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METHODOLOGIES AND APPLICATION

Reassessments of gross domestic product model for fractional derivatives with non-singular and singular kernels

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Abstract

We study gross domestic product (GDP) model utilizing Atangana–Baleanu, Caputo–Fabrizio and Caputo fractional derivatives under the light of real data of the United Kingdom given by World Bank (World development indicators, 2018) between years 1972–2007. We obtain analytical solutions of fractional models by using Laplace transform. We compare the GDP results obtained for different fractional derivatives with real data by simulations and tables with statistical analysis showing the efficiency of fractional models to the integer-order counterpart employing error sum of squares and residual sum of squares.

Keywords Gross domestic product · Atangana–Baleanu fractional derivative · Caputo–Fabrizio fractional derivative · Laplace transform · Modeling problem

1 Introduction

Fractional analysis has led up to born a new ground in applied mathematics, physics, engineering and economy. That case has caused to be a popular subject in recent years. We can observe it in real-world modeling problems. Nevertheless, it has a few disadvantages. One of them is that fractional derivative of a constant is different from zero in Riemann–Liouville definition, and in this case, initial conditions are of fractional order in initial value problems IVPs. It was defined to overcome that problem with the help of Liouville–Caputo definition. Liouville–Caputo fractional derivative of a constant is zero, and so it has initial conditions of integer-order in IVPs. Riemann–Liouville and Liouville–Caputo fractional derivative definitions have singularity in its kernels, so this case sometimes causes important drawbacks in modeling problems. As a solution to that, Caputo and Fabrizio (2015) defined a new fractional derivative definition having non-singularity, by means of exponential function in its kernel. As a generalization of this new definition, Atangana and Baleanu (2016) gave a new fractional definition including Mittag–Leffler kernel.

Some modeling problems with Caputo–Fabrizio fractional derivative are studied by Gómez-Aguilar et al. (2016), Atangana and Baleanu (2017), Alsaedi et al. (2016), Jarad et al. (2017), Uğurlu et al. (2018), Owolabi and Atangana (2017), Atangana and Alqahtani (2016), Al-Refai and Abdeljawad (2017). Also, many scientists study modeling problems with Atangana–Baleanu fractional derivative Sun et al. (2017), Abdeljawad and Baleanu (2017b), Abdeljawad and Baleanu (2017a), Abdeljawad (2017), Gómez-Aguilar et al. (2017), Gómez-Aguilar and Atangana (2017), Yavuz et al. (2018). Various fractional modeling problems are studied by Atangana and Koca (2016), Kanth and Garg (2018), Almeida et al. (2016), Varalta et al. (2014), Kuroda et al. (2017), Chen (2008), Ozarslan et al. (2019), Bas (2015), Bas and Ozarslan (2019), Bas et al. (2019).

GDP shows the economic growth and production of a nation. It is used to determine the economic standard of living of a country. GDP enables to be evaluated increase or decrease in the percentage of economic output in various periods.

GDP model is used to specify the growth rate of a country's economy and defined in two ways as linear and exponential,

$$S'(t) = k, \quad S'(t) = kS(t).$$

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In this study, we consider the GDP model by Atangana–Baleanu, Caputo–Fabrizio and Caputo fractional derivatives with real data of the United Kingdom given by World Bank (2018) between years 1972–2007, and we compare the results obtained with those fractional derivatives with real data by simulations and tables with statistical evaluations. Analytical solutions are obtained by Laplace transform, and results are simulated by figures and tables for showing the efficiency of fractional models relative to the integer model.

