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METHODOLOGIES AND APPLICATION

Option pricing formulas for uncertain exponential Ornstein–Uhlenbeck model with dividends

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Abstract

Uncertain finance is an application to the finance of the uncertainty theory which provides an alternative analysis method from the probability theory under the circumstance that few samples are available. Under the research paradigm of uncertain finance, some financial assets are investigated and modeled with various tools in order to describe the price fluctuation accurately. This paper models the prices of stocks under uncertain finance based on the exponential Ornstein–Uhlenbeck model with periodic dividends. Through the α-path method, the pricing formulas are derived for European call and put options whose underlying assets follow the proposed stock model. In addition, some numerical algorithms to calculate the option prices are designed.

Keywords Uncertainty theory · Uncertain finance · Dividend · Exponential Ornstein–Uhlenbeck model

1 Introduction

An option, as one of the most well-known derivatives, gives its owner the right to trade a certain quantity of underlying assets at a certain price on or before a given date. The call option gives the right of buying the assets, while the put option gives the right of selling. As a result, the value of an option largely depends on the prices of the underlying assets. Many researchers have been devoted to exploring the inner correlation between the option price and the asset price, especially the stock price. In 1900, the Brownian motion was firstly used in modeling the fluctuation of stock price. In order to insure the positive value of stock, the geometric Brownian motion was put forward to describe the change of stock price over time instead of the Brownian motion. Then, Black and Scholes (1973) as well as Merton (1973) constructed the Black–Scholes formula to price the European options.

Although the Black–Scholes formula is generally accepted and applied in estimating the inner-value of derivatives, its rationality relies largely on the estimation of some parameters like the price diffusion and yield rate of the underlying asset which ask for the sufficiency of historical data. But actually in many practical situations, there are not enough historical data available, which means it is unreliable to insist on considering the pricing problems within the framework of the probability theory. An alternative solution is to take some experts' advice to get a reasonable prediction of the underlying assets. However, from the view of Bayesianism, the belief degree cannot be simply dealt with via probability under situations with few samples available. On account of this circumstance, the uncertainty theory was founded by Liu (2007) and refined by Liu (2009) based on normality, duality, subadditivity and product axioms. Nowadays, it has become a branch of axiomatic mathematics to model human's uncertain behavior. In order to describe uncertain variables, Liu (2007) employed the definitions of uncertainty distribution, expected value and variance. Liu and Ha (2010) and Yao (2015) presented some formulas to calculate the expected value and variance of an uncertain variable. Sheng and Kar (2015) generalized these results to the moments of uncertain variables.

The uncertain process, as a sequence of uncertain variables, was presented to put the uncertainty theory into the area of dynamic analysis. Similar to the Brownian motion, the Liu process which is a stationary and independent increment process with normal increments was designed to found the theories of uncertain calculus and uncertain differential

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equation. The existence and uniqueness of the solution of an uncertain differential equation was verified by Chen and Liu (2010) under the linear growth condition and Lipschitz condition, and the stability of the equation was investigated by Yao et al. (2013), Tao and Zhu (2016), Yang et al. (2017) and Liu and Zhang (2020). Most importantly, Yao and Chen (2013) put forward the Yao–Chen formula, which reveals the characteristic of the solution of an uncertain differential equation by ordinary differential equations and constructs the base of many numerical algorithms to solve the pricing problems in uncertain finance.

Based on an assumption that the stock price follows a geometric Liu process, Liu (2009) firstly introduced the uncertainty theory into finance and proposed an uncertain stock model. Then, Chen (2011) and Sun and Chen (2015) derived the pricing formulas for American options and Asian options, respectively. However, Liu's stock model is so simplified that it neglects many characteristics during the fluctuation of stock price, which inspired Dai et al. (2017) to construct a stock model with the reversion behavior by means of an uncertain counterpart of the exponential Ornstein–Uhlenbeck process. There are some other works about uncertain stock models and their derivatives such as Yu and Ning (2019), Gao et al. (2019), Lu et al. (2019), Yang et al. (2019), Zhang et al. (2019) and Wang and Chen (2020). However, all these models fail to take the dividends payments, a benefit to stock owners, into consideration. Theoretically, dividends payment may exert an influence on the yield rate or diffusion of a stock because it reflects some information about the listed company. According to Howatt et al. (2009) and Hussainey et al. (2016), there is a clearly positive relationship between dividend yield and stock price. Hence, the neglect of dividends in these models will lead to an inaccurate estimation of the stock price.

Chen et al. (2013) developed Liu's stock model by taking periodic dividends into consideration. However, the modified model remains failed to describe the stock price in long run, as the mean-reversion behavior is still neglected. For this reason, we aim to present and investigate a new stock model in this paper which considers both the periodic dividends and the reversion property of the stock price. The rest of the paper is organized as follows. In the next section, some preliminaries about uncertain differential equations and uncertain stock models are introduced. Then, the uncertain exponential Ornstein–Uhlenbeck model with periodic dividends is presented in Sect. 3, whose European call and put option pricing formulas are derived in Sects. 4 and 5, respectively. Numerical experiments are also performed in these two sections. Finally, some conclusions are made in Sect. 6.

