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FOUNDATIONS

Basic algebras and L-algebras

Jing Wang[1] · Yali Wu[2] · Yichuan Yang[1]

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Abstract
In this paper, we study the relation between L-algebras and basic algebras. In particular, we construct a lattice-ordered effect algebra which improves an example of Chajda et al. (Algebra Univ 60(1), 63–90, 2009).

Keywords Basic algebras · L-algebras · MV-algebras · Orthomodular lattices · Effect algebras

1 Introduction

Basic algebras, which generalize both MV-algebras and orthomodular lattices, were introduced in Chajda et al. (2009) and Chajda et al. (2007) as a common base for axiomatization of many-valued propositional logics as well as of the logic of quantum mechanics. The relationship between basic algebras, MV-algebras, orthomodular lattices and lattice-ordered effect algebras was considered in Botur (2010), Botur and Halaš (2008), Chajda (2012; 2015), Chajda et al. (2009). One can mention that every MV-algebra is a basic algebra whose induced lattice is distributive (Chajda 2015, P. 18, Lemma 5.2). The sufficient and necessary condition for an orthomodular lattice to be a basic algebra has been obtained in Chajda (2015, P. 17, Theorem 4.3). Relation between lattice-ordered effect algebras and basic algebras was treated in Botur and Halaš (2008), Chajda (2012) by considering their common lattice structure (a lattice with section antitone involutions).

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✉ Yichuan Yang
ycyang@buaa.edu.cn

Jing Wang
wangjing3421@sina.com

Yali Wu
wuyl_hb@163.com

[1] School of Mathematical and Sciences, Beihang University, Beijing 100191, China

[2] School of Mathematics and Physics, Hebei GEO University, Shijiazhuang 050031, China

L-algebras, which are related to algebraic logic and quantum structures, were introduced by Rump (2008). Many examples shown that L-algebras are very useful. Yang and Rump (2012), characterized pseudo-MV-algebras and Bosbach's non-commutative bricks as L-algebras. Wu and Yang (2020) proved that orthomodular lattices form a special class of L-algebras in different ways. It was shown that every lattice-ordered effect algebra has an underlying L-algebra structure in Wu et al. (2019).

In the present paper, we study the relationship between basic algebras and L-algebras. We prove that a basic algebra which satisfies

$$(z \oplus \neg x) \oplus \neg(y \oplus \neg x) = (z \oplus \neg y) \oplus \neg(x \oplus \neg y)$$

can be converted into an L-algebra (Theorem 1). Conversely, if an L-algebra with 0 and relation given by (10) such that it is an involutive bounded lattice can be organized into a basic algebra, it must be a lattice-ordered effect algebra (Theorem 2). Finally, we construct a lattice-ordered effect algebra which improves (Chajda et al. 2009, P. 80, Example 5.3).

2 Preliminaries

Note that basic algebras were introduced in Chajda (2007; 2009), but the axiomatic system was extended by one more axiom which is dependent on the following axioms as shown in Chajda and Kolšík (2009).

Definition 1 A basic algebra is an algebra $\mathcal{B} = (B;\ \oplus,\ \neg, 0)$ of type (2, 1, 0) satisfying the following identities:

$$(BA1)\quad x \oplus 0 = x,$$

$(BA2)\quad \neg\neg x = x,$

$(BA3)\quad \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x,$

$(BA4)\quad \neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = \neg 0.$

For the sake of brevity, we denote by $1 =: \neg 0$.

Let $\mathcal{B} = (B, \oplus, \neg, 0)$ be a basic algebra. The relation $\leq$ defined by

$$x \leq y \text{ if and only if } \neg x \oplus y = 1 \tag{1}$$

is a partial ordering on B such that 0 and 1 are the least and the greatest element of B, respectively.

In what follows, we need the following properties of basic algebras (cf. Chajda 2015; Chajda et al. 2009):

$$x \oplus 1 = 1 = 1 \oplus x. \tag{2}$$

$$0 \oplus x = x. \tag{3}$$

$$\neg x \oplus x = 1. \tag{4}$$

$$x \leq y \Rightarrow \neg x \geq \neg y. \tag{5}$$

$$x \leq y \Rightarrow x \oplus z \leq y \oplus z. \tag{6}$$

$$y \leq x \oplus y. \tag{7}$$

Lemma 1 (Chajda 2015, P. 69, Prop. 3.6) *For every basic algebra $\mathcal{B} = (B, \oplus, \neg, 0)$, the poset $(B, \leq)$ is a bounded lattice in which the supremum $x \vee y$ and the infimum $x \wedge y$ are given by $x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$, respectively.*