2 Preliminaries

Definition 1 (Caputo and Fabrizio 2015) Caputo–Fabrizio fractional derivatives are defined as follows;
left and right derivatives in Caputo sense,

$$ {}_a^{\mathrm{CFC}}D^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t f'(s) \exp\left(\frac{-\alpha}{1-\alpha}(t-s)\right) \mathrm{d}s, \quad (1)$$

$$ {}^{\mathrm{CFC}}D_b^{\alpha} f(t) = \frac{-B(\alpha)}{1-\alpha} \int_t^b f'(s) \exp\left(\frac{-\alpha}{1-\alpha}(s-t)\right) \mathrm{d}s, \quad (2)$$

left and right derivatives in Riemann–Liouville sense

$$ {}_a^{\mathrm{CFR}}D^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t f(s) \exp\left(\frac{-\alpha}{1-\alpha}(t-s)\right) \mathrm{d}s, \quad (3)$$

where $f \in H^1(a,b)$, $a < b$, $\alpha \in [0,1]$.

$$ {}^{\mathrm{CFR}}D_b^{\alpha} f(t) = \frac{-B(\alpha)}{1-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} \int_t^b f(s) \exp\left(\frac{-\alpha}{1-\alpha}(s-t)\right) \mathrm{d}s, \quad (4)$$

where $B(\alpha) > 0$ is a normalization function with $B(0) = B(1) = 1$.

Theorem 1 (Caputo and Fabrizio 2015) *The Laplace transform of fractional definitions with Caputo–Fabrizio fractional derivatives* (1) *and* (3) *is given as follows*

$$ \mathcal{L}\left\{\left({}_a^{\mathrm{CFC}}D^{\alpha} f\right)(t)\right\}(s) = \frac{B(\alpha)}{1-\alpha} \frac{s\mathcal{L}\{f(t)\}(s)}{s+\frac{\alpha}{1-\alpha}} - \frac{B(\alpha)}{1-\alpha} f(a) e^{-as} \frac{1}{s+\frac{\alpha}{1-\alpha}},$$

$$ \mathcal{L}\left\{{}_a^{\mathrm{CFR}}D^{\alpha} f(t)\right\}(s) = \frac{B(\alpha)}{1-\alpha} \frac{s\mathcal{L}\{f(t)\}(s)}{s+\frac{\alpha}{1-\alpha}}.$$

Definition 2 (Caputo and Fabrizio 2015) Left and right integrals for Caputo–Fabrizio fractional derivatives are defined by, respectively

$$ {}_a^{CF}I^{\alpha} f(t) = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)} \int_a^t f(s)\,\mathrm{d}s,$$

$$ {}^{CF}I_b^{\alpha} f(t) = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)} \int_t^b f(s)\,\mathrm{d}s.$$

Definition 3 (Atangana and Baleanu 2016) Atangana–Baleanu fractional derivatives are defined as follows;
left and right derivatives in Caputo sense,

$$ {}_a^{ABC}D^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t f'(s) E_{\alpha}\left(\frac{-\alpha}{1-\alpha}(t-s)^{\alpha}\right) \mathrm{d}s, \quad (5)$$

$$ {}^{ABC}D_b^{\alpha} f(t) = \frac{-M(\alpha)}{1-\alpha} \int_t^b f'(s) E_{\alpha}\left(\frac{-\alpha}{1-\alpha}(s-t)^{\alpha}\right) \mathrm{d}s, \quad (6)$$

left and right derivatives in Riemann–Liouville sense

$$ {}_a^{ABR}D^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t f(s) E_{\alpha}\left(\frac{-\alpha}{1-\alpha}(t-s)^{\alpha}\right) \mathrm{d}s, \quad (7)$$

where $f \in H^1(a,b)$, $a < b$, $\alpha \in [0,1]$.

$$ {}^{ABR}D_b^{\alpha} f(t) = \frac{-M(\alpha)}{1-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} \int_t^b f(s) E_{\alpha}\left(\frac{-\alpha}{1-\alpha}(s-t)^{\alpha}\right) \mathrm{d}s. \quad (8)$$

where $M(\alpha) > 0$ is a normalization function with $M(0) = M(1) = 1$.