2 Preliminary

The uncertainty theory is a branch of axiomatic mathematics and has been put into practice for modeling people's belief degrees. In this section, we introduce some concepts about uncertain differential equations and stock models.

2.1 Uncertain differential equation

Definition 1 (Liu 2007, 2009) Let $\mathcal{L}$ be a σ-algebra on a non-empty set Γ. A set function $\mathcal{M}$: $\mathcal{L} \to [0, 1]$ is called an uncertain measure if it satisfies the following axioms:
Axiom 1: (normality axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.
Axiom 2: (duality axiom) $\mathcal{M}\{\Lambda\}+\mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.
Axiom 3: (subadditivity axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

Axiom 4: (product axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\},$$

where

$$\prod_{k=1}^{\infty} \Lambda_k = \{(\gamma_1, \gamma_2, \ldots) \mid \gamma_k \in \Lambda_k \text{ for } k = 1, 2, \ldots\}.$$

An uncertain variable ξ is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers. The uncertainty distribution $\Phi(x)$ of ξ is defined by $\Phi(x) = \mathcal{M}\{\xi \le x\}$ for any real number x, and its inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution. For example, a normal uncertain variable $\xi \sim \mathcal{N}(e, \sigma)$ has an uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi\,(e - x)}{\sqrt{3}\sigma}\right)\right)^{-1}$$

and an inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = e + \frac{\sqrt{3}\sigma}{\pi} \ln \frac{\alpha}{1 - \alpha}.$$

Theorem 1 (Liu 2007, 2010) *The expected value of an uncertain variable ξ is*

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq x\}\mathrm{d}x - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq x\}\mathrm{d}x = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha,$$

provided that at least one of the two integrals exists and is finite.

In order to model the evolution of uncertain phenomena, the concept of uncertain process is proposed as a sequence of uncertain variables indexed by the time.

Definition 2 (Liu 2009) An uncertain process C_t is called a canonical Liu process if

(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,
(ii) C_t has stationary and independent increments,
(iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with an uncertainty distribution

$$\Phi_t(x) = \left(1 + \exp\left(\frac{-\pi x}{\sqrt{3}t}\right)\right)^{-1}.$$

Let $f(t, x)$ and $g(t, x)$ be two real functions. Then, given an initial value X_0, the differential equation

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t$$

is called an uncertain differential equation. Its α-path $(0 < \alpha < 1)$ is a deterministic function X_t^α with respect to t that solves the corresponding ordinary differential equation

$$\mathrm{d}X_t^\alpha = f(t, X_t^\alpha)\mathrm{d}t + |g(t, X_t^\alpha)|\Phi^{-1}(\alpha)\mathrm{d}t, \quad X_0^\alpha = X_0,$$

where

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}, \qquad 0 < \alpha < 1$$

is the inverse uncertainty distribution of a standard normal uncertain variable.

Theorem 2 (Yao-Chen formula, Yao and Chen 2013) *Assume that X_t and X_t^α are the solution and α-path of the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t$$

with a crisp initial value. Then, we have

$$\mathcal{M}\{X_t \leq X_t^\alpha, \forall t\} = \alpha, \qquad \mathcal{M}\{X_t > X_t^\alpha, \forall t\} = 1 - \alpha.$$

Furthermore, the solution X_t has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = X_t^\alpha.$$

2.2 Uncertain stock models

The uncertainty theory was firstly applied by Liu (2009) to simulating stock price. Denoting the stock price with X_t and the bond price with Y_t, Liu's stock model is formulated as

$$\begin{cases} \mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}C_t \\ \mathrm{d}Y_t = rY_t \mathrm{d}t, \end{cases} \tag{1}$$

where μ is the stock drift, σ is the stock diffusion, C_t is a canonical Liu process and r is the risk-free interest rate. Later on, Dai et al. (2017) presented an uncertain counterpart of the exponential Ornstein–Uhlenbeck stock model as follows:

$$\begin{cases} \mathrm{d}X_t = \mu(1 - c\ln X_t)X_t\mathrm{d}t + \sigma X_t \mathrm{d}C_t \\ \mathrm{d}Y_t = rY_t \mathrm{d}t, \end{cases} \tag{2}$$

where μ, c, σ and r are some positive constants. This nonlinear stock model is more rational in long run as it describes the reversion behavior of stock price, that is, the stock price fluctuates around some value.

