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A constructive sequence algebra for the calculus of indications

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Abstract

In this paper, we investigate some aspects of Spencer–Brown's Calculus of Indications. Drawing from earlier work by Kauffman and Varela, we present a new categorical framework that allows to characterize the construction of infinite arithmetic expressions as sequences taking values in grossone.

Introduction

The Calculus of Indications (Spencer-Brown 1969) provides an elegant formalism to represent fundamental logical operations as nested "distinctions" notated as closed curves in a plane.[1] An especially interesting phenomenon in the Calculus of Indications (CI) is what Spencer Brown calls *the reentry of the form*, understood as the recursive nesting of curves yielding a structure formally equivalent to the Epimenides paradox (or the Liar Paradox (Barwise and Etchemendy 1989)]. As in the classical setting, there are several different attempts made at resolving the paradoxical character of this self-referential or "reentered" form. One, proposed first by Spencer Brown himself and advanced more rigorously in Kauffman and Varela (1980), basically defines an alternative truth value, the "imaginary" value which is neither true nor false. The other, introduced in Varela and Goguen (1978), takes the limit of the nesting process to yield a definite expression that is then taken as a particular solution to a given reentry equation, rather than a distinct general truth value.

We will take an intermediate route between these two approaches, combining the best of the two methods in a category-theoretical setting. We will show first how in the case of expressions consisting of finite nestings, a recursive procedure yields a unique answer as the formal simplification or reduction of the given expression. However, in order to extend our procedure then to the case of infinite nestings, we will have to develop the notion of constructive sequences that build up such expressions part by part. In evaluating such sequences, we will need to deal with infinite sums that, in general, diverge. A finer control on such divergence is needed in order to make the process of evaluating sequences well-defined, and a solution to this problem is offered by the notion of *grossone*, first defined by Yaroslav Sergeyev, with respect to which the natural numbers (including zero) can be represented by the sequence $\mathbb{N} = \{0, 1, 2, 3, \ldots, ① - 2, ① - 1, ①\}$.[2]

In the approach detailed below, incorporating calculations with ①allows for the evaluation function applied to both finite and infinite graphs to yield a distinguished, computable answer, solving the question of how to give a truth value to the reentry of the form with respect to constructive sequences. We therefore aim to capitalize on the basic idea that grossone allows for a "finer" control on infinite denumerable sequences than the classic Cantorian approach (for details, see Sergeyev's survey paper 2017 and, for a more informal treatment, Sergeyev (2013a)). Within the grossone framework, it becomes feasible to deal computationally with infinite quantities, in a way that is both new (in the sense that

[1] This is akin to Peirce's α graphs (Zeman 1974; Gangle et al. 2020), although with a basic but immaterial difference between the representations of *True* and *False* in the two formalisms.

[2] The introduction of the numeral ①may require, in some contexts, the augmentation of the set of natural $\mathbb{N}$ to a larger set $\hat{\mathbb{N}}$, where

$$\hat{\mathbb{N}} = \{0, 1, 2 \ldots, ① - 1, \\ ①, ① + 1, ① + 2, \ldots, 2 \cdot ①, \ldots, ①^2 - 1, ①^2, ①^2 + 1, \ldots\}.$$

For the purpose of this paper we will not need to make use of $\hat{\mathbb{N}}$.

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previously intractable problems become amenable to computation) and natural.[3]

The CI is generated by two simple algebraic rules or "initials" applicable to diagrams called "arithmetic expressions" consisting of collections of non-intersecting closed curves in the Euclidean plane (Fig. 1).

The lack of expression on the right side of initial 2 is merely apparent. For CI, the empty plane (or any empty part of the plane) is also a formal expression (or part of an expression) in the calculus. It is this use of natural topological properties of closed curves in the plane that makes CI particularly interesting as a diagrammatic formal system, as recognized for instance by Louis Kauffman in Kauffman (2012).

Using the two initials as permissions for rewriting equivalent expressions, it becomes possible to simplify such expressions step by step. For instance, the sequence of expressions below represents an application of initial 2, an application of initial 1 and then another application of initial 2 (Fig. 2).

In developing the system of CI, Spencer Brown first considers arithmetic expressions composed of finite collections of closed curves. He calls the system consisting of all such expressions as related algebraically via the two initials schematized above the Primary Arithmetic. He then introduces variables that may take such expressions as values and calls the system of expanded expressions the Primary Algebra, demonstrating standard results of consistency and completeness for the resulting system which is equivalent to that of Classical Propositional Logic. We will be especially concerned in the present paper with the introduction of infinite expressions (infinite collections of non-intersecting closed curves in the Euclidean plane) into the system, which are described in Spencer-Brown (1969) in terms of equations of second degree and the distinctive notion of "reentry of the form".

