model (EW model), and the most diversified model (MD model) can be summarized in the following Table 4.

Table 4 shows that the diversification degree under the equi-weighted model is the largest, followed by the most diversified model and the mean-entropy model, finally by the mean–variance model regardless of which diversification index is used. It means that our mean-entropy model outperforms the mean–variance model in terms of diversification degree, and such conclusion is independent on the diversification index we use. This result shows that our mean-entropy model leads to more diversified investments than the traditional mean–variance model, which echoes the phrase "don't put all your eggs in one basket". However, our mean-entropy model is inferior to the most diversified model and the equi-weighted model. Therefore, we should consider other factors in the portfolio model, such as the diversification ratio, which is defined as the ratio of the portfolio's weighted average volatility to its overall volatility (Choueifaty and Coignard 2008; Choueifaty et al. 2013).

Example 5.2 In order to further illustrate the performance of the mean-entropy portfolio selection model, we consider the situation that the investor has 10 securities for portfolio investment in different industries from Shanghai Stock Exchange in China. Data of the 10 securities from January 2016 to December 2018 are collected. The corresponding distributions for the uncertain random future returns are shown in Table 5, in which the 10 securities are assumed to be uncertain random variables with $\xi_i = \eta_i \tau_i (i = 1, 2, \ldots, 10)$. Note that $\mathcal{U}$ denotes the uniform random distribution and $\mathcal{I}$ represents the linear uncertainty distribution which is shown in Example 2.4. Similar to Example 5.1, we set $k = \frac{1}{2}$.

Based on the data shown in Table 5 and mathematical software MATLAB, we can obtain the portfolio allocation plan in Table 6. To obtain the maximum expected return at the entropy value $\gamma = 1.0$, the investors should select the Securities 3, 8, and 10 whose expected returns are high, and the maximum expected return is 2.8192. Table 6 shows that the lower the preset entropy value, the more diversified the investment allocation. When the preset entropy value achieves 0.5, the investors should invest the four securities with codes 600081, 600591, 600638, and 600886. However, when the preset entropy value is 0.15, the investors should invest the ten securities. These numerical findings are consistent with those in Example 5.1 that people should not put all

Table 4 Diversification degree of Example 5.1 under different diversification indices

Diversification index	ME model	MV model	EW model	MD model
HI^C	0.6230	0.4666	0.7500	0.6481
RI^C	0.7093	0.6451	0.8000	0.7216
CCI^C	0.2810	0.2118	0.4219	0.2995

Table 5 Uncertain random returns of 10 securities of Example 5.2

Security (i)	Security code	Random term	Uncertain term	Inverse uncertainty distribution
1	600030	$\mathcal{U}(-1, 1.2)$	$\mathcal{I}(-1, 2)$	$F_1^{-1}(\alpha, y_1) = y_1(3\alpha - 1)$
2	600050	$\mathcal{U}(-0.8, 3.4)$	$\mathcal{I}(-0.4, 1.4)$	$F_2^{-1}(\alpha, y_2) = y_2(1.8\alpha - 0.4)$
3	600081	$\mathcal{U}(1, 2)$	$\mathcal{I}(1, 3)$	$F_3^{-1}(\alpha, y_3) = y_3(2\alpha + 1)$
4	600111	$\mathcal{U}(0.5, 1.5)$	$\mathcal{I}(-0.2, 2)$	$F_4^{-1}(\alpha, y_4) = y_4(2.2\alpha - 0.2)$
5	600270	$\mathcal{U}(0.4, 2.2)$	$\mathcal{I}(-0.8, 3.2)$	$F_5^{-1}(\alpha, y_5) = y_5(4\alpha - 0.8)$
6	600570	$\mathcal{U}(-0.4, 3)$	$\mathcal{I}(-0.5, 3.1)$	$F_6^{-1}(\alpha, y_6) = y_6(3.6\alpha - 0.5)$
7	600591	$\mathcal{U}(2, 3)$	$\mathcal{I}(0.2, 1.3)$	$F_7^{-1}(\alpha, y_7) = y_7(1.1\alpha + 0.2)$
8	600638	$\mathcal{U}(2.1, 2.4)$	$\mathcal{I}(1, 2.5)$	$F_8^{-1}(\alpha, y_8) = y_8(1.5\alpha + 1)$
9	600713	$\mathcal{U}(0.6, 1.8)$	$\mathcal{I}(2, 4)$	$F_9^{-1}(\alpha, y_9) = y_9(2\alpha + 2)$
10	600886	$\mathcal{U}(-0.6, 3)$	$\mathcal{I}(0.5, 3)$	$F_{10}^{-1}(\alpha, y_{10}) = y_{10}(2.5\alpha + 0.5)$

Table 6 Allocation of money to 10 securities of Example 5.2. (%)

γ	1	2	3	4	5	6	7	8	9	10	Objective value
0.15	2.93	4.56	13.74	6.22	6.64	6.84	13.79	17.64	13.48	14.16	1.8708
0.2	0.00	0.00	15.55	1.76	5.69	7.39	15.19	24.63	12.95	16.84	2.1097
0.4	0.00	0.00	16.36	0.00	0.00	2.08	12.42	47.71	1.09	20.34	2.4413
0.5	0.00	0.00	14.78	0.00	0.00	0.00	9.08	56.19	0.00	19.95	2.5354
1	0.00	0.00	1.89	0.00	0.00	0.00	0.00	86.10	0.00	12.01	2.8192

the eggs into one basket. The consistent findings show that our mean-entropy model outperforms the traditional mean–variance model in selecting diversified portfolios.