3 Exponential Ornstein–Uhlenbeck model with dividends

Based on the assumption in the model of Dai et al. (2017), that is, the stock price follows an uncertain exponential Ornstein–Uhlenbeck process, we construct a new model with the consideration of periodic dividends as follows:

$$\begin{cases} \mathrm{d}X_t = \mu_i(1 - c_i \ln X_t)X_t \mathrm{d}t \\ \quad +\sigma_i X_t \mathrm{d}C_t, \quad t_i \leq t < t_{i+1}, i = 0, 1, 2, 3, \ldots \\ X_{t_{i+1}} = (1 - \delta) \lim_{t \to t_{i+1}^-} X_t, \quad i = 0, 1, 2, 3, \ldots \\ \mathrm{d}Y_t = rY_t \mathrm{d}t, \end{cases} \tag{3}$$

in which t_i is the time to pay the ith dividend, δ is the proportion of the dividend, μ_i, c_i and σ_i are some parameters related to the stock price after the ith dividend and r is the risk-free interest rate.

Now, we discuss the solution and the α-path of the uncertain exponential Ornstein–Uhlenbeck model (3) with dividends.

Theorem 3 *Suppose that the stock price X_t follows the model*

$$\begin{cases} \mathrm{d}X_t = \mu_i(1 - c_i \ln X_t)X_t\mathrm{d}t + \sigma_i X_t\mathrm{d}C_t, & t_i \le t < t_{i+1} \\ X_{t_{i+1}} = (1-\delta)\lim_{t\to t_{i+1}^-} X_t \end{cases} \tag{4}$$

with an initial price X_0, in which $t_1, t_2, t_3, \ldots$ are the times to pay dividends. Then,

$$\begin{cases} X_t = \exp\left(\dfrac{1}{c_i} + \exp\left(-\mu_i c_i (t-t_i)\right)\left(\ln X_{t_i} - \dfrac{1}{c_i}\right)\right. \\ \left. \quad +\sigma_i \int_{t_i}^t \exp\left(\mu_i c_i s - \mu_i c_i t\right)\mathrm{d}C_s\right), \ t_i \le t < t_{i+1} \\ X_{t_{i+1}} = (1-\delta)\lim_{t\to t_{i+1}^-} X_t. \end{cases} \tag{5}$$

Proof From the stock model (4), we have

$$\frac{\mathrm{d}X_t}{X_t} = \mu_i(1 - c_i \ln X_t)\mathrm{d}t + \sigma_i\mathrm{d}C_t, \quad t_i \le t < t_{i+1},$$

which means

$$\mathrm{d}\ln X_t = \mu_i(1 - c_i \ln X_t)\mathrm{d}t + \sigma_i\mathrm{d}C_t, \quad t_i \le t < t_{i+1}.$$

Substituting $\ln X_t$ with Z_t, we get the uncertain differential equation

$$\mathrm{d}Z_t = \mu_i(1 - c_i Z_t)\mathrm{d}t + \sigma_i\mathrm{d}C_t, \quad t_i \le t < t_{i+1}$$

whose solution is

$$Z_t = \frac{1}{c_i} + \exp\left(-\mu_i c_i(t-t_i)\right)\left(Z_{t_i} - \frac{1}{c_i}\right) + \sigma_i \int_{t_i}^t \exp\left(\mu_i c_i s - \mu_i c_i t\right)\mathrm{d}C_s, \quad t_i \le t < t_{i+1}.$$

As $X_t = \exp(Z_t)$, we have

$$X_t = \exp\left(\frac{1}{c_i} + \exp\left(-\mu_i c_i (t-t_i)\right)\left(\ln X_{t_i} - \frac{1}{c_i}\right) + \sigma_i \int_{t_i}^t \exp\left(\mu_i c_i s - \mu_i c_i t\right)\mathrm{d}C_s\right), \ t_i \le t < t_{i+1}.$$

The theorem is proved. □

Theorem 4 *Suppose that the stock price X_t follows the model (4). Then, the α-path of X_t is*

$$\begin{cases} X_t^\alpha = \exp\left(\ln X_{t_i}^\alpha \cdot \exp\left(-\mu_i c_i (t-t_i)\right)\right. \\ \left. \quad +\gamma_i \left(1 - \exp\left(-\mu_i c_i (t-t_i)\right)\right)\right), \ t_i \le t < t_{i+1} \ , \\ X_{t_{i+1}}^\alpha = (1-\delta)\lim_{t\to t_{i+1}^-} X_t^\alpha \end{cases} \tag{6}$$

where

$$\gamma_i = \left(\frac{1}{c_i} + \frac{\sigma_i}{\mu_i c_i} \cdot \frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right).$$