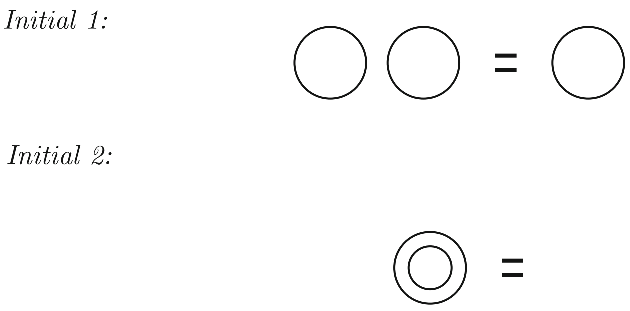


Fig. 1 Algebraic rules

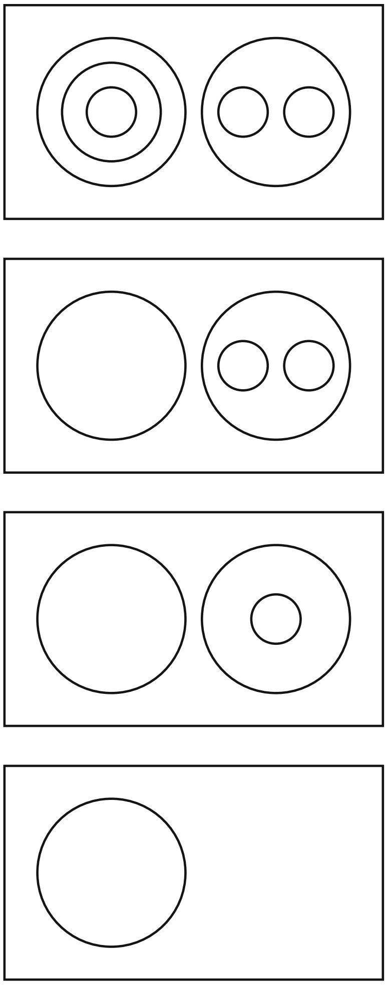


Fig. 2 Reduction process

It may be remarked that the system of CI's Primary Algebra is quite close in spirit to the earlier logical system of Existential Graphs (EG) developed by C.S. Peirce. In particular, the expressions of Spencer Brown's CI Primary Algebra are formally equivalent to those of Peirce's EG-α, which was also designed to express Classical Propositional Logic. However, whereas Peirce elects to consider the empty plane (which he calls the "Sheet of Assertion") to express the value TRUE, Spencer Brown takes the empty sheet to express the value FALSE. In a corresponding manner, a single empty curve or "cut" represents the value FALSE for Peirce and the value TRUE for Spencer Brown. In accordance with this dif-

[3] Evidence of the efficacy of the grossone approach is highlighted by its successful application to many fields of applied mathematics, including optimization (see Cococcioni et al. 2018, 2020; De Cosmis and De Leone 2012; Sergeyev et al. 2018; Iavernaro et al. 2020; Iudin et al. 2012; Sergeyev 2007, 2013b; Zhigljavsky 2012), fractals and cellular automata (see Caldarola 2018; D'Alotto 2012, 2015, 2013) as well as infinite decision-making processes, game theory, and probability (see Fiaschi and Cococcioni 2018; Rizza 2018, 2019), whereas the formal logical foundation of grossone has been investigated in Lolli (2012); Margenstern (2011); Montagna et al. (2015). The approach presented in this paper bears similarities to the application of grossone to Turing machines as described in Sergeyev and Garro (2010). Further investigation of these analogies remains open.

ference, for Peirce the juxtaposition of two or more notational elements (either "cuts" or variables) expresses their conjunction, while for Spencer Brown such a juxtaposition represents their disjunction. For both systems, a curve or "cut" negates the value of its interior. The expressions of the two systems are thus syntactically equivalent but interpreted inversely at the semantic level.[4]

In what follows, we will not further examine the standard algebraic treatment of the Calculus of Indications in which variables in expressions of the Primary Algebra may take basic arithmetic expressions as values (by substituting such expressions diagrammatically for the variables themselves, thus reducing expressions in the Primary Algebra to expressions in the Primary Arithmetic). Instead, we will develop an alternate algebra of *constructive sequences* of arithmetic expressions themselves. This algebra of constructive sequences concerns the extension, proposed by Spencer Brown himself but not fully developed in Spencer-Brown (1969), of expressions in the Primary Arithmetic to include infinite collections of closed curves in the plane. Following the lead of Peirce, we will refer to the closed curves in all such expressions in what follows as "cuts".