Example 5.3 In order to show the application of the mean-entropy model and the robustness of results, we discuss the situation in which the random returns are characterized as normal random distribution and uncertain returns are characterized as various uncertainty distributions such as linear, normal, and lognormal, which are shown in Section 2. The corresponding distributions for the uncertain random future returns are shown in Table 7 with $\xi_i = \eta_i + \tau_i (i = 1, 2, \ldots, 5)$. For random term, $\mathcal{N}$ denotes the normal random distribution. For uncertain term, $\mathcal{I}$, $\mathcal{N}$, and $\mathcal{LOGN}$ represent the linear, normal, and lognormal uncertainty distribution, respectively.

When the predetermined safety level $\gamma = 5$, we can obtain the portfolio allocation plan as follows: $x_1 = 0.6$, $x_2 = 0.1$, $x_3 = 0.1$, $x_4 = 0.15$, and $x_5 = 0.05$, and the maximum expected return is 0.2. That is, the manager should invest Security 1 with proportion 60%, Securities 2 and 3 with proportion 10%, Security 4 with proportion 15%, and Security 5 with proportion 5%. Therefore, Example 5.3 shows that the performance of our mean-entropy model is not relying on the distribution assumption.

Table 7 Uncertain random returns of 5 securities of Example 5.3

Security (i)	Random term	Uncertain term	Inverse uncertainty distribution
1	$\mathcal{N}(0.1, 0.01)$	$\mathcal{I}(-1.5, 1.5)$	$F_1^{-1}(\alpha, y_1) = y_1 + 3\alpha - 1.5$
2	$\mathcal{N}(0.1, 0.04)$	$\mathcal{N}(1.3)$	$F_2^{-1}(\alpha, y_2) = y_2 + 1 + \frac{3\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}$
3	$\mathcal{N}(0.2, 0.09)$	$\mathcal{N}(1.2, 2)$	$F_3^{-1}(\alpha, y_3) = y_3 + 1.2 + \frac{2\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}$
4	$\mathcal{N}(0.3, 0.16)$	$\mathcal{LOGN}(1, 2)$	$F_4^{-1}(\alpha, y_4) = y_4 + \exp(1 + \frac{2\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha})$
5	$\mathcal{N}(0.4, 0.25)$	$\mathcal{LOGN}(2, 3)$	$F_5^{-1}(\alpha, y_5) = y_5 + \exp(2 + \frac{3\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha})$

6 Conclusions

In this paper, we proposed the elliptic entropy of uncertain random variables and applied the elliptic entropy into the portfolio selection problem by using the entropy to measure the investment risk. We first introduced the concept of elliptic entropy for uncertain random variables, and then discussed some mathematical properties of the elliptic entropy. In order to apply the elliptic entropy well, we also investigated the elliptic entropy of function of uncertain random variables. Then we established a mean-entropy portfolio selection model with uncertain random return to test the functionality of the elliptic entropy. We gave some numerical examples to show the application of the mean-entropy model. The numerical results showed that the elliptic entropy had a good performance to reflect risk. We also compared our mean-entropy model with the mean–variance model, the equi-weighted portfolio model, and the most diversified portfolio model by using three kinds of diversification indices, which are the complements of the Herfindahl index, Rosenbluth index, and comprehensive concentration index.

This article contributes to the existing literature by investigating a diversified portfolio selection model in which the security returns are depicted as uncertain random variables. The main contributions of this paper are threefold. First, based on the chance theory, we introduced the concept of the elliptic entropy for uncertain random variables and investigated some mathematical properties of the elliptic entropy for the function of uncertain random variables. Second, we applied the elliptic entropy into the portfolio selection problem and established a mean-entropy portfolio selection model. Finally, we conduct some numerical examples to illustrate the idea of the mean-entropy model and compare our model with the traditional mean–variance model. The comparison results show that our mean-entropy model leads to more diversified investments than the traditional mean–variance model, which echoes the phrase “don’t put all your eggs in one basket.”

There are several issues that should be discussed further. First, we plan to investigate other types of entropies of uncertain random variables such as radical entropy and sine entropy, and we will also study their mathematical properties and possible applications. Second, we plan to design some algorithms to solve large-scale portfolio selection problem with uncertain random variables (Sun et al. 2020). Third, we only discussed the single-period portfolio selection problem in this paper, it will be valuable to investigate the multi-period portfolio selection problem using entropy to measure risk (Gupta et al. 2020). Finally, it is interesting to add variance, diversification ratio which is proposed by Choueifaty and Coignard (2008) and defined as the ratio of the portfolio’s weighted average volatility to its overall volatility, into the mean-entropy model under uncertain random environment. In such situation, we should balance the mean, variance, entropy, and diversification ratio in a portfolio selection model. Considering four factors simultaneously in a model can make it very complicated. We leave it for future research.

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Compliance with ethical standards

Conflict of interest The authors declare that there is no conflict of interests regarding the publication of this paper.

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