Proof From the stock model (4), we have

$$\mathrm{d}X_t^\alpha = \mu_i(1 - c_i \ln X_t^\alpha)X_t^\alpha\mathrm{d}t + \sigma_i X_t^\alpha \Phi^{-1}(\alpha)\mathrm{d}t, \quad t_i \le t < t_{i+1},$$

which means

$$\frac{\mathrm{d}X_t^\alpha}{X_t^\alpha} = \mu_i(1 - c_i \ln X_t^\alpha)\mathrm{d}t + \sigma_i \Phi^{-1}(\alpha)\mathrm{d}t,$$

where

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}.$$

Substituting $\ln X_t^\alpha$ into Z_t^α, we have

$$\mathrm{d}Z_t^\alpha = \mu_i(1 - c_i Z_t^\alpha)\mathrm{d}t + \sigma_i \Phi^{-1}(\alpha)\,\mathrm{d}t, \quad t_i \le t < t_{i+1},$$

whose solution is

$$Z_t^\alpha = Z_{t_i}^\alpha \cdot \exp\left(-\mu_i c_i (t-t_i)\right) + \gamma_i \left(1 - \exp\left(-\mu_i c_i (t-t_i)\right)\right), \ t_i \le t < t_{i+1}$$

with

$$\gamma_i = \left(\frac{1}{c_i} + \frac{\sigma_i}{\mu_i c_i} \cdot \frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right).$$

As $X_t^\alpha = \exp(Z_t^\alpha)$, we have

$$X_t^\alpha = \exp\left(\ln X_{t_i}^\alpha \cdot \exp\left(-\mu_i c_i (t-t_i)\right) + \gamma_i \left(1 - \exp\left(-\mu_i c_i (t-t_i)\right)\right)\right), \ t_i \le t < t_{i+1}.$$

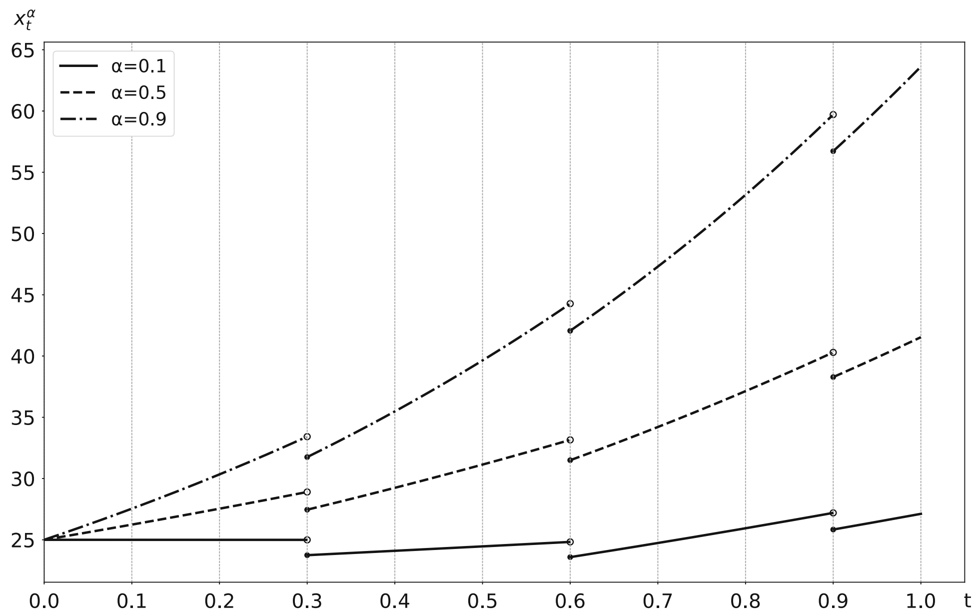
Obviously,

$$X_{t_{i+1}}^\alpha = (1-\delta)\lim_{t\to t_{i+1}^-} X_t^\alpha.$$

The theorem is proved. □

Example 1 Assume the initial value of the stock price is $X_0 = 25$, and in the next periods, the dividends will be paid by 5 % for 3 times at $t_1 = 0.3$, $t_2 = 0.6$, $t_3 = 0.9$. The other parameters of this stock are $\mu_0 = 0.5$, $\mu_1 = 0.7$, $\mu_2 = 1$, $\mu_3 = 1.1$, $c_0 = 0.01$, $c_1 = 0.03$, $c_2 = 0.05$, $c_3 = 0.07$, and $\sigma_0 = 0.4$, $\sigma_1 = 0.4$, $\sigma_2 = 0.3$, $\sigma_3 = 0.3$. Then three α-paths of the possible stock price with $\alpha_1 = 0.1$, $\alpha_2 = 0.5$ and $\alpha_3 = 0.9$, respectively, are shown in Fig. 1.