One of the most interesting features of CI is the idea initially proposed in Spencer-Brown (1969) that solutions for equations such as the following may be expressed as infinite recursive structures.

Consider, for instance, the following "equation" (Fig. 3).

Clearly, neither the substitution of the empty sheet nor the substitution of the empty cut for A (nor any finite expression, since all such expressions reduce to either the empty sheet or the empty cut) will satisfy the above equation. The following expression, however, does formally satisfy it, where the filled innermost circle is meant to represent an infinite sequence of further nested circles (Fig. 4).

Fig. 3 Recursive structure

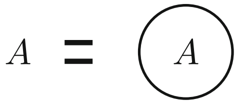
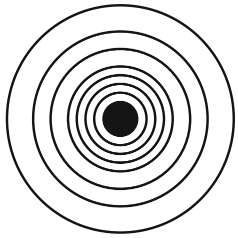


Fig. 4 Infinite nesting



In order to simplify the notation, in what follows we will denote the above expression (an infinite series of single nested cuts) with the symbol I.

The expression above may itself be constructed by a recursive procedure that in effect operationalizes the original equation 1. Informally, I is generated by treating the right side of equation 1 as an operation that draws a circle around A. Starting from A = empty sheet, one then successively plugs in the result of the operation back into the equation and repeats the operation indefinitely. The original equation thus becomes a recipe for generating an expression that is a solution to the equation itself.[5]

This is why Spencer Brown named such recursive constructions "reentry of the form". His own interpretation of this notation was in terms of temporal signals. The solution to the equation above would be conceived as an infinite nesting of circles, and on this interpretation its value would be understood as an *oscillation* between TRUE and FALSE.

The notion of "reentry of the form" has been analyzed and evaluated by various specialists including mathematicians, logicians, and social scientists. Especially noteworthy is the use of CI and reentry as the basis for a comprehensive theory of social communicative systems in Luhmann (1995) and its related research program.

From a more formal point of view, two dominant approaches to reentry in CI may be identified. The first, represented by Kauffman and Varela (1980) and Zalamea (2010) works from the intuition first proposed by Spencer Brown that the infinitely-nested expression that stands as a solution to equation 1 may be understood on analogy with the imaginary value i in complex algebra. Spencer Brown thus calls the value of the CI-expression I the "imaginary value". Somewhat surprisingly, this informal analogy may be made rigorous, as shown in Kauffman and Varela (1980). More recently, Zalamea (2010) has developed this approach (though with primary reference to Peirce's EG rather than Spencer Brown's CI) with respect to holomorphic functions. The second approach is represented by Varela and Goguen (1978). According to this approach, infinite expressions are conceived on analogy with limits of infinite sequences. If the former approach follows the mathematics of complex algebra, this latter approach follows instead that of differential calculus. Our own approach detailed in what follows represents something of a hybrid model between the two previous approaches.[6]

[4] The two systems are slightly different in terms of their rules of deduction (or rewriting). We do not examine this point any further in this paper.

[5] The relevance of *grossone* to this construction will appear in the subsequent sections.

[6] Our approach is similar to that of Varela and Goguen (1978) in that it involves limits of sequences. But by calculating values of the sequences themselves, and not only their limits, the (complex algebra) waveform interpretation of Kauffman and Varela (1980) is partially maintained as well.

1 The category $\mathcal{CI}$ of CI-expressions

In this section, we develop a formalization of the Calculus of Indications in the setting of category theory, in particular, by using categories of presheaves. This approach follows that introduced in Caterina and Gangle (2016) and Caterina and Gangle (2013). The elementary notions of category theory are summarized here for the reader's convenience. For further details, the reader is directed to Mac Lane (1998).

A *category* consists of a class of *objects* together with *morphisms* or arrows between objects. Given two objects a and b a morphism f between them will be denoted either $f : a \to b$ or $a \xrightarrow{f} b$. A category is subject to axioms of *identity* (every object a is equipped with an identity morphism, $a \xrightarrow{1_a} a$), *composition* (two morphisms, $a \xrightarrow{f} b$ and $b \xrightarrow{g} c$ compose to a unique morphism $a \xrightarrow{g\circ f} c$, where $\circ$ indicates the operation of composition) and *associativity* (paths of morphisms compose uniquely, i.e., given three arrows $f : a \to b$, $g : b \to c$ and $h : c \to d$, $h\circ(g\circ f) = (h\circ g)\circ f$, given the same morphism from a to d).