Definition 4 (Abdeljawad and Baleanu 2017a) Left and right fractional integrals for Atangana–Baleanu fractional derivative are defined by, respectively

$$ {}_a^{AB}I^{\alpha} f(t) = \frac{1-\alpha}{M(\alpha)} f(t) + \frac{\alpha}{M(\alpha)}\, {}_aI^{\alpha} f(t),$$

$$ {}^{AB}I_b^{\alpha} f(t) = \frac{1-\alpha}{M(\alpha)} f(t) + \frac{\alpha}{M(\alpha)}\, I_b^{\alpha} f(t).$$

Theorem 2 (Atangana and Baleanu 2016) *The Laplace transform of fractional definitions with Atangana–Baleanu fractional derivative* (5) *and* (7) *is given as follows*

$$ \mathcal{L}\left\{{}_a^{ABC}D^{\alpha} f(t)\right\}(s) = \frac{M(\alpha)}{1-\alpha} \frac{s^{\alpha}\mathcal{L}\{f(t)\}(s) - s^{\alpha-1} f(a)}{s^{\alpha}+\frac{\alpha}{1-\alpha}},$$

$$ \mathcal{L}\left\{{}_a^{ABR}D^{\alpha} f(t)\right\}(s) = \frac{M(\alpha)}{1-\alpha} \frac{s^{\alpha}\mathcal{L}\{f(t)\}(s)}{s^{\alpha}+\frac{\alpha}{1-\alpha}}.$$

Definition 5 (Podlubny 1998) Mittag–Leffler functions with one and two parameters are defined by, respectively

$$E_{\delta}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\delta k+1)},\quad (z\in\mathbb{C},\ Re(\delta)>0),$$

and

$$E_{\delta,\theta}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\delta k+\theta)},\quad (z,\theta\in\mathbb{C},\ Re(\delta)>0).$$

Property 1 (Podlubny 1998) Inverse Laplace transform of some special functions has the following properties;

(i) $\mathcal{L}^{-1}\left\{\frac{a}{s\left(s^{\delta}+a\right)}\right\}=1-E_{\delta}\left(-at^{\delta}\right),$

(ii) $\mathcal{L}^{-1}\left\{\frac{1}{s^{\delta}+a}\right\}=t^{\delta-1}E_{\delta,\delta}\left(-at^{\delta}\right).$

3 Main results

We find analytical solutions of fractional GDP models through direct and inverse Laplace transforms and solutions obtained are evaluated by least-squares error minimization technique with ParametricNDSolve and NMinimize methods in Wolfram Mathematica 11 (Wolfram Research, Inc.) for fitting real data of GDP taken by World Bank (2018).

3.1 Fractional gross domestic product model with Atangana–Baleanu fractional derivative

Let us consider the GDP model equations with Atangana–Baleanu fractional derivative in linear and exponential cases,

$$ {}_{a}^{ABC}D_{t}^{\alpha}S(t)=k, \tag{9}$$
$$ {}_{a}^{ABC}D_{t}^{\alpha}S(t)=kS(t), \tag{10}$$

where $S(t)$ is the GDP per capita at time t and k is a constant having dimension $time^{-\alpha}$.

Applying Laplace transform to both side of Eq. (9) and by the help of Theorem 2, we have

$$\begin{aligned}&\mathcal{L}\left\{{}_{a}^{ABC}D_{t}^{\alpha}S(t)\right\}(s)=\mathcal{L}\{k\}(s),\\&=\frac{M(\alpha)}{1-\alpha}\frac{s^{\alpha}\mathcal{L}\{S(t)\}(s)-s^{\alpha-1}S(a)}{s^{\alpha}+\frac{\alpha}{1-\alpha}}=\frac{k}{s}\\&\Rightarrow\mathcal{L}\{S(t)\}(s)=\frac{k(1-\alpha)}{M(\alpha)}\left(\frac{1}{s}+\frac{1}{s^{\alpha+1}}\frac{\alpha}{1-\alpha}\right)+\frac{S(a)}{s}.\end{aligned}\tag{11}$$

Table 1 Optimal parameter values

	α	k	$M(\alpha)$	$B(\alpha)$
ABC	0.3584	1.4569	2.3773	–
CFC	0.4095	2.3602	–	14.1943
Caputo	0.5008	0.2404	–	–
Integer	–	0.0807	–	–

Now, let us apply inverse Laplace transform to Eq. (11), then we can find the analytical solution of the problem (9),

$$S(t)=\frac{k(1-\alpha)}{M(\alpha)}\left(1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}\frac{\alpha}{1-\alpha}\right)+S(a).$$