Fig. 1 Three α-paths of X_t



4 European call option pricing formula

A European call option is a kind of financial derivatives that can give its buyer the right rather than the obligation to buy a certain kind of financial assets for a specified price at a specified time. Consider a European call option with its underlying stock following

$$\begin{cases} dX_t = \mu_i(1 - c_i \ln X_t)X_t dt + \sigma_i X_t dC_t, & t_i \le t < t_{i+1} \\ X_{t_{i+1}} = (1-\delta) \lim_{t \to t_{i+1}^-} X_t. \end{cases} \quad (7)$$

Suppose the strike price and the expiration time of this option are K and T, respectively. Then, its expected payoff at the expiration time is $E\left[(X_T - K)^+\right]$. Assume the risk-free interest rate remains r during this period. Then, based on the assumption of continuous compounding, the price of a European call option is the present value of its expected payoff, i.e.,

$$C = \exp(-rT)\, E\left[(X_T - K)^+\right]. \quad (8)$$

Theorem 5 (European call option pricing formula) *Suppose a European call option with an underlying stock following model (7) has a strike price K and an expiration time T. Then, the pricing formula for this option is*

$$C = \exp(-rT)\int_0^1 (X_T^\alpha - K)^+ d\alpha, \quad (9)$$

where

$$X_T^\alpha = \exp\left(\ln X_{t_i}^\alpha \cdot \exp\left(-\mu_i c_i\,(T - t_i)\right) + \gamma_i\left(1 - \exp\left(-\mu_i c_i\,(T - t_i)\right)\right)\right), \ \text{if } t_i \le T < t_{i+1}$$

with

$$\gamma_i = \left(\frac{1}{c_i} + \frac{\sigma_i}{\mu_i c_i} \cdot \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right)$$

and $X_{t_i}^\alpha$ is determined by Eq. (6).

Proof Since X_t is an uncertain process with α-path represented by Eq. (6), the inverse uncertainty distribution of X_T is X_T^α. So, the payoff of this option has an inverse uncertainty distribution $(X_T^\alpha - K)^+$. Hence,

$$E\left[(X_T - K)^+\right] = \int_0^1 (X_T^\alpha - K)^+ d\alpha.$$

It follows from Eq. (8) that

$$C = \exp(-rT)\int_0^1 (X_T^\alpha - K)^+ d\alpha.$$

The pricing formula of European call option is verified. □

According to Theorem 5, the algorithm to calculate the European call option price of stock model (3) is designed as follows:

Step 0 Choose a large number N according to the desired precision degree. Set $\alpha_k = k/N$, $k = 1, 2, \ldots, N-1$.

Step 1 Set $k = 0$.

Step 2 Set $k \leftarrow k + 1$.

Step 3 Find the maximum m such that $t_m \le T$, and calculate the stock price at dividend payment times $t_1, t_2, \ldots, t_m$

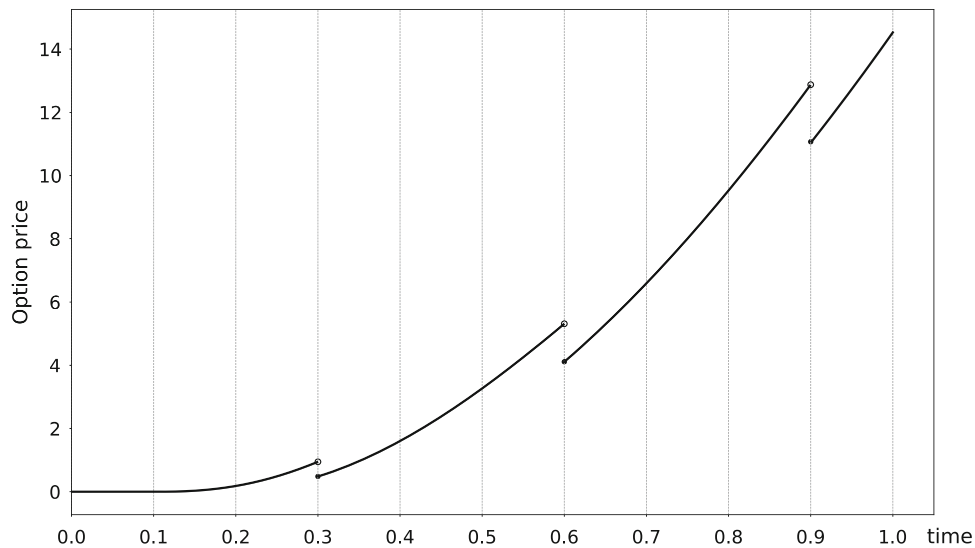


Fig. 2 European call option price with respect to expiration time T in Example 2

successively via

$$\begin{cases} X_{t_1}^{\alpha_k} = \exp\left(\ln X_0 \cdot \exp\left(-\mu_0 c_0 t_1\right) \right. \\ \qquad \left. +\gamma_0^{\alpha_k}\left(1-\exp\left(-\mu_0 c_0 t_1\right)\right)\right) \cdot (1-\delta) \\ X_{t_2}^{\alpha_k} = \exp\left(\ln X_{t_1}^{\alpha_k} \cdot \exp\left(-\mu_1 c_1 \Delta t_2\right)\right. \\ \qquad \left. +\gamma_1^{\alpha_k}\left(1-\exp\left(-\mu_1 c_1 \Delta t_2\right)\right)\right) \cdot (1-\delta) \\ \ldots \\ X_{t_m}^{\alpha_k} = \exp\left(\ln X_{t_{m-1}}^{\alpha_k} \cdot \exp\left(-\mu_{m-1} c_{m-1} \Delta t_m\right)\right. \\ \qquad \left. +\gamma_{m-1}^{\alpha_k}\left(1-\exp\left(-\mu_{m-1} c_{m-1} \Delta t_m\right)\right)\right) \cdot (1-\delta), \end{cases}$$