A functor F is a map between two categories (say $\mathcal{A}$ and $\mathcal{B}$), sending objects to objects and morphisms to morphisms in a way that preserves identities and composition. If for any morphism $a \to a'$ in $\mathcal{A}$, $F(a \to a')$ is mapped to a morphism $F(a) \to F(a')$ in $\mathcal{B}$, F is said to be *covariant*. If instead $F(a \to a')$ maps to $F(a) \leftarrow F(a')$, F is called *contravariant*.

With these core notions in hand, we may now define the notion of an expression in *CI* as a contravariant functor from the category of natural numbers to that of finite sets. More precisely:

Definition 1 A CI-expression is a functor

$$F : \mathbb{N}^{op} \longrightarrow \mathbf{FinSet}$$

where $\mathbb{N}^{op}$ is the category of natural numbers with the usual order reversed[7] and **FinSet** is the category whose objects are finite sets and whose morphisms are functions between them. We call a CI-expression *finite*[8] if and only if there exists some $n \in \mathbb{N}$ such that $F(n) = \emptyset$, otherwise we call it *infinite*.

The collection of all CI-expressions so defined forms a category, with natural transformations between CI-expressions as morphisms. Such a category is typically denoted by $\mathbf{FinSet}^{\mathbb{N}^{op}}$. For the sake of notational simplicity, we label this category $\mathcal{CI}$. Thus, $\mathcal{CI} = \mathbf{FinSet}^{\mathbb{N}^{op}}$.

It is a standard result that, since $\mathcal{CI}$ is a category of presheaves (with the domain category being small), it has the structure of a *topos*. This means, in particular, that it can be endowed with a logical structure. In particular, it possesses a *subobject classifier* that functions in the topos in a manner analogous to the set $\{0, 1\}$ in the category **Set** (also a topos) of sets and functions. Just as morphisms (functions) from any set S into $\{0, 1\}$ correspond to subsets of S, morphisms from any object O in a general topos into the subobject classifier of the topos correspond to monic arrows into O.[9]

Using the notion of monic arrows, it is straightforward to define a preorder $\preceq$ on the objects of $\mathcal{CI}$. Given two such objects O_1 and O_2, $O_1 \preceq O_2$ iff there is a monic arrow $m : O_1 \longrightarrow O_2$.

In the case of the category $\mathcal{CI}$, monic arrows select "parts" of CI-expressions in the sense that they select arbitrary cuts in the target expression subject to the condition that if they select a cut c, they also select all cuts C such that c is inside C. This is illustrated by the following example (Fig. 5).

Given the following two CI-expressions

monic morphisms from the former to the latter correspond to selections of parts of the latter. There are exactly three such morphisms, represented here by thickened lines (Fig. 6).

The mathematical setting of category theory, and in particular presheaves, allows for a rigorous formalization of the somewhat informal approach found in Varela and Goguen (1978).

In any topos of presheaves, each object in the base category gives rise to a canonical presheaf associated with that object called a *representable set*. For details, see Reyes et al. (2004) and Gangle et al. (2020). In the topos $\mathcal{CI}$, we have a *representable set* h_n for every $n \in \mathbb{N}$, which consists of n nested circles.

Morphisms from representable sets h_n into a CI-expression F correspond uniquely to cuts in F at depth n. We may thereby identify cuts $p, q, r \ldots$ in any CI-expression with morphisms $p : h_{n^p} \longrightarrow F, q : h_{n^q} \longrightarrow F, r : h_{n^r} \longrightarrow F \ldots$, where h_{n^x} represents a representable set with as many nested cuts as the depth of x.

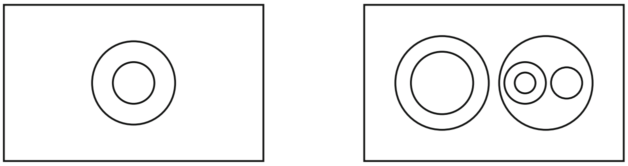


Fig. 5 CI-expressions

[7] Here the objects are the natural numbers, and there is a morphism between two natural numbers $m \to n$ if and only if $m \geq n$. The choice of reversing the natural order is made in order for the functor F to be contravariant.

[8] The condition given here guarantees that all nested cuts are of at most depth $n - 1$.

[9] A *monic arrow* in any category is defined as follows. Given any morphism $f : A \longrightarrow B$, f is said to be *monic* if, for any two morphisms $g, h : Z \longrightarrow A$ such that $f \circ g = f \circ h$, it is the case that $g = h$.