Performing similar operations to both side of Eq. (10), we have

$$\begin{aligned}&\mathcal{L}\left\{{}_{a}^{ABC}D_{t}^{\alpha}S(t)\right\}(s)=k\mathcal{L}\{S(t)\}(s),\\&=\frac{M(\alpha)}{1-\alpha}\frac{s^{\alpha}\mathcal{L}\{S(t)\}(s)-s^{\alpha-1}S(a)}{s^{\alpha}+\frac{\alpha}{1-\alpha}}=k\mathcal{L}\{S(t)\}(s)\\&\Rightarrow\mathcal{L}\{S(t)\}(s)=S(a)\frac{s^{\alpha-1}M(\alpha)}{s^{\alpha}(M(\alpha)-k(1-\alpha))-k\alpha}.\end{aligned}\tag{12}$$

Finally, we can find the analytical solution of Eq. (10),

$$S(t)=\frac{S(a)}{M(\alpha)-k(1-\alpha)}E_{\alpha}\left(\frac{k\alpha}{M(\alpha)-k(1-\alpha)}t^{\alpha}\right).$$

3.2 Fractional gross domestic product model with Caputo–Fabrizio fractional derivative

Let us consider the GDP model equations with Caputo–Fabrizio fractional derivative in linear and exponential cases,

$$ {}_{a}^{\mathrm{CFC}}D_{t}^{\alpha}S(t)=k, \tag{13}$$
$$ {}_{a}^{\mathrm{CFC}}D_{t}^{\alpha}S(t)=kS(t), \tag{14}$$

here k is a constant having dimension $time^{-1}$.

Applying Laplace transform to both side of Eq. (13) and by the help of Theorem 1, then we have

$$\begin{aligned}&\mathcal{L}\left\{{}_{a}^{\mathrm{CFC}}D_{t}^{\alpha}S(t)\right\}(s)=\mathcal{L}\{k\}(s),\\&=\frac{B(\alpha)}{1-\alpha}\frac{s\mathcal{L}\{S(t)\}(s)-S(a)e^{-as}}{s+\frac{\alpha}{1-\alpha}}=\frac{k}{s}\\&\Rightarrow\mathcal{L}\{S(t)\}(s)=\frac{B(\alpha)S(a)e^{-as}}{B(\alpha)s-ks(1-\alpha)-k\alpha}.\end{aligned}\tag{15}$$

Table 2 Comparison of real data with the data obtained by integer and fractional models

Time (year)	ABC ($)	Caputo ($)	CFC ($)	Integer ($)	Real ($)
1979	7392.07	8018.4	6148.54	5780.66	7804.76
1981	8664.93	9297.11	7151.04	6793.54	9599.31
1982	9347.59	9983.6	7712.01	7364.7	9146.08
1987	13,341.4	14,001.7	11,250.2	11,026.8	13,118.6
1989	15,266.6	15,937.6	13,084.5	12,958.9	16,239.3
1992	18,589.9	19,276.3	16,411.7	16,510.	20,487.2
1995	22,530.1	23,229.2	20,585.	21,034.1	23,123.2
1996	24,001.3	24,703.6	22,199.8	22,802.5	24,332.7
1997	25,559.6	26,264.5	23,941.3	24,719.6	26,734.6
1998	27,210.4	27,917.1	25,819.4	26,797.9	28,214.3
2004	39,402.	40,095.4	40,619.8	43,496.7	40,290.3
2005	41,880.4	42,565.9	43,806.3	47,153.7	42,030.3
2006	44,507.6	45,183.1	47,242.7	51,118.1	44,599.7
2007	47,292.6	47,955.8	50,948.7	55,415.8	50,566.8

Table 3 Statistical evaluation of integer and fractional models

Models	SSE	Efficiency (%)
ABC	3.77804×10^6	99.2624
Caputo	4.72894×10^6	99.0767
CFC	5.93409×10^6	98.8414
Integer	1.04124×10^7	97.9671