in which

$$\Delta t_i = t_i - t_{i-1}$$

and

$$\gamma_i^{\alpha_k} = \left(\frac{1}{c_i} + \frac{\sigma_i}{\mu_i c_i} \cdot \frac{\sqrt{3}}{\pi} \ln \frac{\alpha_k}{1-\alpha_k}\right).$$

If $T = t_m$, then go to Step 5 .

Step 4 Calculate the stock price at time T, that is,

$$X_T^{\alpha_k} = \exp\left(\ln X_{t_m}^{\alpha_k} \cdot \exp\left(-\mu_m c_m \left(T-t_m\right)\right)\right. \\ \left. +\gamma_m\left(1-\exp\left(-\mu_m c_m \left(T-t_m\right)\right)\right)\right),$$

where

$$\gamma_m^{\alpha_k} = \left(\frac{1}{c_m} + \frac{\sigma_m}{\mu_m c_m} \cdot \frac{\sqrt{3}}{\pi} \ln \frac{\alpha_k}{1-\alpha_k}\right).$$

Step 5 Calculate the payoff of the European call option

$$(X_T^{\alpha_k} - K)^+ = \max(0, X_T^{\alpha_k} - K).$$

If $k < N-1$, then return to Step 2.

Step 6 Calculate the price of the European call option

$$C \leftarrow \frac{1}{N-1}\exp(-rT)\sum_{i=1}^{N-1}(X_T^{\alpha_k} - K)^+.$$

Example 2 Assume a European call option has an underlying stock following model (7), and the parameters are the same as Example 1. The strike price is $K = 30$ and the risk-free interest rate is $r = 0.03$. Then, the price of this European call option is $C = 14.53$ if the expiration time is $T = 1$. Figure 2 shows the price of this European call option with respect to the expiration time from $T = 0$ to 1, and it is obvious that the option price C is an increasing function with respect to the expiration time T during the period between two successive dividend payment times.

5 European put option pricing formula

A European put option is a kind of financial derivatives that can give its buyer the right rather than the obligation to sell a certain kind of financial assets for a specified price at a specified time. Consider a European put option with its underlying stock following

$$\begin{cases} \mathrm{d}X_t = \mu_i(1 - c_i \ln X_t)X_t\mathrm{d}t + \sigma_i X_t \mathrm{d}C_t, \quad t_i \le t < t_{i+1} \\ X_{t_{i+1}} = (1-\delta)\lim_{t\to t_{i+1}^-} X_t. \end{cases} \qquad (10)$$

Suppose the strike price and the expiration time of a European put option are K and T, respectively. Then, its expected payoff at the expiration time is $E\left[(K - X_T)^+\right]$. Assume the risk-free interest rate remains r during this period. Then,

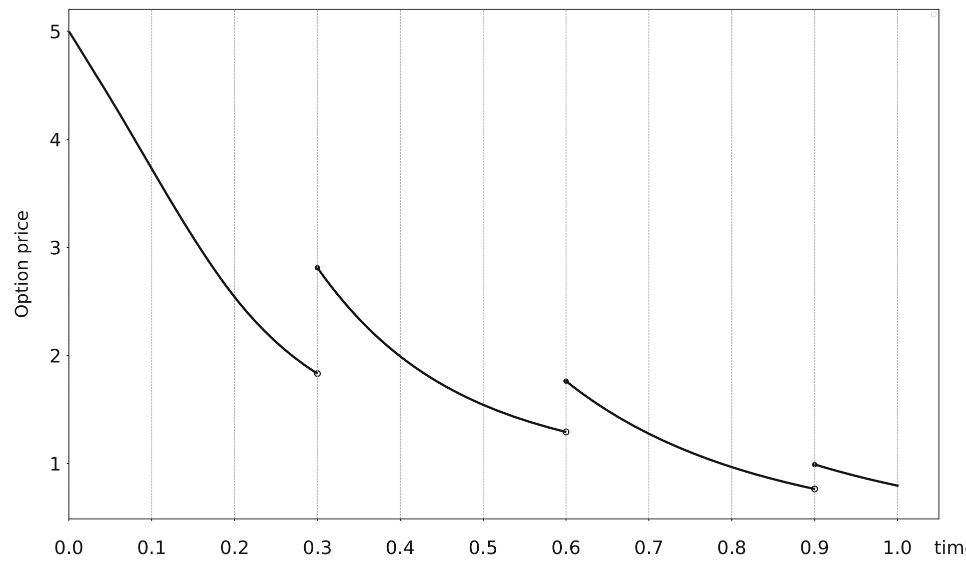


Fig. 3 European put option price with respect to time T in Example 3

based on the assumption of continuous compounding, the price of a European put option is the present value of its expected payoff, i.e.,