An involutive bounded lattice (IBL) (Chiara and Giuntini 2002, P. 191, Def. 12.1) is a structure $(L, \leq, ', 0, 1)$, where $(L, \leq, 0, 1)$ is a lattice with minimum 0 and maximum 1, $'$ is a unary operation on L such that the following conditions are satisfied:

$$\text{(Involutive law)}\quad x = x''. \tag{8}$$

$$\text{(Antitony)}\quad \text{if } x \leq y, \text{ then } y' \leq x'. \tag{9}$$

According to $(BA2)$, (5) and Lemma 1, every basic algebra is an IBL.

Lemma 2 (Chajda 2015, P. 70, Lemma 3.8) *The identity*

$$\neg(\neg(x \oplus y) \oplus y) \oplus y = x \oplus y$$

is true in all basic algebras.

Corollary 1 *The identity*

$$(x \wedge \neg y) \oplus y = x \oplus y$$

is true in all basic algebras.

Proof By Lemmas 1 and 2, $x \oplus y = \neg(\neg(x \oplus y) \oplus y) \oplus y = \neg(\neg x \vee y) \oplus y = (x \wedge \neg y) \oplus y$ is true in all basic algebras. □

Definition 2 (Rump and Yang 2012, P. 122) An L-algebra is an algebra $(L, \to)$ of type (2, 0) satisfying

$(L1)\quad x \to x = x \to 1 = 1,\ 1 \to x = x$

$(L2)\quad (x \to y) \to (x \to z) = (y \to x) \to (y \to z)$

$(L3)\quad x \to y = y \to x = 1 \ \Rightarrow\ x = y$

for all $x, y, z \in L$.

There is a partial ordering by Rump (2008, P. 2332, Prop. 2)

$$x \leq y \ \Leftrightarrow\ x \to y = 1 \tag{10}$$

such that 1 is the greatest element of L. If L admits a smallest element 0, we speak of an L-algebra with 0.

Lemma 3 (Rump and Yang 2012, P. 123, Lemma 2.1) *Let L be an L-algebra. Then, $x \leq y$ implies that $z \to x \leq z \to y$ for all $x, y, z \in L$.*

In particular, if L is an L-algebra with 0 and satisfies (8) for every $x \in L$, then

$$x \leq x' \to y,\ x' \leq x \to y. \tag{11}$$

3 L-algebras and basic algebras

In this section, we are interested in knowing the mutual relation between L-algebras and basic algebras. Assume that they have the same lattice structure. Firstly, we give three types of involutive bounded lattices which can be regarded as both L-algebras and basic algebras: MV-algebras, lattice-ordered effect algebras and orthomodular lattices.

Recall that an MV-algebra Chang (1958) is an algebra $A = (A, \oplus, ', 0)$ of type (2, 1, 0) where $(A, \oplus, 0)$ is a commutative monoid satisfying (8) and the following identities:

$$x \oplus 0' = 0',$$
$$(x' \oplus y)' \oplus y = (y' \oplus x)' \oplus x.$$

MV-algebras are both basic algebras and L-algebras (Chajda et al. 2009; Wu et al. 2019).

An effect algebra (Foulis and Bennett 1994, P. 1333, Def. 2.1) is a system $(E, +, 0, 1)$ consisting of a set E with two special elements $0, 1 \in E$, called the zero and the unit, and with a partially defined binary operation $+$ satisfying the following conditions for all $p, q, r \in E$.

$(E1)$ (Commutative law) If $p+q$ is defined, then $q+p$ is defined and $p+q=q+p$.

$(E2)$ (Associative law) If $p+q$ is defined and $(p+q)+r$ is defined, then $q+r$ and $p+(q+r)$ are defined and $p+(q+r)=(p+q)+r$.

$(E3)$ (Orthosupplement law) For every $p \in E$, there exists a unique $q \in E$ such that $p+q$ is defined and $p+q=1$. The unique element q is written as p' and called the orthosupplement of p.

$(E4)$ (Zero-one law) If $p+1$ is defined, then $p=0$.

Let $(E, +, 0, 1)$ be an effect algebra. Define a binary relation on E by

$$a \leq b \text{ if for some } c \in E,\ c+a=b \tag{12}$$

which is a partial ordering on E such that 0 and 1 are the smallest element and the greatest element of E, respectively. If the poset $(E, \leq)$ is a lattice, then E is called a lattice-ordered effect algebra.