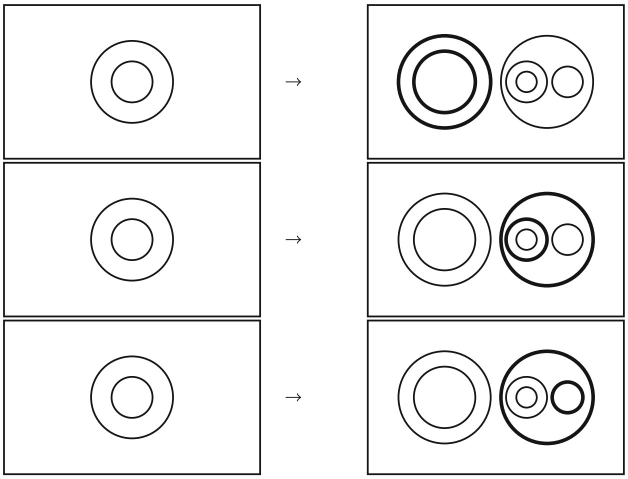


Fig. 6 List of morphisms

Thus with representable sets, we have a straightforward way to characterize formally when one cut is inside another and also when a cut is empty. Given a CI-expression F, a cut q at depth n in F is inside a cut p at depth m in F if and only if the following diagram is well-defined and commutes (Fig. 7).

A cut is empty if no cuts are inside it. Using this framework, it becomes possible to formally define various diagrammatic operations on CI-expressions. We define a pair of such operations, a unary operation CIRCLE and a binary operation JUXTAPOSE:

Definition 2
- *CIRCLE*: given a CI-expression F it generates a new CI-expression $C(F)$ such that $C(F)(1) = \{\emptyset\}$ and for all $n \in \mathbb{N}, C(F)(n+1) = F(n)$ and $C(F)(i_{n+1}) : C(F)(n+2) \longrightarrow C(F)(n+1) = F(i_n) : F(n+1) \longrightarrow F(n)$, where morphisms i_n are the (unique) morphisms from n to $n+1$.
- *JUXTAPOSE*: given two CI-expressions F and G it generates a new CI-expression $J(F, G)$, consisting of the categorical coproduct $F \coprod G$ of F and G.[10]

Remark 1 In any category C the *coproduct* of two objects A and A', denoted by $A \coprod A'$ is an object endowed with two arrows $p_1 : A \coprod A' \leftarrow A$ and $p_2 : A \coprod A' \leftarrow A'$ such that, for all objects T with two arrows f and g respectively from A and A' into T, there exists a unique morphism $!h : T \leftarrow A \coprod A'$ that makes the diagram in Fig. 8 commute.

Fig. 7 Cut inclusion

$$\begin{tikzcd}
h_m \arrow[r, "m<n"] \arrow[dr, "p"'] & h_n \arrow[d, "q"] \\
& F
\end{tikzcd}$$

Fig. 8 Coproduct

$$\begin{tikzcd}
& A \arrow[dl, "f"'] \arrow[d, "p_1"] \\
T & A \coprod A' \arrow[l, "!h"] \\
& A' \arrow[ul, "g"] \arrow[u, "p_2"']
\end{tikzcd}$$

Diagrammatically, CIRCLE generates a new CI-expression by drawing a circle (more generally, a closed curve) around a given CI-expression. Similarly, the operation JUXTAPOSE applied to two CI-expressions generates the CI-expression consisting of both expressions drawn next to one another on the same plane.

2 Evaluating the objects of $\mathcal{CI}$

Each finite object in $\mathcal{CI}$ takes a unique value according the following algorithmic evaluation procedure.

Definition 3 Given a finite CI-expression $F \in \mathrm{Obj}(\mathcal{CI})$, a recursive evaluation of F, denoted by $RE(F)$ is an element of $\{0, 1\}$ generated by a finite sequence $f_1, f_2, \ldots, f_n$ of subobjects of F such that $f_1 = F$ and each step from f_i to f_{i+1} corresponds to an application of the "pruning procedure" algorithm detailed below. Since that algorithm ensures that each of its operations generates a proper subobject of its operand, the sequence $f_1, f_2, \ldots, f_n$ is ordered $f_1 \succ f_2 \succ \cdots \succ f_n$ Additionally, since the elements in $RE(F)$ are ordered by inclusion, they may be thought of as a path in the lattice $Sub(F)$.