$$P = \exp(-rT)\, E\left[(K - X_T)^+\right]. \tag{11}$$

Theorem 6 (European put option pricing formula) *Suppose a European put option with an underlying stock following model (10) has a strike price K and an expiration time T. Then, the pricing formula for this option is*

$$P = \exp(-rT)\int_0^1 (K - X_T^\alpha)^+ \mathrm{d}\alpha, \tag{12}$$

where

$$X_T^\alpha = \exp\left(\ln X_{t_i}^\alpha \cdot \exp(-\mu_i c_i (T - t_i)) + \gamma_i (1 - \exp(-\mu_i c_i (T - t_i)))\right), \ \textit{if}\ t_i \le T < t_{i+1}$$

with

$$\gamma_i = \left(\frac{1}{c_i} + \frac{\sigma_i}{\mu_i c_i}\cdot\frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)$$

and $X_{t_i}^\alpha$ is determined by Eq. (6).

Proof Since X_t is an uncertain process with α-path represented by Eq. (6), the inverse uncertainty distribution of X_T is X_T^α. As the payoff $(K - X_T)^+$ is a decreasing function of X_T, it has an inverse uncertainty distribution $(K - X_T^{1-\alpha})^+$. Hence,

$$E[(K - X_T)^+] = \int_0^1 (K - X_T^{1-\alpha})^+\, \mathrm{d}\alpha.$$

It follows from Eq. (12) that

$$P = \exp(-rT)\int_0^1 (K - X_T^{1-\alpha})^+\, \mathrm{d}\alpha = \exp(-rT)\int_0^1 (K - X_T^{\alpha})^+ \mathrm{d}\alpha.$$

The pricing formula of European put option is verified. □

According to Theorem 6, the algorithm to calculate the European put option price of stock model (3) is designed as follows:

Step 0 Choose a large number N according to the desired precision degree. Set $\alpha_k = k/N$, $k = 1, 2, \ldots, N-1$.
Step 1 Set $k = 0$.
Step 2 Set $k \leftarrow k + 1$.
Step 3 Find the maximum m such that $t_m \le T$, and calculate the stock price at dividend payment times $t_1, t_2, \ldots, t_m$ successively via

$$\begin{cases} X_{t_1}^{\alpha_k} = \exp\left(\ln X_0 \cdot \exp(-\mu_0 c_0 t_1) + \gamma_0^{\alpha_k}(1 - \exp(-\mu_0 c_0 t_1))\right)\cdot(1-\delta) \\ X_{t_2}^{\alpha_k} = \exp\left(\ln X_{t_1}^{\alpha_k} \cdot \exp(-\mu_1 c_1 \Delta t_2) + \gamma_1^{\alpha_k}(1 - \exp(-\mu_1 c_1 \Delta t_2))\right)\cdot(1-\delta) \\ \ldots \\ X_{t_m}^{\alpha_k} = \exp\left(\ln X_{t_{m-1}}^{1-\alpha_k} \cdot \exp(-\mu_{m-1} c_{m-1} \Delta t_m) + \gamma_{m-1}^{\alpha_k}(1 - \exp(-\mu_{m-1} c_{m-1} \Delta t_m))\right)\cdot(1-\delta), \end{cases}$$

in which

$$\Delta t_i = t_i - t_{i-1}$$

and

$$\gamma_i^{\alpha_k} = \left(\frac{1}{c_i} + \frac{\sigma_i}{\mu_i c_i} \cdot \frac{\sqrt{3}}{\pi} \ln \frac{\alpha_k}{1-\alpha_k} \right).$$

If $T = t_m$, then go to Step 5 .
Step 4 Calculate the stock price at time T

$$X_T^{\alpha_k} = \exp\left(\ln X_{t_m}^{\alpha_k} \cdot \exp\left(-\mu_m c_m \left(T - t_m\right)\right) + \gamma_m \left(1 - \exp\left(-\mu_m c_m \left(T - t_m\right)\right)\right)\right),$$

where

$$\gamma_m^{\alpha_k} = \left(\frac{1}{c_m} + \frac{\sigma_m}{\mu_m c_m} \cdot \frac{\sqrt{3}}{\pi} \ln \frac{\alpha_k}{1-\alpha_k} \right).$$

Step 5 Calculate the payoff of the European put option

$$(K - X_T^{\alpha_k})^+ = \max(0, K - X_T^{\alpha_k}).$$

If $k < N - 1$, then return to Step 2.
Step 6 Calculate the price of the European put option

$$P \leftarrow \frac{1}{N-1} \exp(-rT) \sum_{i=1}^{N-1} (K - X_T^{\alpha_k})^+.$$

Example 3 Assume the European put option has an underlying stock following model (10), and the parameters are the same as Example 1. The strike price is $K = 30$ and the risk-free interest rate is $r = 0.03$. Then, the price of this European put option is $P = 0.79$ if the expiration time is $T = 1$. Figure 3 shows the price of this European put option price with respect to the expiration time from $T = 0$ to 1, and it is obvious that the option price P is a decreasing function with respect to the expiration time T during the period between two successive dividend payment times.