Lemmas 4 and 5 show that there is a mutual correspondence between lattice-ordered effect algebras, basic algebras and L-algebras.

Lemma 4 (Chajda 2012, P. 8, Thm. 12) *Let $\mathcal{E}=(E, +, 0, 1)$ be a lattice-ordered effect algebra. Define*

$$x \oplus y := (x \wedge y') + y \ \textit{and} \ \neg x := x'. \tag{13}$$

Then, $\mathcal{B}(E)=(E, \oplus, \neg, 0)$ is a basic algebra (whose lattice order coincides with the original one).

Define $x \to y := (x \wedge y) + x'$.

Lemma 5 (Wu et al. 2019, P106, Thm. 3.3) *Every lattice-ordered effect algebra $(E, +, 0, 1)$ gives rise to an L-algebra $(E, \to)$ with negation such that $x' = x \to 0$ is exactly the orthosupplement of x in $(E, +, 0, 1)$.*

Let $(L, +, 0, 1)$ be a lattice-ordered effect algebra. Define

$$x \oplus y := (x \wedge y') + y,$$

and then, $(L, \oplus, \neg, 0)$ is a basic algebra by Lemma 4. By Lemma 5, $(L, \to, 0, 1)$ is an L-algebra, where

$$x \to y := (x \wedge y) + x'.$$

Then, $x \oplus y = y' \to x$.

An orthomodular lattice (OML) Kalmbach (1983) is an algebra $\mathcal{L}=(L, \vee, \wedge, ', 0, 1)$ of type (2, 2, 1, 0, 0) satisfying (8), (9) and the following axioms: (i) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice. (ii) $x \leq y$ implies $y = x \vee (y \wedge x')$.

In Chajda (2015), the author uses

$$x \oplus y := (x \wedge y') \vee y \ \textit{and} \ \neg x := x' \tag{14}$$

to convert an orthomodular lattice $(L, \vee, \wedge, ', 0, 1)$ into a basic algebra $(L, \oplus, \neg, 0)$.

Define

$$x \to y := x' \vee (x \wedge y), \tag{15}$$

then every orthomodular lattice L gives rise to an L-algebra $(L, \to)$ in [16]. Then, $x \oplus y = y' \to x$.

Now, we will give a basic algebra which is also an L-algebra.

Example 1 Let $\mathcal{B}=(\{0, a, \neg a, 1\}, \oplus, \neg, 0)$ be a basic algebra, where $\oplus$ is given in Table 1.

Table 1 $\oplus$ of Example 1

$\oplus$	0	a	$\neg a$	1
0	0	a	$\neg a$	1
a	a	$\neg a$	1	1
$\neg a$	$\neg a$	1	1	1
1	1	1	1	1

Define $x \to y := y \oplus \neg x$ and $x' = x \to 0 := \neg x$; then, we have Table 2.

Table 2 $\to$ of Example 1

$\to$	0	a	a'	1
0	1	1	1	1
a	a'	a	1	1
a'	a	a'	1	1
1	0	a	a'	1

An easy computation shows that $\mathcal{B}$ is also an L-algebra.

Next, we will give a characterization of basic algebras to be L-algebras.

Theorem 1 *Let $(B, \oplus, \neg, 0)$ be a basic algebra which satisfies the following condition:*

$$(z \oplus \neg x) \oplus \neg(y \oplus \neg x) = (z \oplus \neg y) \oplus \neg(x \oplus \neg y) \tag{LB}$$

Then, $(B, \to)$ is an L-algebra.

Proof Define $x \to y := y \oplus \neg x$.

By (2), $x \to 1 = 1 \oplus \neg x = 1$. $1 \to x = x \oplus \neg 1 = x \oplus 0 = x$. By (4), $x \to x = x \oplus \neg x = 1$. This verifies $(L1)$.

$(x \to y) \to (x \to z) = (x \to z) \oplus \neg(x \to y) = (z \oplus \neg x) \oplus \neg(y \oplus \neg x)$. Similarly, $(y \to x) \to (y \to z) = (z \oplus \neg y) \oplus \neg(x \oplus \neg y)$. By (LB), we have verified $(L2)$ in the definition of an L-algebra.