We define a *successor* of a cut n in a CI-expression F as a cut that appears immediately inside n in F. Note that successors are not necessarily unique. An *empty cut* is a cut that has no successors. An *eliminable cut* is a cut with at least one successor that is empty.

Given a finite CI-expression F, we reduce it to one of the two values 0 or 1 by applying the following iterative pruning procedure:

Procedure 1

(i) *Consider the diagrammatic representation of F as a collection of nested cuts.*
(ii) *Take any eliminable cut n in F.*
(iii) *Erase the cut n and all cuts inside it, yielding a new CI-expression F'.*

[10] $\coprod$ is well defined in $\mathcal{CI}$ since it is a topos.

(iv) *Replace F with F' and repeat (ii) and (iii) until the CI-expression F' is either blank (no cuts at all) or consists of empty cuts only.*

(v) *If the final CI-expression F' is blank, assign the original CI-expression F the value* 1. *Otherwise, assign F the value* 0.[11]

Every time steps (ii) and (iii) of the procedure are applied, a choice must be made among all the eliminable cuts that exist at that stage. But any variant sequence of such choices leads nonetheless invariably to the same final result:

Theorem 1 *Given a finite expression F, the algorithm outlined in Procedure 1 will output either the empty sheet or a finite number of empty cuts, independently of the choices made in step (ii), that is, the order in which the eliminable cuts are chosen.*

Proof We first label all the nodes of F either 0 or 1 according to the following rule:

- All cuts are labeled 0 if they have at least one successor labeled 1 and they are labeled 1 otherwise.

It should be clear that this labeling is well-defined and uniquely determined for any finite expression. We note first that prior to applying the pruning procedure every empty cut of the expression is necessarily labeled 1 since these (by definition) have no successors. We then note that repeated applications of steps (ii) and (iii) in the pruning procedure above will always select and delete a cut labeled 0. It is impossible for a node labeled 1 to have any successors also labeled 1 according to the labeling rule. Since only an eliminable cut can be chosen and deleted in steps (ii) and (iii), no such cut can ever be labeled 1. Thus every cut selected in step (ii) must be labeled 0.

Eventually, *all* cuts labeled 0 must be deleted. This will result in a final expression that has at most depth 1. □

3 Evaluating infinite objects in $\mathcal{CI}$

In this section, we turn our attention to infinite objects in the Calculus of Indications, and following the lead of Varela and Goguen (1978) we consider them from the point of view of limits of sequences of finite expressions that "construct" or "build" them step by step. In the first subsection, the informal presentation in Varela and Goguen (1978) is given a rigorous mathematical setting in the framework of presheaves developed above. We then show how every constructive sequence gives rise to a unique value in the grossone framework. Finally, we define an algebra on the sequences that behaves nicely with respect to valuations in the interval [0, ①].

3.1 Constructive sequences of CI-expressions

Definition 4 A constructive sequence for a CI-expression F is an infinite sequence of finite CI-expressions, $f_1, f_2, f_3, \ldots$ such that for all i $f_i \preceq f_{i+1} \preceq F$ and for any m and any morphism $x : h_m \to F$ there is $n \in \mathbb{N}$ such that the following diagram commutes. (The representable set h_m factors through f_n into F) (Fig. 9).

We designate constructive sequences for CI-expressions $F, G, \ldots$ with Greek letters and capital Roman subscripts $\alpha_F, \beta_F, \ldots, \alpha_G, \beta_G, \ldots,$ dropping the subscripts if the given CI-expression is either obvious from the context or else left indeterminate.

Intuitively, a constructive sequence may be thought of as an *increasingly determined* CI-expression that in the limit converges on F. This is the guiding intuition behind the approach to CI found in Varela and Goguen (1978) which the present paper develops somewhat more rigorously. Each step in the sequence may be thought of as the inscribing of CI-expressions on areas of the CI-expression formed at the previous step. Constructive sequences are thus formal characterizations of the informal notion of "growing" or "deepening" CI-expressions.

Definition 5 An s-evaluation $sval(\alpha)$ for a constructive sequence α is an infinite binary sequence, that is, a function $\mathbb{N} \longrightarrow \{0, 1\}$, such that $sval(\alpha)(n) = RE(\alpha_n)$.

Such a sequence is obviously constructible, given any constructive sequence α, since the calculation of $RE(\alpha_n)$ for every n is itself a finite procedure.