Performing similar operations, we can find the analytical solution of Eq. (13),

$$S(t) = \frac{k(1-\alpha) + k\alpha t}{B(\alpha)} + S(a).$$

Similarly, for Eq. (14), we have

$$\mathcal{L}\left\{{}_{a}^{\text{CFC}}D_t^{\alpha} S(t)\right\}(s) = k\mathcal{L}\{S(t)\}(s),$$
$$= \frac{B(\alpha)}{1-\alpha}\frac{s\mathcal{L}\{S(t)\}(s) - S(a)e^{-as}}{s + \frac{\alpha}{1-\alpha}} = k\mathcal{L}\{S(t)\}(s)$$
$$\Rightarrow \mathcal{L}\{S(t)\}(s) = \frac{k}{s}\frac{1-\alpha}{B(\alpha)}\frac{s+\frac{\alpha}{1-\alpha}}{s} + S(a)\frac{e^{-as}}{s}. \quad (16)$$

Finally, we can find the analytical solution of Eq. (14),

$$S(t) = \frac{B(\alpha)S(a)e^{\frac{\alpha k t}{B(\alpha)-k(1-\alpha)}}}{B(\alpha) - k(1-\alpha)}.$$

3.3 Fractional gross domestic product model with Caputo fractional derivative

Let us consider the GDP model equations with Caputo fractional derivative by linear and exponential cases,

$$ {}_{a}^{C}D_t^{\alpha} S(t) = k, \quad (17)$$
$$ {}_{a}^{C}D_t^{\alpha} S(t) = kS(t), \quad (18)$$

here, k is a constant having dimension $time^{-\alpha}$.

Applying Laplace transform to both side of Eq. (17), then we have

$$\mathcal{L}\left\{{}_{a}^{C}D_t^{\alpha} S(t)\right\}(s) = \mathcal{L}\{k\}(s),$$
$$= s^{\alpha}\mathcal{L}\{S(t)\}(s) - s^{\alpha-1}S(a) = \frac{k}{s}$$
$$\Rightarrow \mathcal{L}\{S(t)\}(s) = \frac{S(a)}{s} + \frac{k}{s^{\alpha+1}}. \quad (19)$$

Performing similar operations, we can find the analytical solution of Eq. (17),

$$S(t) = S(a) + k\frac{t^{\alpha}}{\Gamma(\alpha+1)}.$$

Similarly for Eq. (18), we have

$$\mathcal{L}\left\{{}_{a}^{C}D_t^{\alpha} S(t)\right\}(s) = k\mathcal{L}\{S(t)\}(s),$$
$$= s^{\alpha}\mathcal{L}\{S(t)\}(s) - s^{\alpha-1}S(a) = k\mathcal{L}\{S(t)\}(s)$$
$$\Rightarrow \mathcal{L}\{S(t)\}(s) = \frac{s^{\alpha-1}S(a)}{s^{\alpha}-k}. \quad (20)$$