Assume that $x \to y = y \to x = 1$, then $y \oplus \neg x = x \oplus \neg y = 1$. Since $y \oplus \neg x = 1 \Leftrightarrow \neg y \leq \neg x \Leftrightarrow x \leq y$ by (5) and $(BA2)$, then $x \leq y$, $y \leq x$. Hence, $x = y$. This verifies $(L3)$.

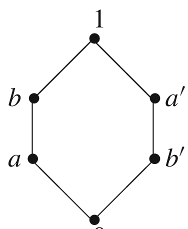
Then, $(B, \to)$ is an L-algebra. □

Table 3 $\oplus$ of Example 2

$\oplus$	0	a	b	$\neg b$	$\neg a$	1
0	0	a	b	$\neg b$	$\neg a$	1
a	a	a	b	$\neg a$	1	1
b	b	a	b	1	1	1
$\neg b$	$\neg b$	b	1	$\neg b$	$\neg a$	1
$\neg a$	$\neg a$	1	1	$\neg b$	$\neg a$	1
1	1	1	1	1	1	1

There are many basic algebras which are not L-algebras with respect to the original involutive bounded lattice structure.

Example 2 Let us consider the ortholattice O_6 with the following Hasse diagram.



O_6

By Corollary 1 and the properties of basic algebras, it is routine to verify that $(O_6, \oplus, \neg, 0)$ is a basic algebra, where $\neg x = x'$ and $\oplus$ is given in Table 3.

Assume O_6 can be converted into an L-algebra with the operation $\rightarrow$. By $(L2)$, $(b \rightarrow a) \rightarrow b' = (a \rightarrow b) \rightarrow a' = 1 \rightarrow a' = a'$. Then, $b \rightarrow a = a'$, since $b' \leq b \rightarrow a$, whence $a' \rightarrow b' = a'$. However, $a \leq a' \rightarrow b' = a'$, which is a contradiction. Thus, O_6 is not an L-algebra.

Conversely, under what conditions can an L-algebra be regarded as a basic algebra? Since every basic algebra is an IBL, we are interested in the L-algebra L with 0 and relation given by (10) such that the L is an IBL. Define $x \oplus y := y' \rightarrow x$, and we have the following theorem:

Theorem 2 *Let $(L, \rightarrow)$ be an L-algebra with 0 and relation given by (10) such that L is an involutive bounded lattice, where $x' = x \rightarrow 0$. Define*

$$x \oplus y := y' \rightarrow x.$$

If $(L, \oplus, \neg, 0)$ is a basic algebra, then L must be a lattice-ordered effect algebra.

Proof Since L is an involutive bounded lattice, then $x'' = x$ and $x \leq y \Rightarrow x' \geq y'$ for every $x, y \in L$. Define $x \oplus y := y' \rightarrow x$, and then, $x \vee y = y' \rightarrow (y' \rightarrow x')'$ by Lemma 1.

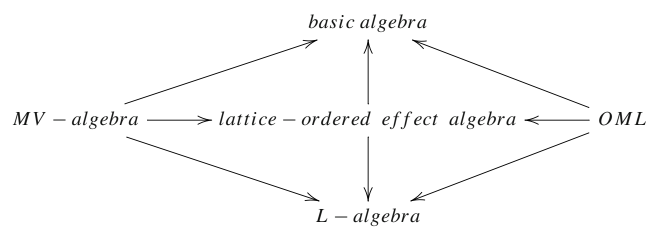
Assume $x \leq y$, then

$$\begin{aligned} y \rightarrow x &= (y \vee x) \rightarrow x \\ &= (x' \rightarrow (x' \rightarrow y')') \rightarrow x \\ &= (x' \rightarrow (x' \rightarrow y')') \rightarrow (x' \rightarrow 0) \quad by\ (L2) \\ &= ((x' \rightarrow y')' \rightarrow x') \rightarrow ((x' \rightarrow y')' \rightarrow 0) \\ &\quad by\ (11)\ and\ (9) \\ &= 1 \rightarrow (x' \rightarrow y') \quad by\ (L1) \\ &= x' \rightarrow y'. \end{aligned}$$

Then by Theorem 3.9 in Wu et al. (2019), L is a lattice-ordered effect algebra. □

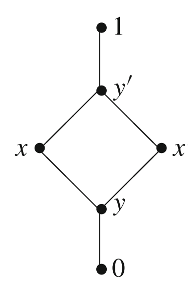
By Rump (2008, P. 2346, Example 1), every partially ordered set with the greatest element 1 can be regarded as an L-algebra. We have already known that every basic algebra $(B, \oplus, \neg, 0)$ is an IBL such that 1 is the greatest element of B, so it can be regarded as an L-algebra. But we are focused on the L-algebra with 0 and relation given by (10) such that it is an involutive bounded lattice, where $x' = x \rightarrow 0$.