Definition 6 Given $k \in \mathbb{N}$ a k-sum $S_k(\alpha)$ for a constructive sequence α is

$$\sum_{n=1}^{k} sval(\alpha)(n)$$

It may be remarked that given any constructive sequence α_F for any finite CI-expression F such that $RE(F) = 1$, the

$$\begin{tikzcd}
h_m \arrow[r] \arrow[dr, "x"'] & f_n \arrow[d, "f_n \preceq F"] \\
 & F
\end{tikzcd}$$

Fig. 9 Constructive sequences

[11] A formal categorical characterization of a variant of this procedure, applied to cuts-only α graphs can found in Gangle et al. (2020).

limit of the k-sum as k approaches infinity will be infinite. That is,

$$\lim_{x\to\infty} S_k(\alpha_F) = \infty$$

Conversely, given any constructive sequence β_G for any finite CI-expression G such that $RE(G) = 0$, the limit of the k-sum as k approaches infinity will be finite or zero. That is,

$$\lim_{x\to\infty} S_k(\beta_G) = n$$

for some $n \in \mathbb{N}$.

In order to normalize k-sums for different values of k, it is natural to define a k-value in the following way:

Definition 7 A k-value $V_k(\alpha)$ for a constructive sequence α and some $k \in \mathbb{N}$ is $\frac{S_k(\alpha)}{k}$.

We now define the following pair of operations on constructive sequences, a unary operation CIRC and a binary operation JUXTA:

1. CIRC: Given a sequence α, the sequence CIRC(α) or $\textcircled{\alpha}$ is the sequence that draws a cut around each expression in the sequence. Formally, $\textcircled{\alpha}$ = CIRCLE(f_1), …, CIRCLE(f_n), …[12]
2. JUXTA: Given two sequences $\alpha = f_1, f_2, \ldots$ and $\beta = g_1, g_2, \ldots$, the sequence JUXTA(α, β) or $\alpha\beta$ is the sequence that juxtaposes α_n and β_n for all $n \in \mathbb{N}$. Formally, $\alpha\beta = J(f_1, g_1), J(f_2, g_2), \ldots$. It should be obvious that JUXTA, like the JUXTAPOSE operation J above, is a commutative operation.

We have the following facts for all $k \in \mathbb{N}$:

1. $S_k\,(\textcircled{\alpha}) = k - S_k(\alpha)$
2. $S_k(\alpha\beta) = \sum_{n=1}^{k} min(sval(\alpha), sval(\beta)) \leq S_k(\alpha) + S_k(\beta)$

Our first result is to show that it is possible to extend these results to the entire sequence taken as a whole by using grossone.

A ①-sum $S_{①}(\alpha)$ for a sequence α is

$$\sum_{n=1}^{①} sval(\alpha)(n)$$

Similarly, using the arithmetic properties of grossone, a ①-value $V_{①}(\alpha)$ for a constructive sequence α is $\frac{S_{①}(\alpha)}{①}$.

In accord with the facts above, the following facts hold:

1. $S_{①}(\textcircled{\alpha}) = ① - S_{①}(\alpha)$
2. $S_{①}(\alpha\beta) \leq S_{①}(\alpha) + S_{①}(\beta)$

In addition, we have that:

3. For any finite expression F and constructive sequence α_F, $V_{①}(\alpha_F)$ is either finite or ①-n for some finite n.

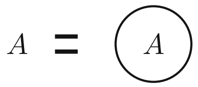
From (3) and elementary divisibility properties of grossone, it follows that any constructive sequence G such that $V_{①}(\alpha_G) = \frac{m}{①} \pm n$ with finite m and n must be the constructive sequence for an infinite expression.

The use of the grossone notation and its associated arithmetic thus allows both for the uniform extension of certain finite determinations of constructive sequences to those sequences taken as a whole (that is, taken as “actual” or “completed” infinities) and also for a new arithmetic characterization (based solely on k-values) of the difference between finite and infinite CI-expressions.

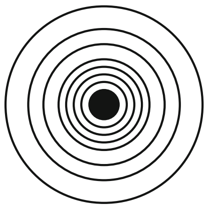
3.2 Evaluating constructive sequences via grossone

We now show how every constructive sequence F gives rise to a unique value F_v such that $0 \leq F_v \leq ①$. Several examples will illustrate how the evaluation works.

Let us first look at the most basic reentry of the form considered in Fig. 3



which gives rise to the infinite nesting expression I as in Fig. 4 (as in Sect. 1, the solid dot stands for an infinite number of concentric circles):



The following three examples show three distinct constructive sequences, all of which construct I but each in a different way:

Example 1 This sequence includes every step of the recursive nesting, in which case each step is evaluated according to the sequence 0, 1, 0, 1, 0, …. Hence, the ①-sum is ①/2 and the ①-value is 1/2, which corresponds to Kauffman's i (Fig. 10).