Finally, we have analytical solution

$$S(t) = S(a)E_{\alpha}\left(kt^{\alpha}\right).$$

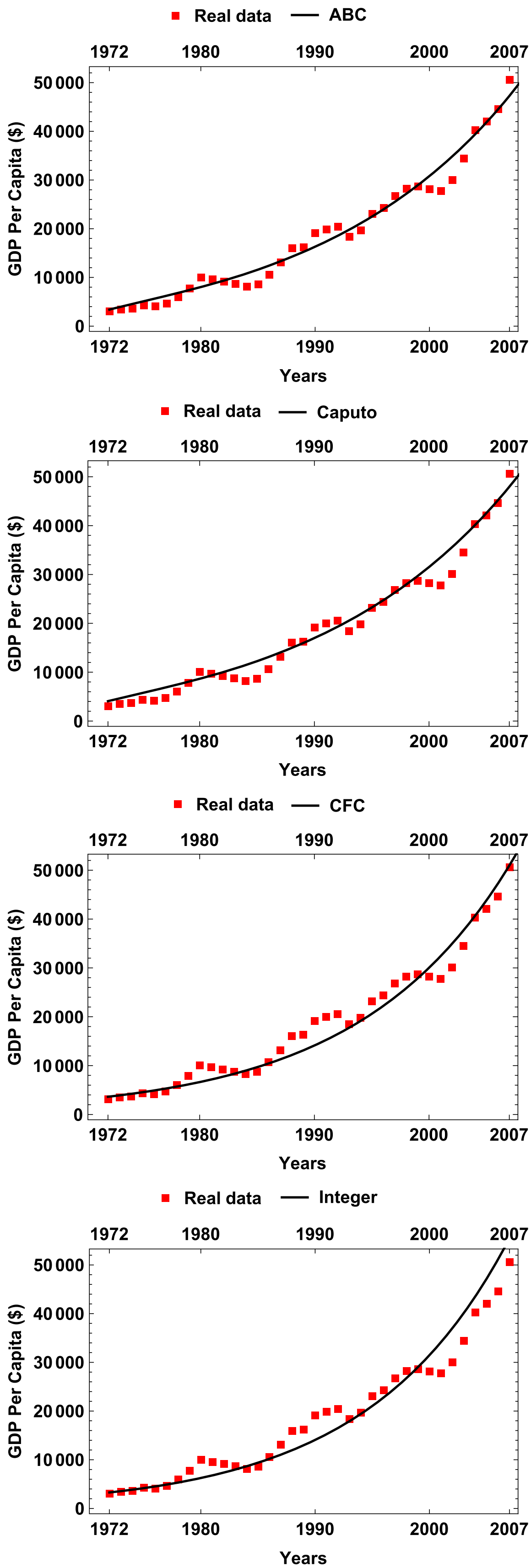

Fig. 1 Comparison of fractional and integer models with real data

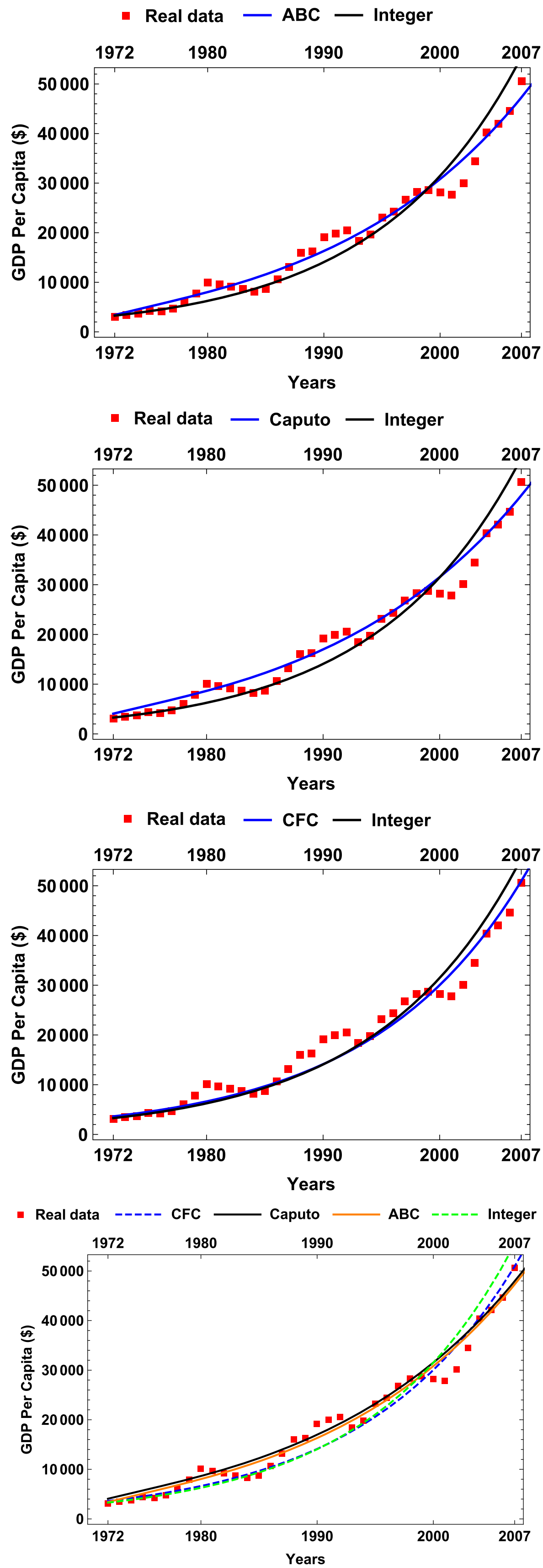

Fig. 2 Comparison of fractional and integer models with each other and real data

Remark 1 Mittag–Leffler function is taken as $E_\delta(z) = \sum_{k=0}^{100} \frac{z^k}{\Gamma(\delta k+1)}$ in all figures to simulate approximate solutions and to find optimum parameter values, also we assume $a = 0$.