In conclusion, we get an interesting relationship diagram as follows:



An involutive bounded lattice which is neither a basic algebra nor an L-algebra (relation given by (10) such that it is an involutive bounded lattice) is given in the following.

Example 3 Let us consider the involutive bounded lattice G_6.



G_6

Assume that G_6 can be converted into an L-algebra with 0 such that $x' := x \rightarrow 0$. By (11), $x' \leq x \rightarrow y$, $y \leq y' \rightarrow x$, then $x \rightarrow y = x'$ or $y'(x \geq y, x \rightarrow y \neq 1)$ and the possible values of $y' \rightarrow x$ are y, x, x', y'.

By (L2) and (L1),

$$(x \to y) \to x' = (y \to x) \to (y \to 0) = 1 \to y' = y' \tag{16}$$

and

$$x' = 1 \to x' = (x \to y') \to x' = (y' \to x) \to y. \tag{17}$$

If $x \to y = x'$, then $1 = x' \to x' = y'$ by (16), a contradiction. Thus, $x \to y = y'$ which implies that $y' \to x' = y'$ by (16).

There are only four possible values of $y' \to x$: $y,\ x,\ x',\ y'$.

- (i) If $y' \to x = y$, then $y \to y = x'$ by (17). However, $y \to y = 1$. Hence, $y' \to x \neq y$.
- (ii) Assume $y' \to x = x$, then $x \to y = x'$ by (17), which contradicts $x \to y = y'$.
- (iii) If $y' \to x = x'$, then $x' \to y = x'$ by (17). Since $x \leq x' \to y$, then $x \leq x' \to y = x'$. However, x is uncomparable with x', and then, $y' \to x \neq x'$.
- (iv) Assume $y' \to x = y'$, then $y' \to y = x'$ by (17). Nevertheless, $x = 1 \to x = (x' \to y') \to x = (y' \to x') \to y = y' \to y = x'$, which is a contradiction.

The above shows that no matter how we define $\to$ on G_6, it cannot be converted into an L-algebra (the induced partial ordering binary relation by (10) is an involutive bounded lattice).

We will verify that G_6 can also not be a basic algebra in the following.

Assume G_6 can be converted into a basic algebra with operation $\oplus$ such that $x' = \neg x$. By Lemma 1,

$$x = x \vee y = \neg(\neg x \oplus y) \oplus y. \tag{18}$$

Since $y \leq \neg x \oplus y$, $y \leq x$ and $y \oplus \neg y = 1$, then the possible values of $\neg x \oplus y$ are $x,\ \neg x,\ \neg y$.

By Lemma 1, we can obtain

$$\neg x = \neg x \vee y = \neg(x \oplus y) \oplus y \tag{19}$$

and

$$\neg y = \neg y \vee y = \neg(y \oplus y) \oplus y. \tag{20}$$

Thus, we get the possible values of $x \oplus y$ and $y \oplus y$ which are also $x,\ \neg x,\ \neg y$.

We will divide into three cases to discuss the values of $\neg x \oplus y$.

- (i) If $\neg x \oplus y = \neg x$, then $x \oplus y = x$ by (18). Since $y \leq x$, then $y \oplus y \leq x \oplus y = x$ by (6). Then, $y \oplus y = x$. By (20), $\neg x \oplus y = \neg y \neq \neg x$, a contradiction.
- (ii) If $\neg x \oplus y = x$ and $x \oplus y = x$, then $\neg x \oplus y = \neg x$ by (19). This contradicts the assumption. If $x \oplus y = \neg x$, since $y \oplus y \leq x \oplus y = \neg x$, then $y \oplus y = \neg x$. Thus by (20), $x \oplus y = \neg y \neq \neg x$. So $x \oplus y = \neg y$, which implies $y \oplus y = \neg x$. But $x = \neg x \oplus y \geq y \oplus y = \neg x$, which is impossible.
- (iii) If $\neg x \oplus y = \neg y$, then $y \oplus y = x$ by (18). Suppose that $x \oplus y = x$, then $\neg x \oplus y = \neg x \neq \neg y$ by (19). If $x \oplus y = \neg y$, then $y \oplus y = \neg x \neq x$. So $x \oplus y = \neg x$. However, $\neg x = x \oplus y \geq y \oplus y = x$, which is absurd.