6 Conclusions

In this paper, we modeled the stock price using the exponential Ornstein–Uhlenbeck process under uncertain financial market and took dividends payment, a general stock behavior, into consideration. Based on the stock model, the pricing formulas of European call options and European put options were derived. In addition, some algorithms to calculate the option prices numerically were designed by using the α-path method, and the relationships between the option price and the expiration time were also discussed based on the numerical experiments. Further research may consider the American and Asian option pricing formulas of the presented stock model. Besides, how to price various types of options for the exponential Ornstein–Uhlenbeck stock models with dividends and floating interest rate is also an interesting and challenging problem.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

References

Black F, Scholes M (1973) The pricing of options and corporate liabilities. J Polit Econ 81(3):637–654
Chen X (2011) American option pricing formula for uncertain financial market. Int J Oper Res 8(2):32–37
Chen X, Liu B (2010) Existence and uniqueness theorem for uncertain differential equations. Fuzzy Optim Decis Mak 9(1):69–81
Chen X, Liu Y, Ralescu DA (2013) Uncertain stock model with periodic dividends. Fuzzy Optim Decis Mak 12(1):111–123
Dai L, Fu Z, Huang Z (2017) Option pricing formulas for uncertain financial market based on the exponential Ornstein–Uhlenbeck model. J Intell Manuf 28(3):597–604
Gao R, Liu K, Li Z, Lv R (2019) American barrier option pricing formulas for stock model in uncertain environment. IEEE Access 7:97846–97856
Howatt B, Zuber RA, Gandar JM, Lamb RP (2009) Dividends, earnings volatility and information. Appl Financ Econ 19(7):551–562
Hussainey K, Oscar Mgbame C, Chijoke-Mgbame AM (2016) Dividend policy and share price volatility: UK evidence. J Risk Finance 12(1):57–68
Liu B (2007) Uncertainty theory, 2nd edn. Springer, Berlin
Liu B (2009) Some research problems in uncertainty theory. J Uncertain Syst 3(1):3–10
Liu B (2010) Uncertainty theory: a branch of mathematics for modeling human uncertainty. Springer, Berlin
Liu Y, Ha M (2010) Expected value of function of uncertain variables. J Uncertain Syst 4(3):181–186
Liu J, Zhang Y (2020) The stability analysis for uncertain heat equations based on p-th moment. Soft Comput 24(4):2833–2839
Lu Z, Yan H, Zhu Y (2019) European option pricing model based on uncertain fractional differential equation. Fuzzy Optim Decis Mak 18(2):199–217
Merton R (1973) Theory of rational option pricing. Bell J Econ Manag Sci 4(1):141–183
Sheng Y, Kar S (2015) Some results of moments of uncertain variable through inverse uncertainty distribution. Fuzzy Optim Decis Mak 14(1):57–76
Sun J, Chen X (2015) Asian option pricing formula for uncertain financial market. J Uncertain Anal Appl 3(1), Article 11
Tao N, Zhu Y (2016) Stability and attractivity in optimistic value for dynamical systems with uncertainty. Int J Gen Syst 45(4):418–433

Tian M, Yang XF, Kar S (2019) Equity warrants pricing problem of mean-reverting model in uncertain environment. Phys A Stat Mech Appl 531, Article 121593

Wang W, Chen P (2020) Valuation of stock loan under uncertain stock model with floating interest rate. Soft Comput 24(3):1803–1814

Yang X, Ni Y, Zhang Y (2017) Stability in inverse distribution for uncertain differential equations. J Intell Fuzzy Syst 32:2051–2059

Yang X, Zhang Z, Gao X (2019) Asian-barrier option pricing formulas of uncertain financial market. Chaos Solitons Fractals 123:79–86

Yao K (2015) A formula to calculate the variance of uncertain variable. Soft Comput 19(10):2947–2953

Yao K, Chen X (2013) A numerical method for solving uncertain differential equations. J Intell Fuzzy Syst 25(3):825–832

Yao K, Gao J, Gao Y (2013) Some stability theorems of uncertain differential equation. Fuzzy Optim Decis Mak 12(1):3–13

Yu S, Ning Y (2019) An interest-rate model with jumps for uncertain financial markets. Phys A Stat Mech Appl 527, Article 121424

Zhang Z, Ke H, Liu W (2019) Lookback options pricing for uncertain financial market. Soft Comput 23(14):5537–5546