We show real data of GDP per capita of the United Kingdom in US$ in the following table and figures. We use the data from the World Bank (2018) between the years 1972–2007. We find the optimal parameter values the order α, normalization constants $B(\alpha)$ and $M(\alpha)$ and proportionality constant k in Table 1. Some data obtained from fractional and integer models are compared to the real data in Table 2, and the results in statistical way are given in Table 3 for showing the efficiency of fractional models. However, we evaluate the data obtained by fractional and integer-order models by using error sum of squares statistical analysis;

$$\text{SSE} = \sum_{i=1}^{n} \frac{(a_i - b_i)^2}{M},$$

here, M is data number, a_i is each real data and b_i is estimated data obtained by models. We also evaluate the results with residual sum of squares for showing the efficiency with percentage value;

$$\text{Efficiency} = \frac{\sum_{i=1}^{n} a_i^2 - \sum_{i=1}^{n} \epsilon_i^2}{\sum_{i=1}^{n} a_i^2} \times 100,$$

here, ϵ_i is error margin between real and estimated data.

4 Discussion of GDP models

GDP model is evaluated by integer and fractional sense within ABC, Caputo and CFC under the light of real data. We use least-squares error minimization technique with ParametricNDSolve and NMinimize methods in Wolfram Mathematica 11 (Wolfram Research, Inc.) for finding optimal parameters of the models like fractional order α, $M(\alpha)$, $B(\alpha)$ normalization constants and proportionality constant k. Optimal parameters are given in Table 1. Using these optimal parameters, comparative results between real data, integer and fractional results are given in Table 2 and the results are evaluated in statistical way in Table 3. We conclude from statistical evaluation that SSE and efficiency of ABC fractional model has the best estimated data and, respectively, estimated data of Caputo and CFC is better than estimated data of integer-order relative to the real data.

We can observe comparatively with fractional order results to the real data in Fig. 1. In Fig. 2, we can observe the results for fractional and integer-order models, also comparatively of all models in the last figure. We can conclude that ABC and Caputo behave similarly, but ABC model is more efficient than Caputo in Table 3. CFC result shows similarity to the integer-order, but it is more efficient for fitting last years' data, and we can see that result also in Table 3.

We can deduce that estimated data of ABC are more efficient approximately %63 than the integer-order, Caputo is more efficient $\tilde{\%}$54 and CFC is more efficient $\tilde{\%}$43 in terms of SSE;

$$\frac{\text{SSE(ABC)} - \text{SSE(Integer)}}{\text{SSE(Integer)}} = \%63.7158,$$
$$\frac{\text{SSE(Caputo)} - \text{SSE(Integer)}}{\text{SSE(Integer)}} = \%54.5834,$$
$$\frac{\text{SSE(CFC)} - \text{SSE(Integer)}}{\text{SSE(Integer)}} = \%43.0091.$$

5 Conclusion

In this study, gross domestic product model is considered by Atangana–Baleanu, Caputo–Fabrizio and Caputo fractional and integer-order derivatives. Analytical solutions are obtained by Laplace transform, and results obtained are simulated by figures comparatively with real data of UK taken by World Bank (2018) between 1972 and 2007. Statistical evaluation is implemented of real and estimated data by using optimal parameter values.

Consequently, we can deduce that fractional models are more efficient in fitting real data than integer-order model. However, ABC is slightly more efficient than Caputo, Caputo and ABC are more efficient than CFC, but CFC model is more successful in capturing last data than integer-order counterpart. These fractional results show that fractional models will be more effective than integer-order ones in truly estimating the GDP results.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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