None of the above cases is satisfied, which means G_6 can also not be considered as a basic algebra.

4 A lattice-ordered effect algebra with different basic algebra structures

In this section, we construct a lattice-ordered effect algebra with two different basic algebra structures and improve (Chajda et al. 2009, P. 80,Example 5.3) which stated as follows:

Let us consider the lattice from Fig. 1 with the antitone involution on the section $[b,\ 1]$ defined by $b^b = 1,\ (\neg b)^b = \neg b,\ (\neg a)^b = \neg a,\ 1^b = b$.

An easy inspection shows that the derived basic algebra $\mathcal{A} = (A,\ \oplus,\ \neg,\ 0)$ is not a lattice-ordered effect algebra [because it does not fulfill (21)], where $A = \{0,\ a,\ b,\ \neg a,\ \neg b,\ 1\}$ and the addition $\oplus$ is given in Table 4:

$$x \leq \neg y \ \ and \ \ x \oplus y \leq \neg z \Rightarrow x \oplus (z \oplus y) = (x \oplus y) \oplus z. \tag{21}$$

It is easily seen that when $x = 0,\ y = b$ and $z = a$, $x \oplus (z \oplus y) = 0 \oplus (a \oplus b) = a \oplus b = \neg a \neq \neg b = b \oplus a = (0 \oplus b) \oplus a = (x \oplus y) \oplus z$. Hence, A does not fulfill (21).

However, using the same $(A,\ \oplus,\ \neg,\ 0)$ as in Fig. 1 and Table 4, we consider

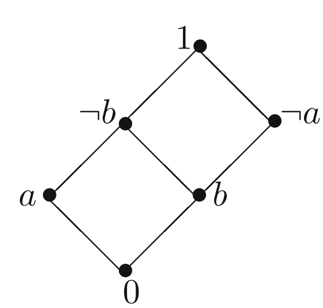


Fig. 1 Lattice of Example 5.3 in Chajda et al. (2009)

Table 4 $\oplus$ of Example 5.3 in Chajda et al. (2009)

$\oplus$	0	a	b	$\neg b$	$\neg a$	1
0	0	a	b	$\neg b$	$\neg a$	1
a	a	a	$\neg a$	$\neg b$	1	1
b	b	$\neg b$	$\neg b$	1	$\neg a$	1
$\neg b$	$\neg b$	$\neg b$	1	1	1	1
$\neg a$	$\neg a$	1	$\neg b$	1	$\neg a$	1
1	1	1	1	1	1	1

Table 5 $\oplus$ of Example 4

$+$	0	a	b	b'	a'	1
0	0	a	b	b'	a'	1
a	a	–	b'	–	1	–
b	b	b'	a'	1	–	–
b'	b'	–	1	–	–	–
a'	a'	1	–	–	–	–
1	1	–	–	–	–	–

Table 6 $\oplus^*$ of Remark 1

$\oplus^*$	0	a	b	$\neg b$	$\neg a$	1
0	0	a	b	$\neg b$	$\neg a$	1
a	a	a	$\neg b$	$\neg b$	1	1
b	b	$\neg b$	$\neg a$	1	$\neg a$	1
$\neg b$	$\neg b$	$\neg b$	1	1	1	1
$\neg a$	$\neg a$	1	$\neg a$	1	$\neg a$	1
1	1	1	1	1	1	1

Example 4 The basic algebra $\mathcal{A} = (A, \oplus, \neg, 0)$ can be converted into a lattice-ordered effect algebra $(\{0, a, b, a', b', 1\}, +, ', 0)$ whose operation is given in Table 5. If $x + y$ is undefined for $x, y \in \{0, a, b, a', b', 1\}$, we denote it by "–."

Remark 1 In Chajda et al. (2009) [P. 75, Prop. 4.5], lattice-ordered effect algebras can be viewed as basic algebras. We can obtain the derived basic algebra of the lattice-ordered effect algebra $(A, +)$ from Example 4.

Define $x \oplus^* y := (x \wedge y') \oplus y$ and $\neg x := x'$. Then, $\mathcal{A}^* = (A^*, \oplus^*, \neg, 0)$ is a basic algebra with $\oplus^*$ given in Table 6.

Hence, we obtain two different basic algebra structures whose operations are given in Tables 4 and 6 from the same lattice-ordered algebra from Example 4.

Compliance with ethical standards

Conflicts of interest The authors declare that they have no conflict of interest.

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