[12] Note that it is important to distinguish this operator $\textcircled{\alpha}$ (as an operation on α) from the circle drawn around 1 in the grossone numeral ①.

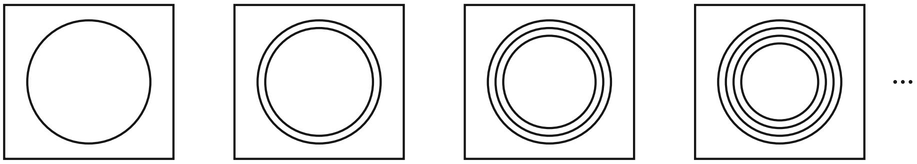


Fig. 10 All steps

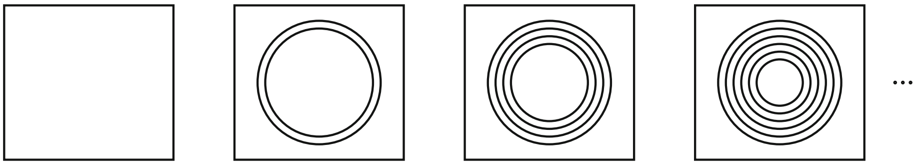


Fig. 11 Only even steps

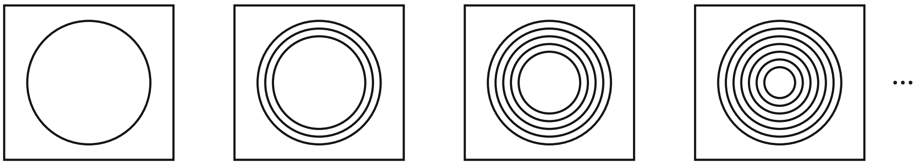


Fig. 12 Only odd steps

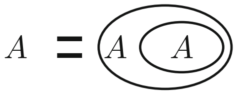


Fig. 13 Varela’s equation

Example 2 This sequence is constituted by the even steps of the recursive nesting only, each of which evaluates to 1, so that the ①-value of the entire sequence is ① (Fig. 11).

Example 3 Similarly, we can consider the sequence that only includes the odd steps of the recursive nesting, each of which evaluates to 0, hence the ①-value of the sequence is 0 (Fig. 12).

Example 4 Let us now consider a different and slightly more complex reentry of the form, via Varela’s equation (Fig. 13)

The first few steps of the recursive construction sequence are given below

Hence, since each step evaluates to 1, the ①-value of the constructive sequence that includes each recursive step is ① (Fig. 14).

4 Conclusion

It has been shown how both finite and infinite expressions in Spencer Brown’s Calculus of Indications may be straightforwardly represented in a category-theoretical setting as presheaves. We have denoted the resulting category of such presheaves (and natural transformations among them) as $\mathcal{CI}$. It was then shown how evaluations of finite expressions correspond to applications of a recursive procedure. In this setting, the concept of an infinite *constructive sequence* determining an expression as its limit was defined and used to define *s-evaluations* and *k-sums* and the latter’s normalized *k-values* for constructive sequences of finite expressions. The extension of these concepts to the case of infinite expressions was then shown to be possible by using the grossone notation and arithmetic framework.

A further, natural generalization is given by adding labels—representing atomic statements—to the Spencer–Brown, cuts-only graphs. In a recent paper Tohmé et al. (2020), we show that the grossone framework, combined with a different sets of tools from category theory, can be

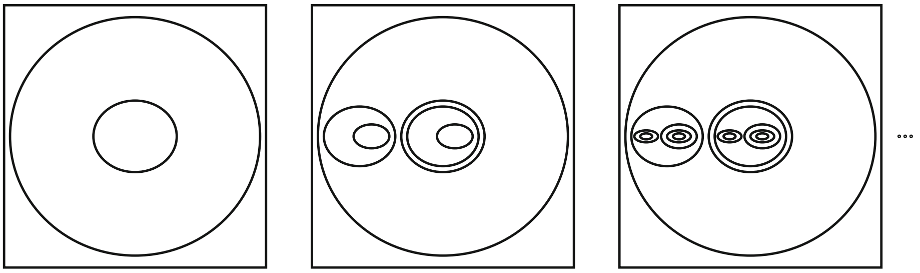


Fig. 14 First three steps of the constructive sequence

successfully employed also to deal with this more general case.

Compliance with ethical standards

Conflict of interest All authors declare that they have no conflict of interest.

Human and animal rights This article does not contain any studies with human participants or animals performed by any of the authors.

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